THEORY OF OPTIMAL HARVESTING FOR A SIZE STRUCTURED MODEL OF FISH

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Abstract. This paper investigates the maximum principle for a nonlinear size structured model that describes the optimal management of the fish resources taking into account harvesting the fish and putting the fry. First, we show the existence of a unique non-negative solution of the system, and give a comparison principle. Next, we prove the existence of optimal policies by using maximizing sequence and Mazur's theorem in convex analysis. Then, we obtain necessary optimality conditions by using tangent-normal cones and adjoint system techniques. Finally, some examples and numerical results demonstrate the effectiveness of the theoretical results in our paper.

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1. INTRODUCTION

Mathematical modelling in ecology and evolution is a rapidly growing area of research, which takes advantage of recently developed mathematical and computational techniques as well as novel methods of data analysis (see [24]). During the past 100 yr, structured population models have played a significant role in the mathematical analysis and control of populations in biology and demography (see [23]). Among the individual structure, there are many structural differences, such as age, body size, gender, gene, and life stage. In [4], the authors studied the stability of equilibrium for the following age structured population model with an increasing recruitment process.

$$\begin{aligned} \frac{\partial u(a, t)}{\partial t} + \frac{\partial u(a, t)}{\partial a} &= -\mu(a)u(a, t), & 0 \leq a \leq a_{\dagger}, \quad t \geq 0, \\ u(a, 0) &= u_0(a), & 0 \leq a \leq a_{\dagger}, \\ u(0, t) &= f(E(t))E(t), & t \geq 0, \end{aligned} \tag{1.1}$$

where $a_{\dagger} \in (0, +\infty)$ is the maximal age of individuals in the population. The state variable $u(a, t)$ represents the density of individuals of age a at time t . $\mu(a)$ is the natural mortality; $E(t) = \int_0^{a_{\dagger}} b(a)u(a, t) da$ represents the total number of new born individuals weighted by the natural fertility $b(a)$; f is the survival probability of the new born individuals. $u_0(a)$ represents the initial age distribution of the population.

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Note that age is only a special kind of size and size is one of the most natural and important variables to describe population dynamics. Sizes can be mass, length, diameter, surface area, volume, maturity, and so on. Further studies show that the size of the individual can determine its prey object and niche, thus it can determine the ability of intra-competition and inter-competition [5]. As a result, modelling population dynamics, it is natural to assume that the vital rates such as fertility, mortality, and growth rates of individuals depended on its body size (see [7, 9]). In [6], the authors investigated the net reproduction functions for nonlinear structured population models. Moreover, construction of a net reproduction function in structured populations is crucial for understanding stability of their stationary states (see [8–10]).

To the best of our knowledge, so far, almost all of the size-structured population models have been established under the assumption that newborn individuals are all alive. However, for the fish, only a fraction of the eggs produced by mature fish can convert to fry. We use a nondecreasing recruitment process to describe this phenomenon. Motivated by these considerations, in this paper, we investigate an exploitation problem of fish incorporating the individual size structure and nondecreasing recruitment process into the model. This paper is concerned with the harvesting of a size structured population of fish.

To build model, we assume that $u(x, t)$ denotes the fish density of individuals at time t with size x . $\mu(x, t)$ and $\beta(x, t)$ are, respectively, the natural mortality and egg-laying rate. $g(x)$ stands for the growth rate of individual's size. $I(t)$ corresponds to the number of eggs deposited at time t . f is the proportion of eggs that survive to become fishes which depend on the parental egg production at time t . $C(t)$ is the amount of fry put by the fisherman at time t . Then the system describing the evolution of fish is given by

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} + \frac{\partial(g(x)u(x, t))}{\partial x} &= -\mu(x, t)u(x, t) - \alpha(x, t)u(x, t), & (x, t) \in Q, \\ g(0)u(0, t) &= C(t) + f(I(t))I(t), & t \in [0, T], \\ u(x, 0) &= u_0(x), & x \in [0, l], \\ I(t) &= \int_0^l \beta(x, t)u(x, t) dx, & t \in [0, T]. \end{aligned} \tag{1.2}$$

Here, $Q = [0, l) \times [0, T]$, $l \in (0, +\infty)$ is the maximum size of any individual in the population and $T \in (0, +\infty)$ is a given time. The control variable α is the harvesting efforts, which belongs to

$$\mathcal{U} = \{\alpha \in L^\infty(Q) \mid 0 \leq \underline{\alpha} \leq \alpha(x, t) \leq \bar{\alpha} \text{ a.e. } (x, t) \in Q\}.$$

Here, positive constants $\underline{\alpha}$ and $\bar{\alpha}$ represent the minimum and maximum harvest effort, respectively.

The major novelties of the current paper include the following: (i) for the fish, only a fraction of the eggs produced by mature fish can convert to fry. We use a nondecreasing recruitment process to describe this phenomenon. (ii) In the ecological environment, the fisherman puts a certain amount of fry in the ocean. (iii) We give some numerical results, which are used to demonstrate the effectiveness of the theoretical results in our paper.

In this context, we consider the problem of finding the optimal harvesting effort that gives the maximal yield. The optimal control problems for age-structured population models have been studied, in the periodic case (see [2, 3]), while in the nonperiodic case (see [11, 13, 22]). The book [1] is a monograph on control of age-structured population models. The optimal control problem of size-structured population models has been studied, in the nonperiodic case (see [12, 19] for optimal birth control, [14–18] for optimal harvesting), in the periodic case (see [25] for optimal harvesting effort). In [19], the authors studied the optimal birth control problems for a nonlinear vermin population model with size-structure. The least cost-size problem and least cost-deviation problem are studied by different methods. In [21], the authors studied the optimal contraception control for a nonlinear population model with size structure and a separable mortality.

