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WELL-POSEDNESS OF A NONLOCAL BOUNDARY VALUE DIFFERENCE ELLIPTIC PROBLEM[*,**]

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Abstract. The second order of approximation two-step difference scheme for the numerical solution of a nonlocal boundary value problem for the elliptic differential equation

$$-v''(t) + Av(t) = f(t) \quad (0 \leq t \leq T), v(0) = v(T) + \varphi, \int_0^T v(s)\mathrm{d}s = \psi$$

in an arbitrary Banach space E with the positive operator A is presented. The well-posedness of the difference scheme in Banach spaces is established. In applications, the stability, almost coercive stability and coercive stability estimates in maximum norm in one variable for the solutions of difference schemes for numerical solution of two type elliptic problems are obtained.

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1. Introduction

The well-posedness in various Banach spaces of the local boundary value problem for the elliptic equation

$$-v''(t) + Av(t) = f(t) \ (0 \leq t \leq T), v(0) = v_0, v(T) = v_T \tag{1.1}$$

in an arbitrary Banach space E with the positive operator A and its related applications have been investigated by many researchers (see, for example, [1, 2, 16, 17, 21] and the references given therein).

Recall that (see, [1]) the operator A is said to be positive if its spectrum $\sigma(A)$ lies in the interior of the sector of angle ϕ , $0 < 2\phi < 2\pi$, symmetric with respect to the real axis and if on the edges of this sector, $S_1(\phi) = \{\rho e^{i\phi} : 0 \leq \rho < \infty\}$ and $S_2(\phi) = \{\rho e^{-i\phi} : 0 \leq \rho < \infty\}$, and outside of it, the resolvent $(\lambda - A)^{-1}$ is

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subject to the bound

$$\left\| (\lambda - A)^{-1} \right\|_{E \to E} \le \frac{M(\phi)}{1 + |\lambda|}$$

The infimum of all such angles ϕ is called the spectral angle of the positive operator A and is denoted by $\phi(A) = \phi(A, E)$.

In mathematical modeling, elliptic equations are used together with local boundary conditions specifying the solution on the boundary of the domain. In some cases, classical boundary conditions cannot describe process or phenomenon precisely. Therefore, mathematical models of various physical, chemical, biological or environmental processes often involve nonclassical conditions. Such conditions usually are identified as nonlocal boundary conditions and reflect situations when the data on the domain boundary cannot be measured directly, or when the data on the boundary depend on the data inside the domain. The well-posedness of various nonlocal boundary value problems for partial differential and difference equations has been studied extensively by many researchers (see, *e.g.*, [5–17, 22–24] and the references given therein).

In the paper [10] the abstract nonlocal boundary value problem for differential equation of elliptic type

$$-v''(t) + Av(t) = f(t) \quad (0 \le t \le T), v(0) = v(T) + \varphi, \int_0^T v(s)\mathrm{d}s = \psi \tag{1.2}$$

in an arbitrary Banach space E with the positive operator A is considered. A function $v(t)$ is called a solution of problem (1.2) if the following conditions are satisfied:

i. $v(t)$ is twice continuously differentiable on the segment $[0, T]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.
ii. The element $v(t)$ belongs to $D(A)$ for all $t \in [0, T]$, and the function $Av(t)$ is continuous on the segment $[0, T]$.
iii. $v(t)$ satisfies the equation and boundary conditions (1.2).

A solution of problem (1.2) defined in this manner will from now on be referred to as a solution of problem (1.2) in the space $C(E) = C([0, T], E)$. Here $C(E)$ stands for the Banach space of all continuous functions $\varphi(t)$ defined on $[0, T]$ with values in E equipped with the norm

$$||\varphi||_{C(E)} = \max_{t \in [0,T]} \|\varphi(t)\|_E.$$

The well-posedness of problem (1.2) in various Banach spaces was established. In applications, the coercive stability estimates in Hölder norms for the solutions of the mixed type nonlocal boundary value problems for elliptic equations were obtained.

In the present paper the second order of approximation two-step difference scheme

$$\begin{cases} -\dfrac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + Au_k = f_k, f_k = f(t_k), t_k = k\tau, 1 \le k \le N-1, N\tau = T, \\ \\ u_0 = u_N + \varphi, \sum\limits_{i=1}^{N} u_i \tau = \psi \end{cases} \tag{1.3}$$

for the approximate solution of problem (1.2) is presented. The well-posedness of the difference scheme (1.3) in Banach spaces is established. In applications, the stability, almost coercive stability and coercive stability estimates in maximum norm in one variable for the solutions of difference schemes for numerical solution of two type elliptic problems are obtained.

2. Auxiliary results

In this section, we give some auxiliary statements from [2] which will be useful in the sequel. We consider the second order of accuracy difference scheme

$$-\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=f_k,\ f_k=f(t_k),\ t_k=k\tau,\ 1\le k\le N-1,\ N\tau=T, \tag{2.1}$$

$$u_0=v_0, u_N=v_T$$

of approximation solution of the boundary value problem (1.1). This problem is uniquely solvable, and the following formula holds

$$\begin{aligned} u_k=(I-R^{2N})^{-1}\Big\{&(R^k-R^{2N-k})u_0+(R^{N-k}-R^{N+k})u_N \\ &-(R^{N-k}-R^{N+k})(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau\Big\} \\ &+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{|k-i|}-R^{k+i})f_i\tau,\ 1\le k\le N-1, \end{aligned} \tag{2.2}$$

where

$$B=B(\tau,A)=\frac{\tau A}{2}+\sqrt{\left(\frac{\tau A}{2}\right)^2+A},\ R=(I+\tau B)^{-1}.$$

Note that $B(\tau,A)\neq A^{\frac{1}{2}}$but then $B(\tau,A)\to A^{\frac{1}{2}}$ as $\tau\to 0$ and it has same spectral properties of $A^{\frac{1}{2}}$ under some assumption for A.

