Math. Model. Nat. Phenom. 14 (2019) 301
https://doi.org/10.1051/mmnp/2018072

Mathematical Modelling of Natural Phenomena
www.mmnp-journal.org

MODELING AND ANALYSIS OF FRACTIONAL NEUTRAL DISTURBANCE WAVES IN ARTERIAL VESSELS

Feixue Song, Zheyuan Yu and Hongwei Yang*

Abstract. The behavior of neutral disturbance in arterial vessels has attracted more and more attention in recent decades because it carries some important information which can be applied to predict and diagnose related heart disease, such as arteriosclerosis and hypertension, etc. Because of the complexity of blood flow in arteries, it is very necessary to construct accurate mathematical model and analyze the mechanical behavior of neutral disturbance in arterial vessels. In this paper, start from the basic equations of blood flow and the two-dimensional Navier–Stokes equation, the vorticity equation describing the disturbance flow is presented. Then, by use of multi-scale analysis and perturbation expansion method, the ZK equation is put forward which can reflect the behavior of the neutral perturbation flow in arterial vessels. Compared with the traditional KdV model, the model established in the paper can show the propagation of the disturbance flow in the radius direction. Furthermore, the time-fractional ZK equation is derived by semi-inverse method and Agrawal's method, which is more convenient and accurate for discussing the feature of neutral disturbance in arterial vessels and can provide more information for analyzing some related heart disease. Meanwhile, with the help of the modified extended tanh method, the above mentioned equation is solved. The results show that neutral disturbance exists in arterial vessels and propagates in the form of solitary waves. By calculating, we find the relation of the stroke volume with vascular radius, blood flow velocity as well as the fractional order parameter α, which is very meaningful for preventing and treating related heart disease because the stroke volume is closely linked with heart disease.

Mathematics Subject Classification. 35Q92.

Received October 22, 2018. Accepted December 3, 2018.

1. Introduction

In recent years, the incidence of cardiovascular diseases has been increasing, and it has become one of the most serious diseases endangering human health. Therefore, the study of human blood flow phenomenon has important practical significance.

There are many nonlinear phenomena in nature. The blood circulation system is a very complex system. It promotes the metabolism of cells, maintains the homeostasis of the body, and ensures the progress of life activities. The nonlinear phenomenon of blood flow has been discovered early by researchers. In 1950s, Womersley

Keywords and phrases: Time-fractional ZK equation, neutral disturbance waves, arterial vessels, multi-scale analysis, modified extend tanh method.

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, PR China.

* Corresponding author: hwyang1979@163.com

[40–42] and McDonald [31, 32] systematically studied the pulsation property of arterial blood flow, established a simplified model of blood flow, and laid the mathematical foundation of bloodresearch. Since then, the dynamic characteristics of blood flow have been studied. Sigeo [36] proposed the KdV equation for blood flow. The propagation of pressure waves in blood vessels was studied by Hashizume [22]. Duan *et al*. [13–15], Demiray [9–12], Yee *et al*. [5] and Gordoa [19] took the blood flow as the research object, and studied and analyzed the nonlinear wave in the blood vessels.

When the body's physical and psychological condition changes, it will affect the flow of blood and certain functions of the heart. We can think of the above effects as a small disturbance to the heart that can spread along with the pulse wave to the distal end of the artery. In 1994, on the assumption of long wave, Le Jiachun [28] has obtained a kind of neutral disturbance of blood flow in arteries by using the theory of hydrodynamic linear stability to analysis the aortic flow. In 2006, Yi Jinqiao derived the KdV model to describe the disturbance flow. But according to the actual situation of blood flow, it is necessary to establish high-dimensional models. So, in this work, we derived the ZK model.

Fractional calculus has attracted more and more attention and has different types of definitions [25, 33, 43]. Fractional calculus has not only been developed in some basic subjects [3, 6, 17], but more importantly, it has been widely used in natural phenomena. Yang *et al*. [45] presented a non-differentiable model of the LC-electric circuit by a local fractional differential equation of fractal dimensional order, Singh *et al*. [37] studied a fractional model chemical kinetic system, Fu *et al*. [18] derived time-fractional NSL equation for describing nonlinear propagation of envelope gravity waves, Singh *et al*. [38] analyzed the El Nino-Southern Oscillation model with a new fractional derivative, Kumar *et al*. [27] investigated a fractional model of the Ambartsumian equation for describing the surface brightness of the Milky Way, Singh *et al*. [39] analyzed fractional diabetes model and so on [47, 48]. Compared with integer order model, fractional order models can better describe wave propagation and analyze various physical phenomena. However, in the process of studying the problem of blood flow, fractional models are rarely derived and established. Therefore, it is necessary to construct a fractional order model to describe the neutral disturbance waves in blood vessels.

With the development of fractional order model, the solution of fractional differential equations is an significant subject. Thus, the exact solutions and the numerical solutions of fractional differential equations are presented by researchers. For instance, iteration method [4, 24], Hirota bilinear method [21, 30], the first integral method [26], the trial function method [20, 46], the (G'/G)-expansion method [29, 35] and others [7, 8]. In this paper, we utilized the modified extended tanh method [34] to obtain the solution of the time-fractional ZK equation.

The content of this paper is as follows. In Section 2, according to the nonlinear condition of blood flow, the vorticity equation of disturbance flow is obtained from the continuity equation and the two-dimensional Navier–Stokes equation of blood flow. Then, by use of multi-scale analysis and perturbation expansion method [44], we establish the ZK model to describe the disturbance wave. In Section 3, on the basis of the integer order model andriemann-Liouville fractional order, the time-fractional ZK equation is derived by employing the semi-inverse method and Agrawal's method [1, 16, 23]. In Section 4, we get the solution of the time-fractional ZK equation by utilising the modified extended tanh method [34]. In the end, based on the solution obtained above, we analyzed and discussed stroke volume.

2. Derivation of the ZK equation

The cardiovascular system is a closed conduit system composed of heart and blood vessels. When the body's physiological function and mental state change, it will affect the blood supply and blood flow of the heart. Researches show that the effects can be seen as a small disturbance to the heart. Next, we started to analyze this disturbance.

To facilitate the study of problems, make the following assumptions:

– Blood vessels are axisymmetric straight tubes, ignoring the elastic effect of the blood vessels, but considering the viscous effect of the blood vessel wall.

– The blood is an incompressible Newtonian fluid. Since the article mainly analyzes the blood flow in the heart, the condition of weak viscosity is considered, and elastic condition is ignored.
– Based on the assumption of weak viscosity, the basic flow in the artery is the Poiseuille flow. Because the viscosity coefficient is very small, the term related to viscosity coefficient is not considered in the Navier–Stokes equation.
– The basic flow of blood is major, some disturbances of the heart are small disturbances compared to the basic flow of blood.
– The basic flow of blood can be thought of as a long wave.

On the basis of the above assumptions, in cylindrical coordinates, take the vascular axis as the coordinate axis, radial coordinates are represented by r. The continuity equation and Navier–Stokes equation [41] satisfied by the blood movement are as follows

$$\frac{\partial u}{\partial x}+\frac{1}{r}\frac{\partial (rv)}{\partial r}=0, \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.3}$$

where u, v are the axial andradial components of blood flow velocity, p is the pressure, ρ is the blood density.

