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ON SPECTRAL AND BOUNDARY PROPERTIES OF THE VOLUME POTENTIAL FOR THE HELMHOLTZ EQUATION☆

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Abstract. In this paper, we study boundary properties and some questions of spectral geometry for certain volume potential type operators (Bessel potential operators) in an open bounded Euclidean domains. In particular, the results can be valid for differential operators, which are related to a nonlocal boundary value problem for the Helmholtz equation, so we obtain isoperimetric inequalities for its eigenvalues as well, namely, analogues of the Rayleigh-Faber-Krahn inequality.

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1. Introduction

The main results of this note consist in showing that (under certain restrictions for indices) the Schatten norms of the Bessel potentials $\mathcal{B}_{\alpha,\Omega}$ over sets of a given measure are maximised on balls. More precisely, we can summarise our results as follows:

– Let $0 < \alpha < d$ and let Ω^* be a ball in $\mathbb{R}^d$. Then for any integer p with $d/\alpha < p \leq \infty$ we have

$$\|\mathcal{B}_{\alpha,\Omega}\|_p \leq \|\mathcal{B}_{\alpha,\Omega^*}\|_p, \tag{1.1}$$

for any domain Ω with $|\Omega| = |\Omega^*|$. Here $\|\cdot\|_p$ is the Schatten p-norm and $|\cdot|$ is the Lebesgue measure. Note that for $p = \infty$ this result gives an analogue of the famous Rayleigh-Faber-Krahn inequality for the Bessel potentials.

– In addition, we construct a new correct boundary value problem for the poly-Helmholtz equation, which is related to the Bessel potential.

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Thus, in $L^2(\Omega)$, $\Omega \subset \mathbb{R}^d$, we consider the Bessel potential operators

$$(\mathcal{B}_{\alpha,\Omega} f)(x) := \int_\Omega \varepsilon_{\alpha,d}(|x-y|) f(y) \mathrm{d}y, \quad f \in L^2(\Omega), \quad 0 < \alpha < d, \tag{1.2}$$

where the kernel is

$$\varepsilon_{\alpha,d}(|x-y|) = c_{\alpha,d} \frac{K_{\frac{d-\alpha}{2}}(|x-y|)}{|x-y|^{\frac{d-\alpha}{2}}}, \tag{1.3}$$

and $c_{\alpha,d}$ is a positive constant,

$$c_{\alpha,d} = \frac{2^{\frac{2-m-\alpha}{2}}}{\pi^{d/2}\Gamma(\alpha/2)}.$$

Here K_ν is the McDonald function (the modified Bessel function of the second kind):

$$K_\nu(z) = \frac{\pi}{2\sin \nu\pi} \left(\mathcal{I}_{-\nu}(z) - \mathcal{I}_\nu(z) \right), \quad \nu \neq 0, \pm 1, \pm 2, \ldots,$$

$$K_n(z) = \lim_{\nu \to n} K_\nu(z), \quad n = 0, \pm 1, \pm 2, \ldots,$$

and

$$\mathcal{I}_\nu = \Sigma_{k=0}^\infty \frac{\left(\frac{z}{2}\right)^{\nu+2k}}{k!\Gamma(\nu+k+1)}.$$

In particular, for an even integer $\alpha = 2m$ with $0 < m < d/2$, the kernel $\varepsilon_{2m,d}(|x|)$ is the fundamental solution to the poly-Helmholtz equation of order $2m$ in $\mathbb{R}^d$:

$$(I - \Delta_x)^{\alpha/2} \varepsilon_{\alpha,d}(|x-y|) = \delta_y,$$

where I is the identity operator, Δ_x is the Laplacian with respect to the variable $x \in \mathbb{R}^d$, and δ_y is the Dirac distribution at the point $y \in \mathbb{R}^d$.

For the Laplacian, under the assumption of a sufficient regularity of the boundary of Ω (for example, piecewise C^1), it is known, see *e.g.* [8], that the integral (the Newton potential)

$$u(x) = \int_\Omega \varepsilon(|x-y|) f(y) \mathrm{d}y \tag{1.4}$$

is equivalent to the Poisson equation

$$-\Delta u(x) = f(x), \quad x \in \Omega, \tag{1.5}$$

with the following (nonlocal) integral boundary condition

$$-\frac{1}{2} u(x) + \int_{\partial\Omega} \frac{\partial \varepsilon(|x-y|)}{\partial n_y} u(y) \mathrm{d}S_y - \int_{\partial\Omega} \varepsilon(|x-y|) \frac{\partial u(y)}{\partial n_y} \mathrm{d}S_y = 0, \ x \in \partial\Omega, \tag{1.6}$$

where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative at the point $y \in \partial\Omega$.

