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ONE-PERIOD STABILITY ANALYSIS OF POLYGONAL SWEEPING PROCESSES WITH APPLICATION TO AN ELASTOPLASTIC MODEL

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Abstract. We offer a finite-time stability result for Moreau sweeping processes on the plane with periodically moving polyhedron. The result is used to establish the convergence of stress evolution of a simple network of elastoplastic springs to a unique cyclic response in just one cycle of the external displacement-controlled cyclic loading. The paper concludes with an example showing that smoothing the vertices of the polyhedron makes finite-time stability impossible.

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1. Introduction

Aiming to design materials with better properties, there has been a great deal of work lately where a discrete structure comes from a certain microstructure formulated through a lattice of elastic springs [13] (metals), [15] (polymers), [6] (titanium alloys), [10] (biological materials). At the same time, recent findings show [4] that a pivotal role in the performance of heterogeneous materials under cyclic loading is played by micro-plasticity. Specifically, the regions of asymptotic concentration of plastic deformations are the likely candidates for fatigue crack initialization [3, 4]. Current methods of computing the asymptotic response (see *e.g.* [5, 16]) run the numeric routine until the difference between the responses corresponding to two successive cycles of loading gets smaller than a prescribed tolerance, without any estimate as for how soon such a prescribed accuracy will be reached. Through a case study, this paper initiates the development of a theory where the distribution of plastic deformations in a network of elastoplastic springs with external loading can be evaluated in just two cycles of the loading.

We stick to the setting of ideal plasticity (the stress of each spring is constrained within so-called *elastic limits* beyond which plastic deformation begins) and address an equivalent problem of finding the asymptotic distribution of stresses $s(t) = (s_1(t), \ldots, s_m(t))$ of a network of m elastoplastic springs. As a benchmark for the development of a general theory, this paper considers a network of elastoplastic springs given by Figure 1. We follow the Moreau sweeping process approach according to which the stress vector $s(t)$ can be computed from

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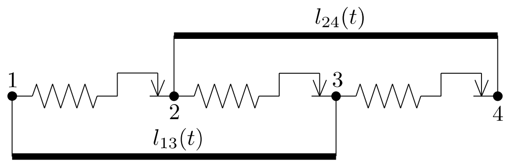

FIGURE 1. Elastoplastic model with two displacement-controlled loadings. Throughout the paper we assume that the stiffnesses of all springs equal 1 and the elastic domains of all springs are $[-1, 1]$. Furthermore, we denote $l(t) = (l_{13}(t), l_{24}(t))^\top$.

the m-dimensional solution $x(t)$ of the differential inclusion

$$-x'(t) \in N_{\mathcal{C}+c(t)}(x(t)), \tag{1.1}$$

where

$$N_C(x) = \begin{cases} \{\zeta \in \mathbb{R}^n : \langle \zeta, c - x\rangle \leqslant 0, \text{ for any } c \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C, \end{cases} \tag{1.2}$$

$\mathcal{C} \subset \mathbb{R}^m$ is a closed convex polyhedron, and $c(t) \in \mathbb{R}^m$ is a T-periodic vector, that we compute in closed-form in the next section of the paper. Moreover, in Moreau's framework the moving set $\mathcal{C} + c(t)$ is always located in some subspace of a lower dimension d thus restricting the dynamics to the subspace. Specifically, in such case (1.1) can be reduced to

$$-y' \in N^V_{\mathcal{C}+c(t)}(y), \tag{1.3}$$

where $N^V_{\mathcal{C}+c(t)}(y)$ is the normal cone in a d-dimensional subspace V of $\mathbb{R}^m$ constructed as follows

$$N^V_{\mathcal{C}+c(t)}(y) = \{\zeta \in V : \langle \zeta, \bar{c} - y\rangle \leqslant 0, \text{ for any } \bar{c} \in \mathcal{C} + c(t)\} = N_{\mathcal{C}+c(t)}(y) \cap V.$$

In the next section of the paper we construct V for the particular model of Figure 1, in which case we get $d = \dim V = 2$. The goal of the paper is, therefore, to obtain conditions under which all the solutions to the planar sweeping process (1.3) with T-periodic $t \mapsto c(t)$ converge to a unique asymptotic regime in exactly one time-interval $[0, T]$.

When $t \mapsto c(t)$ is Lipschitz-continuous (which is the case when the displacement-controlled loadings $l_{13}(t)$ and $l_{24}(t)$ of Figure 1 are Lipschitz-continuous), the existence of solutions to (1.3) and continuous dependence on the initial conditions is a well-known fact, see *e.g.* Kunze and Monteiro Marques [12]. The asymptotic behavior of the sweeping process (1.3) was studied by Krejci [11] who proved that, when $t \mapsto c(t)$ is T-periodic, any solution $x(t)$ of (1.3) converges to a T-periodic solution to the process (1.3) as $t \to \infty$. The uniqueness of such a T-periodic response was established in Gudoshnikov-Makarenkov [9] under the assumption that the set $\mathcal{C}$ is a simplex. The best result towards finite-time stability of solutions to (1.3) is obtained in Adly *et al.* [1], who dealt with a differential inclusion coming from frictional mechanics. According to [1] a solution $x(t)$ of (1.3) is finite-time stable, if the vector $x'(t)$ lies strictly inside the normal cone $N_{\mathcal{C}+c(t)}(x(t))$ for a.a. t such that $x(t)$ is located at a vertex of $\mathcal{C} + c(t)$. Denoting by $\varepsilon > 0$ the corresponding minimal margin between vector $x'(t)$ and the normal cone $N_{\mathcal{C}+c(t)}(x(t))$, the main result of the present paper relates the values of $\varepsilon > 0$ to the time T_0 required for the solution $x(t)$ to reach its asymptotic limit. If T_0 (which depends on ε and $x(0)$) doesn't exceed T for any solution $x(t)$ of (1.3), then any solution $x(t)$ reaches the asymptotic limit in exactly one time-interval $[0, T]$, so that the complete information about the asymptotic dynamics of (1.3) can be gathered by computing $x(t)$ on the time-interval $[T, 2T]$.

The paper is organized as follows. The next section of the paper is devoted to a formulation of the dynamics of the stress-vector $s(t) = (s_1(t), s_2(t), s_3(t))$ in terms of a sweeping process of the form (1.3). Specifically, assuming that the stress of each spring is constrained within $[-1,1]$ and that the Hooke's constants of all springs is 1, we show that the closed convex polygon $\mathcal{C}$ of the associated sweeping process (1.3) has 6 vertexes. Section 4 establishes the main result of the paper (Thm. 3.1) that relates properties of the function $t \mapsto c(t)$ of (1.3) to the duration of finite-time convergence of solutions to the process (1.3) to the asymptotic limit. This relation is then used in Corollary 3.6 of the same section in order to give conditions for one-period stability of sweeping process (1.3), *i.e.* to give conditions which ensure that any solution $x(t)$ of (1.3) merges with its asymptotic limit in a time not exceeding the period T of $t \mapsto c(t)$. Corollary 3.6 is applied to the elastoplastic system of Figure 1 in Section 5, where, for each of the 6 vertexes of $\mathcal{C}$, conditions for the mechanical parameters of the elastoplastic system of Figure 1 are obtained to ensure the one-period convergence to the asymptotic regime given by the vertex under consideration. In Section 6 we give a simple example with $\mathcal{C}$ being a circle, which shows that the polygonal shape of $\mathcal{C}$ is essential for finite-time stability. A conclusions section concludes the paper.