According to the continuity equation, introducing stream function $\Psi(r,x,t)$ to obtain follow formula

$$u=\frac{1}{r}\frac{\partial \Psi}{\partial r},\quad v=-\frac{1}{r}\frac{\partial \Psi}{\partial x}, \tag{2.4}$$

substituting equation (2.4) into equations (2.2) and (2.3), the vorticity equation of blood flow is

$$(\frac{\partial}{\partial t}+\frac{1}{r}\frac{\partial \Psi}{\partial r}\frac{\partial}{\partial x}-\frac{1}{r}\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial r}+\frac{2}{r^2}\frac{\partial \Psi}{\partial x})D^2\Psi=0, \tag{2.5}$$

where $D^2=\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial x^2}$.

Assuming that the basic flow of blood is Poiseuille flow [2, 41]. Since the viscous effect of blood vessel wall is not ignored, the resistance of blood flow near the vessel's boundary layer increases and the velocity of blood flow decreases. So, the basic velocity of blood can be expressed as

$$\overline{u}(r)=u_{\max}\left[\frac{2r^2}{R_0^2}\ln\frac{r}{R_0}+(1-\frac{r^2}{R_0^2})\right], \tag{2.6}$$

where $u_{\max}=\frac{\Delta pR^2}{4\mu\Delta L}$ is the maximum velocity of vascular central axis. And ΔL is the length of blood fluid was studied, $\Delta p=p_2-p_1$ is the pressure difference between the two ends of the blood was studied, μ is the blood viscosity coefficient, R_0 is the radius of blood vessel.

From the equation (2.6), the boundary conditions can be derived as follows

$$\begin{cases} u(R_0)=0, \\ \lim\limits_{r\to +\ 0}\overline{u}(r)=u_{\max}. \end{cases} \tag{2.7}$$

We assume that the form of the original stream function is

$$\Psi(x,r,t)=\overline{\psi}(r)+\epsilon\psi(x,r,t), \tag{2.8}$$

where $\overline{\psi}$ is the basic flow function of the flow of blood, ψ is the disturbance flow function and used to indicate some disturbance at the arterial entrance, ϵ is the small parameter.

By the equation (2.7), equation (2.4) can be expressed as

$$u=\overline{u}+\frac{1}{r}\frac{\partial\psi}{\partial r},\quad v=-\frac{1}{r}\frac{\partial\psi}{\partial x}, \tag{2.9}$$

according to the equations (2.7) and (2.8), equation (2.5) can take the following form

$$(\frac{\partial}{\partial t}+\frac{\epsilon}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x}-\frac{\epsilon}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}+\frac{2\epsilon}{r^2}\frac{\partial\psi}{\partial x}+\overline{u}\frac{\partial}{\partial x})D^2\psi-a\frac{\partial\psi}{\partial x}=0, \tag{2.10}$$

where $a=\frac{\partial 4u_{\max}}{\partial R^2}$ is the constant, the equation (2.10) is a vorticity equation with only perturbed flow function ψ that reflects the effect of the human body physiological or psychological changes on the heart.

Under the assumption of long wave approximation, introduce the following coordinate transformation

$$X=\epsilon^{\frac{1}{2}}(x-ct),\quad \hat{\mathrm{r}}=\epsilon^{\frac{1}{2}}r,\quad T=\epsilon^{\frac{3}{2}}t, \tag{2.11}$$

substituting equation (2.11) into equation (2.10), equation (2.10) can be deformed to

$$\begin{aligned}&\left[(\overline{u}-c)\frac{\partial}{\partial X}+\epsilon\frac{\partial}{\partial T}+\frac{\epsilon^{\frac{3}{2}}}{r}\frac{\partial\psi}{\partial R}\frac{\partial}{\partial X}+\frac{\epsilon}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial X}-\frac{\epsilon^{\frac{3}{2}}}{r}\frac{\partial\psi}{\partial X}\frac{\partial}{\partial R}-\frac{\epsilon}{r}\frac{\partial\psi}{\partial X}\frac{\partial}{\partial R}+\frac{2\epsilon}{r^2}\frac{\partial\psi}{\partial X}\right]\\&\left(\epsilon\frac{\partial^2\psi}{\partial R^2}+2\epsilon^{\frac{1}{2}}\frac{\partial^2\psi}{\partial R\partial r}+\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\epsilon^{\frac{1}{2}}\frac{\partial\psi}{\partial R}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\epsilon\frac{\partial^2\psi}{\partial X^2}\right)-a\frac{\partial\psi}{\partial X}=0.\end{aligned} \tag{2.12}$$

Introduce small parameter ϵ, and the disturbance flow function is expanded

$$\psi=\psi_0+\epsilon^{\frac{1}{2}}\psi_1+\epsilon\psi_2+\epsilon^{\frac{3}{2}}\psi_3+\epsilon^2\psi_4+\cdots, \tag{2.13}$$

substituting equation (2.13) into equation (2.12), we can get

$$\epsilon^0:(\overline{u}-c)\frac{\partial}{\partial X}\left(\frac{\partial^2\psi_0}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_0}{\partial r}\right)-a\frac{\partial\psi_0}{\partial X}=0, \tag{2.14}$$

$$\epsilon^{\frac{1}{2}}:(\overline{u}-c)\frac{\partial}{\partial X}\left(\frac{\partial^2\psi_1}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right)-a\frac{\partial\psi_1}{\partial X}+(\overline{u}-c)\frac{\partial}{\partial X}\left(2\frac{\partial^2\psi_0}{\partial r\partial R}-\frac{1}{r}\frac{\partial\psi_0}{\partial R}\right)=0, \tag{2.15}$$

$$\begin{aligned}\epsilon:(\overline{u}-c)\frac{\partial}{\partial X}\left(\frac{\partial^2\psi_2}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_2}{\partial r}\right)-a\frac{\partial\psi_2}{\partial X}+(\overline{u}-c)\frac{\partial}{\partial X}\left(2\frac{\partial^1\psi_0}{\partial r\partial R}-\frac{1}{r}\frac{\partial\psi_1}{\partial R}\right)+\left(\frac{\partial}{\partial T}+\frac{1}{r}\frac{\partial\psi_0}{\partial r}\frac{\partial}{\partial X}\right.\\\left.-\frac{1}{r}\frac{\partial\psi_0}{\partial X}\frac{\partial}{\partial r}+\frac{2}{r^2}\frac{\partial\psi_0}{\partial X}\right)\left(\frac{\partial\psi_0^2}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_0}{\partial r}\right)+(\overline{u}-c)\frac{\partial}{\partial X}\left(\frac{\partial\psi_0^2}{\partial R^2}+\frac{\partial\psi_0^2}{\partial X^2}\right)=0.\end{aligned} \tag{2.16}$$

Suppose ψ_0 have the following form of separated variables

$$\psi_0 = A(X,R,T)\phi_0(r), \tag{2.17}$$

by boundary conditions $\phi_0|_{r=R_0} = 0$, $\frac{\mathrm{d}\phi_0}{\mathrm{d}r}|_{r=0} = 0$, equation (2.14) can be expressed as

$$(\overline{u}-c)\frac{\partial A}{\partial X}(\frac{\partial^2\phi_0}{\partial r^2} - \frac{1}{r}\frac{\partial\phi_0}{\partial r}) - a\frac{\partial A}{\partial X}\phi_0 = 0, \tag{2.18}$$

under normal physiological conditions, the velocity of blood flow is much less than that of blood, so $\overline{u} \ll c$. And $\frac{\partial A}{\partial X} \neq 0$, equation (2.18) can be rewritten as

$$\frac{\partial^2\phi_0}{\partial r^2} - \frac{\partial\phi_0}{\partial r} + \frac{a}{c}\phi_0 = 0. \tag{2.19}$$