In a bounded connected domain $\Omega \subset \mathbb{R}^d$ with a piecewise C^1 boundary $\partial\Omega$, as an analogue to (1.5) we consider the poly-Helmholtz equation

$$\mathcal{L}u(x) := (I - \Delta_x)^m u(x) = f(x), \quad x \in \Omega, \quad m \in \mathbb{N}. \tag{1.7}$$

To relate the (polyharmonic) volume potential (1.2) ($\alpha = 2m$ and $0 < m < d/2$) to the boundary value problem (1.7) in Ω, we show that for any $f \in L^2(\Omega)$, the (polyharmonic) volume potential (1.2) belongs to the functional class $H^{2m}(\Omega)$ and satisfies the following nonlocal integral boundary conditions

$$-\frac{1}{2}\mathcal{L}^i u(x) + \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \mathcal{L}^{m-i-1-j} \varepsilon_{2(m-i),d}(|x-y|) \mathcal{L}^{j+i} u(y) \mathrm{d}S_y - \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{m-i-1-j} \varepsilon_{2(m-i),d}(|x-y|) \frac{\partial}{\partial n_y} \mathcal{L}^{j+i} u(y) \mathrm{d}S_y = 0, \ x \in \partial\Omega, \tag{1.8}$$

for $i = 0, 1, \ldots, m-1$. Conversely, if $u \in H^{2m}(\Omega)$ satisfies (1.7) and the boundary conditions (1.8) for $i = 0, 1, \ldots, m-1$, then it coincides with the (polyharmonic) volume potential (defined by the formula (1.2)).

Therefore, our analysis of the (polyharmonic) volume potential (1.2) yields corresponding result for the boundary value problem (1.7) and (1.8). Note that the analogue of the boundary value problem (1.7) and (1.8) for polyharmonic Newton potentials was studied in [9], for the Kohn Laplacian and its powers on the Heisenberg group in [15] and for the stratified group sub-Laplacians in [18].

In the recent paper [13], spectral geometry results as mentioned were obtained for the Riesz potentials. A similar spectral geometric extremum problem for the logarithmic potential has been investigated in [14]. However, we believe that because of the importance of the Bessel potential the results should be presented separately and, in addition, as we mentioned we also study some connections with the boundary value problems of the (poly-)Helmholtz equation. We also note that the obtained results can be extended to more general Bessel potential type operators (*cf.* [16, 17]). The basic reason why these type of results are useful, beyond the intrinsic interest of geometric extremum problems, is that they give an exact *a priori* bounds for spectral invariants of operators on arbitrary domains.

In Section 2 we discuss some preliminary properties of the Bessel potential operator and formulate the main results of this note. Their proofs will be given in Section 3.

2. Main results

2.1. Preliminaries

We consider the spectral problem for the Bessel potentials

$$\mathcal{B}_{\alpha,\Omega} u = \int_\Omega \varepsilon_{\alpha,d}(|x-y|) u(y) \mathrm{d}y = \lambda u, \ u \in L^2(\Omega), \tag{2.1}$$

where

$$\varepsilon_{\alpha,d}(|x-y|) := c_{\alpha,d} \frac{K_{\frac{d-\alpha}{2}}(|x-y|)}{|x-y|^{\frac{d-\alpha}{2}}}$$

and $0 < \alpha < d$. The indices α, d and Ω in the notations can be sometimes dropped if it does not cause a confusion. As discussed in the introduction it is equivalent to considering the spectrum of the operator corresponding to the boundary value problem (1.7) and (1.8), which we call $\mathcal{L} = \mathcal{L}_{2m,\Omega}$ in a bounded connected domain $\Omega \subset \mathbb{R}^d$ with a piecewise C^1 continuous boundary $\partial\Omega$, thus,

$$\mathcal{L}^m u(x) = \lambda^{-1} u(x),\ x \in \Omega,\ m \in \mathbb{N}, \tag{2.2}$$

with the nonlocal boundary conditions (1.8).

Since Ω has finite Lebesgue measure, by using the Schur test we see that $\mathcal{B}_{\alpha,\Omega}$ is bounded in $L^2(\Omega)$. Moreover, this operator is compact in $L^2(\Omega)$ as well and belongs to certain Schatten classes $\mathfrak{S}^p$, *cf.* (2.4). Since the Bessel kernel is symmetric, the operator $\mathcal{B}_{\alpha,\Omega}$ is also self-adjoint.