Let us denote by $F_\tau(E)=F([0,T]_\tau,E)$ the space of grid functions $\varphi^\tau=\{\varphi_k\}_{k=1}^{N-1}$ for fixed $\tau=\frac{T}{N}$. Thus, $F_\tau(E)$ is the vector space whose elements are ordered $(N-1)$-tuples of elements of E. The space $F_\tau(E)$ can be equipped with various norms and thus become a normed space. For instance, the vector space $F_\tau(E)$ generates the normed space $C_\tau(E)=C([0,T]_\tau,E)$ with the norm

$$\|\varphi^\tau\|_{C_\tau(E)}=\max_{1\le k\le N-1}\|\varphi_k\|_E.$$

Let us reduce the difference scheme (2.1) to an operator problem in the space $F_\tau(E)$. In addition to the operator D_τ^2, acting from the space $E\times F_\tau(E)\times E$ of vectors $w^\tau=\{w_k\}_{k=0}^N$ into the space $F_\tau(E)$ of vectors $v^\tau=\{v_k\}_{k=1}^{N-1}$ by the rule

$$v^\tau=D_\tau^2u^\tau, v_k=\frac{1}{\tau^2}(w_{k+1}-2w_k+w_{k-1}), k=1,\dots,N-1,$$

define an operator A_τ from the space $E\times F_\tau(E)\times E$ of vectors $w^\tau=\{w_k\}_{k=0}^N$ into the space $F_\tau(E)$ of vectors $v^\tau=\{v_k\}_{k=1}^{N-1}$ by the rule

$$v^\tau=A_\tau u^\tau,\ v_k=Aw_k,\ k=1,\dots,N-1.$$

Next, let us introduce the continuation operator $\Pi(u_0,u_N)$, which acts from $E\times F_\tau(E)\times E$ to $F_\tau(E)$ according to the rule

$$\Pi(u_0,u_N)(u_1,\dots,u_{N-1})=(u_0,u_1,\dots,u_{N-1},u_N).$$

Then the difference scheme (2.1) can obviously be rewritten as the equivalent operator equation

$$-D_\tau^2\Pi(u_0,u_N)u^\tau + A_\tau\Pi(u_0,u_N)u^\tau = f^\tau.$$

Here f^τ is defined by the formula

$$f^\tau = (f_1,\dots,f_{N-1}).$$

The last operator problem will be considered in the space $F_\tau(E)$. From its unique solvability for any u_0, $u_N \in E$ and $f^\tau \in F_\tau(E)$ it follows that its solution u^τ that defines an additive and homogeneous operator $u^\tau(f^\tau, u_0, u_N)$ is continuous.

The boundary value problem (2.1) is said to be stable in $F_\tau(E)$ if we have the estimate

$$\| u^\tau(f^\tau,u_0,u_N) \|_{F_\tau(E)} \le M[\| f^\tau \|_{F_\tau(E)} + \| u_0\|_E + ||u_N\|_E],$$

where M is independent not only of f^τ, u_0, u_N, but also of τ.

The boundary value difference problem (2.1) is said to be well-posed (coercively stable) in $F_\tau(E)$ if we have the coercive inequality

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{F_\tau(E)} + \| \{Au_k\}_1^{N-1} \|_{F_\tau(E)}$$

$$\le M[\| f^\tau \|_{F_\tau(E)} + \| Au_0\|_E + ||Au_N\|_E],$$

where M is independent not only of f^τ, u_0, u_N, but also of τ.

From formula (2.3) it follows that the investigation of the stability and well-posedness of difference scheme (2.1) relies in an essential manner on a number of properties of the powers of the operator $R = (I+\tau B)^{-1}$ in the general cases of operator A. We begin by deriving some estimates for norms of $\exp\{-tA_l\}$ and powers of the operator $(I+\tau B)^{-1}$ (see, [2]).

Lemma 2.1. *Let A be a strongly positive operator in a Banach space E with spectral angle $\phi(A,E) < \pi/2$. Then $-A$ is a generator of the analytic semigroup exp$\{-tA\}$ $(t \ge 0)$ with exponentially decreasing norm, when $t \longrightarrow +\infty$, i.e. we have the following estimates*

$$\|\exp\{-tA\}\|_{E\longrightarrow E} \le M\ e^{-t\delta(A)}\ (t>0), \tag{2.3}$$

$$\|tA\exp\{-tA\}\|_{E\longrightarrow E} \le M\ (t>0) \tag{2.4}$$

for some $1 \le M < +\infty$, $0 < \delta(A) < +\infty$.

Lemma 2.2. *Let $-A$ be the generator of the analytic semigroup exp$\{-tA\}$ $(t \ge 0)$ with exponentially decreasing norm, when $t \longrightarrow +\infty$. Then the following estimates hold for any $k \ge 1$:*

$$\left\|(\lambda I+\tau B)^{-k}\right\|_{E\to E} \le M[\lambda+\tau a(A)]^{-k},\ a(A) = \sqrt{\delta(A)}, \tag{2.5}$$

$$||k\tau B(I+\tau B)^{-k}||_{E\to E} \le M, \tag{2.6}$$

where M does not depend on τ.

We have the following results.

Theorem 2.3. *Let A be a strongly positive operator in a Banach space E. Then difference problem (2.1) is stable in $C_\tau(E)$; its solutions satisfy the stability inequality*

$$\| u^\tau \|_{C_\tau(E)} \leq M[\| f^\tau \|_{C_\tau(E)} + \| u_0\|_E + ||u_N\|_E],$$

where M does not depend on f^τ, u_0, u_N and τ.