2. The sweeping process of the benchmark elastoplastic system

We consider a system of 3 elastoplastic springs connected as shown in Figure 1. We assume that the coordinate of each of the 4 nodes is described by a single scalar and that the distance between node 1 and node 3 is locked and controlled to be equal $l_{13}(t)$ (called *displacement-controlled loading*). Similarly, a displacement-controlled loading $l_{24}(t)$ locks the distance between nodes 2 and 4. If $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)^\top$ are the coordinates of the nodes of the elastoplastic system of Figure 1, then introducing

$$D\xi = \begin{pmatrix} \xi_2 - \xi_1 \\ \xi_3 - \xi_2 \\ \xi_4 - \xi_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix},$$

the total elongations $x = (x_{12}, x_{23}, x_{34})$ of the 3 springs of Figure 1 can be computed as

$$x = D\xi.$$

Note, that solving

$$D\xi = 0,$$

leads to

$$\operatorname{Ker} D = \operatorname{span}\left((1,1,1,1)^\top\right). \tag{2.1}$$

The displacement-controlled loadings $l_{13}(t)$ and $l_{24}(t)$ lead to the following constraints

$$\begin{array}{rcl} (1,1,0)x & = & l_{13}, \\ (0,1,1)x & = & l_{24}. \end{array} \tag{2.2}$$

Thus $x \in U + g$, where U is a subspace of $\mathbb{R}^3$ such that $U \perp \operatorname{span}\left((1,1,0)^\top, (0,1,1)^\top\right)$ and g is a suitable (shift) vector of $\mathbb{R}^3$ such that $g \perp U$. Since $(1-1,1)(1,1,0)^\top = (1-1,1)(0,1,1)^\top = 0$, we get

$$U = \operatorname{span}\left((1,-1,1)^\top\right).$$

To find g, observe, that $x - g \in U$ means that

$$\begin{aligned} (1,1,0)(x-g)^\top &= 0, \\ (0,1,1)(x-g)^\top &= 0, \end{aligned}$$

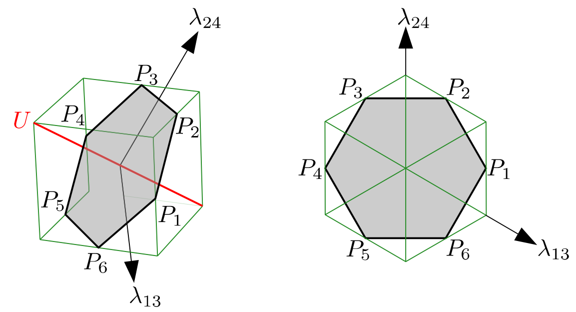

FIGURE 2. Moving set of the sweeping process (1.1) shown in two projections.

whose difference with (2.2) yields

$$\begin{aligned}(1,1,0)g &= l_{13},\\ (0,1,1)g &= l_{24}.\end{aligned}$$

On the other hand, $g \perp U$ implies that

$$(1,-1,1)g = 0,$$

so that we can solve the above three equations for g obtaining

$$g = \frac{1}{3}(2l_{13} - l_{24}, l_{13} + l_{24}, -l_{13} + 2l_{24}) = \frac{1}{3}\left(\begin{pmatrix}2\\1\\-1\end{pmatrix} l_{13} + \begin{pmatrix}-1\\1\\2\end{pmatrix} l_{24}\right) = \frac{1}{3}\Lambda l, \tag{2.3}$$

where

$$\Lambda = \begin{pmatrix}\lambda_{13} & \lambda_{24}\end{pmatrix}, \qquad \lambda_{13} = \begin{pmatrix}2\\1\\-1\end{pmatrix}, \qquad \lambda_{24} = \begin{pmatrix}-1\\1\\2\end{pmatrix}, \qquad l(t) = \begin{pmatrix}l_{13}(t)\\l_{24}(t)\end{pmatrix}. \tag{2.4}$$

When the stiffnesses of all springs equal 1 and the elastic domains are $[-1,1]$ (meaning that -1 and 1 are elastic limits for all springs), the equation for plastic elongations $p = (p_1, p_2, p_3)^\top$ of springs takes the form

$$p' \in N_C(s), \quad C = [-1,1] \times [-1,1] \times [-1,1],$$

where $s = (s_1, s_2, s_3)$ are stresses of the springs. According to Moreau [14] (see also Gudoshnikov-Makarenkov [9]), if, at each time t, the elastoplastic system under consideration attains a static equilibrium, then the evolution (termed *quasi-static evolution*) of the variable

$$y(t) = s(t) - g(t), \tag{2.5}$$

can be described by the differential inclusion (1.3) with

$$\mathcal{C} + c(t) = C \cap V - g(t) \quad \text{and} \quad V = (1,-1,1)^\perp, \tag{2.6}$$

(see Fig. 2) from which we conclude that $d = \dim V = 2$.

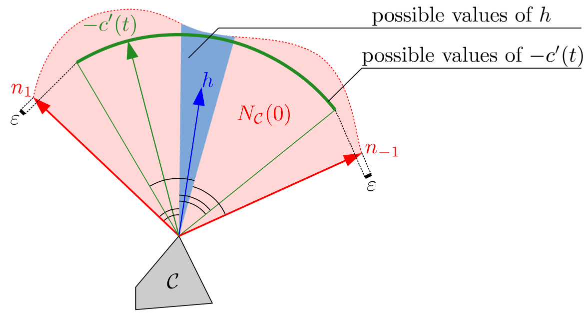

FIGURE 3. Illustration of conditions of Theorem 3.1.

3. ONE-PERIOD STABILITY RESULT FOR PLANAR SWEEPING PROCESSES

In this section we restrict ourselves to a 2-dimensional linear space V equipped with an inner product $\langle\cdot,\cdot\rangle$ and the consider sweeping process (1.3) with a closed convex polyhedron of a general form

$$\mathcal{C} := \bigcap_{i\in\mathcal{I}} \{x \in V : \langle x, n_i\rangle \leqslant c_i\} \tag{3.1}$$

where $\mathcal{I}$ is a finite set of indices and $\{n_i\}_{i\in\mathcal{I}}$ are unit vectors. Without loss of generality, we will assume that $0 \in \mathcal{C}$ is a vertex of $\mathcal{C}$, *i.e.* there exist non-parallel unit vectors $n_1, n_{-1} \in V$ such that

$$N_{\mathcal{C}}(0) = \{\alpha n_1 + \beta n_{-1} : \alpha, \beta \geqslant 0\}$$

see Figure 3. In what follows we will assume that $c : [0,T] \to V$ in (1.3) is a Lipschitz-continuous function. This allows to get the existence and uniqueness of solutions to the process (1.3) by applying the standard existence and uniqueness theorem (see *e.g.* Kunze and Monteiro Marques [12]) successively on each of the intervals of continuous differentiability of $c : [0,T] \to V$. The main result of this paper is the following theorem.

Theorem 3.1. *Let $c : [0,T] \to V$ be a Lipschitz-continuous function such that the following conditions hold for all $t \in [0,T]$ where $c'(t)$ exists:*

(A1) $\|c'(t)\| = 1$,
(A2) $-c'(t) \in N_{\mathcal{C}}(0)$,
(A3) there exists $0 < \varepsilon < 1$ (independent of t) such that

$$\langle -c'(t), n_1\rangle < 1-\varepsilon \quad \text{and} \quad \langle -c'(t), n_{-1}\rangle < 1-\varepsilon.$$

Fix an arbitrary $h \in V$, satisfying

$$\langle -c'(t), h\rangle \geqslant \langle n_1, h\rangle > 0, \quad \textit{and} \quad \langle -c'(t), h\rangle \geqslant \langle n_{-1}, h\rangle > 0, \quad \textit{for all } t, \ \textit{where } c'(t) \ \textit{exists}, \tag{3.2}$$

see Figure 3. Consider a solution y to the sweeping process (1.3) with an initial condition $y(0) = y_0 \in \mathcal{C} + c(0)$. Put

$$T_{y_0} = \frac{\langle c(0) - y_0, h\rangle}{\mu} \quad \text{with} \quad \mu = \varepsilon \min\{\langle h, n_1\rangle, \langle h, n_{-1}\rangle\}. \tag{3.3}$$

If $T_{y_0} \leq T$, then $c(t) = y(t)$, for all $t \geq T_{y_0}$.

Remark 3.2. It is always possible to find h satisfying (3.2). Indeed, one can use Figure 3 to show that a possible choice for h is *e.g.* $h = n_1 + n_{-1}$, see also the proofs below.

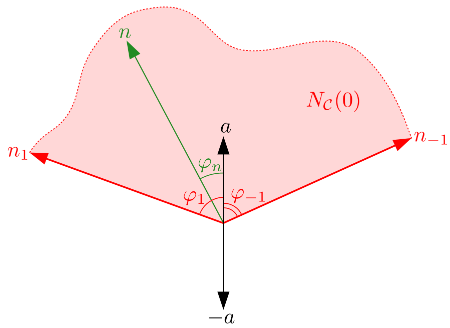

FIGURE 4. An illustration to Lemma 3.3.

The following two lemmas are required for the proof of Theorem 3.1.

Lemma 3.3. *Let $a \in N_{\mathcal{C}}(0), a \neq 0$ and $n \in V, \|n\| = 1$. If*

$$\langle n - n_1, a\rangle \geqslant 0 \quad \text{and} \quad \langle n - n_{-1}, a\rangle \geqslant 0,$$

then $n \in N_{\mathcal{C}}(0)$. The notations of Lemma 3.3 are illustrated at Figure 4.

