Math. Model. Nat. Phenom. 14 (2019) 504
https://doi.org/10.1051/mmnp/2019036
Mathematical Modelling of Natural Phenomena
www.mmnp-journal.org

FRONTS IN SUBDIFFUSIVE FITZHUGH–NAGUMO SYSTEMS

A.A. Nepomnyashchy[1,2] and V.A. Volpert[2,*]

Abstract. Front solutions of subdiffusive FitzHugh–Nagumo equations are studied for a piece-wise linear nonlinearity. Multiple solutions of the problem and the dependence of the propagation velocity on the parameters are discussed.

Mathematics Subject Classification. 35C07, 92C20, 60J90.

Received January 4, 2019. Accepted August 2, 2019.

1. Introduction

The significance of FitzHugh–Nagumo equations [7, 12]

$$\partial_t v = \partial_x^2 v + r(v) - w, \quad \partial_t w = \epsilon(v - bw)$$

has long outgrown its original application to nerve conduction. The equations have become a classical example of an excitable system that exhibits plethora of propagation phenomena, including pulses of various structures, fronts, and N-front traveling waves [1, 5, 8, 9, 14, 15, 18]. From the analysis viewpoint the system is attractive as it can be treated for small ϵ as a singular perturbation of a bistable Nagumo equation with a cubic nonlinearity [2], which is well understood [6], or a piecewise linear problem for a special choice of $r(v)$ [9, 14, 15].

Lately there is some interest in considering these equations when the diffusion process is anomalous [10, 11]. In particular, traveling waves in a superdiffusive medium are considered in [16, 17], while traveling pulses in a subdiffusive version of the problem are qualitatively discussed and numerically computed in [3, 4].

In this paper we study a subdiffusive version of the FitzHugh–Nagumo system with the piecewise linear reaction function $r(v) = -v + H(v - a)$ (*cf.* [9, 14, 15]), where H is the Heaviside function:

$$\partial_t^\alpha v = \partial_x^2 v - v + H(v - a) - w, \tag{1.1a}$$

$$\partial_t w = \epsilon(v - bw). \tag{1.1b}$$

Here ∂_t^α is the Caputo fractional derivative,

$$\partial_t^\alpha v = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{v_\tau(x,\tau)}{(t-\tau)^\alpha}\, \mathrm{d}\tau, \quad 0 \le \alpha < 1,$$

Keywords and phrases: Traveling wave, subdiffusion, FitzHugh–Nagumo equations, exact solution.

[1] Technion – Israel Institute of Technology, Haifa 32000, Israel.
[2] Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston IL 60208-3125, USA.

[*] Corresponding author: v-volpert@northwestern.edu

and $\epsilon > 0$ and $b \geq 0$ are problem parameters. We assume that for $\alpha > 0$ parameter a is such that $0 < a < b/(b+1)$. (The case $\alpha = 0$ is somewhat different, see below.) This condition guarantees that the problem has a critical point $v = v_* = b/(1+b)$, $w = w_* = 1/(1+b)$ (referred to as the excited state), in addition to the critical point $v = w = 0$ (the rest state).

We seek traveling wave solutions that connect the rest state and the excited state.

2. Traveling wave solutions

Suppose the rest state is located at $-\infty$ while the excited state is located at $+\infty$. Let us consider the wave in which the rest state is displaced by the excited state. In the coordinate system moving to the left together with the wave with the speed $c > 0$

$$v = v(z), \quad w = w(z); \quad z = x + ct, \quad c > 0 \tag{2.1}$$

we have (*cf.* [13])

$$c^\alpha \partial_z^\alpha v = \partial_z^2 v - v + H(v - a) - w, \tag{2.2a}$$

$$c\partial_z w = \epsilon(v - bw), \tag{2.2b}$$

$$z \to -\infty: \quad v \to 0, \quad w \to 0; \quad z \to \infty: \quad v \to v_*, \quad w \to w_*. \tag{2.2c}$$

We remark that it is sufficient to consider the $c > 0$ case because the $c < 0$ case, *i.e.*, when the excited state is displaced by the rest state can be obtained from the former one by a simple transformation. Indeed, introducing

$$v(z) = \frac{b}{1+b} - V(Z), \quad w(z) = \frac{1}{1+b} - W(Z), \quad Z = -z = (-x) + (-c)t,$$

we obtain

$$(-c)^\alpha \partial_Z^\alpha V = \partial_Z^2 V - V + H\left(V + a - \frac{b}{1+b}\right) - W,$$

$$(-c)\partial_Z W = \epsilon(V - bW),$$

$$Z \to -\infty: \quad V \to 0, \quad W \to 0; \quad Z \to \infty: \quad V \to v_*, \quad W \to w_*.$$

This calculation shows that it is sufficient to consider solutions with $c > 0$ because any solution with $c = -c_0 < 0$ and some value of $a = a_0$ can be obtained by the above transformation from the solution with $c = c_0 > 0$ and $a = b/(1+b) - a_0$.

We will seek solutions such that there is only one spatial location at which $v = a$. Due to the translation invariance of the problem we can assume that $v = a$ at $z = 0$ with $v < a$ for $z < 0$ and $v > a$ for $z > 0$. Let us denote $v = v_1$, $w = w_1$ in the region $z \leq 0$ and $v = v_2$, $w = w_2$ in the region $z \geq 0$.