Theorem 2.4. *Let A be a strongly positive operator in the Banach space E and u_0, $u_N \in D(A)$. Then the solutions of the difference problem (2.1) in $C_\tau(E)$ obey the almost coercivity inequality*

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C_\tau(E)} + \| \{Au_k\}_1^{N-1} \|_{C_\tau(E)}$$

$$\leq M[\min\left\{\ln\frac{1}{\tau}, |\ln\|A\|_{E\to E}|\right\} \| f^\tau \|_{C_\tau(E)} + \| Au_0\|_E + ||Au_N\|_E],$$

where M is independent not only of f^τ, u_0, u_N but also of τ.

Theorem 2.5. *Let A be a strongly positive operator in a Banach space E and $Au_0, Au_N \in E'_\alpha$. Then the solutions of the difference problem (2.1) in $C_\tau(E'_\alpha)$ obey the coercivity inequality*

$$\| \{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1} \|_{C_\tau(E'_\alpha)} + \| \{Au_k\}_1^{N-1} \|_{C_\tau(E'_\alpha)}$$

$$\leq M\left[\frac{1}{\alpha(1-\alpha)}\| f^\tau \|_{C_\tau(E'_\alpha)} + \|Au_0\|_{E'_\alpha} + \|Au_N\|_{E'_\alpha}\right],$$

where M is independent not only of f^τ, u_0, u_N, α, but also of τ.

Here, the Banach space $E'_\alpha = E'_\alpha(B, E)$ $(0 < \alpha < 1)$ consists of those $v \in E$ for which the norm

$$\| v \|_{E'_\alpha} = \sup_{z>0} z^\alpha \| B(zI + B)^{-1}v \|_E + \| v \|_E$$

is finite.

3. Well-posedness of difference problem (1.3)

We consider the difference problem (1.3). Using formula (2.3) and nonlocal conditions

$$u_0 = u_N + \varphi, \sum_{i=1}^N u_i\tau = \psi,$$

we get

$$\begin{aligned} u_0 = {} & (2I + \tau B)^{-1}(I + R^N)\left(I - R^N\right)^{-1}\left\{B\psi - B^{-1}(I + \tau B)\sum_{i=1}^{N-1} f_i\tau\right\} \\ & + (I + \tau B)(2I + \tau B)^{-1}\left(I - R^N\right)^{-1}\left(I - R^{N+1}\right)\varphi \\ & + B^{-1}(I + \tau B)(2I + \tau B)^{-1}\left(I - R^N\right)^{-1}\left\{\sum_{i=1}^{N-1} R^{N-i}f_i\tau + \sum_{i=1}^{N-1} R^i f_i\tau\right\}, \end{aligned} \quad (3.1)$$

$$u_N = (2I+\tau B)^{-1}(I+R^N)\left(I-R^N\right)^{-1}\left\{B\psi - B^{-1}(I+\tau B)\sum_{i=1}^{N-1} f_i\tau\right\} \tag{3.2}$$
$$-(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left(I-R^{N-1}\right)\varphi$$
$$+B^{-1}(I+\tau B)(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left\{\sum_{i=1}^{N-1} R^{N-i} f_i\tau + \sum_{i=1}^{N-1} R^i f_i\tau\right\}.$$

Actually, applying formula (2.3), we get

$$\psi = u_N\tau + \sum_{k=1}^{N-1} u_k\tau = (I-R^{2N})^{-1}\left\{\sum_{k=1}^{N-1}(R^k - R^{2N-k})u_0\tau + \sum_{k=1}^{N}(R^{N-k}-R^{N+k})u_N\tau\right.$$
$$\left.-\sum_{k=1}^{N-1}(R^{N-k}-R^{N+k})(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau^2\right\}$$
$$+(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{k=1}^{N-1}\sum_{i=1}^{N-1}(R^{|k-i|}-R^{k+i})f_i\tau^2.$$

By computing and interchanging the order of summation, we obtain

$$\psi = (I-R^{2N})^{-1}\left\{R\left(I+\cdots+R^{N-2}\right) - R^{N+1}\left(R^{N-2}+\cdots+I\right)\right\}u_0\tau$$
$$+(I-R^{2N})^{-1}\left\{\left(R^{N-1}+\cdots+I\right) - R^{N+1}\left(I+\cdots+R^{N-1}\right)\right\}u_N\tau$$
$$-(I-R^{2N})^{-1}\left\{R\left(R^{N-2}+\cdots+I\right) - R^{N+1}\left(I+\cdots+R^{N-2}\right)\right\}$$
$$\times(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau^2 + (I+\tau B)(2I+\tau B)^{-1}B^{-1}$$
$$\times\sum_{i=1}^{N-1}\left\{\left(R^{i-1}+\cdots+I\right)+\left(R+\cdots+R^{N-i-1}\right)-\left(R^{1+i}+\cdots+R^{N+i-1}\right)\right\}f_i\tau^2.$$

Therefore,

$$\psi = (I-R^{2N})^{-1}R\left(I-R^N\right)(I-R)^{-1}\left(I-R^{N-1}\right)u_0\tau + (I-R^{2N})^{-1}\left(I-R^{N+1}\right)(I-R)^{-1}\left(I-R^N\right)u_N\tau$$
$$-(I-R^{2N})^{-1}R\left(I-R^N\right)(I-R)^{-1}\left(I-R^{N-1}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau^2$$
$$+(I+\tau B)(2I+\tau B)^{-1}B^{-1}(I-R)^{-1}\sum_{i=1}^{N-1}\left\{\left(I-R^i\right)+R\left(I-R^{N-i-1}\right)-R^{1+i}\left(I-R^{N-1}\right)\right\}f_i\tau^2.$$