Letting $r = \sqrt{\frac{c}{a}}y$ and $\phi_0(r) = yf(y)$, equation (2.20) can be shown as

$$y^2\frac{\mathrm{d}^2 f}{\mathrm{d}f^2} + y\frac{\mathrm{d}f}{y} + (y^2-1)f = 0, \tag{2.20}$$

the above formula is a first order Bessel equation, the solution is

$$f(y) = C_1 J_1(y) + C_2 N_1(y), \tag{2.21}$$

where

$$J_1(y) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+2)}(\frac{y}{2})^{2m+1}, \quad N_1(y) = \frac{J_1(y)\cos(\pi) - J_{-1}(y)}{\sin(\pi)},$$

$J_1(y)$ is the first order Bessel function, $N_1(y)$ is the first order Neumann function, C_1 and C_2 are constants. By the boundary conditions and $\lim_{y\to 0} N_1(y) = -\infty$, we might as well let $C_2 = 0$.

On the basis of the above conclusions, we can get

$$\phi_0(r) = C_1 y J_1(y). \tag{2.22}$$

Similarly, suppose ψ_1 have the following form of separated variables

$$\psi_1 = B(X,R,T)\phi_1(r), \tag{2.23}$$

according to the boundary conditions $\phi_1|_{r=R_0} = 0$, $\frac{\mathrm{d}\phi_1}{\mathrm{d}r}|_{r=0} = 0$ and equation (2.17), equation (2.15) can be expressed as

$$(\overline{u}-c)\frac{\partial B}{\partial X}\left(\frac{\partial^2\phi_1}{\partial r^2} - \frac{1}{r}\frac{\partial\phi_1}{\partial r}\right) - a\frac{\partial B}{\partial X}\phi_1 + (\overline{u}-c)\frac{\partial^2 A}{\partial r\partial R}\left(2\frac{\partial\phi_0}{\partial r} - \frac{1}{r}\phi_0\right) = 0, \tag{2.24}$$

by $\overline{u} \ll c$ and setting

$$\frac{\partial B}{\partial X} = \frac{\partial^2 A}{\partial X\partial R}, \tag{2.25}$$

then, equation (2.24) can be simplified as

$$\frac{\partial^2\phi_1}{\partial r^2}-\frac{\partial\phi_1}{\partial r}+\frac{a}{c}\phi_1=-(2\frac{\partial\phi_0}{\partial r}-\frac{1}{r}\phi_0). \tag{2.26}$$

Assuming that $\phi_0=0$, equation (2.26) can be shown as

$$\frac{\partial^2\phi_1}{\partial r^2}-\frac{\partial\phi_1}{\partial r}+\frac{a}{c}\phi_1=0, \tag{2.27}$$

based on the calculation method and process of ϕ_0, we can get

$$\phi_1(r)=C_1yJ_1(y). \tag{2.28}$$

By the equation (2.17), equation (2.23) and $\overline{u}\ll c$, equation (2.16) can be shown as

$$-c\frac{\partial A}{\partial X}\left(\frac{\partial^2\psi_2}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_2}{\partial r}\right)-a\frac{\partial A}{\partial X}\psi_2=F, \tag{2.29}$$

where

$$F=\frac{a}{c}\phi_0\frac{\partial A}{\partial T}+\left[\frac{a}{rc}\phi_0\phi_0'+\frac{\phi_0}{r}\left(\phi_0'''+\frac{a}{rc}\phi_0\right)+\frac{2\phi_0^2}{r^2c}\right]A\frac{\partial A}{\partial X}+c\left(2\phi_1'-\frac{1}{r}\phi_1+\phi_0\right)\frac{\partial^3A}{\partial X\partial R^2}+c\phi_0\frac{\partial^3A}{\partial X^3}.$$

Multiply both sides of equation (2.29) by $\frac{\phi_0}{r}$, then integrating it over r from 0 to R_0, we obtain

$$\begin{aligned}&\int_0^{R_0}\frac{\phi_0}{r}\left[-c\frac{\partial A}{\partial X}\left(\frac{\partial^2\psi_2}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_2}{\partial r}\right)-a\frac{\partial A}{\partial X}\psi_2\right]\mathrm{d}r\\&=\int_0^{R_0}\frac{\phi_0}{r}\left[\frac{a}{c}\phi_0\frac{\partial A}{\partial T}+\left[\frac{a}{rc}\phi_0\phi_0'+\frac{\phi_0}{r}\left(\phi_0'''+\frac{3a}{rc}\phi_0\right)\right]A\frac{\partial A}{\partial X}+c\left(2\phi_1'-\frac{1}{r}\phi_1+\phi_0\right)\frac{\partial^3A}{\partial X\partial R^2}+c\phi_0\frac{\partial^3A}{\partial X^3}\right]\mathrm{d}r,\end{aligned} \tag{2.30}$$

according to the boundary conditions, We get the following relation

$$\begin{aligned}&\int_0^{R_0}\frac{1}{r}\phi_0\left[-c\frac{\partial A}{\partial X}\left(\frac{\partial^2\psi_2}{\partial r^2}-\frac{1}{r}\frac{\partial\psi_2}{\partial r}\right)-a\frac{\partial A}{\partial X}\psi_2\right]\mathrm{d}r\\&\quad=-c\frac{\partial}{\partial X}\int_0^{R_0}\frac{\partial}{\partial r}\left(\frac{\phi_0}{r}\frac{\partial\psi_2}{\partial r}-\frac{1}{r}\psi_2\frac{\partial\phi_0}{\partial r}\right)\mathrm{d}r+c\frac{\partial}{\partial X}\int_0^{R_0}\frac{\psi_2}{r}\left(\frac{1}{r}\frac{\partial\phi_0}{\partial r}-\frac{\partial^2\phi_0}{\partial r^2}-\frac{a}{c}\phi_0\right)\mathrm{d}r=0,\end{aligned} \tag{2.31}$$

substituting equation (2.31) into equation (2.30), we obtain

$$\begin{aligned}&\frac{a}{c}\int_0^{R_0}\frac{\phi_0^2}{r}\mathrm{d}r\frac{\partial A}{\partial T}+\int_0^{R_0}\frac{\phi_0}{r}[\frac{a}{rc}\phi_0\phi_0'+\frac{\phi_0}{r}(\phi_0'''+\frac{3a}{rc}\phi_0)]\mathrm{d}rA\frac{\partial A}{\partial X}\\&\quad+c\int_0^{R_0}\frac{\phi_0}{r}(2\phi_1'-\frac{1}{r}\phi_1+\phi_0)\mathrm{d}r\frac{\partial^3A}{\partial X\partial R^2}+c\int_0^{R_0}\frac{\phi_0^2}{r}\mathrm{d}r\frac{\partial^3A}{\partial X^3}=0.\end{aligned} \tag{2.32}$$