In the region $z < 0$, we obtain the system of equations

$$c^\alpha \partial_z^\alpha v_1 = \partial_z^2 v_1 - v_1 - w_1, \tag{2.3a}$$

$$c\partial_z w_1 = \epsilon(v_1 - bw_1), \tag{2.3b}$$

with the boundary conditions

$$v_1(-\infty) = 0,\ w_1(-\infty) = 0,\ v_1(0) = a. \tag{2.3c}$$

We seek solution in the form

$$v_1 = A\exp(Qz),\ w_1 = B\exp(Qz). \tag{2.4}$$

Substituting (2.4) into (2.3) we obtain

$$v_1 = a\exp(Qz), \quad w_1 = \frac{\epsilon a}{cQ + \epsilon b}\exp(Qz), \tag{2.5}$$

where Q is solution of

$$\Delta(Q) \equiv \begin{vmatrix} (cQ)^\alpha - Q^2 + 1 & 1 \\ -\epsilon & cQ + \epsilon b \end{vmatrix} = 0$$

or, equivalently,

$$\Delta(Q) \equiv -cQ^3 - \epsilon bQ^2 + (cQ)^{\alpha+1} + cQ + \epsilon b(cQ)^\alpha + \epsilon(1+b) = 0, \tag{2.6}$$

with a positive real part. It can be shown that for any parameter values this equation has one positive real root and does not have any other roots with positive real parts. It is easy to see that this is true for some parameter values. For example, for $\alpha = 0$, $b = 0$ and $\epsilon = 4c$, equation (2.6) reduces to

$$Q^3 - 2Q - 4 = 0,$$

the roots of which are $Q = 2$ and $Q = -1 \pm i$. On the other hand, equation (2.6) does not have pure imaginary roots for any parameter values, so the number of roots with positive real part is the same for all parameters, namely, one (and the root must be real, otherwise, there would be a second, complex conjugate root). To show the absence of pure imaginary roots, we seek imaginary roots of the form $Q = Re^{i\pi/2}$. Substituting into (2.6) and separating real and imaginary parts we obtain

$$\epsilon bR^2 - (cR)^{\alpha+1}\sin\frac{1}{2}\pi\alpha + \epsilon b(cR)^\alpha\cos\frac{1}{2}\pi\alpha + \epsilon(1+b) = 0,$$

$$cR^3 + (cR)^{\alpha+1}\cos\frac{1}{2}\pi\alpha + cR + \epsilon b(cR)^\alpha\sin\frac{1}{2}\pi\alpha = 0.$$

Expressing the sine from the first equation and substituting into the second gives

$$cR^3 + (cR)^{\alpha+1}\cos\frac{1}{2}\pi\alpha + cR + \frac{\epsilon b(cR)^\alpha}{(cR)^{\alpha+1}}\left[\epsilon bR^2 + \epsilon b(cR)^\alpha\cos\frac{1}{2}\pi\alpha + \epsilon(1+b)\right] = 0,$$

which does not have any solutions because all the terms on the left hand side including the cosine are positive for $0 \le \alpha < 1$. For $\alpha = 1$ the cosine drops out but the left hand side is still positive.

Let us now consider the region $z>0$. Similarly to [13], we obtain the following system of equations

$$\frac{c^\alpha}{\Gamma(1-\alpha)}\left[\int_{-\infty}^{0}\frac{v_{1,\zeta}\mathrm{d}\zeta}{(z-\zeta)^\alpha}+\int_{0}^{z}\frac{v_{2,\zeta}\mathrm{d}\zeta}{(z-\zeta)^\alpha}\right]=\partial_z^2 v_2-v_2+1-w_2, \tag{2.7a}$$

$$c\partial_z w_2=\epsilon(v_2-bw_2), \tag{2.7b}$$

with the boundary conditions

$$v_2(0)=v_1(0)=a,\ \partial_z v_2(0)=\partial_z v_1(0)=aQ,\ v_2(\infty)=v_*,\ w_2(\infty)=w_*,$$

$$w_2(0)=w_1(0)=\frac{\epsilon a}{cQ+\epsilon b}. \tag{2.8}$$

We have replaced $H(v_2-a)$ with 1 in (2.7a) because $v_2-a>0$ for $z>0$.

Let us substitute (2.5) into (2.7):

$$\int_{-\infty}^{0}\frac{v_{1,\zeta}\mathrm{d}\zeta}{(z-\zeta)^\alpha}=aQ\int_{-\infty}^{0}\frac{\exp(Q\zeta)\mathrm{d}\zeta}{(z-\zeta)^\alpha}=aQ^\alpha e^{Qz}\int_{Qz}^{\infty}e^{-y}y^{-\alpha}\mathrm{d}y,$$

where the substitution $y\equiv Q(z-\zeta)$ was done at the last step. Thus, equations (2.7) can be rewritten as

$$\frac{c^\alpha}{\Gamma(1-\alpha)}\int_{0}^{z}\frac{v_{2,\zeta}\mathrm{d}\zeta}{(z-\zeta)^\alpha}-\partial_z^2 v_2+v_2+w_2=-\frac{a(cQ)^\alpha}{\Gamma(1-\alpha)}e^{Qz}\int_{Qz}^{\infty}e^{-y}y^{-\alpha}\mathrm{d}y+1, \tag{2.9a}$$

$$-\epsilon v_2+(c\partial_z+\epsilon b)w_2=0. \tag{2.9b}$$

Let us apply the Laplace transform to equations (2.9) using the relations (*cf.* [13]):