From that it follows that

$$\psi = (I+R^N)^{-1}R(I-R)^{-1}\left(I-R^{N-1}\right)u_0\tau + (I+R^N)^{-1}\left(I-R^{N+1}\right)(I-R)^{-1}u_N\tau$$
$$-(I+R^N)^{-1}(I-R)^{-1}\left(I-R^{N-1}\right)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}(R^{N-i}-R^{N+i})f_i\tau^2$$
$$+(I+\tau B)(2I+\tau B)^{-1}B^{-1}(I-R)^{-1}\sum_{i=1}^{N-1}\left(I-R^i+R-R^{N-i}-R^{i+1}+R^{N+i}\right)f_i\tau^2.$$

Thus

$$\psi=(I+R^N)^{-1}R\,(I-R)^{-1}\left(I-R^{N-1}\right)u_0\tau+(I+R^N)^{-1}\left(I-R^{N+1}\right)(I-R)^{-1}u_N\tau$$
$$-B^{-1}\,(I-R)^{-1}\,(I+R^N)^{-1}\left\{\sum_{i=1}^{N-1}R^{N-i}f_i\tau^2+\sum_{i=1}^{N-1}R^i f_i\tau^2\right\}$$
$$+B^{-1}(I+\tau B)(2I+\tau B)^{-1}\,(I-R)^{-1}\,(I+R)\sum_{i=1}^{N-1}f_i\tau^2.$$

Since $u_N=u_0-\varphi$, we have that

$$\psi=\ (I-R)^{-1}\,(I+R^N)^{-1}\,(I+R)\left(I-R^N\right)u_0\tau-(I-R)^{-1}\,(I+R^N)^{-1}\left(I-R^{N+1}\right)\varphi\tau$$
$$-B^{-1}\,(I-R)^{-1}\,(I+R^N)^{-1}\left\{\sum_{i=1}^{N-1}R^{N-i}f_i\tau^2+\sum_{i=1}^{N-1}R^i f_i\tau^2\right\}+B^{-1}\,(I-R)^{-1}\sum_{i=1}^{N-1}f_i\tau^2.$$

From that follow formulas (3.1), (3.2).

Theorem 3.1. *Let A be a strongly positive operator in a Banach space E and $\psi=A^{-1}\sum_{i=1}^{N-1}f_i\tau$. Then difference problem (1.3) is stable in $C_\tau(E)$, its solutions satisfy the stability inequality*

$$\|\ u^\tau\ \|_{C_\tau(E)}\leq M\left[\|\ f^\tau\ \|_{C_\tau(E)}+\|\ \varphi\ \|_E\right],$$

where M_1 does not depend on f^τ,φ and τ.

Proof. By Theorem 2.3 we have the following estimate

$$\|\ u^\tau\ \|_{C_\tau(E)}\leq M[\|\ f^\tau\ \|_{C_\tau(E)}+\|\ u_0\|_E+||u_N\|_E]$$

for solution of problem (2.1). Therefore, to prove the theorem it suffices to establish the estimate for $||u_0\|_E$ and $||u_N\|_E$. Applying condition $\psi=A^{-1}\sum_{i=1}^{N-1}f_i\tau$, formula (3.1), we get

$$u_0=(I+\tau B)(2I+\tau B)^{-1}\left(I-R^{N+1}\right)\left(I-R^N\right)^{-1}\varphi$$
$$+B^{-1}(I+\tau B)(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left\{\sum_{i=1}^{N-1}R^{N-i}f_i\tau+\sum_{i=1}^{N-1}R^i f_i\tau\right\}. \tag{3.3}$$

Using formula (3.3) and the triangle inequality, we get

$$\|u_0\|_E\leq\left\|(I+\tau B)(2I+\tau B)^{-1}\left(I-R^{N+1}\right)\left(I-R^N\right)^{-1}\right\|_{E\to E}\|\varphi\|_E$$
$$+\left\|B^{-1}(I+\tau B)(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\right\|_{E\to E}$$
$$\times\sum_{i=1}^{N-1}\left\|R^{N-i}\right\|_{E\to E}\|f_i\|_E\,\tau+\sum_{i=1}^{N-1}\left\|R^i\right\|_{E\to E}\|f_i\|_E\,\tau.$$

Using estimate (2.5), (2.6), we get

$$\|u_0\|_E\leq M_1\,\|\varphi\|_E+M_2\max_{1\leq i\leq N-1}\|f_i\|_E\sum_{i=1}^{N-1}\left(\frac{1}{1+\tau a\,(A)}\right)^i\tau$$

$$\leq M_3\left(\| f^\tau \|_{C_\tau(E)}+\|\varphi\|_E\right).$$

From that and formula $u_0=u_N+\varphi$ it follows

$$||u_N\|_E\leq M_3\| f^\tau \|_{C_\tau(E)}+(M_3+1)\| \varphi \|_E.$$

Theorem 3.1 is proved. □

Note that for the unbounded operator B the following estimate

$$\left\|B\psi-B^{-1}\left(I+\tau B\right)\sum_{i=1}^{N-1}f_i\tau\right\|_E\leq M\left(\|\psi\|_H+\|f^\tau\|_{C_\tau(E)}\right)$$

is not take place. Therefore, under the conditions of Theorem 3.1 we can not able to get the stability inequality

$$\| u^\tau \|_{C_\tau(E)}\leq M\left[\| f^\tau \|_{C_\tau(E)}+\| \varphi \|_E+\| \psi \|_E\right]$$

for the solution of difference problem (1.3).