Let $u(x, t)$ be the solution of system (1.2) corresponding to $\alpha \in \mathcal{U}$. In this paper, we investigate the following optimization problem

$$\max_{\alpha \in \mathcal{U}} J(\alpha), \quad (1.3)$$

where

$$J(\alpha) = \int_0^T \int_0^l \omega(x, t) \alpha(x, t) u(x, t) dx dt - \int_0^T r(t) C(t) dt.$$

Here, function $\omega(x, t)$ is the economic values of an individual with size x at time t . $r(t)$ is the price of fry at time t and $r_1 \leq r(t) \leq r_2$ for $t \in [0, T]$, where r_1 and r_2 are positive constants. The first term of functional J represents the benefits from harvesting the fish, the second term represents the cost of putting the fry. Therefore, $J(\alpha)$ represents the total net economic benefit yielded from harvesting during a time of T .

Let $R_+ \doteq [0, \infty)$, $L_+^1 = L^1(0, l; R_+)$, and $L_+^\infty = L^\infty(0, l; R_+)$. Throughout this paper, the parameters in the model satisfy the following conditions.

(A₁) $g \in C^1[0, l]$ is a bounded function. $g(0) = 1$, $\lim_{x \uparrow l} g(x) = 0$, and $g(x) > 0$, $g'(x) \leq 0$ for $x \in [0, l]$. Moreover, there is a positive constant L_V such that

$$|g(x_1) - g(x_2)| \leq L_V |x_1 - x_2| \quad \text{for } x_1, x_2 \in [0, l].$$

(A₂) There exists $\bar{\beta} \in R_+$ such that $0 < \beta(x, t) \leq \bar{\beta}$ for $(x, t) \in Q$.

(A₃) $\mu : [0, l] \times [0, T] \rightarrow R_+$ is a measurable function and $\mu(x, t) + g'(x) \geq 0$ for any $(x, t) \in Q$.

(A₄) $f \in C(R_+, R_+)$ is bounded, $F(s) \doteq sf(s) \in C^1(R_+, R_+)$ and $F'(s) \geq 0$ for $s \in R_+$. Moreover, there exists constant k such that

$$|F(s_1) - F(s_2)| \leq k |s_1 - s_2| \quad \text{for } s_1, s_2 \in R_+.$$

(A₅) $C \in L^1[0, T]$ and there exists $c_1, c_2 \in R_+$ such that $c_1 \leq C(t) \leq c_2$ for $t \in [0, T]$. $u_0 \in L_+^1$ and there exists $\bar{u} \in R_+$ such that $0 \leq u_0(x) \leq \bar{u}$ for all $x \in [0, l]$.

(A₆) The weighted functions $\omega(x, t)$ and $r(t)$ are all non-negative.

According to [8], the survival rate of the individual is

$$S(x, t) = \exp \left\{ - \int_0^x \frac{\mu(s, \varphi^{-1}(s; t, x)) + g'(s)}{g(s)} ds \right\}.$$

Thus, assumption $\mu(x, t) + g'(x) \geq 0$ implies $0 < S(x, t) \leq 1$. Hence, in a biological sense, assumption $\mu(x, t) + g'(x) \geq 0$ is reasonable. In addition, the biological meaning of $F'(s) \geq 0$ is that the number of eggs converted into fishes increases with the increase of the egg-laying amount.

The remaining parts of this paper are organized as follows. In Section 2, we give the existence of solutions and comparison principle. Then we prove the existence of optimal policies by using maximizing sequence and Mazur's theorem in convex analysis in Section 3. The adjoint system of the state system is derived in Section 4, while Section 5 derives the necessary optimality conditions, which are finished by tangent-normal cones and adjoint system techniques. Section 6 contains examples and numerical results, which are used to demonstrate the effectiveness of the theoretical results in our paper. The paper ends with a conclusion section.

2. WELL-POSEDNESS OF THE STATE SYSTEM

This section is devoted to the well-posedness of system (1.2).

Definition 2.1. The unique solution $x = \varphi(t; t_0, x_0)$ of the initial-valued problem $x'(t) = g(x(t))$, $x(t_0) = x_0$ is said to be a characteristic curve of system (1.2). Let $z(t) = \varphi(t; 0, 0)$ be the characteristic curve through $(0, 0)$ in the $x - t$ plane.

Definition 2.2. The derivative of the function $u(x, t)$ at (x, t) along the characteristic curve φ is given by

$$D_\varphi u(x, t) = \lim_{h \rightarrow 0} \frac{u(\varphi(t+h; t, x), t+h) - u(x, t)}{h}.$$

For an arbitrary point (x, t) in the first quadrant of the $x - t$ plane such that $x \leq z(t)$, that is, $\varphi(t; t, x) \leq z(t)$, define the initial time $\tau \doteq \tau(x, t)$ implicitly by the relation $\varphi(t; \tau, 0) = x$ if and only if $\varphi(\tau; t, x) = 0$. We then show the existence, uniqueness, and boundedness of solutions to (1.2). Without loss of generality, we assume that $\alpha(x, t) \equiv 0$. Utilizing the characteristic curve technique, we have

$$u(x, t) = \begin{cases} u(0, \tau) \exp \left\{ - \int_\tau^t \mu(\varphi(s; t, x), s) + g'(\varphi(s; t, x)) \, ds \right\}, & x \leq z(t), \\ u_0(\varphi(0; t, x)) \exp \left\{ - \int_0^t \mu(\varphi(s; t, x), s) + g'(\varphi(s; t, x)) \, ds \right\}, & x > z(t). \end{cases} \quad (2.1)$$

A similar discussion as that in [20], we have the following theorem with the proof being omitted.