According to the $r = \sqrt{\frac{c}{a}}y$ and $\phi_0(r) = yf(y)$, let I, I_1, I_2, I_3 be in the following form

$$\begin{cases} I = C_1^2 \frac{a}{c} \int_0^{\sqrt{\frac{a}{c}}R_0} yJ_1(y)\mathrm{d}y, \\ I_1 = C_1^3 \frac{a}{c}^2 \int_0^{\sqrt{\frac{a}{c}}R_0} J_1(y)(J_1(y)\frac{\mathrm{d}}{\mathrm{d}y}yJ_1(y) + J_1(y)\frac{\mathrm{d}^3}{\mathrm{d}y^3}yJ_1(y) + 3J_1^2(z)), \\ I_2 = C_1^2 \int_0^{\sqrt{\frac{a}{c}}R_0} cyJ_1(y)\mathrm{d}y, \\ I_3 = C_1^2 \int_0^{\sqrt{\frac{a}{c}}R_0} cJ_1(2\sqrt{\frac{a}{c}}\frac{\mathrm{d}}{\mathrm{d}y}yJ_1(y) - \sqrt{\frac{a}{c}}J_1(y) + yJ_1(y))\mathrm{d}y, \end{cases} \tag{2.33}$$

so, equation (2.32) can be expressed as

$$A_T + a_1 AA_X + a_2 A_{XXX} + a_3 A_{XRR} = 0, \tag{2.34}$$

where the coefficients a_1, a_2, a_3 satisfy the following form

$$a_1 = \frac{I_1}{I}, \quad a_2 = \frac{I_2}{I}, \quad a_3 = \frac{I_3}{I}. \tag{2.35}$$

Remark: The above equation is ZK equation, which is used to show the propagation of neutral disturbance flow. Different from the previous KdV model, the model can describe the propagation of the disturbance flow in the radius direction. In addition, the conclusion shows that neutral disturbance exists in arterial vessels and propagates in the form of solitary waves.

3. Derivation of time-fractional ZK equation

In Section 2, we obtained the following ZK equation

$$A_T + a_1 AA_X + a_2 A_{XXX} + a_3 A_{XRR} = 0,$$

where $[X, R] \subseteq \Omega$ is the space coordinate, $T \subseteq T^*$ is the time. Letting $B_X = A(X, R, T)$, $B(X, R, T)$ is a potential function, and the above equation can be expressed as

$$B_{XT} + a_1 B_X B_{XX} + a_2 B_{XXXX} + a_3 B_{XXRR} = 0. \tag{3.1}$$

Then, by use of the semi-inverse method [1, 23], the Lagrange form of ZK equation is obtained

$$J(B) = \iint_\Omega \mathrm{d}X\mathrm{d}R \int_{T^*} \mathrm{d}TB[c_1 B_{XT} + c_2 a_1 B_X B_{XX} + c_3 a_2 B_{XXXX} + c_4 a_3 B_{XXRR}], \tag{3.2}$$

where c_1, c_2, c_3, are the undetermined constants. By partial integral and letting $B_T|_{T^*} = B_X|_{X^*} = B_X|_{T^*} = B_X|_{R^*} = {}_{XRR}|_{X^*} = B_{XXX}|_{X^*} = 0$, equation (3.2) can be rewritten as

$$J(B) = \iint_\Omega \mathrm{d}X\mathrm{d}R \int_{T^*} \mathrm{d}T[-c_1 B_T B_X - \frac{1}{2}c_2 a_1 B_X^3 + c_3 a_2 B_{XX}^2 + c_4 a_3 B_{XR}^2], \tag{3.3}$$

according to the property of functional function, the Euler–Lagrangian equation of the above functional is

$$\begin{aligned} L(X,R,T,B,B_T,B_X,B_{XR},B_{XX}) &= -\frac{\partial}{\partial T}(\frac{\partial F}{\partial B_T}) - \frac{\partial A}{\partial X}(\frac{\partial F}{\partial B_X}) + \frac{\partial^2}{\partial X^2}(\frac{\partial F}{\partial B_{XX}}) + \frac{\partial^2}{\partial X \partial R}(\frac{\partial F}{\partial B_{XR}}) \\ &= 2c_1 B_{XT} + 3c_2 a_1 B_X B_{XX} + 2c_3 a_2 B_{XXXX} + 2c_4 a_3 B_{XXRR} = 0. \end{aligned} \tag{3.4}$$

Obviously, equations (3.4) and (3.1) are equivalent, we get

$$c_1 = \frac{1}{2},\ c_2 = \frac{1}{3},\ c_3 = \frac{1}{2},\ c_4 = \frac{1}{2}, \tag{3.5}$$

substituting equation (3.6) into equation (3.1), the Lagrange form of the ZK equation is

$$L(B_T,B_X,B_{XX},B_{XR}) = -\frac{1}{2}B_T B_X - \frac{1}{6}a_1 B_X^3 + \frac{1}{2}a_2 B_{XX}^2 + \frac{1}{2}a_3 B_{XR}^2. \tag{3.6}$$

Likewise, the Lagrangian form of the time fractional ZK equation can be written as

$$F({}_0D_T^\alpha B, B_X, B_{XX}, B_{XR}) = -\frac{1}{2}[{}_0D_T^\alpha B]B_X - \frac{1}{6}a_1 B_X^3 + \frac{1}{2}a_2 B_{XX}^2 + \frac{1}{2}a_3 B_{XR}^2, \tag{3.7}$$

where, ${}_aD_t^\alpha f(t)$ is the left Riemann–Liouville fractional derivative of function $f(t)$ [25, 33]

$${}_0D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t (t-\tau)^{n-\alpha-1}A\mathrm{d}t, \quad n-1 \le \alpha < n, t \in [a,b], \tag{3.8}$$

therefore, the functional form of time fractional ZK equation is

$$J(B) = \iint_\Omega \mathrm{d}X\mathrm{d}R \int_{T^*} \mathrm{d}T F({}_0D_T^\alpha B, B_X, B_{XX}, B_{XR}). \tag{3.9}$$

Via Agrawal's method [1, 16], equation (3.9) can become

$$\delta J(B) = \iint_\Omega \mathrm{d}X\mathrm{d}R \int_{T^*} \mathrm{d}T[\frac{\partial F}{\partial_0 D_T^\alpha B}\delta_0 D_T^\alpha B + (\frac{\partial F}{\partial B_X})\delta B_X + (\frac{\partial F}{\partial B_{XX}})\delta B_{XX} + (\frac{\partial F}{\partial B_{XR}})\delta B_{XR}]. \tag{3.10}$$

The partial integral criterion of fractional integral is

$$\int_a^b \mathrm{d}t f(t)_a D_t^\alpha g(t) = \int_a^b \mathrm{d}t g(t)_t D_b^\alpha f(t), \quad f(t), g(t) \in [a,b], \tag{3.11}$$

where, ${}_tD_b^\alpha f(t)$ is the right Riemann-Liouville fractional derivative of function $f(t)$ [25, 33]

$${}_tD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^b (\tau-t)^{n-\alpha-1} f(t)\mathrm{d}t, \quad n-1 \le \alpha < n, t \in [a,b]. \tag{3.12}$$