Proof. Observe, that $N_{\mathcal{C}}(0)$ is a sector between vectors n_{-1}, n_1 of an angle less than π. We take the direction of a as the zero direction and parametrize two unit semicircles from a to $-a$ by an angle $\varphi \in [0, \pi]$. Let $\varphi_{-1}, \varphi_1, \varphi_n$ be the angles of the vectors n_1, n_{-1}, n respectively. Notice that if $\varphi_n \leqslant \min\{\varphi_1, \varphi_{-1}\}$ then $n \in N_{\mathcal{C}}(0)$. Recall that

$$\langle n, a\rangle = \|a\| \cos\varphi_n, \qquad \langle n_1, a\rangle = \|a\| \cos\varphi_1, \qquad \langle n_{-1}, a\rangle = \|a\| \cos\varphi_{-1};$$

thus

$$\langle n - n_1, a\rangle = \|a\|(\cos\varphi_n - \cos\varphi_1) \geqslant 0 \quad \text{and} \quad \langle n - n_1, a\rangle = \|a\|(\cos\varphi_n - \cos\varphi_1) \geqslant 0.$$

Therefore, since cos is decreasing on $[0, \pi]$, we have $\varphi_n \leqslant \min\{\varphi_{-1}, \varphi_1\}$ and $n \in N_{\mathcal{C}}(0)$. □

Remark 3.4. Due to isomorphism of a finite-dimensional inner product space with Euclidean space $\mathbb{R}^n$ of the same dimension (see *e.g.* Anton-Rorres [2], Sect. 8.2, p. 796) we can justify the use of a geometric argument in the proof above (as well as in the proof of Thm. 3.1) for arbitrary inner product $\langle\cdot,\cdot\rangle$.

Lemma 3.5. *Let $y(t)$ be a solution to a (not necessarily 2-dimensional) sweeping process of the type (1.3) with a polyhedral set $\mathcal{C}$, described as (3.1) with n_i being unit vectors. Assume that a point $t^* \in (0, T)$ is such that $c'(t^*)$ and $y'(t^*)$ exist, the inclusion (1.3) holds and $y(t^*)$ lays on a single facet of the moving set, i.e. there is exactly one $i \in \mathcal{I}$ such that*

$$\langle n_i, y(t^*)\rangle = \langle n_i, c(t^*)\rangle + c_i. \tag{3.4}$$

Then $\langle n_i, c'(t^*)\rangle \leqslant 0$ and $y'(t^*) = \langle n_i, c'(t^*)\rangle n_i$.

Proof. Condition (3.4) means that

$$-y'(t^*) \in N_{\mathcal{C}+c(t^*)}(y(t^*)) = \{\alpha n_i : \alpha \geqslant 0\}.$$

In other words,

$$y'(t^*) = \alpha n_i, \qquad \text{where } \alpha = \langle n_i, y'(t^*)\rangle \leqslant 0.$$

Let $\beta := \langle n_i, c'(t^*)\rangle$.

We need to show that $\alpha = \beta$. Since $y'(t^*)$ and $c'(t^*)$ exist we can use the definitions of derivative and limit, *i.e.* for each $\varepsilon_1 > 0$ there is $\delta_1 > 0$ such that for all t satisfying $0 < |t - t^*| < \delta_1$

$$\left| \frac{\langle n_i, y(t) - y(t^*)\rangle}{t - t^*} - \alpha \right| < \varepsilon_1,$$

and for each $\varepsilon_2 > 0$ there is $\delta_2 > 0$ such that for all t satisfying $0 < |t - t^*| < \delta_2$

$$\left| \frac{\langle n_i, c(t) - c(t^*)\rangle}{t - t^*} - \beta \right| < \varepsilon_2.$$

Assume that $\alpha > \beta$, choose $\varepsilon_1 = \varepsilon_2 = \frac{\alpha-\beta}{2}$ and observe that for all t such that $0 < |t - t^*| < \min\{\delta_1, \delta_2\}$

$$\frac{\langle n_i, y(t) - y(t^*)\rangle}{t - t^*} > \frac{\alpha + \beta}{2} > \frac{\langle n_i, c(t) - c(t^*)\rangle}{t - t^*}.$$

Choose $t > t^*$ and conclude that

$$\langle n_i, y(t) - y(t^*)\rangle > \langle n_i, c(t) - c(t^*)\rangle,$$
$$\langle n_i, y(t)\rangle > \langle n_i, c(t)\rangle + \langle n_i, y(t^*) - c(t^*)\rangle = \langle n_i, c(t)\rangle + c_i,$$

which means the contradiction $y(t) \notin \mathcal{C} + c(t)$.

If we assume $\alpha < \beta$ we proceed similarly and choose $t < t^*$. Therefore $\alpha = \beta \leqslant 0$ which is the statement of the lemma. □

Proof of Theorem 3.1. At first, notice, that $0 \in \mathcal{C}$ and (A2) guarantee, that $c(t)$ is the solution to (1.3) for $y_0 = c(0)$

$$-c'(t) \in N_{\mathcal{C}+c(t)}(c(t)),$$

i.e. the point $c(t)$ is swept by two edges, corresponding to n_1 and n_{-1}. Condition (A3) means that $n_1 \neq c'(t) \neq n_{-1}$ hence by (A1) and (A2) we necessarily have

$$-c'(t) \in \operatorname{int} N_{\mathcal{C}}(0).$$

By monotonicity of a normal cone this implies that for any other vertex $v \neq 0$ of $\mathcal{C}$ we have

$$-c'(t) \notin N_{\mathcal{C}}(v) = N_{\mathcal{C}+c(t)}(v + c(t)),$$

therefore a solution which will happen to be at any vertex of C other than 0 will leave the vertex immediately.

Choose any $y_0 \in (\mathcal{C} \setminus \{0\}) + c(0)$ and consider the corresponding solution $y(t)$ of (1.3). For almost all $t \in [0, T]$ derivatives $c'(t)$ and $y'(t)$ exist and (1.3) holds as an inclusion. Therefore for almost all $t \in [0, T]$ either

(a) $y(t)$ is not swept, *i.e.* $y'(t) = 0$, or
(b) $y(t)$ is swept by a single edge, *i.e.* $y'(t) = \langle n_i, c'(t)\rangle n_i$ for some $i \in \mathcal{I}$ and $\langle -c'(t), n_i\rangle > 0$ (see lemma 3.5).

Notice, that

$$h \in \operatorname{int} N_{\mathcal{C}}(0).$$

Indeed, (3.2) implies that

$$\|h\| \cos \varphi_t \geqslant \max\{\|h\| \cos \varphi_1, \|h\| \cos \varphi_{-1}\},$$

where $\varphi_t, \varphi_1, \varphi_{-1} \in [0, \pi]$ are the angles between h and $-c'(t), n_1, n_{-1}$ respectively. The function $\phi \mapsto \cos\phi$ is decreasing on $[0, \pi]$, so we have $\varphi_t \leqslant \min\{\varphi_1, \varphi_{-1}\}$. Since $-c'(t) \in N_{\mathcal{C}}(0)$, vector h cannot be outside of the sector $N_{\mathcal{C}}(0)$. Moreover, $h \neq \|h\| n_i, i \in \{-1, 1\}$, otherwise $\langle -c'(t), n_i\rangle \geqslant 1$, which is impossible due to (A3).

