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FRACTIONAL ORDER PREY-PREDATOR MODEL WITH INFECTED PREDATORS IN THE PRESENCE OF COMPETITION AND TOXICITY

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Abstract. In this paper, we propose a fractional-order prey-predator model with reserved area in the presence of the toxicity and competition. We prove different mathematical results like existence, uniqueness, non negativity and boundedness of the solution for our model. Further, we discuss the local and global stability of these equilibria. Finally, we perform numerical simulations to prove our results.

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1. Introduction

Each population within an ecosystem does not exist in isolation, and there must be some relationships between these different populations [3]. The relationship between them is divided into types: mutualism, parasitism, competition and predation. The dynamic relationship between predator and prey is long established and will remain among the crucial topics in ecology and mathematical ecology because of its universal existence and its importance [30].

In recent years, the fractional calculations have developed rapidly and have shown broad application prospects in many areas. Useful results can be obtained by extracting a dynamic behaviour of biological systems presented by a mathematical model of integer derivatives. However, most biological systems also have memory. In this case, the modelling in fractional order, unlike the classical mode. The existence of the memory is taken into account. The fractional derivative of a biological process at a point is affected by all the information and behaviour of the model at all previous times, while the classical derivative at a point is only affected by information from the local neighbourhood of that point. For this reason, many papers study the theory of differential fractional equations [19, 21, 23, 24, 26].

Recently, many applications have been developed in areas such as laser physics, chemical reactors, secure communication, biomedicine, epidemiology, signal processing, control theory, mechanics, etc. It has been found that various applications can be modelled using fractional derivatives.

Keywords and phrases: Prey–predator system, toxicity, equilibria, stability, competition, fractional order.

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TABLE 1. Variables and parameters descriptions.

Parameters and variables	Explanation
x	Biomass densities of the unreserved areas
y	Biomass densities of the reserved areas
S	Susceptible predator
I	Infected predator
E_1, E_2, E_3	The effort applied for harvesting in the unreserved area, the susceptible predator populations, the infected predator populations, respectively
r_1, r_2	The growth rates of fish population inside reserved and the unserved areas
q_1, q_2	The catchability coefficient in the unreserved area and the predator species
σ_1, σ_2	Migration rate from unreserved area to reserved area and reserved area to unreserved area
n_1, n_2	The competition coefficients
γ	The strength of intra-specific between prey and infected predator
δ	The disease transmission coefficient
β	The search rate of the prey toward susceptible predator
μ	The death rate of susceptible predator
η	The death rate of infected predator
α'	Saturation constant while susceptible predators attack the prey
σ	The conversion rate of susceptible predator due to prey
ux^2, vy^2	The reduction terms, in the unreserved area and reserved area respectively, where u and v the coefficients of toxicity
$\frac{\beta x S}{\alpha'+x}$	The functional response of feeding prey by susceptible predator

Not long ago, many researchers began to study fractional biological models [1, 10, 20, 25]. In article [14], a dynamic system modelling a prey-predator with harvest area and reserve for prey in the presence of competition and toxicity. In article [25], let us introduce a fractional prey-predator model with two types of susceptible and infected predators. In our paper it has been supposed that the prey are divided into two areas reserved and free and reserved zone, as well as the predators are divided into two categories, susceptible and infected predators. Now the basic model based on [14, 25] is governed by the following fractional system (Fig. 1):

$$\left\{\begin{array}{lcl} D^\alpha x & = & r_1 x\left(1-\frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - ux^2 - \frac{\beta x S}{\alpha'+x} - q_1 E_1 x - n_1 xy - \gamma x I, \\ D^\alpha y & = & (r_2-\sigma_2)y + \sigma_1 x - vy^2 - n_2 xy, \\ D^\alpha S & = & \frac{\sigma\beta x S}{\alpha'+x} - \delta S I - \mu S - q_2 E_2 S, \\ D^\alpha I & = & \delta S I + \sigma\gamma x I - q_3 E_3 I - \eta I \end{array}\right. \tag{1.1}$$

where D^α is in the sense of Caputo fractional derivative and $0 < \alpha \leq 1$ defined by [22]: $D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(x)}{(t-x)^\alpha}\mathrm{d}x$.

Where f is defined by : $f : \Omega \longrightarrow \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$.

The detailed description of the model (1.1) is illustrated in the following schema: the explanation an Units of this parameters and variables given by the tables (Tabs. 1 and 2):

From [7], if there is no migration of fish population from reserved area to unreserved area ($\sigma_2 = 0$) and ($r_1 - \sigma_1 - q_1 E_1 < 0$), we find that $D^\alpha x < 0$. Similarly, if there is no migration of fish population from unreserved area to reserved area ($\sigma_1 = 0$) and $r_2 - \sigma_2 < 0$, then $D^\alpha y < 0$.

If $\sigma\beta - \mu - q_2 E_2 < 0$, then $D^\alpha S < 0$.