By equations (3.11) and (3.10), the Euler-Lagrange equation of time-fractional ZK equation is

$${}_\tau D_{T_0}^\alpha(\frac{\partial F}{\partial_0 D_T^\alpha B}) - \frac{\partial}{\partial X}(\frac{\partial F}{\partial B_X}) + \frac{\partial^2}{\partial X^2}(\frac{\partial F}{\partial B_{XX}}) + \frac{\partial^2}{\partial X \partial R}(\frac{\partial F}{\partial B_{XR}}) = 0, \tag{3.13}$$

substituting equation (3.7) into equation (3.13), we can obtain

$$-\frac{1}{2}{}_0D_T^\alpha B_X+\frac{1}{2}{}_\tau D_{T_0}^\alpha B_X+a_1B_XB_{XX}+a_2B_{XXXX}+a_3B_{XXRR}=0, \tag{3.14}$$

according to the potential function $B_X=A(X,R,T)$, equation (3.14) can shown as

$$-\frac{1}{2}{}_0D_T^\alpha A+\frac{1}{2}{}_\tau D_{T_0}^\alpha A+a_1AA_X+a_2A_{XXX}+a_3A_{XRR}=0, \tag{3.15}$$

then,

$${}_0^RD_{T_0}^\alpha A+a_1AA_X+a_2A_{XXX}+a_3A_{XRR}=0, \tag{3.16}$$

where ${}_0^RD_t^\alpha f(t)$ is the Riesz fractional derivative of function $f(t)$ [25, 33]

$${}_0^RD_t^\alpha f(t)=\frac{1}{2}\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^b|t-\tau|^{n-\alpha-1}f(t)\mathrm{d}t,\quad n-1\le\alpha<n,t\in[a,b]. \tag{3.17}$$

4. Solutions of the time-fractional ZK equation

In this section, we use the modified extended tanh method [34] to solve the time-fractional ZK equation.

$${}_0^RD_{T_0}^\alpha A+a_1AA_X+a_2A_{XXX}+a_3A_{XRR}=0.$$

The following fractional complex transformation is given

$$A(X,R,T)=U(\xi),\xi=X+Y-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}-X_0, \tag{4.1}$$

where ω is the arbitrary constant. Substituting equation (4.1) into the time-fractional ZK equation, we can obtain the following ordinary differential equation

$$-\omega U'+a_1UU'+(a_2+a_3)U'''=0, \tag{4.2}$$

where $U'=\frac{\mathrm{d}U}{\mathrm{d}\xi}$. Integrating equation (4.2) once with respect to ξ, we can obtain

$$-\omega U'+\frac{1}{2}a_1U^2+(a_2+a_3)U'''=0, \tag{4.3}$$

where the integral constant is considered to be zero.

The equilibrium of the highest derivative term U'' and the nonlinear term U^2 in equation (4.3), we get $N+2=2N$. That means $N=2$. The solution of the equation (4.3) can be expressed as

$$U(\xi)=l_0+l_1\varphi(\xi)+l_2\varphi(\xi)+k_1\varphi(\xi)^{-1}+k_2\varphi(\xi)^{-2}, \tag{4.4}$$

where l_0,l_1,l_2,k_1,k_2 are unknown constants. $\varphi(\xi)$ satisfies the Riccati equation

$$\varphi'=\varphi^2+b, \tag{4.5}$$

substituting equations (4.3) and (4.5) into equation (4.4), letting the sum of the coefficients of each power of $\varphi(\xi)$ be equal to zero, we can obtain

$$\begin{cases} \varphi^0: -\omega l_0 + \frac{1}{2}l_0^2 + a_1 l_1 k_1 + a_1 l_2 k_2 + 2(a_2+a_3)b^2 l_2 + 2(a_2+a_3)k_2 = 0, \\ \varphi: -\omega l_1 + a_1 l_0 + l_1 + a_1 l_2 k_1 + 2(a_2+a_3)b l_1 = 0, \\ \varphi^{-1}: -\omega k_1 + a_1 l_0 k_1 + a_1 l_1 k_2 + 2(a_2+a_3)b k_1 = 0, \\ \varphi^2: -\omega l_2 + \frac{1}{2}a_1 l_1^2 + a_1 l_2 l_0 + 8(a_2+a_3)b l_2 = 0, \\ \varphi^{-2}: -\omega k_2 + \frac{1}{2}a_1 k_1^2 + a_1 k_2 l_0 + 8(a_2+a_3)b k_2 = 0, \\ \varphi^3: a_1 l_1 l_2 + 2(a_2+a_3)l_1 = 0, \\ \varphi^{-3}: a_1 k_1 k_2 + 2(a_2+a_3)b^2 k_1 = 0, \\ \varphi^4: \frac{1}{2}a_1 l_2^2 + 6(a_2+a_3)l_2 = 0, \\ \varphi^{-4}: \frac{1}{2}a_1 k_2^2 + 6(a_2+a_3)k_2 b^2 = 0, \end{cases} \tag{4.6}$$

by solving the above equations, six groups of solutions are obtained as follows:

Case 1:

$$l_1 = k_1 = k_2 = 0,\ l_0 = -\frac{\omega}{a_1},\ l_2 = -\frac{12(a_2+a_3)}{a_1},\ b = \frac{\omega}{4(a_2+a_3)}, \tag{4.7}$$

$$\begin{cases} A_1(X,R,T) = -\frac{\omega}{a_1} - \frac{\omega}{3a_1}\tan^2(\frac{1}{2}\sqrt{\frac{\omega}{a_2+a_3}}\xi), \\ A_2(X,R,T) = -\frac{\omega}{a_1} - \frac{\omega}{3a_1}\cot^2(\frac{1}{2}\sqrt{\frac{\omega}{a_2+a_3}}\xi), \end{cases} \tag{4.8}$$

where $\xi = X + R - \frac{\omega T^\alpha}{\Gamma(1+\alpha)} - X_0$.

Case 2:

$$l_1 = k_1 = k_2 = 0, l_0 = \frac{3\omega}{a_1}, l_2 = -12\frac{12(a_2+a_3)}{a_1}, b = -\frac{\omega}{4(a_2+a_3)}, \tag{4.9}$$

$$\begin{cases} A_3(X,R,T) = \frac{3\omega}{a_1} - \frac{3\omega}{a_1}\tanh^2(\frac{1}{2}\sqrt{\frac{\omega}{a_2+a_3}}\xi), \\ A_4(X,R,T) = \frac{3\omega}{a_1} - \frac{3\omega}{a_1}\coth^2(\frac{1}{2}\sqrt{\frac{\omega}{a_2+a_3}}\xi), \end{cases} \tag{4.10}$$

where $\xi = X + R - \frac{\omega T^\alpha}{\Gamma(1+\alpha)} - X_0$.

Case 3:

$$l_1 = k_1 = 0, l_0 = \frac{\omega}{2a_1}, l_2 = \frac{12(a_2+a_3)}{a_1}, k_2 = -\frac{3\omega^2}{64a_1(a_2+a_3)}, b = \frac{\omega}{16(a_2+a_3)}, \tag{4.11}$$

$$A_5(X,R,T) = \frac{\omega}{2a_1} - \frac{3\omega}{4a_1}\tan^2(\frac{1}{4}\sqrt{\frac{\omega}{a_2+a_3}}\xi) - \frac{3\omega}{4a_1}\cot^2(\frac{1}{4}\sqrt{\frac{\omega}{a_2+a_3}}\xi), \tag{4.12}$$

where $\xi = X + R - \frac{\omega T^\alpha}{\Gamma(1+\alpha)} - X_0$.