We consider the function

$$f(t) = \langle y'(t) - c'(t), h\rangle$$

and go over the cases:
Case (a) $f(t) = \langle -c'(t), h\rangle \geqslant \langle n_1, h\rangle$.
Case (b1), when $n_i = n_1$ or $n_i = n_{-1}$:

$$\begin{aligned} f(t) = \langle \langle n_i, c'(t)\rangle n_i - c'(t), h\rangle = \langle n_i, c'(t)\rangle\langle n_i, h\rangle - \langle c'(t), h\rangle > -(1-\varepsilon)\langle n_i, h\rangle - \langle c'(t), h\rangle = \\ = \varepsilon\langle n_i, h\rangle - \langle n_i, h\rangle - \langle c'(t), h\rangle \geqslant \varepsilon\, \langle n_i, h\rangle. \end{aligned}$$

Case (b2), when $\langle n_i, h\rangle \leqslant 0$. Then

$$f(t) = \langle n_i, c'(t)\rangle\langle n_i, h\rangle - \langle c'(t), h\rangle \geqslant \langle -c'(t), h\rangle \geqslant \langle n_1, h\rangle.$$

Case (b3), when $\langle n_i, n_1\rangle$ and $n_i \neq n_{-1}$ but $\langle n_i, h\rangle > 0$. Observe that $n_i \notin N_{\mathcal{C}}(0)$ and by lemma 3.3 (applied to $a = h, n = n_i$) we have that

$$\text{either} \quad \langle n_i - n_1, h\rangle < 0 \quad \text{or} \quad \langle n_i - n_{-1}, h\rangle < 0.$$

Similarly for $c'(t) \notin N_{\mathcal{C}}(0)$: lemma 3.3 applied to $a = -c'(t), n = n_i$ leads to

$$\text{either} \quad \langle n_i - n_1, -c'(t)\rangle < 0 \quad \text{or} \quad \langle n_i - n_{-1}, -c'(t)\rangle < 0.$$

Combining the two statements above we get that there exist $j_1, j_2 \in \{1, -1\}$ such that

$$\langle n_{j_1}, h\rangle > \langle n_i, h\rangle > 0 \quad \text{and} \quad \langle n_{j_2}, c'(t)\rangle < \langle n_i, c'(t)\rangle.$$

From this, taking in account (3.2), we get that

$$\begin{aligned} f(t) = \langle n_i, c'(t)\rangle\langle n_i, h\rangle - \langle c'(t), h\rangle > \langle n_{j_2}, c'(t)\rangle\langle n_i, h\rangle - \langle c'(t), h\rangle > -(1-\varepsilon)\langle n_i, h\rangle - \langle c'(t), h\rangle \\ > -(1-\varepsilon)\langle n_{j_1}, h\rangle - \langle c'(t), h\rangle = \varepsilon\langle n_{j_1}, h\rangle - \langle n_{j_1} + c'(t), h\rangle \geqslant \varepsilon\, \langle n_{j_1}, h\rangle. \end{aligned}$$

Summing up the analysis in cases a and b, we conclude that

$$f(t) \geqslant \varepsilon \min\{\langle h, n_1\rangle, \langle h, n_{-1}\rangle\} = \mu.$$

Since $h \in \operatorname{int} N_{\mathcal{C}}(0)$ we have $\langle y_0 - c(0), h\rangle = \langle y_0 - c(0) - 0, h\rangle < 0$ since $y_0 \in \mathcal{C} + c(0)$ and $0 \in \mathcal{C}$. Moreover, the point 0 is the only point $y^* \in V$ such that $h \in N_{\mathcal{C}}(y^*)$, hence

$$\langle h, x\rangle = 0 \quad \text{if and only if} \quad x = 0.$$

Now, considering

$$\langle y(t) - c(t), h\rangle = \left\langle y_0 - c(0) + \int_0^t y'(s) - c'(s)\mathrm{d}s, h \right\rangle = \langle y_0 - c(0), h\rangle + \int_0^t f(s)\mathrm{d}s \geqslant \langle y_0 - c(0), h\rangle + t\mu,$$

we have $\langle y(T_0) - c(T_0), h\rangle = 0$, *i.e.* $y(T_0) - c(T_0) = 0$, which concludes the proof.

Theorem 3.1 finally allows us to formulate the following result about one-period stability of the solutions to the sweeping process (1.3).

Corollary 3.6. *Assume that $c : [0, +\infty) \to V$ is a τ-periodic Lipschitz-continuous function. Assume that conditions (A1)–(A3) of Theorem 3.1 hold with some $T \in (0, \tau]$ and h is chosen according to (3.2). Let T_{y_0} be as given by (3.3). If $T_{y_0} \le T$ for all $y_0 \in \mathcal{C} + c(t)$, then the solution $\hat{y}$ of (1.3) with the initial condition $\hat{y}(T) = c(T)$ is a one-period globally stable τ-periodic solution. Specifically, any other solution y of (1.3) with $y(0) \in \mathcal{C} + c(0)$ merges with $\hat{y}$ in time T (i.e. $y(t) = \hat{y}(t)$, for all $t \ge T$).*

4. Application of the one-period stability result to the sweeping process of the initial elastoplastic system

In this section we apply Theorem 3.1 and Corollary 3.6 to the sweeping process of the elastoplastic system of Figure 1 derived in Section 2.

Let us denote the vertices of hexagon $\mathcal{C}$ by $P_1, \dots, P_6$. The coordinates of the vertices in $\mathbb{R}^3$ are listed in the table below. One can verify that each of the vertices $P_1, \dots, P_6$ simultaneously belongs to the edges of the cube C and to the plane V as defined in (2.6). In the same table we list the coordinates of $P_1, \dots, P_6$ in the basis Λ given by (2.3).

Table 1 The coordinates of the vertices of the hexagon $\mathcal{C}$ of Figure 5 in bases $\mathbb{R}^3$ and Λ.

Basis	P_1	P_2	P_3	P_4	P_5	P_6
$\mathbb{R}^3$	$(1,1,0)$	$(0,1,1)$	$(-1,0,1)$	$(-1,-1,0)$	$(0,-1,-1)$	$(1,0,-1)$
$\lambda_{13}, \lambda_{24}$	$(\frac{2}{3}, \frac{1}{3})$	$(\frac{1}{3}, \frac{2}{3})$	$(-\frac{1}{3}, \frac{1}{3})$	$(-\frac{2}{3}, -\frac{1}{3})$	$(-\frac{1}{3}, -\frac{2}{3})$	$(\frac{1}{3}, -\frac{1}{3})$

Observing that

$$\left\|\Lambda(1,0)^\top\right\| = \left\|\Lambda(1,1)^\top\right\| = \left\|\Lambda(0,1)^\top\right\| = \sqrt{6},$$

one can also write down the coordinates in the basis Λ of the outward normals $N_1, \dots, N_6$ as follows.

Table 2 The normals of the hexagon $\mathcal{C}$ of Figure 5 in the basis Λ.

N_1^Λ	N_2^Λ	N_3^Λ	N_4^Λ	N_5^Λ	N_6^Λ
$\frac{1}{\sqrt{6}}(1,1)$	$\frac{1}{\sqrt{6}}(0,1)$	$\frac{1}{\sqrt{6}}(-1,0)$	$\frac{1}{\sqrt{6}}(-1,-1)$	$\frac{1}{\sqrt{6}}(0,-1)$	$\frac{1}{\sqrt{6}}(1,0)$

and the actual vectors are

$$N_i = \Lambda N_i^\Lambda \in \mathbb{R}^3. \tag{4.1}$$

As the vector h required by Theorem 3.1 we take the position-vectors of the vertices:

$$H_i := P_i, \tag{4.2}$$

see Figure 5. Observe from Figures 3 and 5 that the requirement (3.2) is always satisfied for $h = H_i, i \in \overline{1,6}$.

We apply Theorem 3.1 to the 2-dimensional subspace $V = (1,-1,1)^\perp$ of the space of elongations $\mathbb{R}^3$ with the inner product $\langle \cdot, \cdot \rangle$ being the usual dot product from $\mathbb{R}^3$ induced to the subspace.

We fix an arbitrary $i \in \overline{1,6}$, translate the corresponding vertex P_i to the origin, and redefine $\mathcal{C}$ and $c(t)$ as

$$\mathcal{C} = [-1,1] \times [-1,1] \times [-1,1] \cap (1,-1,1)^\perp - P_i, \qquad c(t) = -g(t) + P_i = -\frac{1}{3}\Lambda l(t) + P_i. \tag{4.3}$$

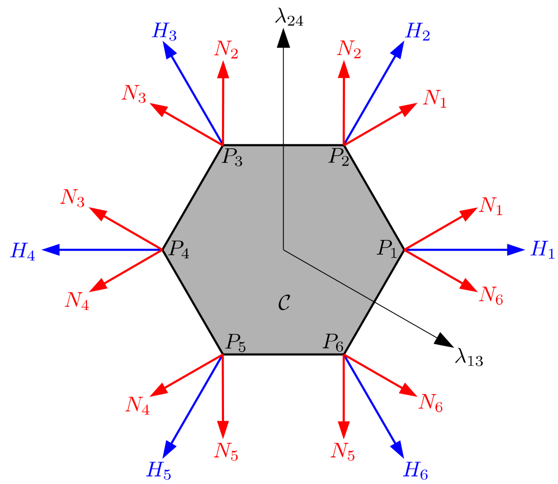

FIGURE 5. The moving set of the sweeping process from Section 2 and the elements which we use to apply Theorem 3.1.