Theorem 2.3. Let assumptions (A_1) – (A_5) hold. Then for any $\alpha \in \mathcal{U}$, system (1.2) has a unique solution $u \in L^\infty(Q)$, which is non-negative and bounded, $|u(x, t)| \leq B$. Moreover, for any $t \in [0, T]$ and $\alpha_1, \alpha_2 \in \mathcal{U}$, there exists positive constant B_1 such that

$$\|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^1[0, l]} \leq B_1 \int_0^t \|\alpha_1(\cdot, s) - \alpha_2(\cdot, s)\|_{L^1[0, l]} \, ds, \quad (2.2)$$

where $u_1(x, t)$ and $u_2(x, t)$ are solutions of system (1.2) corresponding to α_1 and $\alpha_2 \in \mathcal{U}$, respectively.

Theorem 2.4. Suppose that (A_1) – (A_5) hold. Let $u_i(x, t)$ be the solutions of system (1.2) corresponding to $\alpha_i(x, t) \in \mathcal{U}$, $i = 1, 2$. If $\alpha_1(x, t) \geq \alpha_2(x, t)$ for all $(x, t) \in Q$, then $u_1(x, t) \leq u_2(x, t)$ for all $(x, t) \in Q$.

Proof. Note that $b(t) = g(0)u(0, t) = C(t) + F(I(t))$. It follows from $g(0) = 1$ that

$$b(t) = C(t) + F \left(\int_0^{z(t)} K(x, t) b(\varphi^{-1}(0; t, x)) \, dx + \phi(t) \right), \quad (2.3)$$

where

$$K(x, t) = \beta(x, t)E(t; t, x) \quad \text{and} \quad \phi(t) = \int_{z(t)}^l \beta(x, t)u_0(\varphi(0; t, x))\Pi(t; t, x) \, dx.$$

Here

$$\begin{aligned} E(t; t, x) &\doteq \exp \left\{ - \int_\tau^t \mu(\varphi(s; t, x), s) + g'(\varphi(s; t, x)) + \alpha(\varphi(s; t, x), s) \, ds \right\}, \\ \Pi(t; t, x) &\doteq \exp \left\{ - \int_0^t \mu(\varphi(s; t, x), s) + g'(\varphi(s; t, x)) + \alpha(\varphi(s; t, x), s) \, ds \right\}. \end{aligned}$$

Then, for $\int_0^{z(t)} K(x, t) b(\varphi^{-1}(0; t, x)) dx$ in (2.3), we use the transformation $s = \varphi^{-1}(0; t, x)$. By Definition 2.1, $s = t$ when $x = 0$ while $s = 0$ when $x = z(t)$. Note that dx/ds is the solution of the initial-valued problem

$$\begin{cases} \frac{dz}{dt} = g'(\varphi(t; s, 0))z, \\ z(s) = -g(0) = -1. \end{cases} \quad (2.4)$$

It is clear that the solution of (2.4) is $z(t) = -\exp \left\{ \int_s^t g'(\varphi(r; s, 0)) dr \right\}$. That is,

$$dx = -\exp \left\{ \int_s^t g'(\varphi(r; s, 0)) dr \right\} ds. \quad (2.5)$$

Thus,

$$\int_0^{z(t)} K(x, t) b(\varphi^{-1}(0; t, x)) dx = \int_0^t \bar{K}(\varphi(t; s, 0), t) b(s) ds,$$

where

$$\bar{K}(\varphi(t; s, 0), t) = K(\varphi(t; s, 0), t) \exp \left\{ \int_s^t g'(\varphi(r; s, 0)) dr \right\}, \quad s \in [0, t], \quad t \in [0, T].$$

Therefore, equation (2.3) is equivalent to

$$b(t) = C(t) + F \left(\int_0^t \bar{K}(\varphi(t; s, 0), t) b(s) ds + \phi(t) \right). \quad (2.6)$$

For $i = 1, 2$, let

$$\begin{aligned} K_i(t, x) &= \beta(x, t) E_i(t; t, x), \\ \phi_i(t) &= \int_{z(t)}^l \beta(x, t) u_0(\varphi(0; t, x)) \Pi_i(t; t, x) dx, \\ \bar{K}_i(\varphi(t; s, 0), t) &= K_i(\varphi(t; s, 0), t) \exp \left\{ \int_s^t g'(\varphi(r; s, 0)) dr \right\}, \end{aligned}$$

where

$$E_i(t; t, x) \doteq \exp \left\{ - \int_{\tau}^t \mu(\varphi(s; t, x), s) + g'(\varphi(s; t, x)) + \alpha_i(\varphi(s; t, x), s) ds \right\}, \quad (2.7)$$

$$\Pi_i(t; t, x) \doteq \exp \left\{ - \int_0^t \mu(\varphi(s; t, x), s) + g'(\varphi(s; t, x)) + \alpha_i(\varphi(s; t, x), s) ds \right\}. \quad (2.8)$$