Case 4:

$$l_1 = k_1 = 0, l_0 = \frac{3\omega}{2a_1}, l_2 = -\frac{12(a_2+a_3)}{a_1}, k_2 = -\frac{3\omega^2}{64a_1(a_2+a_3)}, b = -\frac{\omega}{16(a_2+a_3)}, \tag{4.13}$$

$$A_6(X,R,T) = \frac{3\omega}{2a_1} - \frac{3\omega}{4a_1}\tanh^2(\frac{1}{4}\sqrt{\frac{\omega}{a_2+a_3}}\xi) - \frac{3\omega}{4a_1}\cosh^2(\frac{1}{4}\sqrt{\frac{\omega}{a_2+a_3}}\xi), \tag{4.14}$$

where $\xi = X + R - \frac{\omega T^\alpha}{\Gamma(1+\alpha)} - X_0$.

Case 5:

$$k_1 = l_1 = l_2 = 0, l_0 = -\frac{\omega}{a_1}, k_2 = -\frac{3\omega^2}{4a_1(a_2+a_3)}, b = \frac{\omega}{4(a_2+a_3)}, \tag{4.15}$$

note that the result is identical to the case 1.

Case 6:

$$k_1 = l_1 = l_2 = 0, l_0 = \frac{3\omega}{a_1}, k_2 = -\frac{3\omega^2}{4a_1(a_2+a_3)}, b = -\frac{\omega}{4(a_2+a_3)}, \tag{4.16}$$

note that the result is identical to the case 2.

5. Analysis and discussion of stroke volume

Next, we select one of the solutions for analysis with $A_3(X,R,T)$.
Assuming that the basic flow in the human aorta is Poiseuille flow [2], $\overline{\psi}$ can be shown as

$$\overline{\psi} = u_{\max}(\frac{r^2}{2} - \frac{r^4}{4R_0^2}). \tag{5.1}$$

According to the equations (2.8), (2.13) and $A_3(X, R, T)$, the original stream function can be expressed as

$$\begin{aligned}\Psi = u_{\max}(\frac{r^2}{2} - \frac{r^4}{4R_0^2}) + C_1\frac{3\omega}{a_1}\sqrt{\frac{a}{c}}rJ_1(\sqrt{\frac{a}{c}}r)[1-\epsilon^{\frac{1}{2}}\sqrt{\frac{\omega}{a_2+a_3}}\tanh(\frac{1}{2}\sqrt{\frac{\omega}{a_2+a_3}}(\epsilon^{\frac{1}{2}}(x-ct+r) \\ -\frac{\omega(\epsilon^{\frac{3}{2}}t)^\alpha}{\Gamma(1+\alpha)} - X_0))]\operatorname{sech}^2[\frac{1}{2}\sqrt{\frac{\omega}{a_2+a_3}}(\epsilon^{\frac{1}{2}}(x-ct+r) - \frac{\omega(\epsilon^{\frac{3}{2}}t)^\alpha}{\Gamma(1+\alpha)} - X_0)].\end{aligned} \tag{5.2}$$

We assume that the coordinates of the central position at cardiac outlet are $x = 0, r = 0$ with the stream function is Ψ_1, and the coordinates of the junction of the heart and the aorta are $x = 0, r = R_0$ with the stream function is Ψ_2.

$$\begin{aligned}&\Psi_1 = 0, \\ &\Psi_2 = \frac{u_{\max}R_0^2}{2} + C_1\frac{3\omega}{a_1}\sqrt{\frac{a}{c}}R_0J_1(\sqrt{\frac{a}{c}}R_0)[1-\epsilon^{\frac{1}{2}}\sqrt{\frac{\omega}{a_2+a_3}}\tanh(\frac{1}{2}\sqrt{\frac{\omega}{a_2+a_3}}(\epsilon^{\frac{1}{2}}(-ct+R_0) \\ &\quad -\frac{\omega(\epsilon^{\frac{3}{2}}t)^\alpha}{\Gamma(1+\alpha)} - X_0))]\operatorname{sech}^2[\frac{1}{2}\sqrt{\frac{\omega}{a_2+a_3}}(\epsilon^{\frac{1}{2}}(-ct+R_0) - \frac{\omega(\epsilon^{\frac{3}{2}}t)^\alpha}{\Gamma(1+\alpha)} - X_0)].\end{aligned} \tag{5.3}$$

In cylindrical coordinates, based on the calculation method of rotating volume, the blood flow of the cardiac outlet is

$$\begin{aligned}Q = \frac{\pi u_{\max}R_0^2}{2} + C_1\frac{6\pi\omega}{a_1}\sqrt{\frac{a}{c}}R_0J_1(\sqrt{\frac{a}{c}}R_0)[1-\epsilon^{\frac{1}{2}}\sqrt{\frac{\omega}{a_2+a_3}}\tanh(\frac{1}{2}\sqrt{\frac{\omega}{a_2+a_3}}(\epsilon^{\frac{1}{2}}(-ct+R_0) \\ -\frac{\omega(\epsilon^{\frac{3}{2}}t)^\alpha}{\Gamma(1+\alpha)} - X_0))]\operatorname{sech}^2[\frac{1}{2}\sqrt{\frac{\omega}{a_2+a_3}}(\epsilon^{\frac{1}{2}}(-ct+R_0) - \frac{\omega(\epsilon^{\frac{3}{2}}t)^\alpha}{\Gamma(1+\alpha)} - X_0)].\end{aligned} \tag{5.4}$$

As we know, stroke volume is the amount of blood emitted from one side of the ventricle during a cardiac cycle. The formula for calculating the stroke volume is

$$V = \int_0^{T_0} Q\mathrm{d}t,$$

where, T is the cardiac cycle, Q is the blood flow of the cardiac outlet.

The physiological parameters at the entrance of the human aorta are as follows

$$R_0 = 0.0125\,\mathrm{m}, u_{\max} = 0.4\ \mathrm{m}, c = 5.5\,\mathrm{m/s},$$

letting $C_1 = 1, \epsilon = 0.01, \omega = 3, X_0 = 4$. According to the equations (2.33) and (2.35), we can obtain the following result

$$a_1 = 1817.3644, a_2 = 2.9541 \times 10^3, a_3 = 0.92528.$$

Figure 1 shows that the relationship between blood flow volume Q at cardiac outlet and time (t) and fractional order (α). In a cardiac cycle, the blood flow volume Q decreases over time t and tends to flatten. In addition, with the increase of fractional order α, the blood flow is increasing and the increasing trend is slowing down.