Moreover, for the solution of difference problem (1.3) the coercivity inequality

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{C_\tau(E)}+ \ \| \{Au_k\}_1^{N-1} \|_{C_\tau(E)}$$

$$\leq M_C[\|f\|_{C_\tau(E)}+ \ ||A\varphi||_E+||A\psi||_E]$$

fails. Nevertheless, we have the following two results.

Theorem 3.2. *Let A be a strongly positive operator in a Banach space E and $\psi=A^{-1}\sum\limits_{i=1}^{N-1}f_i\tau$. Then the solutions of the difference problem (1.3) in $C_\tau(E)$ obey the almost coercivity inequality*

$$\| \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1} \|_{C_\tau(E)}+\| \{Au_k\}_1^{N-1} \|_{C_\tau(E)}$$

$$\leq M\left[\min\left\{\ln\frac{1}{\tau},|\ln\|A\|_{E\to E}|\right\}\| f^\tau \|_{C_\tau(E)}+\| A\varphi \|_E\right],$$

where M_4 is independent not only of f^τ,φ but also of τ.

Proof. By Theorem 2.4 we have the following estimate

$$\| Au^\tau \|_{C_\tau(E)}\leq M[\min\left\{\ln\frac{1}{\tau},|\ln\|A\|_{E\to E}|\right\}\| f^\tau \|_{C_\tau(E)}+\| Au_0\|_E+||Au_N\|_E]$$

for solution of problem (2.1). Therefore, to prove the theorem it suffices to establish the estimate for $||Au_0\|_E$ and $||Au_N\|_E$. Using formula (3.3), and formula $A=B^2R$,we get

$$\begin{aligned}Au_0&=(I+\tau B)(2I+\tau B)^{-1}\left(I-R^{N+1}\right)\left(I-R^N\right)^{-1}A\varphi\\&+(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left[\sum_{i=1}^{N-1}BR^{N-i}f_i\tau+\sum_{i=1}^{N-1}BR^if_i\tau\right].\end{aligned}\tag{3.4}$$

Applying the triangle inequality, we get

$$||Au_0\|_E \leq ||(I+\tau B)(2I+\tau B)^{-1}\left(I-R^{N+1}\right)\left(I-R^N\right)^{-1}\|_{E\to E}||A\varphi\|_E$$
$$+||(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\|_{E\to E}\left\{\sum_{i=1}^{N-1}\left\|BR^{N-i}\right\|_{E\to E}\|f_i\|_E\,\tau+\sum_{i=1}^{N-1}\left\|BR^i\right\|_{E\to E}\|f_i\|_E\,\tau\right\}.$$

Using estimate (2.5), (2.6), we get

$$||Au_0\|_E \leq M_5||A\varphi\|_E + M_6\max_{1\leq i\leq N-1}\|f_i\|_E\ln\frac{1}{\tau},$$

$$||Au_0\|_E \leq M_5||A\varphi\|_E + M_6\max_{1\leq i\leq N-1}\|f_i\|_E\left[1+|\ln\|B\|_{E\to E}|\right].$$

Hence

$$\|\ Au_0\|_E \leq M_7\left[\|\ A\varphi\ \|_E+\min\left\{\ln\frac{1}{\tau},1+|\ln\|B\|_{E\to E}|\right\}\|\ f^\tau\ \|_{C_\tau(E)}\right].$$

From that and formula $u_N=u_0-\varphi$ it follows

$$||Au_N\|_E \leq (M_7+1)\,\|\ A\varphi\ \|_E + M_7\min\left\{\ln\frac{1}{\tau},1+|\ln\|B\|_{E\to E}|\right\}\|\ f^\tau\ \|_{C_\tau(E)}.$$

Theorem 3.2 is proved. □

Theorem 3.3. *Let A be a strongly positive operator in a Banach space E and $\psi = A^{-1}\sum_{i=1}^{N-1} f_i\tau$. Then the solutions of the difference problem (1.3) in $C_\tau(E'_\alpha)$ obey the coercivity inequality*

$$\|\ \{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\ \|_{C_\tau(E'_\alpha)}+\ \|\ \{Au_k\}_1^{N-1}\ \|_{C_\tau(E'_\alpha)}$$

$$\leq M\left[\frac{1}{\alpha(1-\alpha)}\|\ f^\tau\ \|_{C_\tau(E'_\alpha)}+\|A\varphi\|_{E'_\alpha}\right],$$

where M_5 is independent not only of f^τ, φ,α, but also of τ.

Proof. By Theorem 2.5 we have the following estimate

$$\|\ \{Au_k\}_1^{N-1}\ \|_{C^\alpha_\tau(E)} \leq M\left[\frac{1}{\alpha(1-\alpha)}\|\ f^\tau\ \|_{C_\tau(E'_\alpha)}+\|Au_0\|_{E'_\alpha}+\|Au_N\|_{E'_\alpha}\right]$$

for solution of problem (2.1). Therefore, to prove the theorem it suffices to establish the estimate $\|Au_0\|_{E'_\alpha}$, $\|Au_N\|_{E'_\alpha}$. To see again, we get

$$Au_0=(I+\tau B)(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left(I-R^{N+1}\right)A\varphi$$
$$+(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left\{\sum_{i=1}^{N-1}(I-R)\,R^{N-i-1}f_i+\sum_{i=1}^{N-1}(I-R)\,R^{i-1}f_i\right\}.$$