Recall that the norm in the Schatten class $\mathfrak{S}^p$ (the Schatten p-norm) of a compact operator T is defined as

$$\|T\|_p = \left(\sum_{j=1}^{\infty} s_j(T)^p \right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \tag{2.3}$$

for $s_1 \ge s_2 \ge \cdots > 0$ being the singular values of T. For $p = \infty$, it is usual to set

$$\|T\|_\infty := \|T\|,$$

i.e., the operator norm of T in $L^2(\Omega)$. For compact self-adjoint operators the singular values are equal to the moduli of (nonzero) eigenvalues, and the corresponding eigenfunctions form a (complete orthogonal) basis on L^2. Since the Bessel operators are non-negative, the words 'moduli of' in the previous sentence can be deleted in our case.

Thus, the eigenvalues of $\mathcal{B}_{\alpha,\Omega}$ may be enumerated in the descending order,

$$\lambda_1 \ge \lambda_2 \ge \cdots$$

where λ_j is repeated in this series according to its multiplicity. We denote the corresponding eigenfunctions by $u_1, u_2, \ldots$, so that for each eigenvalue λ_j one and only one corresponding (normalised) eigenfunction u_j is fixed,

$$\mathcal{B}_{\alpha,\Omega} u_j = \lambda_j u_j, \quad j = 1, 2, \ldots$$

Recall the operator $\mathcal{B}_{\alpha,\Omega}$ is nonnegative, means, in particular, that all eigenvalues are nonnegative and equal to its singular values

$$\lambda_j \equiv \lambda_j(\mathcal{B}_{\alpha,\Omega}) = |\lambda_j(\mathcal{B}_{\alpha,\Omega})| = s_j.$$

Moreover, for each eigenvalue λ_j holds the inequality (see, *e.g.* [6], Eq. (1))

$$\lambda_j \le C|\Omega|^{\vartheta} j^{-\vartheta}, \tag{2.4}$$

where $\vartheta = \alpha/d$. In particular, this implies the compactness of the operator in $\Omega \subset \mathbb{R}^d$, which is a measurable set with finite Lebesgue measure (throughout this note).

This also means that the Bessel operator $\mathcal{B}_{\alpha,\Omega}$ belongs to all Schatten classes $\mathfrak{S}^p$ with $p > \alpha/d$ and

$$\|\mathcal{B}_{\alpha,\Omega}\|_p = \left(\sum_{j=1}^{\infty} \lambda_j(\Omega)^p \right)^{\frac{1}{p}},\ 1 \le p < \infty. \tag{2.5}$$

Furthermore, we apply Theorem 2.4 in [7] to the Bessel operator kernel $K(x,y) = \varepsilon_{\alpha,d}(|x-y|)$, $x, y \in \Omega$, that is, since the measure of Ω is finite, the kernel $K(x,y)$ belongs to $L^{p',p}(\Omega\times\Omega)$ for any $p > \frac{d}{\alpha}$. Therefore, for $s > \frac{d}{\alpha}$ we have

$$\sum \lambda_j(\mathcal{B}_{\alpha,\Omega})^s = \mathrm{Tr}(\mathcal{B}^s_{\alpha,\Omega}) = \int\limits_{\Omega^s} \left(\prod_{k=1}^{s} K(x_k, x_{k+1})\right) \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_s,\; x_{s+1} \equiv x_1. \tag{2.6}$$

In fact, for the Bessel operator $\mathcal{B}_{\alpha,\Omega}$, the inequality (2.4) is accompanied by an asymptotic formula with an explicitly given constant. For a bounded set, this asymptotics is a particular case of general results of Birman and Solomyak [4] concerning integral operators with weak polarity in the kernel (*cf.* [13], Rem. 2.2). One can easily dispose of this boundedness condition, using the estimate (2.4) and the asymptotic approximation procedure, like this was done many times since early seventeenth (see, for instance, in [5, 12]).

2.2. Formulation of main results

We now formulate the main results of this note.