Note that $\alpha_1(x, t) \geq \alpha_2(x, t)$ for all $(x, t) \in Q$. This implies that $E_1(t; t, x) \leq E_2(t; t, x)$ and $\Pi_1(t; t, x) \leq \Pi_2(t; t, x)$. Further,

$$\begin{aligned} K_1(x, t) &\leq K_2(x, t) \quad \text{and} \quad \phi_1(t) \leq \phi_2(t), \quad (x, t) \in Q. \\ \bar{K}_1(\varphi(t; s, 0), t) &\leq \bar{K}_2(\varphi(t; s, 0), t), \quad s \in [0, t], \quad t \in [0, \infty). \end{aligned}$$

It is clear that $I_i(t) = \int_0^t \bar{K}_i(\varphi(t; s, 0), t)[C(s) + F(I_i(s))] ds + \phi_i(t)$ $i = 1, 2$. Then,

$$\begin{aligned} I_1(0) - I_2(0) &= 0, \\ I_1(t) - I_2(t) &= \int_0^t [\bar{K}_1(\varphi(t; s, 0), t)F(I_1(s)) - \bar{K}_2(\varphi(t; s, 0), t)F(I_2(s))] ds \\ &\quad + \int_0^t C(s) [\bar{K}_1(\varphi(t; s, 0), t) - \bar{K}_2(\varphi(t; s, 0), t)] ds + \phi_1(t) - \phi_2(t). \end{aligned}$$

Next we prove that $I_1(t) \leq I_2(t)$ for all $t \in [0, T]$. Suppose that there exists $t_0 > 0$ such that $I_1(t_0) - I_2(t_0) > 0$. Then, there exists $t_1 \geq 0$ such that

$$t_1 = \inf_{t \in [0, t_0]} \{t | I_1(t) - I_2(t) > 0\}.$$

Clearly, $I_1(t_1) = I_2(t_1)$, and $I_1(t) \leq I_2(t)$ for any $t \in [0, t_1]$. Assume that $I_1(t) - I_2(t) > 0$ for any $t \in (t_1, t_0]$. Let $\gamma = \sup_{\Delta t \in [0, \delta]} \{I_1(t_1 + \Delta t) - I_2(t_1 + \Delta t)\}$, where $\delta = \min \{1/2\bar{\beta}k, t_0 - t_1\}$. Then $\gamma > 0$. From assumption (A₄), we know that F is non-decreasing. For any $\Delta t \in (0, \delta]$, denote $\hat{t} = t_1 + \Delta t$. Then,

$$\begin{aligned} I_1(\hat{t}) - I_2(\hat{t}) &= \int_0^{\hat{t}} [\bar{K}_1(\varphi(\hat{t}; s, 0), \hat{t})F(I_1(s)) - \bar{K}_2(\varphi(\hat{t}; s, 0), \hat{t})F(I_2(s))] ds \\ &\quad + \int_0^{\hat{t}} C(s) [\bar{K}_1(\varphi(\hat{t}; s, 0), \hat{t}) - \bar{K}_2(\varphi(\hat{t}; s, 0), \hat{t})] ds + \phi_1(\hat{t}) - \phi_2(\hat{t}) \\ &\leq \int_0^{\hat{t}} \bar{K}_2(\varphi(\hat{t}; s, 0), \hat{t}) [F(I_1(s)) - F(I_2(s))] ds \\ &\leq \int_{t_1}^{\hat{t}} \bar{K}_2(\hat{t}; s, 0), \hat{t}) [F(I_1(s)) - F(I_2(s))] ds \\ &\leq \int_{t_1}^{\hat{t}} \bar{\beta}k [I_1(s) - I_2(s)] ds \leq \int_{t_1}^{\hat{t}} \bar{\beta}k\gamma ds \leq \frac{\gamma}{2}. \end{aligned}$$

Hence, $\gamma \leq \frac{\gamma}{2}$, that is, $\gamma \leq 0$, which contradicts $\gamma > 0$. Thus, $I_1(t) \leq I_2(t)$ for all $t \in [0, T]$. Note that $b_i = C + F(I_i)$ and F is non-decreasing. Hence, $b_1(t) \leq b_2(t)$ for $t \in [0, T]$. It follows from (2.1) that $u_1(x, t) \leq u_2(x, t)$ for all $(x, t) \in Q$. The proof is complete. $\square$

3. EXISTENCE OF OPTIMAL CONTROL POLICY

The purpose of this section is to prove the existence of the optimal control policy.