In what follows, we apply Theorem 3.1 with n_1 and n_{-1} equal N_{i-1} and N_i respectively (defining $N_0 = N_6$) on the subspace V. The reduction of the normal cone $N_{\mathcal{C}}(0)$ to V takes the form

$$N_{\mathcal{C}}^V(0) = \{x : x \in V, \langle x, H_{i+1}\rangle \geqslant 0, \langle x, H_{i-1}\rangle \geqslant 0\},$$

that we use as $N_{\mathcal{C}}(0)$ when applying Theorem 3.1. Finally, since the angles between the vectors N_i and H_i are the same for every vertex, the constant μ of (3.3) computes as

$$\mu = \varepsilon \min\{\langle H_i, N_{i-1}\rangle, \langle H_i, N_i\rangle\} = \varepsilon\langle H_i, N_i\rangle = \varepsilon\langle H_i, N_{i-1}\rangle = \varepsilon\langle H_1, N_1\rangle. \tag{4.4}$$

To summarize, the following proposition follows directly from Theorem 3.1.

Proposition 4.1. *Assume that, for some $\varepsilon \in (0,1)$ and for some $i \in \overline{1,6}$, the Lipschitz-continuous function $l : [0,T] \to \mathbb{R}^2$ satisfies*

$$\begin{cases} \|c'(t)\| = 1, \\ \langle H_{i+1}, -c'(t)\rangle \geqslant 0, \\ \langle H_{i-1}, -c'(t)\rangle \geqslant 0, \\ \langle N_i, -c'(t)\rangle < 1-\varepsilon, \\ \langle N_{i-1}, -c'(t)\rangle < 1-\varepsilon, \end{cases} \qquad c'(t) = -\frac{1}{3}\Lambda l'(t), \qquad t \in \{t \in [0,T] : l'(t) \text{ exists}\}. \tag{4.5}$$

where Λ is given by (2.4). Consider the solution y of sweeping process (1.3) with the initial condition $y(0) = y_0 \in \mathcal{C} + c(0)$, where $\mathcal{C}$ and $c(t)$ are as defined in (4.3), and $V = (1,-1,1)^\perp$. Let

$$T_{y_0} = \frac{\langle c(0) - y_0, H_i\rangle}{\varepsilon\langle H_1, N_1\rangle}, \tag{4.6}$$

where H_i is given by (4.2) with P_i given by the $\mathbb{R}^3$-basis line of Table 1, and N_i is given by (4.1) with N_i^Λ given by Table 2. If $T_{y_0} \leqslant T$, then the solution $y(t)$ remains at the vertex $P_i - \frac{1}{3}\Lambda l(t)$ of $\mathcal{C} + c(t)$ for all $t \geqslant T_0$.

Likewise, the Corollary 3.6 leads to the following result.

Corollary 4.2. *Consider a sweeping process (1.3) with $\mathcal{C}$ and $c(t)$ given by (4.3), and $V = (1,-1,1)^\perp$. Assume that, for some $\varepsilon \in (0,1)$, for some $i \in \overline{1,6}$, and for some $T \le \tau$, the τ-periodic Lipschitz-continuous function $l : [0,\infty) \to \mathbb{R}^2$ satisfies the condition (4.5) of Proposition 4.1. Let T_{y_0} and P_i be as defined in Proposition 4.1. If $T_{y_0} \le T$ for all $y_0 \in \mathcal{C} + c(0)$, then the solution $\hat{y}$ of (1.3) with the initial condition $\hat{y}(T) = P_i + c(T)$ is a one-period globally stable τ-periodic solution. Specifically, any other solution y of (1.3) with $y(0) \in \mathcal{C} + c(0)$ merges with $\hat{y}$ in time T (i.e. $y(t) = \hat{y}(t)$, for all $t \ge T$).*

5. Formulation of the results of the previous section in terms of the displacement-controlled loading explicitly

In this section we rewrite (4.5) in terms of $l(t)$ only. Proceeding with (4.5) we could directly compute all $N_i = \Lambda N_i^\Lambda \in \mathbb{R}^3$ and plug them into (4.5) but it is more convenient to work using the coordinates of basis Λ in V. To do so we define an auxiliary inner product in $\mathbb{R}^2$, which takes its values on the pairs of coordinates of a vector as equal to the values of $\langle \cdot, \cdot \rangle$ on actual vectors:

$$\langle \cdot, \cdot \rangle_\Lambda : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

$$\langle x, y \rangle_\Lambda := \langle \Lambda x, \Lambda y \rangle = \Lambda x \cdot \Lambda y, \tag{5.1}$$

where the dot product $\cdot$ is from $\mathbb{R}^3$. Moreover, we can write

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_\Lambda = \left(\Lambda^\top \Lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right)^\top \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= 6(x_1 y_1 + x_2 y_2) - 3(x_1 y_2 + x_2 y_1).$$

We plug (5.1) into (4.5) and obtain the conditions for each $i \in \overline{1,6}$ on the velocities of the displacement-controlled loadings $l'(t)$:

$$\begin{cases} \left\| \frac{1}{3} \Lambda l'(t) \right\| = 1, \\ \langle H_{i+1}^\Lambda, l'(t) \rangle_\Lambda \geqslant 0, \\ \langle H_{i-1}^\Lambda, l'(t) \rangle_\Lambda \geqslant 0, & t \in \{ t \in [0,T] : l'(t) \text{ exists} \}, \\ \frac{1}{3} \langle N_i^\Lambda, l'(t) \rangle_\Lambda < 1 - \varepsilon, \\ \frac{1}{3} \langle N_{i-1}^\Lambda, l'(t) \rangle_\Lambda < 1 - \varepsilon, \end{cases} \tag{5.2}$$

where H_i^Λ, N_i^Λ are coordinate vectors in the basis Λ of the respective vectors H_i, N_i taken from the tables above and $\varepsilon > 0$.

The first equation in (5.2) is the same for all $i \in \overline{1,6}$ and can be simplified as

$$l_{13}'^2(t) - l_{13}'(t) l_{24}'(t) + l_{24}'^2(t) = \frac{3}{2}.$$

If we plug all the values in the expressions $\langle H_i^\Lambda, l'(t)\rangle_\Lambda \leqslant 0$ and $\frac{1}{3}\langle N_i^\Lambda, l'(t)\rangle_\Lambda < 1-\varepsilon$ then we obtain, respectively:

$$\begin{array}{llll}
\text{for } i=1: & l'_{24} \geqslant 0 & \text{and} & l'_{13}+l'_{24} < \sqrt{6}(1-\varepsilon);\\
\text{for } i=2: & l'_{24} \geqslant 0 & \text{and} & 2l'_{24}-l'_{13} < \sqrt{6}(1-\varepsilon);\\
\text{for } i=3: & l'_{24} \geqslant l'_{13} & \text{and} & l'_{24}-2l'_{13} < \sqrt{6}(1-\varepsilon);\\
\text{for } i=4: & l'_{13} \leqslant 0 & \text{and} & -l'_{13}-l'_{24} < \sqrt{6}(1-\varepsilon);\\
\text{for } i=5: & l'_{24} \leqslant 0 & \text{and} & l'_{13}-2l'_{24} < \sqrt{6}(1-\varepsilon);\\
\text{for } i=6: & l'_{13} \geqslant l'_{24} & \text{and} & 2l'_{13}-l'_{24} < \sqrt{6}(1-\varepsilon).
\end{array}$$

where a pair of inequalities must be satisfied for all $t \in [0,T]$ such that $l'(t)$ exists. Therefore the condition (5.2) becomes (for all t as before)

for $i=1$:

$$\begin{cases} l'^2_{13} - l'_{13}l'_{24} + l'^2_{24} = \frac{3}{2},\\ l'_{13} \geqslant l'_{24} \geqslant 0,\\ l'_{13}+l'_{24} < \sqrt{6}(1-\varepsilon),\\ 2l'_{13}-l'_{24} < \sqrt{6}(1-\varepsilon). \end{cases} \tag{5.3a}$$

for $i=2$:

$$\begin{cases} l'^2_{13} - l'_{13}l'_{24} + l'^2_{24} = \frac{3}{2},\\ l'_{24} \geqslant l'_{13} \geqslant 0,\\ 2l'_{24}-l'_{13} < \sqrt{6}(1-\varepsilon),\\ l'_{13}+l'_{24} < \sqrt{6}(1-\varepsilon). \end{cases} \tag{5.3b}$$