Theorem 2.1. *For any integer p with $\frac{d}{\alpha} < p \leq \infty$, we have*

$$\|\mathcal{B}_{\alpha,\Omega}\|_p \leq \|\mathcal{B}_{\alpha,\Omega^*}\|_p, \tag{2.7}$$

for any domain Ω with $|\Omega| = |\Omega^*|$, where Ω^* is a ball in $\mathbb{R}^d$.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with a piecewise smooth boundary $\partial\Omega \in C^1$. For $m \in \mathbb{N}$, we denote

$$\mathcal{L}^m := \mathcal{L}\mathcal{L}^{m-1}, \quad m = 2, 3, \dots,$$

with

$$\mathcal{L} = I - \Delta.$$

Then for $m = 1, 2, \dots$, we consider the equation

$$\mathcal{L}^m u(x) = f(x),\; x \in \Omega, \tag{2.8}$$

for a given $f \in L^2(\Omega)$ and

$$u(x) = \int_\Omega f(y)\varepsilon_{2m}(|x-y|)\mathrm{d}y \tag{2.9}$$

in $\Omega \subset \mathbb{R}^d$, where $\varepsilon_{2m}(|x-y|)$ is a fundamental solution of (2.8).

A simple calculation shows that the generalised volume potential (2.9) is a solution of (2.8) in Ω. Also it is known that if $f \in L^2(\Omega)$, then $u \in H^{2m}(\Omega)$ (see, *e.g.* [1]). One of our aims is to find a boundary condition on $\partial\Omega$ such that with this boundary condition the equation (2.8) has a unique solution in $H^{2m}(\Omega)$, which coincides with (2.9). Here and after $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative at the point $y \in \partial\Omega$.

Theorem 2.2. *For any $f \in L^2(\Omega)$, the generalised volume potential (2.9) is a unique solution of the equation (2.8) in $H^{2m}(\Omega) \cap H^{2m-1}(\overline{\Omega})$ with m boundary conditions*

$$-\frac{\mathcal{L}^i u(x)}{2}+\sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathcal{L}^{j+i}u(y)\frac{\partial}{\partial n_y}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\mathrm{d}S_y$$
$$-\sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{j+i}u(y)\mathrm{d}y=0,\quad x\in\partial\Omega,\quad (2.10)$$

for all $i=0,1,\dots,m-1$.

3. Proofs of Theorem 2.1 and Theorem 2.2

3.1. Proof of Theorem 2.1

Since the integral kernel of $\mathcal{B}_{\alpha,\Omega}$ is positive, the statement, sometimes called Jentsch's theorem, applies, see, *e.g.*, [12].

Lemma 3.1. *The eigenvalue* λ_1 *of* $\mathcal{B}_{\alpha,\Omega}$ *with the largest modulus is positive and simple; the corresponding eigenfunction* u_1 *is positive, and any other eigenfunction* $u_j,\ j>1$, *is sign changing in* Ω.

Note that the positivity of λ_1 is already known, since the operator $\mathcal{B}_{\alpha,\Omega}$ is nonnegative; what is important is the positivity of u_1.

Now we prove the following analogue of the famous Rayleigh-Faber-Krahn theorem for the operator $\mathcal{B}_{\alpha,\Omega}$. See [3] for a general discussion of this subject.

Lemma 3.2. *The ball* Ω^* *is a maximiser of the first eigenvalue of the operator* $\mathcal{B}_{\alpha,\Omega}$ *among all domains of a given volume, i.e.*

$$0<\lambda_1(\Omega)\le\lambda_1(\Omega^*)$$

for an arbitrary domain $\Omega\subset\mathbb{R}^d$ *with* $|\Omega|=|\Omega^*|$.

Remark 3.3. In other words Lemma 3.2 says that the operator norm of $\mathcal{B}_{\alpha,\Omega}$ is maximised in the ball among all Euclidean domains of a given volume.

Proof of Lemma 3.2. Let Ω be a bounded measurable set in $\mathbb{R}^d$. Its symmetric rearrangement Ω^* is an open ball centred at 0 with the measure equal to the measure of Ω, *i.e.* $|\Omega^*|=|\Omega|$. Let u be a nonnegative measurable function in Ω, such that all its positive level sets have finite measure. With the definition of the symmetric-decreasing rearrangement of u we can use the layer-cake decomposition [11], which expresses a nonnegative function u in terms of its level sets as

$$u(x)=\int_0^\infty \chi_{\{u(x)>t\}}\mathrm{d}t,\tag{3.1}$$

where χ is the characteristic function of the corresponding domain. The function

$$u^*(x):=\int_0^\infty \chi_{\{u(x)>t\}^*}\mathrm{d}t\tag{3.2}$$

is called the (radially) symmetric-decreasing rearrangement of a nonnegative measurable function u.