Theorem 3.1. *Assume that (A₁) – (A₆) hold. Then the optimal control problem (1.2)–(1.3) has at least one solution.*

Proof. Denote $d = \sup_{\alpha \in \mathcal{U}} J(\alpha)$. It follows from Theorems 2.3 and 2.4 that

$$-r_2 c_2 T \leq d \leq \bar{\omega} \bar{\alpha} \int_0^T \int_0^l u^\alpha(x, t) dx dt - r_1 c_1 T,$$

where $\bar{\omega}$ is the upper bound of the weight function $\omega(x, t)$ and $u^\alpha(x, t)$ is the solution of system (1.2) corresponding to $\alpha = \underline{\alpha}$. Let $\{\alpha_n : n \geq 1\}$ be a maximizing sequence of $J(\alpha)$ in $\mathcal{U}$ satisfying

$$d - \frac{1}{n} < J(\alpha_n) \leq d, \quad \text{for all } n \geq 1.$$

Note that $\{u^{\alpha_n}\}$ is uniformly bounded about the control variable $\alpha \in \mathcal{U}$. Then there exists a subsequence of $\{\alpha_n\}$, still denoted by $\{\alpha_n\}$, such that $u^{\alpha_n} \rightharpoonup u^*$ weakly in $L^2(Q)$ as $n \rightarrow \infty$ for some $u^* \in L^2(Q)$. For $\{u^{\alpha_n}\}$, it follows from the Mazur theorem that there exists the following convex combination of $\{u^{\alpha_n}\}$,

$$\tilde{u}_n = \sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}, \quad \lambda_i^n \geq 0, \quad \sum_{i=n+1}^{k_n} \lambda_i^n = 1, \quad k_n \geq n+1, \quad (3.1)$$

such that $\tilde{u}_n \rightarrow u^*$ in $L^2(Q)$ as $n \rightarrow \infty$. Then there exists a subsequence of $\{\alpha_n\}$, still denoted by $\{\alpha_n\}$, such that $I^{\alpha_n} \rightarrow I^*$ as $n \rightarrow \infty$ and $I^{\alpha_n}(t) \rightarrow I^*(t)$ for almost every $t \in [0, T]$. Consequently, $I^*(t) = \int_0^l \beta(x, t) u^*(x, t) dx$.

Define the function sequence of the control variable as

$$\tilde{\alpha}_n(x, t) = \begin{cases} \frac{\sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}(x, t) \alpha_i(x, t)}{\sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}(x, t)} & \text{if } \sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}(x, t) \neq 0, \\ 0 & \text{if } \sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}(x, t) = 0. \end{cases} \quad (3.2)$$

For any $n+1 \leq i \leq k_n$, note that $\underline{\alpha} \leq \alpha_i(x, t) \leq \bar{\alpha}$. Combining this with $\lambda_i^n \geq 0$ and $u^{\alpha_i}(x, t) \geq 0$, we have $\tilde{\alpha}_n \in \mathcal{U}$.

From the boundedness of $\{\tilde{\alpha}_n\}$ and the weak compactness of the bounded sequence, we obtain that there exists a subsequence of $\{\tilde{\alpha}_n\}$, still denoted by $\{\tilde{\alpha}_n\}$, such that $\tilde{\alpha}_n \rightharpoonup \alpha^*$ weakly in $L^2(Q)$ as $n \rightarrow \infty$.

Note that u^{α_i} is the solution of system (1.2) corresponding to $\alpha = \alpha_i$. Then, from (3.1)–(3.2),

$$\begin{aligned} D_\varphi \tilde{u}_n(x, t) &= -[\mu(x, t) + g'(x)] \tilde{u}_n - \sum_{i=n+1}^{k_n} \lambda_i^n \alpha_i u^{\alpha_i}(x, t), & (x, t) \in Q, \\ g(0) \tilde{u}_n(0, t) &= C(t) + \sum_{i=n+1}^{k_n} \lambda_i^n F(I^{\alpha_i}(t)), & t \in [0, T], \\ \tilde{u}_n(x, 0) &= u_0(x), & x \in [0, l], \\ I^{\alpha_i}(t) &= \int_0^l \beta(x, t) u^{\alpha_i}(x, t) dx, & t \in [0, T]. \end{aligned} \quad (3.3)$$

Using $I^{\alpha_n}(t) \rightarrow I^*(t)$ as $n \rightarrow \infty$ and the continuity of F yields

$$\sum_{i=n+1}^{k_n} \lambda_i^n F(I^{\alpha_i}(t)) \rightarrow \sum_{i=n+1}^{k_n} \lambda_i^n F(I^*(t)) = F(I^*(t)).$$

Passing to the limit $n \rightarrow \infty$ in (3.3), we produce the following system

$$\begin{aligned} D_\varphi u^*(x, t) &= -[\mu(x, t) + g'(x) + \alpha^*(x, t)]u^*(x, t), & (x, t) \in Q, \\ g(0)u^*(0, t) &= C(t) + F(I^*(t)), & t \in (0, T], \\ u^*(x, 0) &= u_0(x), & x \in [0, l], \\ I^*(t) &= \int_0^l \beta(x, t)u^*(x, t) dx, & t \in [0, T], \end{aligned}$$

which implies that $u^*(x, t) = u^{\alpha^*}(x, t)$ a.e. $(x, t) \in Q$ and $I^*(t) = I^{\alpha^*}(t)$ a.e. $t \in [0, T]$.

Next, we show that the control $\alpha^* \in \mathcal{U}$ is an optimal policy. On the one hand, since $d - \frac{1}{n} < J(\alpha_i) \leq d$ for any $\alpha_i \in \mathcal{U}$, using $\lambda_i^n \geq 0$ and $\sum_{i=n+1}^{k_n} \lambda_i^n = 1$, we have

$$d - \frac{1}{n} < \sum_{i=n+1}^{k_n} \lambda_i^n J(\alpha_i) \leq d.$$

Thus, we have $\sum_{i=n+1}^{k_n} \lambda_i^n J(\alpha_i) \rightarrow d$ as $n \rightarrow \infty$.