for $i=3$:

$$\begin{cases} l'^2_{13} - l'_{13}l'_{24} + l'^2_{24} = \frac{3}{2},\\ l'_{13} \leqslant 0,\\ l'_{24} \geqslant 0,\\ l'_{24}-2l'_{13} < \sqrt{6}(1-\varepsilon),\\ 2l'_{24}-l'_{13} < \sqrt{6}(1-\varepsilon). \end{cases} \tag{5.3c}$$

for $i=4$:

$$\begin{cases} l'^2_{13} - l'_{13}l'_{24} + l'^2_{24} = \frac{3}{2},\\ l'_{13} \leqslant l'_{24} \leqslant 0,\\ -l'_{13}-l'_{24} < \sqrt{6}(1-\varepsilon),\\ l'_{24}-2l'_{13} < \sqrt{6}(1-\varepsilon). \end{cases} \tag{5.3d}$$

for $i=5$:

$$\begin{cases} l'^2_{13} - l'_{13}l'_{24} + l'^2_{24} = \frac{3}{2},\\ l'_{24} \leqslant l'_{13} \leqslant 0,\\ l'_{13}-2l'_{24} < \sqrt{6}(1-\varepsilon),\\ -l'_{13}-l'_{24} < \sqrt{6}(1-\varepsilon). \end{cases} \tag{5.3e}$$

for $i=6$:

$$\begin{cases} l'^2_{13} - l'_{13}l'_{24} + l'^2_{24} = \frac{3}{2},\\ l'_{13} \geqslant 0,\\ l'_{24} \leqslant 0,\\ 2l'_{13}-l'_{24} < \sqrt{6}(1-\varepsilon),\\ l'_{13}-2l'_{24} < \sqrt{6}(1-\varepsilon). \end{cases} \tag{5.3f}$$

Furthermore, (4.4) computes explicitly as follows.

$$\mu = \varepsilon\langle H_1, N_1\rangle = \varepsilon\langle H_1^\Lambda, N_1^\Lambda\rangle_\Lambda = \varepsilon\left\langle \left(\frac{2}{3},\frac{1}{3}\right), \frac{1}{\sqrt{6}}(1,1)\right\rangle_\Lambda = \varepsilon\frac{\sqrt{3}}{\sqrt{2}}.$$

We can finally formulate Proposition 4.1 and Corollary 4.2 of the previous section referring to mechanical parameters of the elastoplastic system of Figure 1 only.

Proposition 5.1. *Let $s(t) = (s_1(t), s_2(t), s_3(t))^\top$ be the stress response of the quasi-static evolution of the elastoplastic system of Figure 1 subject to a Lipschitz-continuous displacement-controlled loading $l(t) = (l_{13}(t), l_{24}(t))^\top$. Assume that, for some $i \in \overline{1,6}$, the respective set of conditions of (5.3) holds on an interval $[0,T]$ with a fixed $\varepsilon \in (0,1)$. Define $T_0 > 0$ as*

$$T_0 = \frac{\sqrt{2}}{\varepsilon\sqrt{3}}(P_i - s(0))^\top H_i \tag{5.4}$$

where P_i and H_i are as defined in Proposition 4.1. Finally, assume that $T_0 \leqslant T$. Then the stress-vector $s(t)$ remains equal to P_i for all $t \geqslant T_0$.

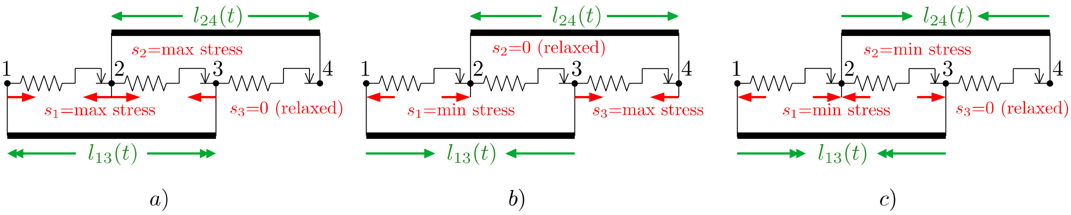

FIGURE 6. Interpretations of the convergence of stress to P_1 (a), P_3 (b), P_4 (c). The top and bottom arrows surrounding l_{13}, l_{24} represent the directions of controlled displacements (double arrow represents the greater rate) and the arrows in between show the resulting state of the stresses.

To conclude the statement of Proposition 5.1 from Proposition 4.1, one uses formula (2.5) connecting $s(t)$ and $y(t)$ and observes that the setting of quasi-static evolution implies that the initial value of the stress satisfies (see Moreau [14] or Gudoshnikov-Makarenkov [9])

$$s(0) \in [-1,1] \times [-1,1] \times [-1,1] \cap \left(\Lambda\mathbb{R}^2\right) = \operatorname{conv}\{P_i, i \in \overline{1,6}\}$$

(where conv stands for a convex hull), which gives $y(0) \in \mathcal{C} + c(0)$ required for Proposition 4.1.

Corollary 5.2. *Consider quasi-static evolution of the elastoplastic system of Figure 1 subject to a τ-periodic Lipschitz-continuous displacement-controlled loading $l(t) = (l_{13}(t), l_{24}(t))^\top$ with $\tau > \frac{4\sqrt{2}}{\sqrt{3}}$. Assume that, for some $\frac{4\sqrt{2}}{\tau\sqrt{3}} < \varepsilon < 1$ and for some $i \in \overline{1,6}$, the function $l(t)$ satisfies the respective set of conditions of (5.3) on the interval $[0,T]$ with $T = \frac{4\sqrt{2}}{\varepsilon\sqrt{3}}$. Then the stress evolution $\hat{s}$ of Figure 1 defined by the initial condition $\hat{s}(T) = P_i$ (where P_i is as defined in Prop. 5.1) is a one-period globally stable τ-periodic stress response. Specifically, any stress evolution s of elastoplastic system of Figure 1 takes the values $s(t) = \hat{s}(t)$ for all $t \geq T$.*

To prove the corollary we refer to Figure 5 and estimate (5.4) as follows.

$$T_0 = \frac{\sqrt{2}}{\varepsilon\sqrt{3}}(P_i - s(0))^\top H_i \leqslant \frac{\sqrt{2}}{\varepsilon\sqrt{3}}(P_i - P_{i-3})^\top P_i = \frac{\sqrt{2}}{\varepsilon\sqrt{3}}(P_1 - P_4)^\top P_1 = \frac{\sqrt{2}}{\varepsilon\sqrt{3}}4,$$

where in the expression after the first inequality we use $H_i = P_i$ and assume $P_0 = P_6, P_{-1} = P_5, P_{-2} = P_4$.

We can interpret condition (5.3a) of convergence to $P_1 = (1,1,0) \in \mathbb{R}^3$ as follows: if both controlled displacements are expanding, but l_{13} is expanding faster then l_{24} then (excluding a non-boundary case using ε) elastic elongations e_1, e_2 and stresses s_1, s_2 will reach their maximal value (so they will produce maximal stress) and spring 3 will have zero elastic elongation (*i.e.* it will be relaxed) by the time T_0.

Analogously, for $i = 3$ we conclude from (5.3c) and $P_3 = (-1,0,1)$ that if the controlled displacement l_{13} is shrinking and l_{24} is expanding, then by the time T_0 the variables e_1, s_1 will be minimal (so it will produce minimal stress), e_3, s_3 will be maximal, and spring 2 will be relaxed.

For $i = 4$ we conclude from (5.3d) and $P_4 = (-1,-1,0)$ that, in case of both controlled displacements shrinking and l_{13} doing it faster, then by the time T_0 elastic elongations e_1, e_2 and stresses s_1, s_2 will be minimal but spring 3 will be relaxed.

These three cases are shown in Figure 6. The rest of the cases $i = 2,5,6$ are the same up to the spatial symmetry.