Recalling the Riesz inequality [11] and the fact that $\varepsilon_\alpha(|x-y|)$ is a symmetric-decreasing function, we obtain

$$\int_\Omega\int_\Omega u_1(y)\varepsilon_\alpha(|y-x|)u_1(x)\mathrm{d}y\mathrm{d}x \le \int_{\Omega^*}\int_{\Omega^*} u_1^*(y)\varepsilon_\alpha(|y-x|)u_1^*(x)\mathrm{d}y\mathrm{d}x. \tag{3.3}$$

In addition, for each nonnegative function $u \in L^2(\Omega)$ we have

$$\|u\|_{L^2(\Omega)} = \|u^*\|_{L^2(\Omega^*)}. \tag{3.4}$$

Therefore, from (3.3), (3.4) and the variational principle for $\lambda_1(\Omega^*)$, we get

$$\begin{aligned}\lambda_1(\Omega) &= \frac{\int_\Omega\int_\Omega u_1(y)\varepsilon_\alpha(|y-x|)u_1(x)\mathrm{d}y\mathrm{d}x}{\int_\Omega |u_1(x)|^2\mathrm{d}x} \le \frac{\int_{\Omega^*}\int_{\Omega^*} u_1^*(y)\varepsilon_\alpha(|y-x|)u_1^*(x)\mathrm{d}y\mathrm{d}x}{\int_{\Omega^*}|u_1^*(x)|^2\mathrm{d}x}\\ &\le \sup_{v\in L^2(\Omega^*), v\neq 0}\frac{\int_{\Omega^*}\int_{\Omega^*} v(y)\varepsilon_\alpha(|y-x|)v(x)\mathrm{d}y\mathrm{d}x}{\int_{\Omega^*}|v(x)|^2\mathrm{d}x} = \lambda_1(\Omega^*),\end{aligned}$$

completing the proof. □

Now we can finish the Proof of Theorem 2.1. For integer values of $p > \frac{d}{\alpha}$ by the formula (2.5) we have

$$\sum_{j=1}^\infty \lambda_j^p(\Omega) = \int_\Omega \cdots \int_\Omega \varepsilon_\alpha(|y_1-y_2|)\cdots\varepsilon_\alpha(|y_p-y_1|)\mathrm{d}y_1\cdots\mathrm{d}y_p, \quad p > \frac{d}{\alpha},\ p\in\mathbb{N}. \tag{3.5}$$

It follows from the Brascamp-Lieb-Luttinger inequality [2] that

$$\int_{\Omega^*}\cdots\int_{\Omega^p} \varepsilon_\alpha(|y_1-y_2|)\ldots\varepsilon_\alpha(|y_p-y_1|)\mathrm{d}y_1\ldots\mathrm{d}y_p \le \int_{\Omega*} \varepsilon_\alpha(|y_1-y_2|)\ldots\varepsilon_\alpha(|y_p-y_1|)\mathrm{d}y_1\ldots\mathrm{d}y_p, \tag{3.6}$$

which proves

$$\sum_{j=1}^\infty \lambda_j^p(\Omega) \le \sum_{j=1}^\infty \lambda_j^p(\Omega^*),\ \ p\in\mathbb{N},\ \ p > \frac{d}{\alpha}, \tag{3.7}$$

for $\Omega \subset \mathbb{R}^d$ with $|\Omega| = |\Omega^*|$. Here we have used that the kernel ε_α is a symmetric-decreasing function in $\Omega^*\times\Omega^*$, *i.e.*

$$\varepsilon_\alpha^*(|x-y|) = \varepsilon_\alpha(|x-y|), \quad x,y \in \Omega^*\times\Omega^*.$$

The case of $p=\infty$ is already proved in Lemma 3.2. The proof is complete.