On the other hand, by the definition of $J(\cdot)$, (3.1), and (3.2), we have

$$\begin{aligned} \sum_{i=n+1}^{k_n} \lambda_i^n J(\alpha_i) &= \sum_{i=n+1}^{k_n} \lambda_i^n \left[\int_0^T \int_0^l \omega(x, t) \alpha_i(x, t) u^{\alpha_i}(x, t) dx dt - \int_0^T r(t) C(t) dt \right] \\ &= \int_0^T \int_0^l \omega(x, t) \tilde{\alpha}_n(x, t) \tilde{u}_n(x, t) dx dt - \int_0^T r(t) C(t) dt \\ &\rightarrow \int_0^T \int_0^l \omega(x, t) \alpha^*(x, t) u^{\alpha^*}(x, t) dx dt - \int_0^T r(t) C(t) dt \\ &= J(\alpha^*). \end{aligned}$$

This means that α^* is an optimal control policy for the control problem (1.2)–(1.3). This completes the proof. $\square$

4. THE ADJOINT SYSTEM

In this section, we will derive the adjoint system of system (1.2). Here and following, we denote by $\mathcal{T}_{\mathcal{U}}(\alpha)$ and $\mathcal{N}_{\mathcal{U}}(\alpha)$ the tangent cone and the normal cone of $\mathcal{U}$ at α , respectively.

Lemma 4.1. *Let $u^*(x, t)$ be the solution of system (1.2) corresponding to $\alpha^* \in \mathcal{U}$. For each $v \in T_{\mathcal{U}}$ such that $\alpha^* + \varepsilon v \in \mathcal{U}$ for sufficiently small $\varepsilon > 0$, we have*

$$\frac{1}{\varepsilon} [u^\varepsilon - u^*] \rightarrow z(x, t)$$

as $\varepsilon \rightarrow 0$, where u^ε is the solution of (1.2) corresponding to $\alpha^* + \varepsilon v \in \mathcal{U}$ and $z(x, t)$ is the solution of the following system

$$\begin{aligned} D_\varphi z(x, t) &= -[\mu(x, t) + g'(x) + \alpha^*(x, t)]z - v(x, t)u^*(x, t), & (x, t) \in Q, \\ g(0)z(0, t) &= f'(I^*(t))I^*(t)Z(t) + f(I^*(t))Z(t), & t \in [0, T], \\ z(x, 0) &= 0, & x \in [0, l], \\ Z(t) &= \int_0^l \beta(x, t)z(x, t) dx, & t \in [0, T]. \end{aligned} \quad (4.1)$$

Proof. The existence and uniqueness of solution to system (4.1) can be established by a similar way as that in the proofs of Theorems 1 and 2 in [20]. According to Lemma 3.13 in [1], $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}[u^\varepsilon - u^*]$ makes sense. Note that $u^\varepsilon \doteq u^{\alpha^* + \varepsilon v}$ and u^* are solutions of system (1.2) corresponding to $\alpha^* + \varepsilon v$ and $\alpha^* \in \mathcal{U}$, respectively. Then $\frac{1}{\varepsilon}[u^\varepsilon - u^*]$ must be the solution of the following system

$$\begin{aligned} D_\varphi \frac{1}{\varepsilon}[u^\varepsilon - u^*] &= -[\mu + g']\frac{1}{\varepsilon}[u^\varepsilon - u^*] - \frac{1}{\varepsilon}[(\alpha^* + \varepsilon v)u^\varepsilon - \alpha^*u^*], \\ g(0)\frac{1}{\varepsilon}[u^\varepsilon - u^*](0, t) &= \frac{1}{\varepsilon}[f(I^\varepsilon(t))I^\varepsilon(t) - f(I^*(t))I^*(t)], \\ \frac{1}{\varepsilon}[u^\varepsilon - u^*](x, 0) &= 0, \\ \frac{1}{\varepsilon}[I^\varepsilon(t) - I^*(t)] &= \int_0^l \beta(x, t)\frac{1}{\varepsilon}[u^\varepsilon - u^*](x, t) dx. \end{aligned} \quad (4.2)$$

From Theorem 2.3 and assumption (A₄), it follows that, as $\varepsilon \rightarrow 0$,

$$\frac{1}{\varepsilon}[f(I^\varepsilon(t))I^\varepsilon(t) - f(I^*(t))I^*(t)] \rightarrow f'(I^*(t))I^*(t)Z(t) + f(I^*(t))Z(t). \quad (4.3)$$

Passing to the limit as $\varepsilon \rightarrow 0$ in system (4.2) and using (4.3) produces the required result. The proof is complete. $\square$

Next we consider the following adjoint system of system (1.2).

$$\begin{aligned} D_\varphi \xi(x, t) &= \mu(x, t)\xi(x, t) + [\omega(x, t) + \xi(x, t)]\alpha(x, t) \\ &\quad - \beta(x, t)\xi(0, t)[f'(I^\alpha(t))I^\alpha(t) + f(I^\alpha(t))], & (x, t) \in Q, \\ \xi(l, t) &= 0, & t \in [0, T], \\ \xi(x, T) &= 0, & x \in [0, l], \\ I^\alpha(t) &= \int_0^l \beta(x, t)u^\alpha(x, t) dx, & t \in [0, T], \end{aligned} \quad (4.4)$$

where $\xi(x, t)$ is an unknown dual variable and u^α is the solution of (1.2) corresponding to $\alpha \in \mathcal{U}$.