$$\mathcal{L}\left\{\int_{0}^{z}\frac{v_{2,\zeta}\mathrm{d}\zeta}{(z-\zeta)^\alpha}\right\}=\Gamma(1-\alpha)s^{\alpha-1}[s\tilde{v}_2(s)-a],$$

$$\mathcal{L}\left\{e^{Qz}\int_{Qz}^{\infty}e^{-y}y^{-\alpha}\mathrm{d}y\right\}=\frac{\Gamma(1-\alpha)\left(Q^{1-\alpha}-s^{1-\alpha}\right)}{(Q-s)s^{1-\alpha}},$$

$$\mathcal{L}\{\partial_z w_2\}=s\tilde{w}_2(s)-w_2(0)=s\tilde{w}_2(s)-\frac{\epsilon a}{cQ+\epsilon b}.$$

$$\mathcal{L}\{\partial_z^2 v_2\}=s^2\tilde{v}_2(s)-sv_2(0)-\partial_z v_2(0)=s^2\tilde{v}_2(s)-sa-aQ.$$

Thus, we obtain the following system of linear algebraic equations for $\tilde{v}_2(s)$, $\tilde{w}_2(s)$:

$$(c^\alpha s^\alpha-s^2+1)\tilde{v}_2(s)+\tilde{w}_2(s)=F(s), \tag{2.10a}$$

$$-\epsilon\tilde{v}_2(s)+(sc+\epsilon b)\tilde{w}_2(s)=G, \tag{2.10b}$$

where

$$F(s) = ac^\alpha s^{\alpha-1}\left[1 - Q^\alpha \frac{s^{1-\alpha} - Q^{1-\alpha}}{s-Q}\right] - a(s+Q) + \frac{1}{s},$$

$$G = \frac{c\epsilon a}{cQ + \epsilon b}.$$

We find

$$\tilde{v}_2(s) = \frac{\Delta_v(s)}{\Delta(s)}, \quad \tilde{w}_2(s) = \frac{\Delta_w(s)}{\Delta(s)}, \tag{2.11}$$

where the determinant $\Delta(s)$ is given by formula (2.6) and

$$\Delta_v(s) = F(s)(sc + \epsilon b) - G, \quad \Delta_w(s) = [(cs)^\alpha - s^2 + 1]G + \epsilon F(s).$$

Functions $\tilde{v}_2(s)$ and $\tilde{w}_2(s)$ should have no poles for positive s, which would correspond to functions $v_2(z)$ and $w_2(z)$ exponentially growing at $z \to \infty$. Because $\Delta(s)$ has a zero at $s = Q$, we get the conditions

$$\Delta_v(Q) = (Qc + \epsilon b)F(Q) - G = 0,$$

$$\Delta_w(Q) = \epsilon F(Q) + [(cQ)^\alpha - Q^2 + 1]G = 0.$$

Because the determinant of the homogeneous linear system $\Delta(Q) = 0$, both conditions are equivalent, *i.e.*, actually we get one condition for vanishing of both functions at infinity, say,

$$(Qc + \epsilon b)F(Q) - G = 0. \tag{2.12}$$

Taking the limit of the expression $F(s)$ at $s \to Q$, we find

$$F(Q) = a\alpha c^\alpha Q^{\alpha-1} - 2aQ + \frac{1}{Q}.$$

Thus, equation (2.12) has the form

$$(Qc + \epsilon b)^2\left[a\alpha(cQ)^\alpha - 2aQ^2 + 1\right] = c\epsilon a Q. \tag{2.13}$$

Finally, we get a system of two equations, (2.6) and (2.13), for finding Q and c.

Introducing the notation $Qc = p$ we rewrite equations (2.6) and (2.13) as

$$Q = \sqrt{p^\alpha + \frac{\epsilon}{p + \epsilon b} + 1}, \tag{2.14a}$$

$$c = \frac{p}{Q} = \frac{p}{\sqrt{p^\alpha + \frac{\epsilon}{p+\epsilon b} + 1}} \equiv g(p), \tag{2.14b}$$

$$a = \left[\frac{p\epsilon}{(p+\epsilon b)^2} + 2Q^2 - \alpha p^\alpha\right]^{-1} = \left[2 + (2-\alpha)p^\alpha + \frac{p\epsilon}{(p+\epsilon b)^2} + \frac{2\epsilon}{p+\epsilon b}\right]^{-1} \equiv \frac{1}{f(p)}. \tag{2.14c}$$

These equations define a parametric curve $(a(p), c(p))$ for fixed values of b, ϵ and α.

3. The $\alpha = 0$ case

Let us first consider the case when $\alpha = 0$, which can be thought of as the case of the most pronounced subdiffusion. As in the general case, the traveling wave connects two critical points, $(0,0)$ (approached by the solution as $z \to -\infty$) and $(v_{*0}, w_{*0}) = (b/(1+2b), 1/(1+2b))$ as $z \to \infty$. Note that the latter critical point for $\alpha = 0$ is different from that for $\alpha > 0$.