Stroke volume is an important index to measure the blood circulation efficiency. The relationship between stroke volume V and fractional order α is shown in Figure 2. As the fractional order α increases, the stroke volume increases and the trend slows down. By calculation, when $\alpha = 0.1$, $V = 79.3\,\mathrm{ml}$. But under normal physiological

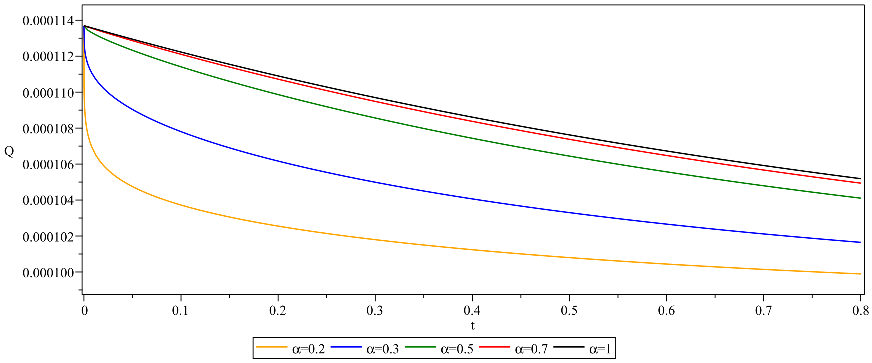

FIGURE 1. Blood flow volume of the cardiac outlet at different values of α and t.

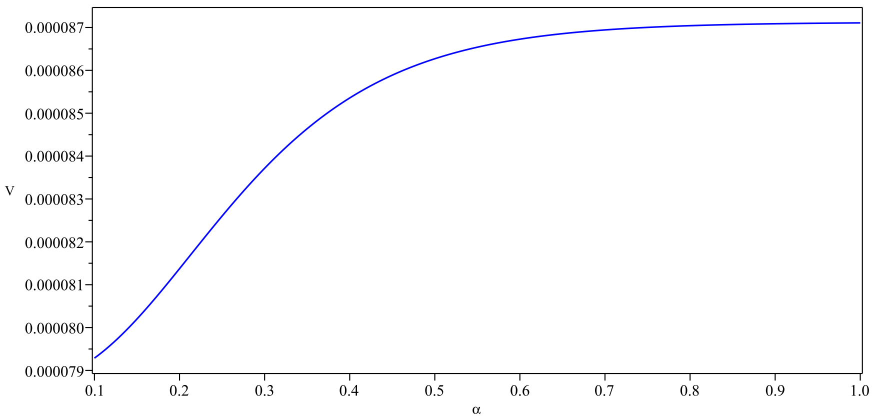

FIGURE 2. Stroke volume of different value α.

conditions, stroke volume is $60 \sim 70\,\text{ml}$. As a result, certain disturbances (such as some physiological diseases or dramatic change of mood etc.) to the heart can have an effect on stroke volume. This situation is often due to increased in terminal ventricular diastolic volume and venous return volume, which leads to increased blood flow into the aorta during systolic phase, the increased blood volume in the aorta and the main artery, and the greater tension on the blood vessels. It could eventually lead to some cardiovascular disease.

Figure 3 displays the variation of stroke volume V with the vascular radius R for different $u_{\max}$, where $u_{\max}$ is the blood flow velocity. The result shows that stroke volume V with the increase of the vascular radius R. In the case of the same vascular radius, the increase of the blood flow velocity $u_{\max}$ also added stroke volume V.

6. CONCLUSIONS

In this paper, based on the basic equations of blood flow and the two-dimensional Navier–Stokes equation, and by use of multi-scale analysis and perturbation expansion method, the ZK equation is derived in describing the neutral disturbance flow. Compared with the traditional KdV model, the ZK model derived in this paper can describe the propagation of disturbance flow in the radius direction. Then, the time-fractional ZK equation is obtained, which is more in accordant with the actual condition. We solved the time-fractional ZK equation and analyzed the effects of fractional order, vascular radius and blood flow velocity on stroke volume.

The conclusion shows that neutral disturbance exists in arterial vessels, and the disturbance propagates in the from of solitary waves. This disturbance is not produced in the arterial vessels, but is caused by the heart. So, neutral disturbance waves are associated with certain functions inside the heart. Due to the complexity

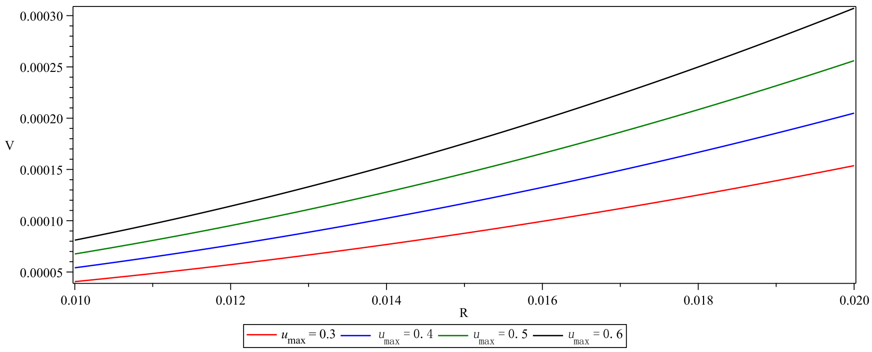

FIGURE 3. Stroke volume of different values $u_{\max}$ and R with $\alpha = 0.5$.

of human blood circulation, pulse wave is affected by the blood pressure, blood velocity or vascular branch during the process of propagation, which causes the waveform to change and can not completely reflect some information within the heart. As we all know, the solitary waves can keep its energy basically unchanged during the propagation process, it can spread almost without attenuation, and some of the information in the heart can be transmitted the superficial parts of the body through the neutral disturbance wave of the heart. If we can detect the pulse information of the superficial part of the human body through some medical methods, and separate neutral disturbances from complex pulse waves. We can obtain some information of the heart, and prevent and diagnose some cardiovascular diseases more effectively.

REFERENCES

[1] O.P. Agrawal, Fractional of Euler-Lagrange equations for fractional variational problems. *J. Math. Anal. Appl.* **272** (2002) 368–379.

[2] S.P. Sutera and R. Skalak, The history of Poiseuille's law. *Annu. Rev. Fluid Mech.* **25** (1993) 1–20.

[3] Z.B. Bai, W.C. Sun and W.H. Zhang, Positive solutions for boundary value problems of singular fractional differential equations. *Abst. Appl. Anal.* **2013** (2013) 129640.

[4] Z.B. Bai, S. Zhang, S.J. Sun and C. Yin, Monotone iterative method for fractional differential equations. *Electr. J. Differ. Equ.* **2016** (2016) 1–8.

[5] Y.Y. Choy, K.G. Tay and C.T. Ong, Modulation of Nonlinear Waves in an Inviscid Fluid (Blood) Contained in a Stenosed Artery. *Appl. Math. Sci.* **7** (2013) 5003–5012.

[6] Y.J. Cui, Uniqueness of solution for boundary value problems for fractional differential equations. *Appl. Math. Lett.* **51** (2016) 48–54.

[7] Y.J. Cui and Y. M. Zou, Exitence results and the monotone iterative technique for nonlinear fractional differential systems with coupled four-point boundary value problems. *Abst. Appl. Anal.* **51** (2014) 242591.

[8] Y.J. Cui, W.J. Ma, Q. Sun and X.W. Su, New uniqueness results for boundary value problem of fractional differential equation. *Nonlinear Anal. Modell. Cont.* **23** (2018) 31–39.

[9] H. Demiray, Solitary waves in prestressed elastic tubes. *Bull. Math. Biol.* **58** (1996) 939–955.

[10] H. Demiray, Solitary waves in fluid-filled elastic tubes: weakly dispersive case. *Int. J. Eng. Sci.* **39** (2001) 439–451.