6. An example of asymptotic stability in a sweeping process

In this section we show that increasing the magnitude of a uni-directional loading is no longer capable to create finite-time stability when the boundary of the moving set is smooth. Indeed, let $B_r(0) \subset \mathbb{R}^2$ be the open ball of radius $r > 0$ centered at 0. For arbitrary $r > 0$ and $T > 0$, introduce the moving set $C(t)$ as a T-periodic

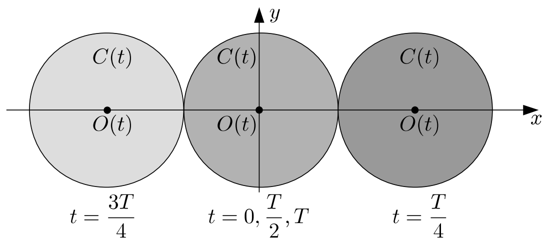

FIGURE 7. Sample positions of $C(t)$ during a period.

horizontal displacement of $\overline{B_r(0)}$ back and forth, *i.e.* (6.1), extended to $[0,\infty)$ by T-periodicity. Sample positions of the set $C(t)$ during one period are illustrated at Figure 7.

$$C(t) = \overline{B_r(0)} + O(t), \quad O(t) = (x(t), 0)^\top, \quad x(t) = \begin{cases} Lt, & \text{if } t \in \left[0, \frac{T}{4}\right], \\ -L(t - \frac{T}{2}), & \text{if } t \in \left[\frac{T}{4}, \frac{3T}{4}\right], \\ L(t-T), & \text{if } t \in \left[\frac{3T}{4}, T\right]. \end{cases} \quad L = \frac{8r}{T}, \tag{6.1}$$

Consider the sweeping process

$$\begin{cases} -u'(t) \in N_{C(t)}(u(t)), & t \in [0,T]; \\ u(0) = u_0 \in C(0). \end{cases} \tag{6.2}$$

Let $u(t) = (v(t), h(t))$ be a solution corresponding to an initial condition $u_0 = (v_0, h_0)$ such that $h_0 \neq 0$ and u_0 is close to the point $(-r, 0)$. Observe that for such u_0 and $x(t)$ given by (6.1) there is an interval $[t_0, t_0 + \Delta] \subset [0, \frac{T}{4}]$ such that the solution $u(t)$ remains on boundary $\partial C(t)$ of $C(t)$ for all $t \in [t_0, t_0 + \Delta]$. Indeed, set $\left\{t \in \left[0, \frac{T}{4}\right] : u(t) \in \partial C(t)\right\}$ is nonempty (otherwise $u(t)$ is constant, which is impossible by the choice of u_0) and compact, therefore there exists its minimum t_0. Since u and O are both Lipschitz-continuous with constant L we can choose $\Delta \in \left[0, \frac{\pi}{4} - t_0\right]$ and a radius r such that for all $t \in [t_0, t_0 + \Delta]$ and $x \in \overline{B_r(u(t_0) + O(t) - O(t_0))} \cap \partial C(t)$ we have an obtuse angle between velocity $O'(t)$ of the moving set and an outward normal vector $n(x)$ to boundary $\partial C(t)$:

$$n(x) \cdot O'(t) < 0, \tag{6.3}$$

while $u(t)$ remains within the distance r from the point $u(t_0) + O(t) - O(t_0)$ for all $t \in [t_0, t_0 + \Delta]$.

The following lemma shows that property (6.3) indeed guarantees us that $u(t)$ stays on the boundary of $C(t)$ for all $t \in [t_0, t_0 + \Delta]$.

Lemma 6.1. *Let u be a solution to the sweeping process (6.2) with a moving set of the type $C(t) = \mathcal{C} + c(t)$, where $\mathcal{C}$ is a nonempty closed convex set and c is a function, differentiable on an interval $[t_0, t_0 + \tau)$. Let $u(t_0) \in \partial C(t_0)$ and let a neighborhood U of $u(t_0) - c(t_0)$ be such that $u(t) \in U + c(t)$ for all $t \in [t_0, t_0 + \tau]$ and for all $x \in U \cap \partial\mathcal{C}$ there is $n_{(x,t)} \in N_{\mathcal{C}}(x)$ such that*

$$\langle n_{(x,t)}, c'(t) \rangle < 0. \tag{6.4}$$

Then $u(t) \in \partial C(t)$ for all $t \in [t_0, t_0 + \Delta]$.

Proof. Assume the contrary, *i.e.* that there is $t_1 \in [t_0, t_0 + \Delta]$ such that $u(t_1) \in \operatorname{int} C(t_1)$. Take $t^* := \max\{t \in [t_0, t_1] : u(t) \in \partial C(t)\}$, *i.e.* the last moment when $u(t)$ remains on the boundary before t_1. It follows that for all $t \in [t^*, t_1]$ solution $u(t)$ remains at place: $u(t) \equiv u(t^*)$.

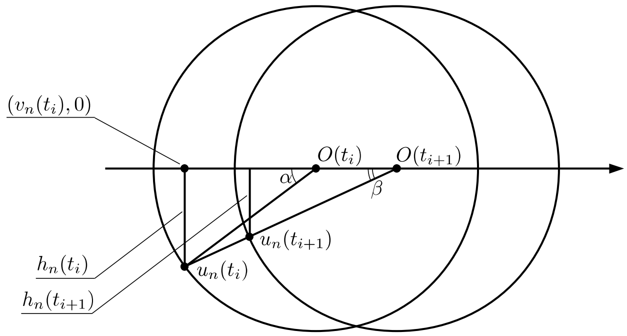

FIGURE 8. Angles α, β and other values in the sets $C(t_i), C(t_{i+1})$.

Let $n^* = n_{(u(t^*),t^*)}$. Using Heine's definition of limit and the definition of derivative we can show from (6.4) that there is $\tau > 0$ small enough for

$$\langle n^*, c(t^* + \tau) - c(t^*)\rangle < 0. \tag{6.5}$$

From the other side, notice that $u(t^*) = u(t^* + \tau) \in C(t^* + \tau) = \mathcal{C} + c(t^* + \tau)$, *i.e.* $u(t^*) - c(t^* + \tau) \in \mathcal{C}$. Moreover, $n^* \in N_{C(t^*)}(u(t^*)) = N_{\mathcal{C}}(u(t^*) - c(t^*))$ and by the definition of normal cone

$$\langle n^*, u(t^*) - c(t^* + \tau) - u(t^*) + c(t^*)\rangle \leqslant 0,$$

$$\langle n^*, c(t^* + \tau) - c(t^*)\rangle \geqslant 0.$$

The contradiction with (6.5) proves our assumption wrong and finishes the proof of the lemma. □

In what follows, we will approximate the solution $u(t)$ on the interval $t \in [t_0, t_0 + \Delta]$ using a so-called *catching-up algorithm* [14],[12]. Specifically, for every $n \in \mathbb{N}$ we equally partition the interval by the points $t_0 < t_1 < t_2 < \ldots < t_i < t_{i+1} < \ldots < t_n = t_0 + \Delta$, where $|t_{i+1} - t_i| = \frac{1}{n}$, and define an approximation $u_n(t)$ of the solution $u(t)$ of (6.2) as

$$\begin{array}{rcll} u_n(t) & = & u_{n,i} + \left(\frac{t-t_i}{t_{i+1}-t_i}\right)(u_{n,i+1} - u_{n,i}), & t \in [t_i, t_{i+1}], \\ u_{n,i+1} & = & \mathrm{proj}(u_{n,i}, C(t_{i+1})), & i \in \overline{0, n-1}, \\ u_{n,0} & = & u(t_0). & \end{array}$$

According to the catching-up algorithm (see *e.g.* [12], Thm. 2), it holds that $u_n(t) \to u(t)$ as $n \to \infty$, pointwise on the interval $[t_0, t_0 + \Delta]$. Let $(v_n(t), h_n(t))$ be the components of $u_n(t)$.