3.2. Proof of Theorem 2.2

Proof. By applying Green's second formula for each $x\in\Omega$, we obtain

$$\begin{aligned}u(x) &= \int_\Omega f(y)\varepsilon_{2m}(|x-y|)\mathrm{d}y = \int_\Omega \mathcal{L}^m u(y)\varepsilon_{2m}(|x-y|)\mathrm{d}y\\ &= \int_\Omega \mathcal{L}^{m-1}u(y)\mathcal{L}\varepsilon_{2m}(|x-y|)\mathrm{d}y - \int_{\partial\Omega}\mathcal{L}^{m-1}u(y)\frac{\partial}{\partial n_y}\varepsilon_{2m}(|x-y|)\mathrm{d}S_y\end{aligned}$$

$$
\begin{aligned}
&+\int_{\partial\Omega}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{m-1}u(y)\mathrm{d}S_y=\int_{\Omega}\mathcal{L}^{m-2}u(y)\mathcal{L}^2\varepsilon_{2m}(|x-y|)\mathrm{d}y\\
&-\int_{\partial\Omega}\mathcal{L}^{m-2}u(y)\frac{\partial}{\partial n_y}\mathcal{L}\varepsilon_{2m}(|x-y|)\mathrm{d}S_y+\int_{\partial\Omega}\mathcal{L}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{m-2}u(y)\mathrm{d}S_y\\
&-\int_{\partial\Omega}\mathcal{L}^{m-1}u(y)\frac{\partial}{\partial n_y}\varepsilon_{2m}(|x-y|)\mathrm{d}S_y+\int_{\partial\Omega}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{m-1}u(y)\mathrm{d}S_y=\dots\\
=\;&u(x)-\sum_{j=0}^{m-1}\int_{\partial\Omega}\mathcal{L}^{j}u(y)\frac{\partial}{\partial n_y}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\mathrm{d}S_y\\
&+\sum_{j=0}^{m-1}\int_{\partial\Omega}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{j}u(y)\mathrm{d}S_y,\quad x\in\Omega,
\end{aligned}
$$

where $\frac{\partial}{\partial n_y}=n_1\frac{\partial}{\partial y_1}+\cdots+n_n\frac{\partial}{\partial y_n}$ is the outer normal derivative at the point $y\in\partial\Omega$ and $n_1,\dots,n_n$ are the coordinates of the unit outer normal.

This implies the identity

$$
\begin{aligned}
&\sum_{j=0}^{m-1}\int_{\partial\Omega}\mathcal{L}^{j}u(y)\frac{\partial}{\partial n_y}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\mathrm{d}S_y\\
&\qquad-\sum_{j=0}^{m-1}\int_{\partial\Omega}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{j}u(y)\mathrm{d}S_y=0,\quad x\in\Omega.
\end{aligned}\tag{3.8}
$$

All these integrals are continuous (single layer) integrals except the first one, that is,

$$
\int_{\partial\Omega}u(y)\frac{\partial}{\partial n_y}\mathcal{L}^{m-1}\varepsilon_{2m}(|x-y|)\mathrm{d}S_y
$$

is the double layer potential for the Helmholtz equation (see [9] for the iterated Laplace equations and [19] for the interated heat equations). By using the properties of the double and single layer potentials as x approaches the boundary $\partial\Omega$ from the interior (see [10]), from (3.8) we obtain

$$
\begin{aligned}
&-\frac{u(x)}{2}+\sum_{j=0}^{m-1}\int_{\Omega}\mathcal{L}^{j}u(y)\frac{\partial}{\partial n_y}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\mathrm{d}S_y\\
&\qquad-\sum_{j=0}^{m-1}\int_{\partial\Omega}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{j}u(y)\mathrm{d}S_y=0,\quad x\in\partial\Omega.
\end{aligned}
$$

Thus, this relation is one of the boundary conditions of (2.9). Let us derive the remaining boundary conditions. To this end, we write

$$
\mathcal{L}^{m-i}\mathcal{L}^{i}u=f,\quad i=0,1,\dots,m-1,\quad m=1,2,\dots,\tag{3.9}
$$

and carry out similar calculations just as above. This yields

$$
\mathcal{L}^{i}u(x)=\int_{\Omega}f(y)\mathcal{L}^{i}\varepsilon_{2m}(|x-y|)\mathrm{d}y=\int_{\Omega}\mathcal{L}^{m-i}\mathcal{L}^{i}u(y)\mathcal{L}^{i}\varepsilon_{2m}(|x-y|)\mathrm{d}y
$$