The parametric representation (2.14) reduces to

$$Q_0 = \sqrt{\frac{1}{b(q+1)} + 2}, \tag{3.1a}$$

$$C = \frac{bq}{Q_0} = \frac{bq}{\sqrt{\frac{1}{b(q+1)} + 2}} \equiv g_0(q), \tag{3.1b}$$

$$a = \left[4 + \frac{q}{b(q+1)^2} + \frac{2}{b(q+1)}\right]^{-1} \equiv \frac{1}{f_0(q)}, \tag{3.1c}$$

where $q = p/(\epsilon b)$ and $C = c/\epsilon$. Both C and a are monotonically increasing functions of q, so that C as a function of a is monotonically increasing as illustrated in Figure 1. The curves start at $a = v_{*0}/2$, $C = 0$ at $q = 0$. As $q \to \infty$, a goes to $1/4$ and C to infinity for all b.

It has to be noted that there is a condition that the solution must satisfy. Namely, for all $z > 0$ the solution $v_2(z)$ must be greater than a (which has been assumed when treating the Heaviside function in the original equations). It implies, in particular, that $a < v_{*0}$, so that we must analyze whether the parametric dependence (3.1) is such that this condition is satisfied. In order to check if solution v_2 dips below a we find the solution as follows.

For $\alpha = 0$ the system of differential equations in the moving coordinate system has the form

$$\partial_z^2 v_1 - 2v_1 - w_1 = 0,$$

$$c\partial_z w_1 = \epsilon(v_1 - bw_1),$$

with the boundary conditions

$$v_1(-\infty) = 0,\ w_1(-\infty) = 0,\ v_1(0) = a$$

for $z < 0$ and

$$\partial_z^2 v_2 - 2v_2 - w_2 = -1,$$

$$c\partial_z w_1 = \epsilon(v_1 - bw_1),$$

with the boundary conditions

$$v_2(0) = a, \quad v_2'(0) = aQ_0, \quad w_2(0) = w_1(0) = \frac{a\epsilon}{cQ_0 + \epsilon b},$$

$$v_2(\infty) = v_{*0} \equiv \frac{b}{1+2b}, \quad w_2(\infty) = w_{*0} \equiv \frac{1}{1+2b}$$

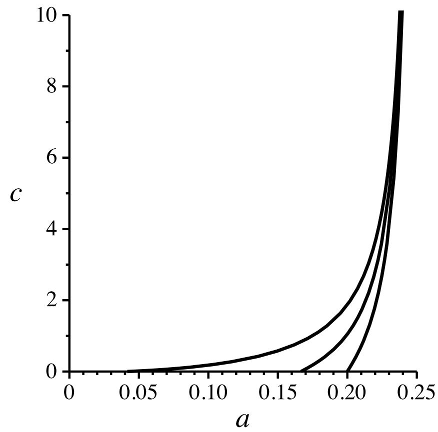


FIGURE 1. Dependence of the scaled propagation velocity $C = c/\epsilon$ on a for three values of b: $b = 0.1$, $b = 1$ and $b = 2$, from left to right. Here $\alpha = 0$.

for $z > 0$. Solution of this problem that can be found either directly or by inverting the Laplace transform in (2.11) for $z > 0$, leading to the same result, can be written as

$$v_1(z) = ae^{Q_0 z}, \quad w_1(z) = a(Q_0^2 - 2)e^{Q_0 z} \quad (z < 0),$$

$$v_2(z) = v_{*0} + Ae^{Q_1 z} + Be^{Q_2 z}, \quad w_2(z) = w_{*0} + A(Q_1^2 - 2)e^{Q_1 z} + B(Q_2^2 - 2)e^{Q_2 z} \quad (z > 0),$$

where

$$A = -\frac{(v_{*0} - a)Q_2 + aQ_0}{Q_2 - Q_1}, \quad B = \frac{(v_{*0} - a)Q_1 + aQ_0}{Q_2 - Q_1}.$$

Here Q_0, Q_1 and Q_2 are solutions of the cubic equation

$$\Delta_0(Q) \equiv -cQ^3 - \epsilon bQ^2 + 2cQ + \epsilon(1 + b) = 0 \tag{3.2}$$

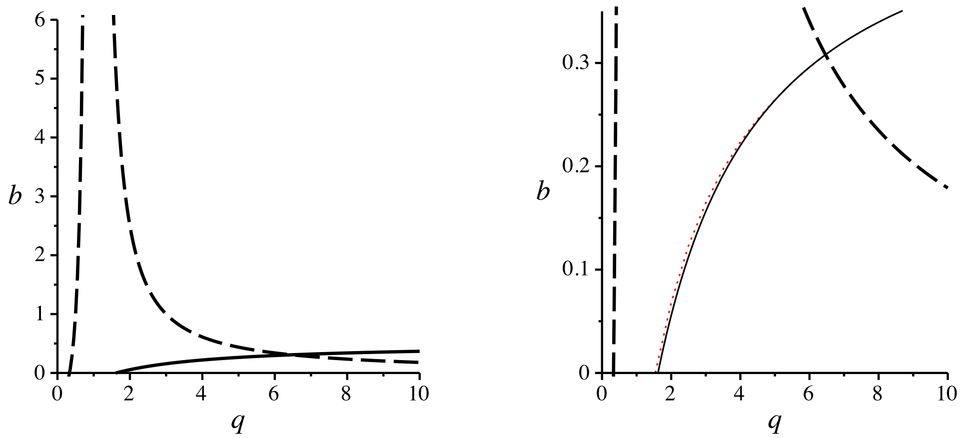
(*cf.* (2.6) with $\alpha = 0$). The root Q_0 which enters parametrization (3.1) is positive while Q_1 and Q_2 are either both negative or complex conjugate with negative real part, $Q_{1,2} = Q_r \pm iQ_i$, $Q_r < 0$. Solution for $v_2(z)$ can be written in the case of complex roots in the form convenient for the forthcoming analysis as

$$v_2(z) - a = (v_{*0} - a)\left[1 + e^{Q_r z}\left(\frac{Q_r}{Q_i}\sin(Q_i z) - \cos(Q_i z)\right)\right] + \frac{aQ_0}{Q_i}e^{Q_r z}\sin(Q_i z). \tag{3.3}$$