Denoting $\Delta_i = \frac{\Delta}{n}$, we have

$$L\Delta_i = \frac{L\Delta}{n} = |x(t_{i+1}) - x(t_i)| = \|O(t_{i+1}) - O(t_i)\| = \frac{|x(t_0 + \Delta) - x(t_0)|}{n}.$$

Let α be the angle formed by the line segments $[u_n(t_i), O(t_i)]$ and $[O(t_i), (v_n(t_i), 0)]$ and let β be the angle formed by the line segments $[u_n(t_i), O(t_{i+1})]$ and $[O(t_{i+1}), (v_n(t_i), 0)]$ (see Fig. 8). Then, by the definition of cotangent in the triangle formed by the vertices $u_n(t_i)$, H and $O(t_{i+1})$,

$$\cot\beta = \frac{r\cos\alpha + L\Delta_i}{r\sin\alpha}$$

and

$$\frac{1}{\sin^2\beta} = 1+\cot^2\beta = 1+\frac{(r\cos\alpha+L\Delta_i)^2}{r^2\sin^2\alpha} = \frac{r^2+2\,L\Delta_i r\cos\alpha+L^2\Delta_i^2}{r^2\sin^2\alpha}.$$

Hence

$$\frac{\sin^2\alpha}{\sin^2\beta} = \frac{r^2+2\,L\Delta_i r\cos\alpha+L^2\Delta_i^2}{r^2} = \frac{r^2+L\Delta_i(2r\cos\alpha+L\Delta_i)}{r^2}.$$

Since $u(0)$ was chosen to be close to $(-r,0)$, we have $\cos\alpha \geq \frac{1}{2}$ and $2r\cos\alpha+L\Delta_i \geqslant r$ which gives

$$\frac{\sin^2\alpha}{\sin^2\beta} \geqslant \frac{r^2+L\Delta_i\, r}{r^2} = 1+\frac{L\Delta_i}{r}.$$

On the other hand,

$$\frac{\sin^2\alpha}{\sin^2\beta} = \frac{r^2+2\,L\Delta_i r\cos\alpha+L^2\Delta_i^2}{r^2} \leqslant \frac{r^2+2\,L\Delta_i r+L^2\Delta_i^2}{r^2} = \left(\frac{r+L\Delta_i}{r}\right)^2 = \left(1+\frac{L\Delta_i}{r}\right)^2.$$

Therefore we can estimate the ratio $\frac{h_n(t_{i+1})}{h_n(t_i)} = \frac{r\sin\beta}{r\sin\alpha} = \sqrt{\frac{\sin^2\beta}{\sin^2\alpha}}$ as

$$\sqrt{1+\frac{L\Delta_i}{r}} \leqslant \frac{h_n(t_i)}{h_n(t_{i+1})} \leqslant \sqrt{\left(1+\frac{L\Delta_i}{r}\right)^2}$$

or

$$\left(1+\frac{L\Delta_i}{r}\right)^{-1} \leqslant \frac{h_n(t_{i+1})}{h_n(t_i)} \leqslant \left(1+\frac{L\Delta_i}{r}\right)^{-\frac{1}{2}}.$$

Collecting these inequalities through $i \in \overline{0, n-1}$, we get

$$\left(\left(1+\frac{L\Delta}{nr}\right)^{-1}\right)^n \leqslant \frac{h_n(t_0+\Delta)}{h_n(t_0)} \leqslant \left(\left(1+\frac{L\Delta}{nr}\right)^{-\frac{1}{2}}\right)^n$$

or

$$\left(\left(1+\frac{L\Delta}{nr}\right)^n\right)^{-1} \leqslant \frac{h_n(t_0+\Delta)}{h_n(t_0)} \leqslant \left(\left(1+\frac{L\Delta}{nr}\right)^n\right)^{-\frac{1}{2}}.$$

Since $e^x = \lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n$, we can pass to the limit as $n \to \infty$ obtaining

$$0 < e^{-\frac{L\Delta}{r}} \leqslant \frac{h(t_0+\Delta)}{h(t_0)} \leqslant e^{-\frac{L\Delta}{2r}} < 1. \tag{6.6}$$

Recall that $[t_0, t_0+\Delta] \subset \left[0, \frac{T}{4}\right]$. Arguments similar to above can be made for each of the two remaining intervals of monotonicity of the function $x(t)$. Specifically, there are intervals $\left[\widetilde{t}_0, \widetilde{t}_0+\widetilde{\Delta}\right] \subset \left[\frac{T}{4}, \frac{3T}{4}\right]$, $\left[\widetilde{\widetilde{t}}_0, \widetilde{\widetilde{t}}_0+\widetilde{\widetilde{\Delta}}\right] \subset \left[\frac{3T}{4}, T\right]$ along which $u(t)$ remains on the boundary of $C(t)$. We may partition these intervals in the same manner and obtain the estimates analogous to (6.6):

$$0 < e^{-\frac{L\widetilde{\Delta}}{r}} \leqslant \frac{h\left(\widetilde{t}_0 + \widetilde{\Delta}\right)}{h\left(\widetilde{t}_0\right)} \leqslant e^{-\frac{L\widetilde{\Delta}}{2r}} < 1, \qquad 0 < e^{-\frac{L\widetilde{\widetilde{\Delta}}}{r}} \leqslant \frac{h\left(\widetilde{\widetilde{t}}_0 + \widetilde{\widetilde{\Delta}}\right)}{h\left(\widetilde{\widetilde{t}}_0\right)} \leqslant e^{-\frac{L\widetilde{\widetilde{\Delta}}}{2r}} < 1. \tag{6.7}$$

Since there are only 3 intervals of monotonicity of $x(t)$ on $[0,T]$, we can be sure that for

$$t \in [0,T] \setminus \left([t_0, t_0+\Delta] \cup \left[\widetilde{t}_0, \widetilde{t}_0 + \widetilde{\Delta}\right] \cup \left[\widetilde{\widetilde{t}}_0, \widetilde{\widetilde{t}}_0 + \widetilde{\widetilde{\Delta}}\right]\right),$$

the solution $u(t)$ is located in the interior of $C(t)$ and it is not moving ($u'(t) = 0$). Therefore,

$$h(0) = h(t_0), \quad h(t_0+\Delta) = h\left(\widetilde{t}_0\right), \quad h\left(\widetilde{t}_0 + \widetilde{\Delta}\right) = h\left(\widetilde{\widetilde{t}}_0\right), \quad h\left(\widetilde{\widetilde{t}}_0 + \widetilde{\widetilde{\Delta}}\right) = h(T),$$

so that we can combine (6.6) and (6.7), and write the following estimate for the entire period $[0,T]$:

$$e^{-8} = e^{-\frac{8r}{T}\frac{T}{r}} \leqslant e^{-\frac{L\left(\Delta+\widetilde{\Delta}+\widetilde{\widetilde{\Delta}}\right)}{r}} \leqslant \frac{h(T)}{h(0)} \leqslant e^{-\frac{L\left(\Delta+\widetilde{\Delta}+\widetilde{\widetilde{\Delta}}\right)}{2r}} < 1. \tag{6.8}$$

The same estimate can be obtained for each period $[T(l-1), Tl]$, $l \in \mathbb{N}$. It follows from the right-hand side of (6.8) that the distance from the solution to the x-axis decreases. The left-hand side implies that on the l-th period we have

$$0 < e^{-8l} \leqslant \frac{h(Tl)}{h(0)}.$$

Since $h(0) \neq 0$, we have that $h(Tl) \neq 0$, $l \in \mathbb{N}$, *i.e.* the solution $u(t)$ never reaches the x-axis while approaching it arbitrary closely. In other words, the following proposition takes place.

Proposition 6.2. *Consider a planar sweeping process (6.2) with a moving set given by (6.1), where $r > 0$ and $T > 0$ are arbitrary given constants. Then, any solution $u(t) = (v(t), h(t))$ of (6.2) asymptotically approaches the solution $u_0(t) = (v_0(t), 0)$, where*

$$v_0(t) = \begin{cases} -r + Lt, & \text{if } 0 \le t \le \frac{T}{4}, \\ r, & \text{if } \frac{T}{4} \le t \le \frac{T}{2}, \\ r - L\left(t - \frac{T}{2}\right), & \text{if } \frac{T}{2} \le t \le \frac{3T}{4}, \\ -r, & \text{if } \frac{3T}{4} \le t \le T. \end{cases}$$

Furthermore, if $h(0) \neq 0$, then the solution $u(t)$ never reaches $u_0(t)$ in finite time.

7. Conclusions

The paper investigated finite-time stability of a network of 3 elastoplastic springs on 4 nodes subject to 2 cyclic displacement-controlled loadings. Though this looks simple, such a network led us to an interesting mathematical problem within the theory of sweeping processes. Specifically, in order to prove finite-time convergence of stresses of the springs of the network, a theory of finite-time stability for planar sweeping processes with periodically moving polyhedron has been developed. Our analysis clarified how different typical loading scenarii can lead to 6 different types of stress response corresponding to the 6 vertices of the associated polyhedron.

In order to emphasize the crucial role of the nonsmoothness of the boundary of the convex moving set for the finite-time stability, we showed that replacing the periodically moving polyhedron of the sweeping process by a circle is no longer a finite-time stable system.

Our approach is intrinsically two-dimensional. The idea of Adly *et al.*[1] (a Lyapunov function approach) seems to be promising for further generalizations of our work to higher dimensions.

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