Treating (4.4) in the same manner as that in Theorems 1 and 2 in [20], we can get the following result with the proof being omitted.

Theorem 4.2. *Let assumptions (A₁) – (A₅) hold. For each $\alpha \in \mathcal{U}$, the adjoint system (4.4) has a unique bounded solution $\xi(x, t) \in L^\infty(Q)$. Moreover, there exists a positive constant B_2 such that*

$$\|\xi_1 - \xi_2\|_{L^1(Q)} \leq B_2 T \|\alpha_1 - \alpha_2\|_\infty,$$

where ξ_1 and ξ_2 are solutions of (4.4) corresponding to α_1 and $\alpha_2 \in \mathcal{U}$, respectively.

5. OPTIMALITY CONDITIONS

In this section, we will derive the first-order necessary conditions of optimality in the form of an Euler–Lagrange system.

Theorem 5.1. *Let $\alpha^*(x, t)$ be an optimal policy for the control problem (1.2)–(1.3). Under the conditions of Theorem 2.3, we have*

$$\alpha^*(x, t) = \begin{cases} \underline{\alpha}, & \text{if } \omega(x, t) + \xi(x, t) < 0, \\ \bar{\alpha}, & \text{if } \omega(x, t) + \xi(x, t) > 0, \end{cases} \quad (5.1)$$

where $\xi(x, t)$ is the solution of the following system

$$\begin{aligned} D_\varphi \xi(x, t) &= \mu(x, t)\xi(x, t) + [\omega(x, t) + \xi(x, t)]\alpha^*(x, t) \\ &\quad - \beta(x, t)\xi(0, t)[f'(I^*(t))I^*(t) + f(I^*(t))], \quad (x, t) \in Q, \\ \xi(l, t) &= 0, \quad t \in [0, T], \\ \xi(x, T) &= 0, \quad x \in [0, l], \\ I^*(t) &= \int_0^l \beta(x, t)u^*(x, t) dx, \quad t \in [0, T], \end{aligned} \quad (5.2)$$

in which $f'(I^*(t))$ is the derivatives of f with respect to I^* .

Proof. For any element of tangent cone $v \in \mathcal{T}_U(\alpha^*)$, we have $\alpha^\varepsilon \doteq \alpha^* + \varepsilon v \in \mathcal{U}$ for sufficiently small $\varepsilon > 0$. Let $u^\varepsilon(x, t)$ be the solution of system (1.2) corresponding to $\alpha^\varepsilon \doteq \alpha^* + \varepsilon v$. From the optimality of α^* , it follows that

$$\int_0^T \int_0^l \omega(x, t)\alpha^\varepsilon(x, t)u^\varepsilon(x, t) dx dt \leq \int_0^T \int_0^l \omega(x, t)\alpha^*(x, t)u^*(x, t) dx dt.$$

By Lemma 4.1, as $\varepsilon \rightarrow 0^+$, we have

$$\int_0^T \int_0^l \omega(x, t)\alpha^*(x, t)z(x, t) dx dt + \int_0^T \int_0^l \omega(x, t)v(x, t)u^*(x, t) dx dt \leq 0, \quad (5.3)$$

where $z(x, t)$ is the solution of (4.1).

Now, we use the structure of the corresponding normal cone to derive the optimality conditions. Multiplying the first equation of (5.2) by $z(x, t)$ and integrating on Q , we obtain

$$\int_0^T \int_0^l (D_\varphi z)\xi dx dt = - \int_0^T \int_0^l [\mu + g']z\xi dx dt - \int_0^T \int_0^l \alpha^*[\omega + \xi]z dx dt. \quad (5.4)$$

Next, multiplying the first equation of (4.1) by $\xi(x, t)$ and integrating on Q , we have

$$\int_0^T \int_0^l (D_\varphi z)\xi dx dt = - \int_0^T \int_0^l [\mu + g']z\xi dx dt - \int_0^T \int_0^l [\alpha^*z + vu^*]\xi dx dt. \quad (5.5)$$

From (5.4) and (5.5), it follows that

$$\int_0^T \int_0^l \omega(x, t)\alpha^*(x, t)z(x, t) dx dt = \int_0^T \int_0^l v(x, t)u^*(x, t)\xi(x, t) dx dt.$$

FIGURE 1. Population density in Example 6.1 with $\alpha = 0$ (left); with $\alpha = \alpha^*$ (right).

Substituting this into (5.3) yields

$$\int_0^T \int_0^l [\omega(x, t) + \xi(x, t)] u^*(x, t) v(x, t) dx dt \leq 0,$$

for each $v \in \mathcal{T}_{\mathcal{U}}(\alpha^*)$. Thus, $[\omega(x, t) + \xi(x, t)] u^*(x, t) \in N_{\mathcal{U}}(\alpha^*)$, where $N_{\mathcal{U}}(\alpha^*)$ is the normal cone of $\mathcal{U}$ at α^* , which implies the conclusion of this theorem. $\square$

Remark 5.2. From Theorems 3.1 and 5.1, there exists at least one optimal control for the control problem (1.2)–(1.3) and the optimal policy is given as (5.1). The conditions in Theorem 5.1 for the optimal policy provide a threshold condition, which depends on $\omega(x, t) + \xi(x, t)$.