Applying the triangle inequality, we get

$$\left\|\lambda^\alpha B(\lambda I+B)^{-1}Au_0\right\|_E \le \left\|(I+\tau B)(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\left(I-R^{N+1}\right)\right\|_{E\to E}\left\|\lambda^\alpha B(\lambda I+B)^{-1}A\varphi\right\|_E$$

$$+\left\|(2I+\tau B)^{-1}\left(I-R^N\right)^{-1}\right\|_{E\to E}$$

$$\times\left\{\sum_{i=1}^{N-1}\left\|\lambda^\alpha B(\lambda I+B)^{-1}(I-R)R^{N-i-1}f_i\right\|_E+\sum_{i=1}^{N-1}\left\|\lambda^\alpha B(\lambda I+B)^{-1}(I-R)R^{i-1}f_i\right\|_E\right\}.$$

To estimate the last two sums we use the following Cauchy-Riesz representation formula for these two operators $\lambda^\alpha B(\lambda I+B)^{-1}(I-R(\tau B))R^{N-i-1}(\tau B)f_i$, $\lambda^\alpha B(\lambda I+B)^{-1}(I-R(\tau B))R^{i-1}(\tau B)f_i$.

$$\begin{aligned}&B(\lambda I+B)^{-1}(I-R(\tau B))R^{N-i-1}(\tau B)f_i\\&=\int_{S_1\cup S_2}(\lambda+s)^{-1}(I-R(\tau s))R^{N-i-1}(\tau s)B(sI-B)^{-1}f_i\mathrm{d}s\\&=\int_{S_1\cup S_2}(\lambda\tau+z)^{-1}(I-R(z))R^{N-i-1}(z)B\left(\frac{z}{\tau}I-B\right)^{-1}f_i\mathrm{d}z\end{aligned}$$

where $S_1=\left\{\rho e^{i\psi},0\le\rho<\infty\right\}$ and $S_2=\left\{\rho e^{-i\psi},0\le\rho<\infty\right\}$, $0\le\psi<\frac{\pi}{2l}$. Since $z=\rho e^{\pm i\psi}$, with $|\psi|<\frac{\pi}{2l}$, from the strongly positivity of A it follows that

$$\left|\frac{z}{\tau}\right|^\alpha\left\|B\left(\frac{z}{\tau}I-B\right)^{-1}f_i\right\|_E\le\left|\frac{\rho}{\tau}\right|^\alpha\left\|B\left(\frac{\rho}{\tau}I+B\right)^{-1}f_i\right\|_E.$$

From this estimate and the estimates

$$\frac{1}{|\lambda\tau+z|}\le\frac{M_9}{\lambda\tau+\rho},$$

$$\left|(I-R(z))R^{N-i-1}(z)\right|=\rho\left(1+2\rho\cos\psi+\rho^2\right)^{\frac{-N+i}{2}}$$

it follows that

$$\begin{aligned}&\sum_{i=1}^{N-1}\left\|\lambda^\alpha B(\lambda I+B)^{-1}(I-R)R^{N-i-1}f_i\right\|_E\\&\le M_{10}\int_0^\infty\sum_{i=1}^{N-1}\frac{\rho^{1-\alpha}}{(1+2\rho\cos\psi+\rho^2)^{\frac{N-i}{2}}}\frac{(\lambda\tau)^\alpha}{\lambda\tau+\rho}\left(\frac{\rho}{\tau}\right)^\alpha\left\|B\left(\frac{\rho}{\tau}I+B\right)^{-1}f_i\right\|_E\mathrm{d}\rho\\&\le M_{10}\int_0^\infty\frac{\left(\frac{\lambda\tau}{\rho}\right)^\alpha}{\lambda\tau+\rho}\mathrm{d}\rho\max_{1\le i\le N-1}\|f_i\|_{E'_\alpha}=M_{11}\frac{1}{\alpha(1-\alpha)}\|f^\tau\|_{C_\tau(E'_\alpha)}.\end{aligned}$$

In a similar manner,

$$\sum_{i=1}^{N-1}\left\|\lambda^\alpha B(\lambda I+B)^{-1}(I-R)R^{i-1}f_i\right\|_E\le M_{12}\frac{1}{\alpha(1-\alpha)}\|f^\tau\|_{C_\tau(E'_\alpha)}.$$

Hence

$$\left\|\lambda^{\alpha}B(\lambda I+B)^{-1}Au_0\right\|_E \leq M_{13}\left\{\|A\varphi\|_{E'_\alpha}+\frac{1}{\alpha(1-\alpha)}\|\ f^{\tau}\ \|_{C_\tau(E'_\alpha)}\right\}.$$

Thus

$$\|\ Au_0\ \|_{E'_\alpha} \leq M_{13}\left\{\|A\varphi\|_{E'_\alpha}+\frac{1}{\alpha(1-\alpha)}\|\ f^{\tau}\ \|_{C_\tau(E'_\alpha)}\right\}.$$

From that and formula $u_0=u_N+\varphi$ it follows

$$\|\ Au_N\ \|_{E'_\alpha} \leq M_{13}\frac{1}{\alpha(1-\alpha)}\|\ f^{\tau}\ \|_{C_\tau(E'_\alpha)}+(M_{13}+1)\,\|A\varphi\|_{E'_\alpha}.$$