The root Q_0 which enters parametrization (3.1) is positive while the roots Q_1 and Q_2 of the cubic equation (3.2) can be found in terms of Q_0 as follows:

$$\Delta_0(Q) = \Delta_0(Q) - \Delta_0(Q_0) = -cQ^3 + cQ_0^3 - \epsilon bQ^2 + \epsilon bQ_0^2 + 2cQ - 2cQ_0$$

$$= -c(Q - Q_0)\left[Q^2 + Q_0 Q\left(1 + \frac{1}{q}\right) + Q_0^2\left(1 + \frac{1}{q}\right) - 2\right] = 0.$$



(a) (b)

FIGURE 2. Functions $f_1(q)$ (*dashed curve*) and $f_2(q)$ (*solid curve*).

Thus,

$$Q_{1,2} = \frac{Q_0}{2}\left[-\left(1+\frac{1}{q}\right) \pm \sqrt{\left(1+\frac{1}{q}\right)^2 - 4\left(1+\frac{1}{q}\right) + \frac{8}{Q_0^2}}\right].$$

Condition for the roots $Q_{1,2}$ to be complex, *i.e.*,

$$\left(1+\frac{1}{q}\right)^2 - 4\left(1+\frac{1}{q}\right) + \frac{8}{Q_0^2} < 0$$

can be written as

$$b < \frac{3q-1}{2(q-1)^2} \equiv f_1(q),$$

while the condition $a < v_{*0}$ reduces to

$$b > \frac{q^2-q-1}{2(q+1)^2} \equiv f_2(q).$$

The last condition is always satisfied if $b \geq 1/2$; for $b < 1/2$ it reduces to

$$0 \leq q < q_*(b) \equiv \frac{1}{2(1-2b)}\left[1+4b+\sqrt{5+8b}\right],$$

so that for $b < 1/2$ large velocities in the $c(a)$ figure that correspond to $q > q_*(b)$ are not allowed.

As to the $v_2(z) > a$ condition, we first note that a necessary condition for v_2 to dip below a is that $Q_{1,2}$ are complex. Indeed, if v_2 goes below a, it must have at least two extrema, which is not possible in the case of real roots as a linear combination of two exponentials has at most one extremum. If the roots are complex (*i.e.*, b is below the dashed curve in Fig. 2a) this indeed can happen if (q, b) is very close to the solid curve. At this curve $a = v_{*0}$ so that the terms proportional to $(v_{*0} - a)$ in (3.3) drop out and we end up with an oscillatory solution going below a. If (q, b) is sufficiently close to the curve, then the last term in (3.3) still prevails. It turns out, however, that (q, b) must be extremely close to the solid curve for this to happen. The region of such (q, b) is between the red dotted line and the solid line in Figure 2b.

4. The $0 < \alpha < 1$ case

Unlike the $\alpha = 0$ case, the dependence of the propagation velocity c on a for $\alpha > 0$ may be multi-valued for some b, ϵ and α. Let us analyze this dependence. First, the function $g(p)$ is monotonically increasing because

$$g'(p) = \left(p^\alpha + \frac{\epsilon}{p+\epsilon b} + 1\right)^{-3/2} \left[\left(1 - \frac{1}{2}\alpha\right) p^\alpha + 1 + \frac{\epsilon}{p+\epsilon b} + \frac{\epsilon p}{2(p+\epsilon b)^2}\right] > 0.$$

Next, the function $f(p)$ is not necessarily monotonic. We have

$$f'(p) = \alpha(2-\alpha)p^{\alpha-1} - \frac{\epsilon}{(p+\epsilon b)^2} - 2\frac{\epsilon p}{(p+\epsilon b)^3}.$$

Equation $f'(p) = 0$ can be written as

$$h(q) \equiv \frac{q^{1-\alpha}(3q+1)}{(q+1)^3} = K,$$

where

$$q = \frac{p}{\epsilon b}, \quad K = \alpha(2-\alpha)\epsilon^\alpha b^{1+\alpha}.$$

The graph of $h(q)$ is shown in Figure 3 for three values of α. For all α we have $h(1) = 1/2$. For a fixed value of $q < 1$ the function monotonically increases with α, while for $q > 1$ it decreases. It has one maximum at $q = q_m$ where q_m is a solution of the quadratic equation

$$3(1+\alpha)q_m^2 - 4(1-\alpha)q_m - (1-\alpha) = 0$$

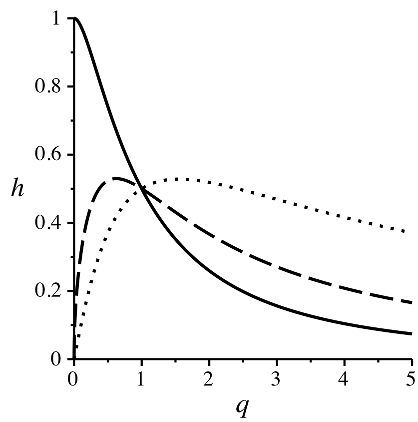


FIGURE 3. $h(q)$ for three values of α: $\alpha = 0$ (*dotted curve*), $\alpha = 0.5$ (*dashed curve*) and $\alpha = 1$ (*solid curve*).