6. NUMERICAL SIMULATIONS

In this section, we give the numerical results to illustrate our results. The numerical simulations of population dynamics are carried out for the academic tests with the arbitrary values of parameters which do not correspond to some specific biological populations. Note that the habitat of the fish undergoes some periodic changes because of the such effects as seasoning variations. Thus, it is reasonable to assume that all the parameters in (1.2)–(1.3) are periodic with respect to time t . And

$$\begin{aligned} \beta(x, t) &= 20x^2(1-x)(1+\sin \pi t), \quad \mu(x, t) = e^{-4x}(1-x)^{-1.4}(2+\cos \pi t), \\ g(x) &= 1-x, \quad C(t) = 0.8(1+\sin(\pi t)), \quad \omega(x, t) = \frac{1}{20}(2\pi x + \sin(\pi t) + 1), \\ f(I(t)) &= 0.8, \quad \underline{\alpha} = 0, \quad \bar{\alpha} = 1, \quad l = 1, \quad T = 10. \end{aligned}$$

Example 6.1. Take $u_0(x) = (1-x)(1+x)$ (Figs. 1, 2).

Example 6.2. Take $u_0(x) = (1-x)^2(1+x)$ (Figs. 3, 4).

Example 6.3. Compare the Examples 6.1 and 6.2 (Fig. 5).

Note that $\alpha^* \geq 0$. As can be seen from the right figure of Figures 2 and 4, we know that population density u is monotonically decreasing with respect to α , which is consistent with the comparison principle in Theorem 2.4.

FIGURE 2. Optimal harvesting efforts $\alpha^*(x, t)$ in Example 6.1 (left); trend of the total population in the case of harvest and no harvest in Example 6.1 (right).

FIGURE 3. Population density in Example 6.2 with $\alpha = 0$ (left); with $\alpha = \alpha^*$ (right).

Furthermore, with the increase of time, if all the parameters in (1.2)–(1.3) are periodic with respect to the time t , then the population size changed periodically.

It can be seen from the left of Figure 2 and the left of Figure 4 that the optimal harvesting effort basically owns a bang–bang structure. We observe a significant similarity in the structure of optimal trajectories. It leads to the conclusion that the bang–bang structure of solutions is much more common in optimal population management. Moreover, for different initial values, the optimal control strategy is the same as long as the parameters of the model are the same.

From Figure 5, with the evolution of time, the solutions of system (1.2) tend to the same period solution for different initial datum u_0 . Moreover, comparing the left figure and right figure of Figure 5, we can see that fishing can shorten the time that the solution of the model tends to a periodic solution.

FIGURE 4. Optimal harvesting efforts $\alpha^*(x, t)$ in Example 6.2 (left); trend of the total population in the case of harvest and no harvest in Example 6.2 (right).

FIGURE 5. Trend of the total population in the case of harvest (left) and no harvest (right).

Remark 6.4. In order to simulate the solution of the following system

$$\begin{aligned}
 \frac{\partial u(x, t)}{\partial t} + \frac{\partial(g(x)u(x, t))}{\partial x} &= -\mu(x, t)u(x, t) - \alpha(x, t)u(x, t), & (x, t) \in Q, \\
 g(0)u(0, t) &= C(t) + f(I(t))I(t), & t \in [0, \infty), \\
 u(x, t) &= u(x, t + T), & (x, t) \in Q, \\
 I(t) &= \int_0^t \beta(x, t)u(x, t) dx, & t \in [0, \infty)
 \end{aligned}$$

with T -periodic parameters. First, we take the numerical solution of system (1.2), corresponding to an arbitrary positive initial datum u_0 , on some interval $[kT, (k+1)T]$, where k is large enough. Then, we can get the periodic solution of above system by extending the numerical solution.

7. CONCLUSIONS AND DISCUSSIONS

This paper is concerned with the harvesting of a size structured population of fish. To the best of our knowledge, for the fish, only a fraction of the eggs produced by mature fish can convert to fry. Thus, we take the survival rate of the new born individuals into account when modeling the fish dynamics with size structure. In the previous sections, we prove the existence of optimal policies by using maximizing sequence and Mazur's theorem in convex analysis. Following that is a careful derivation of necessary optimality conditions, which is finished by tangent-normal cones and adjoint system techniques. Moreover, some examples and numerical results are presented, which are used to demonstrate the effectiveness of the theoretical results.

From (5.1), we can see that the optimal harvesting strategy has a bang-bang structure. The conditions in Theorem 5.1 for the optimal harvesting policy provide a threshold condition, which depends on $\omega(x, t) + \xi(x, t)$. Roughly speaking, if this quantity is less than 0, that is, if the economic values of an individual with size x at time t is very small, then the harvesting effort is equal to its minimum $\underline{\alpha}$; otherwise, one needs to apply the harvesting efforts to its maximum $\bar{\alpha}$. This result has obvious practical significance. In reality, if the price of fish is high, more fish are caught; if the price of fish is low, fewer fish are caught. However, since we do not have a clear biological meaning of the solution $\xi(x, t)$ to the adjoint system, it is hard to give a precise interpretation of the above threshold condition.

To conclude this paper, we mention the practical application of the model in fish resource development. First, we need to fit the parameters of the model by the observation data. Then, combining (1.2) and (1.3) with (5.2), we can calculate the optimal harvesting policy, optimal population density. Furthermore, we can get the optimal index x^* (that is, the minimum size of fish to be caught) by the numerical calculation. Then, the size of the mesh can be designed by using the optimal harvesting index x^* . This provides a theoretical basis for the optimal exploitation of fish resources.

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