Theorem 3.3 is proved. □

4. Applications

Finally, we consider applications of Theorems 3.1–3.3 to elliptic equations. First, we consider the nonlocal boundary value problem for two dimensional elliptic equations

$$\begin{cases} -\frac{\partial^2 u}{\partial y^2}-a(x)\frac{\partial^2 u}{\partial x^2}+\delta u=f(y,x), 0<y<T, 0<x<1, \\ u(0,x)=u(T,x)+\varphi(x), \int\limits_0^T u(s,x)\mathrm{d}s=0,\ 0\leq x\leq 1, \\ u(y,0)=u(y,1),\ \ u_x(y,0)=u_x(y,1),\ \ 0\leq y\leq T, \end{cases} \tag{4.1}$$

where $a(x)$, $\varphi(x)$ and $f(y,x)$ are given sufficiently smooth functions and $a(x)>0$, $\delta>0$ is a sufficiently large number. The discretization of problem (4.1) is carried out in two steps. In the first step, let us define the grid sets

$$[0,1]_h=\{x_n=nh, 0\leq n\leq M, Mh=1\}.$$

We introduce the Banach spaces $C_h=C\,[0,1]_h$ and $C_h^{\alpha}=C^{\alpha}\,[0,1]_h\,, 0\leq\alpha\leq 1$ of the grid functions $\varphi^h(x)=\{\varphi_n\}_{n=0}^{M}$ defined on $[0,1]_h$ equipped with the norms

$$\left\|\varphi^h\right\|_{C_h}=\max_{x\in[0,1]_h}\left|\varphi^h(x)\right|,$$

$$\left\|\varphi^h\right\|_{C_h^{\alpha}}=\left\|\varphi^h\right\|_{C_h}+\sup_{0\leq n<n+r\leq M}\frac{|\varphi_{n+r}-\varphi_n|}{rh},$$

respectively. For the differential operator A defined by (4.1), we assign the difference operator A_h^x defined by the formula

$$A_h^x\varphi^h\left(x\right)=\left\{-a\left(x_n\right)\frac{\varphi_{n+1}-2\varphi_n+\varphi_{n-1}}{h^2}+\delta\varphi_n\right\}_{n=1}^{M-1}, \tag{4.2}$$

acting in the space of grid functions $\varphi^h\left(x\right)=\{\varphi_n\}_{n=0}^{M}$ defined on $[0,1]_h$ satisfying the conditions $\varphi_0=\varphi_M$, $\varphi_1-\varphi_0=\varphi_M-\varphi_{M-1}$.

With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2u^h(y,x)}{dy^2} + A_h^x u^h(y,x) = f^h(y,x), \\ x \in [0,1]_h, 0 < y < T, \\ u^h(0,x) = u^h(T,x) + \varphi^h(x), \int\limits_0^T u^h(s,x)ds = 0,\ x \in [0,1]_h. \end{cases} \tag{4.3}$$

In the second step we replace problem (4.3) by the difference scheme

$$-\frac{1}{\tau^2}(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) + A_h^x u_k^h(x) = f_k^h(x), f_k^h(x) = f^h(y_k,x), y_k = k\tau,$$

$$1 \le k \le N-1, N\tau = T, u_0^h(x) = u_N^h(x) + \varphi^h(x),\ \sum_{i=1}^N u_i^h(x) = 0,\ x \in [0,1]_h. \tag{4.4}$$

Theorem 4.1. *Assume that* $\sum\limits_{i=1}^N f_i^h(x) = 0,\ x \in [0,1]_h$. *Let* τ *and* h *be sufficiently small numbers. Then, the solutions of difference scheme (4.4) satisfy the following estimates*

$$\max_{0\le k\le N} \left\|u_k^h\right\|_{C_h} \le M_1 \left[\max_{1\le k\le N-1} \left\|f_k^h\right\|_{C_h} + ||\varphi^h||_{C_h}\right],$$

$$\|\ \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1}\ \|_{C_\tau(C_h)} + ||\{A_h^x u_k^h\}_1^{N-1}||_{C_\tau(C_h)}$$

$$\le M_1 \left[\ln \frac{1}{\tau+h} ||\{f_k^h\}_1^{N-1}||_{C_\tau(C_h)} + ||\varphi^h||_{C_h^2}\right],$$

$$\|\ \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1}\ \|_{C_\tau\left(C_h^{2\alpha}\right)} + ||\{A_h^x u_k^h\}_1^{N-1}||_{C_\tau\left(C_h^{2\alpha}\right)}$$

$$\le M_2(\alpha)||\{f_k^h\}_1^{N-1}||_{C_\tau\left(C_h^{2\alpha}\right)} + M_1||\varphi^h||_{C_h^{2+2\alpha}}, 0 < \alpha < \frac{1}{2}.$$

Here $M_1, M_2(\alpha)$ *do not depend on* τ, h *and* $f_k^h, 1 \le k \le N-1$ *and* φ^h.

Proof. It is known that (see, [2, 6]) A_h^x is a stronly positive operator in C_h. Therefore, we can replace difference scheme (4.4) by difference problem (2.1). Then, the proof of Theorem 4.1 is based on the abstract Theorems 3.1–3.3, the positivity of the operator A_h^x in C_h, the structure of the fractional spaces $E_\alpha^{'}((A_h^x)^{\frac{1}{2}}, C_h)$ and the following estimate (see, [2, 6])

$$\min\left\{\ln\frac{1}{\tau}, \left|\ln \|A_h^x\|_{C_h\to C_h}\right|\right\} \le M_2 \ln \frac{1}{\tau+h}.$$