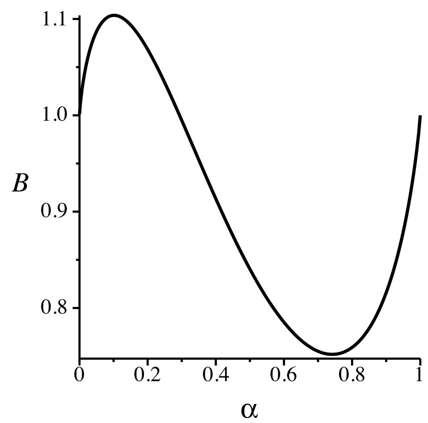
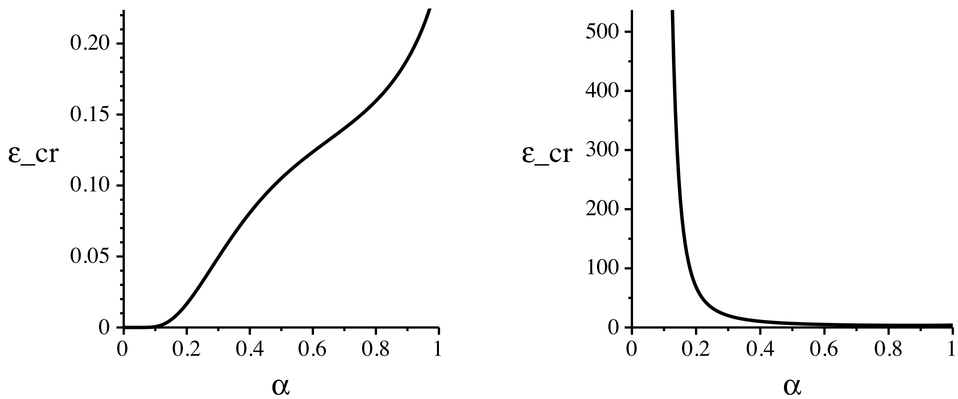


FIGURE 4. The function $B(\alpha)$ varies between 0.752 and 1.102 for $0 < \alpha < 1$.



(a) $b = 2$ (b) $b = 0.5$

FIGURE 5. ϵ_{cr} as a function of α for two values of b.

given by

$$q_m = \frac{1}{3(1+\alpha)}\left[2(1-\alpha) + \sqrt{4(1-\alpha)^2 + 3(1-\alpha^2)}\right].$$

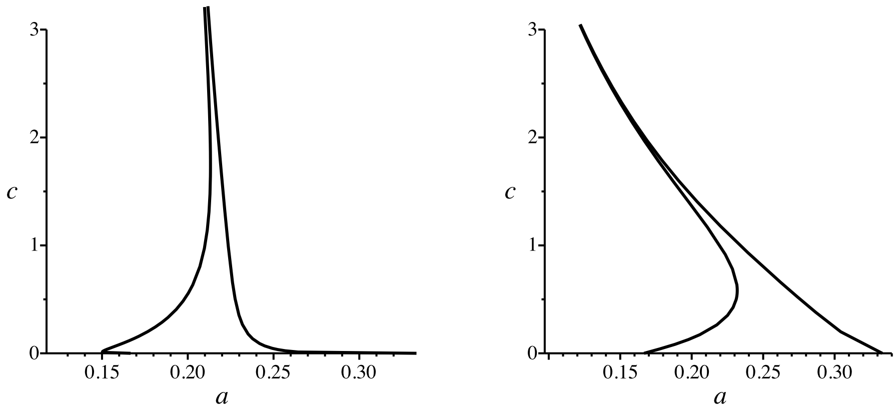
Thus, if $h(q_m) < K$, then the equation $f'(p) = 0$ does not have a solution, so that $f(p)$ is a monotonic function and the dependence of the propagation velocity c on a is single-valued. If, however, $h(q_m) > K$, *i.e.*,

$$h(q_m) > \alpha(2-\alpha)\epsilon^{\alpha} b^{1+\alpha}, \tag{4.1}$$

there is a range of a where three traveling wave solutions (with different positive speeds) exist. Inequality (4.1) can be rewritten as

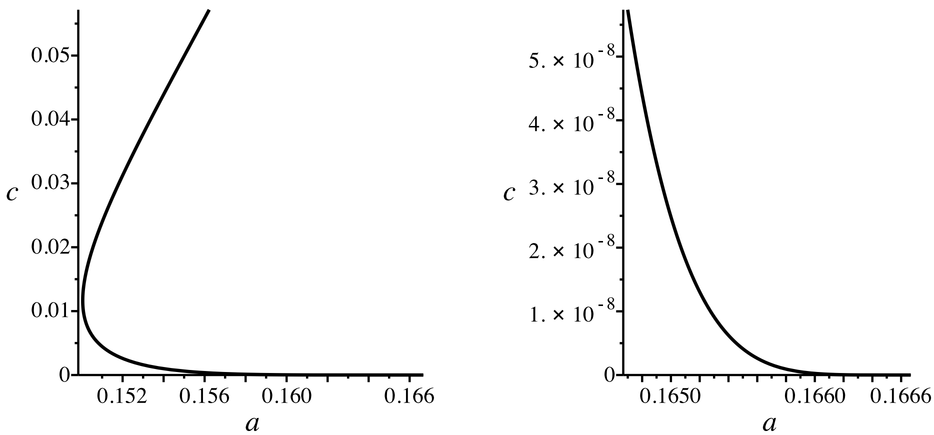
$$\epsilon < \epsilon_{cr} = \frac{B(\alpha)}{b^{1+1/\alpha}}, \quad B(\alpha) \equiv \left[\frac{h(q_m)}{\alpha(2-\alpha)}\right]^{\alpha}.$$