$$
\begin{aligned}
&= \int_\Omega \mathcal{L}^{m-i-1}\mathcal{L}^i u(y)\mathcal{L}\mathcal{L}^i\varepsilon_{2m}(|x-y|)\mathrm{d}y - \int_{\partial\Omega}\mathcal{L}^{m-i-1}\mathcal{L}^i u(y)\frac{\partial}{\partial n_y}\mathcal{L}^i\varepsilon_{2m}(|x-y|)\mathrm{d}S_y \\
&\quad + \int_{\partial\Omega}\mathcal{L}^i\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{m-i-1}\mathcal{L}^i u(y)\mathrm{d}S_y = \int_\Omega \mathcal{L}^{m-i-2}\mathcal{L}^i u(y)\mathcal{L}^2\mathcal{L}^i\varepsilon_{2m}(|x-y|)\mathrm{d}y \\
&\quad - \int_{\partial\Omega}\mathcal{L}^{m-i-2}\mathcal{L}^i u(y)\frac{\partial}{\partial n_y}\mathcal{L}\mathcal{L}^i\varepsilon_{2m}(|x-y|)\mathrm{d}S_y + \int_{\partial\Omega}\mathcal{L}\mathcal{L}^i\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{m-i-2}\mathcal{L}^i u(y)\mathrm{d}S_y \\
&\quad - \int_{\partial\Omega}\mathcal{L}^{m-i-1}\mathcal{L}^i u(y)\frac{\partial}{\partial n_y}\mathcal{L}^i\varepsilon_{2m}(|x-y|)\mathrm{d}S_y + \int_{\partial\Omega}\mathcal{L}^i\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{m-i-1}\mathcal{L}^i u(y)\mathrm{d}S_y \\
&= \cdots = \int_\Omega \mathcal{L}^i u(y)\mathcal{L}^{m-i}\mathcal{L}^i\varepsilon_{2m}(|x-y|)\mathrm{d}y \\
&\quad - \sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathcal{L}^j\mathcal{L}^i u(y)\frac{\partial}{\partial n_y}\mathcal{L}^{m-i-1-j}\mathcal{L}^i\varepsilon_{2m}(|x-y|)\mathrm{d}S_y \\
&\quad + \sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathcal{L}^{m-i-1-j}\mathcal{L}^i\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^j\mathcal{L}^i u(y)\mathrm{d}S_y \\
&= \mathcal{L}^i u(x) - \sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathcal{L}^{j+i}u(y)\frac{\partial}{\partial n_y}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\mathrm{d}S_y \\
&\quad + \sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{j+i}u(y)\mathrm{d}S_y, \quad x\in\Omega,
\end{aligned}
$$

where, $\mathcal{L}^i\varepsilon_m$ is a fundamental solution of the equation (3.9), *i.e.*,

$$\mathcal{L}^{m-i}\mathcal{L}^i\varepsilon_{2m} = \delta, \qquad i = 0, 1, \ldots, m-1.$$

From the previous relations, we obtain the identities

$$\sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathcal{L}^{j+i}u(y)\frac{\partial}{\partial n_y}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\mathrm{d}S_y - \sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{j+i}u(y)\mathrm{d}S_y = 0$$

for any $x \in \Omega$ and $i = 0, 1, \ldots, m-1$. By using the properties of the double and single layer potentials as x approaches the boundary $\partial\Omega$ from the interior of Ω, we find that

$$
\begin{aligned}
-\frac{\mathcal{L}^i u(x)}{2} &+ \sum_{j=0}^{m-i-1}\int_\Omega \mathcal{L}^{j+i}u(y)\frac{\partial}{\partial n_y}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\mathrm{d}S_y \\
&- \sum_{j=0}^{m-i-1}\int_{\partial\Omega}\mathcal{L}^{m-1-j}\varepsilon_{2m}(|x-y|)\frac{\partial}{\partial n_y}\mathcal{L}^{j+i}u(y)\mathrm{d}S_y = 0, \quad x \in \partial\Omega,
\end{aligned}
$$

are all boundary conditions of (2.9) for each $i = 0, 1, \ldots, m-1$.

Conversely, let us show that if a function $w \in H^{2m}(\Omega)\cap H^{2m-1}(\overline{\Omega})$ satisfies the equation $\mathcal{L}^m w = f$ and the boundary conditions (2.10), then it coincides with the solution (2.9). Indeed, otherwise the function

$$v = u - w \in H^{2m}(\Omega)\cap H^{2m-1}(\overline{\Omega}),$$

where u is the generalised volume potential (2.9), satisfies the homogeneous equation

$$\mathcal{L}^m v = 0 \tag{3.10}$$

and the boundary conditions (2.10), *i.e.*

$$I_i(v)(x) := -\frac{\mathcal{L}^i v(x)}{2} + \sum_{j=0}^{m-i-1} \int_{\Omega} \mathcal{L}^{j+i} v(y) \frac{\partial}{\partial n_y} \mathcal{L}^{m-1-j} \varepsilon_{2m}(|x-y|) \mathrm{d}S_y$$
$$- \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{m-1-j} \varepsilon_{2m}(|x-y|) \frac{\partial}{\partial n_y} \mathcal{L}^{j+i} v(y) \mathrm{d}S_y = 0, \quad i = 0, 1, \dots, m-1,$$