□

Second, we consider the boundary value problem on the range $\{0 \leq y \leq T,\ x \in \mathbb{R}^n\}$ for $2m-$order multidimensional elliptic equations

$$\begin{cases} -\frac{\partial^2 u}{\partial y^2} + \sum\limits_{|r|=2m} a_r(x) \frac{\partial^{|\tau|} u}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} + \delta u(y,x) = f(y,x), \\ 0 < y < T,\ x, r \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n, \\ u(0,x) = u(T,x) + \varphi(x), \int\limits_0^T v(s,x) \mathrm{d}s = 0, x \in \mathbb{R}^n, \end{cases} \tag{4.5}$$

where $a_r(x)$ and $f(y,x)$, $\varphi(x)$ are given sufficiently smooth functions and $\alpha_r(x) > 0$, $\delta > 0$ is the sufficiently large number. We will assume that the symbol

$$B^x(\xi) = \sum_{|r|=2m} a_r(x) \left(i\xi_1\right)^{r_1} \cdots \left(i\xi_n\right)^{r_n}, \xi = (\xi_1, \cdot\cdot\cdot, \xi_n) \in R^n$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \tag{4.6}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1 |\xi|^{2m} \leq (-1)^m B^x(\xi) \leq M_2 |\xi|^{2m} < \infty$$

for $\xi \neq 0$. The discretization of problem (5.27) is carried out in two steps. Let us define the grid space $R_h^n\,(0 < h \leq h_0)$ as the set of all points of the Euclidean space R^n whose coordinates are given by

$$x_k = s_k h, \qquad s_k = 0, \pm 1, \pm 2, \ldots,\ k = 1, \cdot\cdot\cdot, n.$$

We introduce the Banach spaces $C_h = C\left(R_h^n\right)$ and $C_h^\alpha = C^\alpha\left(R_h^n\right), 0 \leq \alpha \leq 1$ of the grid functions $\varphi^h(x)$ defined on R_h^n equipped with the norms

$$\left\|\varphi^h\right\|_{C_h} = \max_{x \in R_h^n} \left|\varphi^h(x)\right|,$$

$$\left\|\varphi^h\right\|_{C_h^\alpha} = \left\|\varphi^h\right\|_{C_h} + \sup_{x,y \in R_h^n, x \neq y} \frac{\left|\varphi^h(x) - \varphi^h(y)\right|}{|x-y|^\alpha},$$

respectively. To the differential operator A let us give the difference operator A_h^x by the formula

$$A_h^x u_x^h = \sum_{2m \leq |r| \leq S} b_r^x D_h^r u_x^h + \delta u_x^h.$$

The coefficients are chosen in such a way that the operator A_h^x approximates in a specified way the operator [2]

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} + \delta.$$

We assume that for $|\xi_k h| \leq \pi$ the symbol $A(\xi h, h)$ of the operator $A_h^x - \delta$ satisfies the inequalities

$$(-1)^m A^x(\xi h, h) \geq M_1 |\xi|^{2m}, |\arg A^x(\xi h, h)| \leq \phi < \phi_0 \leq \frac{\pi}{2l}.$$

With the help of A_h^x we arrive at the boundary value problem

$$-\frac{d^2 v^h(y,x)}{\mathrm{d}y^2} + A_h^x v^h(y,x) = f^h(y,x), 0 < y < T, \tag{4.7}$$

$$v^h(0,x) = v^h(T,x) + \varphi^h(x), \int_0^T v^h(s,x)\mathrm{d}s = 0, x \in \mathbb{R}_h^n,$$

for an infinite system of ordinary differential equations. In the second step we replace problem (4.7) by the difference scheme

$$-\frac{1}{\tau^2}(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)) + A_h^x u_k^h(x) = f_k^h(x), f_k^h(x) = f^h(y_k, x), y_k = k\tau,$$

$$1 \leq k \leq N-1, N\tau = T, u_0^h(x) = u_N^h(x) + \varphi^h(x), \ \sum_{i=1}^N u_i^h(x) = 0, \ x \in R_h^n. \tag{4.8}$$

Theorem 4.2. *Assume that* $\sum_{i=1}^N f_i^h(x) = 0, \ x \in R_h^n$.*Let* τ *and* h *be sufficiently small numbers. Then, the solutions of difference scheme (4.8) satisfy the following estimates*

$$\max_{0 \leq k \leq N} \|u_k^h\|_{C_h} \leq M_1 \left[\max_{1 \leq k \leq N-1} \|f_k^h\|_{C_h} + ||\varphi^h||_{C_h}\right],$$

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{C_\tau(C_h)} + ||\{A_h^x u_k^h\}_1^{N-1}||_{C_\tau(C_h)}$$

$$\leq M_1 \left[\ln \frac{1}{\tau + h} ||\{f_k^h\}_1^{N-1}||_{C_\tau(C_h)} + \ln \frac{1}{h} ||\varphi^h||_{C_h^{2m}}\right],$$

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{C_\tau\left(C_h^{2m\alpha}\right)} + ||\{A_h^x u_k^h\}_1^{N-1}||_{C_\tau\left(C_h^{2m\alpha}\right)}$$

$$\leq M_2(\alpha) \left[||\{f_k^h\}_1^{N-1}||_{C_\tau\left(C_h^{2m\alpha}\right)} + ||\varphi^h||_{C_h^{2m+2m\alpha}}\right], 0 < \alpha < \frac{1}{2m}.$$

Here $M_1, M_2(\alpha)$ *do not depend on* τ, h *and* $f_k^h, 1 \leq k \leq N-1$ *and* φ^h.

Proof. It is known that (see, [1, 22]) A_h^x is a stronly positive operator in C_h. Therefore, we can replace difference scheme (4.8) by difference problem (2.1). Then, the proof of Theorem 4.2 is based on the abstract Theorems 3.1–3.3, the positivity of the operator A_h^x in C_h, and on the almost coercivity inequality for an elliptic operator A_h^x in C_h and the following estimate

$$\min\left\{\ln \frac{1}{\tau}, \left|\ln \|A_h^x\|_{C_h \to C_h}\right|\right\} \leq M_2 \ln \frac{1}{\tau + h},$$

and the structure of the fractional spaces $E'_\alpha((A_h^x)^{\frac{1}{2}}, C_h)$ and on the coercivity inequality for an elliptic operator A_h^x in C_h^β (see, [2, 24]). □

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