The function B is equal to 1 for $\alpha = 0$ and $\alpha = 1$ and does not vary too much over the entire interval $0 < \alpha < 1$ (see Fig. 4). The dependence of ϵ_{cr} on b is more interesting, especially for small α. Indeed, if $\alpha \ll 1$, then $\epsilon_{cr} \gg 1$



(a) $\alpha = 0.2$; $b = 0.5$ (left curve), $b = 2$ (right curve)   (b) $\alpha = 0.8$; $b = 0.5$ (left curve), $b = 2$ (right curve)

FIGURE 6. c as a function of a for $\epsilon = 0.5$ and two values of α: $\alpha = 0.2$ (*left figure*) and $\alpha = 0.8$ (*right figure*).



(a)   (b)

FIGURE 7. c as a function of a for $\epsilon = 0.5$, $\alpha = 0.2$ and $b = 0.5$, on two different scales.

for $b < 1$ (so that multiple traveling waves exist for any ϵ, while $\epsilon_{cr} \ll 1$ for $b > 1$ (so that the traveling wave solution is unique) – see Figure 5.

We illustrate these results by plotting parametric curves $(a(p), c(p))$ for various parameter values in Figure 6. For the parameter values used in the figure, ϵ is less than ϵ_{cr} for $b = 0.5$ (left curves in both subfigures), but greater than ϵ_{cr} for $b = 2$ (right curves in both subfigures). As a result, the left curves represent a multi-valued dependence while the right curves are single-valued. In fact, the multi-valued dependence is triple-valued, which is not well seen in the figure because of its scale. Figure 7 is plotted for the same parameters as the left curve in Figure 6a, but on two different scales. Both graphs demonstrate the presence of a low speed branch in the $a(c)$ dependence. In addition, Figure 7b shows that $c = 0$ at $a = v_*/2$ (see (2.14c)), which is 1/6 for $b = 0.5$ (though the velocity c for all these values of a is so small that even on this scale we cannot clearly see this result).

For large values of p, parametrization (2.14) gives

$$c \sim p^{1-\frac{\alpha}{2}}, \quad a \sim \frac{1}{2-\alpha} p^{-\alpha} \quad p \gg 1,$$

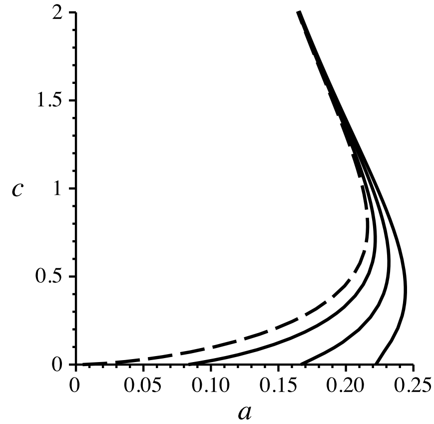


FIGURE 8. Dependence of the propagation speed c on a for $\alpha = 0.8$, $\epsilon = 0.5$ and four values of b. From right to left, $b = 0.8$, $b = 0.5$, $b = 0.2$ and $b = 0$ (*dashed curve*).

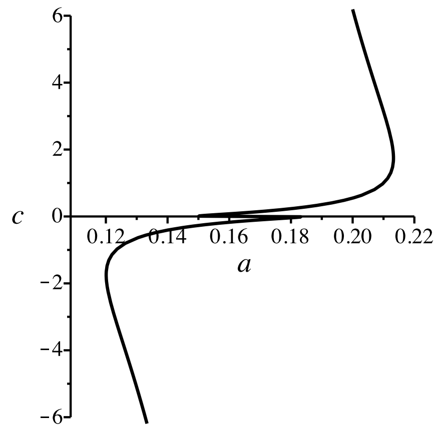


FIGURE 9. A complete dependence of the propagation speed c on a that includes both $c > 0$ (the excited state displaces the rest state) and $c < 0$ (the rest state displaces the excited state). Here $\alpha = 0.2$, $\epsilon = 0.5$ and $b = 0.5$.

so that

$$c \sim [(2-\alpha)a]^{-\frac{2-\alpha}{2\alpha}}, \quad c \gg 1.$$

This asymptotics does not depend on the values of b and ϵ, which explains why the curves in both subfigures in Figure 6 converge as c increases.

As discussed in the case of $\alpha = 0$, we must have $a < v_*$. This inequality is satisfied for all the parameter values used above, but is not necessarily true for other parameter values. Indeed, consider Figure 8 where the dependence of c on a is shown for several values of b. Once again, there is also a lower branch in each curve which is not seen on the scale of the figure because the speed c there is very small. As b decreases, the maximum value of a (call it $a_{\max}$) converges to $a_{\max}$ for $b = 0$ which for these parameter values is approximately equal to 0.215. The quantity $v_* = b/(1+b)$ goes to zero as $b \to 0$, so that for sufficiently small b we have $a_{\max} > v_*$. In this case the propagation speed c obtained for the range $v_* < a < a_{\max}$ is not a valid solution. We remark that there is another condition that must be satisfied, namely, $v_2(z) > a$. In the $\alpha = 0$ case the parameter regime where this condition is not satisfied is very small and can be disregarded and so we intend to disregard it for $\alpha > 0$ as well. One could speculate that if this condition is not satisfied for some z, it probably means that there exists a solution of different structure, with more oscillatory behavior.