for $x \in \partial\Omega$. By applying the Green formula to the function $v \in H^{2m}(\Omega) \cap H^{2m-1}(\overline{\Omega})$ and by following the lines of the above argument, we obtain

$$\begin{aligned}
0 &= \int_{\Omega} \mathcal{L}^m v(x) \mathcal{L}^i \varepsilon_{2m}(|x-y|) \mathrm{d}y \\
&= \int_{\Omega} \mathcal{L}^{m-i} \mathcal{L}^i v(x) \mathcal{L}^i \varepsilon_{2m}(|x-y|) \mathrm{d}y \\
&= \int_{\Omega} \mathcal{L}^{m-1} v(x) \mathcal{L}\mathcal{L}^i \varepsilon_{2m}(|x-y|) \mathrm{d}y \\
&\quad - \int_{\partial\Omega} \mathcal{L}^{m-1} v(x) \frac{\partial}{\partial n_y} \mathcal{L}^i \varepsilon_{2m}(|x-y|) \mathrm{d}S_y \\
&\quad + \int_{\partial\Omega} \mathcal{L}^i \varepsilon_{2m}(|x-y|) \frac{\partial}{\partial n_y} \mathcal{L}^{m-1} v(x) \mathrm{d}S_y = \dots \\
&= \mathcal{L}^i v(x) - \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{j+i} v(y) \frac{\partial}{\partial n_y} \mathcal{L}^{m-1-j} \varepsilon_{2m}(|x-y|) \mathrm{d}S_y \\
&\quad + \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{m-1-j} \varepsilon_{2m}(|x-y|) \frac{\partial}{\partial n_y} \mathcal{L}^{j+i} v(y) \mathrm{d}S_y, \quad i = 0, 1, \dots, m-1.
\end{aligned}$$

If we pass to the limit as $x \to \partial\Omega$, only the integral with the integrant $\frac{\partial}{\partial n_y} \mathcal{L}^{m-1} \varepsilon_{2m}(|x-y|)$ gives the jump relation since this integral is the double layer potential and the all other integrals are continuous functions of x. Thus, by passing to the limit as $x \to \partial\Omega$ we obtain the relations

$$0 = \mathcal{L}^i v(x) + \frac{\mathcal{L}^i v(x)}{2} - \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{j+i} v(y) \frac{\partial}{\partial n_y} \mathcal{L}^{m-1-j} \varepsilon_{2m}(|x-y|) \mathrm{d}S_y$$
$$+ \sum_{j=0}^{m-i-1} \int_{\partial\Omega} \mathcal{L}^{m-1-j} \varepsilon_{2m}(|x-y|) \frac{\partial}{\partial n_y} \mathcal{L}^{j+i} v(y) \mathrm{d}S_y, \quad x \in \partial\Omega, \quad i = 0, 1, \dots, m-1.$$

That is

$$\mathcal{L}^i v(x) \mid_{x \in \partial\Omega} = I_i(v)(x) \mid_{x \in \partial\Omega} = 0, \quad x \in \partial\Omega, \quad i = 0, 1, \dots, m-1. \tag{3.11}$$

Assuming for the moment the uniqueness of the solution of the boundary value problem

$$\mathcal{L}^m v = 0, \tag{3.12}$$

$$\mathcal{L}^i v \mid_{\partial\Omega} = 0, \quad i = 0, 1, \ldots, m-1,$$

we get that $v = u - w \equiv 0$, for all $x \in \Omega$, *i.e.* w coincides with u in Ω. Thus (2.9) is the unique solution of the boundary value problem (2.8), (2.10) in Ω.

It remains to argue that the boundary value problem (3.12) has a unique solution in $H^{2m}(\Omega) \cap H^{2m-1}(\overline{\Omega})$. Denoting $\tilde{v} := \mathcal{L}^{m-1} v$, this follows by induction from the uniqueness in $C^2(\Omega) \cap C^1(\overline{\Omega})$ of the problem

$$\mathcal{L}\tilde{v} = 0, \quad \tilde{v} \mid_{\partial\Omega} = 0.$$

The proof of Theorem 2.2 is complete. □

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