[11] H. Demiray, Solitary waves in a tapered prestressed fluid-filled elastic tube. *Z. Angew. Math. Phys.* **55** (2004) 282–294.

[12] H. Demiray, Variable coefficient modified KdV equation in fluid-filled elastic tubes with stenosis: solitary waves. *Chaos Solitons Fract.* **42** (2009) 358–364.

[13] W.S. Duan, B.R. Wang and R.J. Wei, The decay of soliton in small blood artery. *J. Phys. Soc. Jpn.* **65** (1996) 945–947.

[14] W.S. Duan, B.R. Wang and R.J. Wei, The cubic nonlinear Schrödinger equation in a fluid-filled elastic tube. *Phys. Lett. A* **224** (1997) 154–158.

[15] W.S. Duan, B.R. Wang and R.J. Wei, Nonlinear waves propagating in an inhomogeneous blood vessel. *Chin. Phys. B* **6** (1997) 801–810.

[16] S.A. El-Wakil, E.M. Abulwafa, M.A. Zahran and A.A. Mahmoud, Time-fractional kdv equation: formulation and solution using variational methods. *Nonlinear Dyn.* **65** (2011) 55–63.

[17] G.S.F. Frederico and D.F.M. Torres, Fractional conservation laws in optimal control theory. *Nonlinear Dyn.* **53** (2008) 215–222.

[18] C. Fu, C.N. Lu and H.W. Yang, Time-space fractional (2+1) dimensional nonlinear Schrödinger equation for envelope gravity waves in baroclinic atmosphere and conservation laws as well as exact solutions. *Adv. Differ. Equ.* **2018** (2018) 56.

[19] P.R. Gordoa, A note on solutions of an equation modelling arterial deformation. *Chaos Solitons Fract.* **33** (2007) 1505–1511.

[20] M. Guo, Y. Zhang, M. Wang, Y.D. Chen and H.W. Yang, A new ZK-ILW equation for algebraic gravity solitary waves in finite depth stratified atmosphere and the research of squall lines formation mechanism. *Comput. Math. Appl.* **75** (2018) 3589–3603.

[21] M. Guo, C. Fu, Y. Zhang, J.X. Liu and H.W. Yang, Study of ion-acoustic solitary waves in a magnetized plasma using the three-dimensional time-space fractional Schamel-KdV equation. *Complexity* (2018) 6852548.

[22] Y. Hanshizume, Nonlinear pressure wave propagation in arteries. *J. Phys. Soc. Jpn.* **57** (1988) 4160–4168.

[23] J.H. He, Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbo-machinery aerodynamics. *Int. J. Turbo Jet Engines* **14** (1997) 23–28.

[24] J.H. He, Variational-iteration a kind of nonlinear analytical technique: some examples. *Int. J. Nonlinear Mech.* **34** (1999) 699–708.

[25] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and applications of fractional differential equations. Elsevier, Amsterdam (2006).

[26] B. Kilic and M. Inc, The first integral method for the time fractional Kaup-Boussinesq system with time dependent coefficient. *Appl. Math. Comput.* **254** (2015) 70–74.

[27] D. Kumar, J. Singh, D. Baleanu and S. Rathore, Analysis of a fractional model of the Ambartsumian equation. *Eur. J. Phys. Plus* **133** (2018) 159.

[28] J.C. Le, A neutral disturbance of the blood flow in the arteries. *Chin. Q. Mech.* **15** (1994) 40–49.

[29] C.N. Lu, C. Fu and H.W. Yang, Time-fractional generalized Boussinesq equation for Rossby solitary waves with dissipation effect in stratified fluid and conservation laws as well as exact solutions. *Appl. Math. Comput.* **327** (2018) 104–116.

[30] W.X. Ma and Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms. *J. Differ. Equ.* **264** (2018) 2633–2659.

[31] D.A. McDonald, The velocity of blood flow in the rabbit aorta studied with high-speed cinematography. *J. Physiol.* **118** (1952) 328–339.

[32] D.A. McDonald, The relation of pulsatile pressure to flow in arteries. *J. Physiol.* **127** (1955) 533–552.

[33] I. Podlubny, Fractional differential equations. Mathematics in Science and Engineering, San Diego (1999).

[34] K.P. Raslan, K.K. Ali and M.A. Shallal, The modified extended tanh method with the Riccati equation for solving the space-time fractional EW and MEW equations. *Chaos Solitons Fract.* **103** (2017) 404–409.

[35] N. Shang and B. Zheng, Exact solutions for three fractional partial differential equations by the (G'/G) method. *Int. J. Appl. Math.* **43** (2013) 1–6.

[36] Y. Sigeo, Solitary waves in large blood vessels. *J. Phys. Soc. Jpn.* **56** (1987) 506–520.

[37] J. Singh, D. Kumar and D. Baleanu, On the analysis of chemical kinetics system pertaining to a fractional derivative with Mittag-Leffler type kernel. Chaos **27** (2017) 103113.

[38] J. Singh, D. Kumar and J.J. Nieto, Analysis of an El Nino-Southern Oscillation model with a new fractional derivative. *Chaos Solitons Fract.* **99** (2017) 109–115.

[39] J. Singh, D. Kumar and D. Baleanu, On the analysis of fractional diabetes model with exponential law. *Adv. Differ. Equ.* **2018** (2018) 231.

[40] J.R. Womersley, Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known. *J. Physiol.* **127** (1955) 553–563.

[41] J.R. Womersley, On the oscillatory motion of a viscous liquid in thin-walled elastic tube: I. The linear approximation for long waves, The London, Edinburgh. *Dublin Philos. Mag. J. Sci.* **46** (1955) 199–221.

[42] J.R. Womersley, Oscillatory flow in arteries: the constrained elastic tube as a model of arterial flow and pulse transmission. *Phys. Med. Biol.* **2** (1957) 178–187.

[43] X.J. Yang, *Local fractional integral equations and their applications. Adv. Comput. Sci. Appl.* **1** (2012) 234–239.

[44] H.W. Yang, Z.H. Xu, D.Z. Yang, X.R. Feng and H.H. Dong, Zk-burgers equation for three-dimensional Rossby solitary waves and its solutions as well as chirp effect. *Adv. Differ. Equ.* **2016** (2016) 167.

[45] X.J. Yang, J.A. Tenreiro Machado, C. Cattani and F. Gao, On a fractal LC-electric circuit modeled by local fractional calculus. *Commun. Nonlinear Sci. Numer. Simulat.* **47** (2017) 200–206.

[46] H.W. Yang, X. Chen, M. Guo and Y.D. Chen, A new ZK-BO equation for three-dimensional algebraic Rossby solitary waves and its solution as well as fission property. *Nonlinear Dyn.* **91** (2018) 2019–2032.

[47] H.W. Yang, M. Guo and H.L. He, Conservation Laws of Space-Time Fractional mZK Equation for Rossby Solitary Waves with Complete Coriolis Force. To appear in: *Int. J. Nonlinear Sci. Numer. Simulat.* DOI: https://doi.org/10.1515/ijnsns-2018-0026 (2018).

[48] H.W. Yang, J.C. Sun and C. Fu, Time-fractional Benjamin-Ono equation for algebraic gravity solitary waves in baroclinic atmosphere and exact multi-soliton solution as well as interaction. *Commun. Nonlinear Sci. Numer. Simulat.* **71** (2019) 187–201.