5. Summary

We have considered front propagation in a subdiffusive FitzHugh–Nagumo system (1.1). The subdiffusion exponent α varies from zero (which can be thought of as the most pronounced subdiffusion case) to α equal to one, which is the normal diffusion case. The traveling wave connects the rest state $(0, 0)$ located at minus infinity with the excited state (v_*, w_*) located at plus infinity (in the case of $\alpha = 0$ it is denoted by (v_{*0}, w_{*0})). The propagation velocity $c > 0$ corresponds to the wave propagating to the left, *i.e.*, the excited state displacing the rest state while in the $c < 0$ case the rest state displaces the excited state. We studied the $c > 0$ case because the $c < 0$ case can be obtained from the former one by a simple transformation discussed in the paper. An important assumption about the form of the solution that we seek is that there is only one spatial location at which $v = a$. Thus, we exclude from consideration multi-front traveling waves known to exist in the normal diffusion case.

In the $\alpha = 0$ case for each value of b there exists a range of a, $v_{*0}/2 < a < v_m \leq 1/4$ such that for these values of a there exists a unique wave. Its propagation velocity c is positive. There is also a range of $a < v_{*0}/2$ where a single wave with $c < 0$ exists. For $a = v_{*0}/2$ the propagation speed equals zero (*i.e.*, the solution is a standing wave). There are no other traveling waves of the desired form.

For $\alpha > 0$ the situation is quite different in that multiple solutions can occur for some parameter regimes. As an illustration consider Figure 9 where the dependence of the propagation velocity c on a is shown for both $c > 0$ and $c < 0$ (the curve is symmetric with respect to the point $(v_*/2, 0)$). We see that multiple solutions, both with $c > 0$ and $c < 0$ can occur for the same value of a (in particular, there are solutions with large $|c|$ that are not seen in the figure because of the chosen scale). We remark that some of these solutions are unstable, but analysis of their stability is beyond the scope of this paper. In the paper we present conditions for the multiplicity depending on the parameters of the problem.

Acknowledgements. This work has been partially supported by the US-Israel Binational Science Foundation (grant No. 2016322).

References

[1] D.C. Bell and B. Deng, Singular perturbation of N-front travelling waves in the FitzHugh–Nagumo equations. *Nonlin. Anal. Real World Appl.* **3** (2002) 515–541.
[2] G.A. Carpenter, A geometric approach to singular perturbation problems with applications to nerve impulse equations. *J. Differ. Equ.* **23** (1977) 335–367.
[3] B. Datsko and V. Gafiychuk, Mathematical modeling of traveling autosolitons in fractional-order activator-inhibitor systems. *B. Pol. Acad. Sci. Tech. Sci.* **66** (2018) 411–418.
[4] B. Datsko, V. Gafiychuk and I. Podlubny, Solitary travelling auto-waves in fractional reaction–diffusion systems. *Commun. Nonlin. Sci. Numer. Simul.* **23** (2015) 378–387.
[5] B. Deng, The existence of infinitely many traveling front and back waves in the FitzHugh–Nagumo equations. *SIAM J. Math. Anal.* **22** (1991) 1631–1650.
[6] P.C. Fife and J.B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Ration. Mech. Anal.* **65** (1977) 335–361.
[7] R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane. *Biophys. J.* **1** (1961) 445–466.
[8] C.K.R.T. Jones, Stability of the travelling wave solution of the FitzHugh-Nagumo system. *Trans. Am. Math. Soc.* **286** (1984) 431–469.
[9] H.P. McKean Jr, Nagumo's equation. *Adv. Math.* **4** (1970) 209–223.
[10] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **339** (2000) 1–77.
[11] R. Metzler and J. Klafter, The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics. *J. Phys. A* **37** (2004) R161–R208.
[12] J. Nagumo, S. Arimoto and S. Yoshizawa, An active pulse transmission line simulating nerve axon. *Proc. Inst. Radio Eng.* **50** (1962) 2061–2070.

[13] A.A. Nepomnyashchy and V.A. Volpert, An exactly solvable model of subdiffusion–reaction front propagation. *J. Phys. A* **46** (2013) 065101.

[14] J. Rinzel and J.B. Keller, Traveling wave solutions of a nerve conduction equation. *Biophys. J.* **13** (1973) 1313–1337.

[15] J. Rinzel and D. Terman, Propagation phenomena in a bistable reaction-diffusion system. *SIAM J. Appl. Math.* **42** (1982) 1111–1137.

[16] V.A. Volpert, Y. Nec and A.A. Nepomnyashchy, Exact solutions in front propagation problems with superdiffusion. *Physica D* **239** (2010) 134–144.

[17] V.A. Volpert, Y. Nec and A.A. Nepomnyashchy, Fronts in anomalous diffusion–reaction systems. *Philos. Trans. Royal Soc. A* **371** (2013) 20120179.

[18] E. Yanagida, Stability of travelling front solutions of the FitzHugh-Nagumo equations. *Math. Comput. Model.* **12** (1989) 289–301.