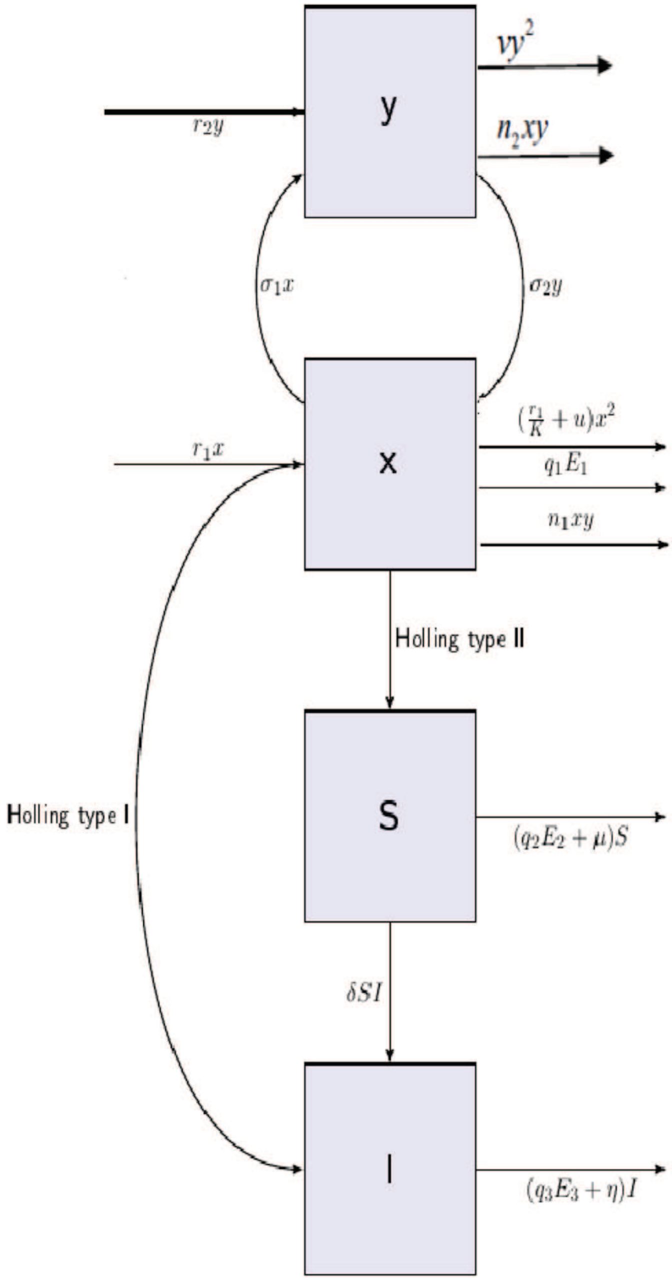

FIGURE 1. Schematic diagram of the model (1.1).

So, finally we conclude that:

$$r_1 - \sigma_1 - q_1 E_1 > 0,\ r_2 - \sigma_2 > 0,\ \sigma\beta - \mu - q_2 E_2 > 0. \tag{1.2}$$

Our paper is organized as follows. In the following section, we prove the of the existence and uniqueness solutions of the system (1.1), in Section 3, we show the boundedness and positivity of the solutions.

TABLE 2. Units of variables and parameters.

Parameters and variables	Units
x, y, S, I, α'	Number per unit area
E_1, E_2, E_3	Catch Per Unit Effort
q_1, q_2	Number per unit of fishing effort,
σ_1, σ_2, r_1, r_2, μ, η, β, u, v, δ, n_1, n_2	Per day
γ, σ	Constant

In Section 4, we study the existence and stability of all the equilibria of our model (1.1). Finally, we present the numerical simulations to study the stability of the equilibria.

2. Basic properties and equilibria

Theorem 2.1. *The sufficient condition for the existence and uniqueness of the solution of system (1.1) in the region* $\Omega \times [t_0, T]$ *with initial conditions*
$X(0) = (x(0), y(0), S(0), I(0))$ *and* $t \in [t_0, T]$ *is:*
$L = \max((r_1 - q_1E_1 + M(n_1 + \gamma + \beta(1 + \sigma) + 2(\frac{r_1}{K} + u))), (r_2 + M(n_2 + 2v)),$
$(\beta(1 + \sigma) + \mu + q_2E_2 + 2\delta M), (\sigma\gamma M + \eta + q_3E_3))$.

Proof. Let $X = (x, y, S, I)^T$ and $X^{'} = (x^{'}, y^{'}, S^{'}, I^{'})^T$ the system (1.1) can be is written in this form:

$$D^\alpha X = F(X), \tag{2.1}$$

where $F(X) = \begin{pmatrix} r_1x\left(1 - \frac{x}{K}\right) - \sigma_1x + \sigma_2y - ux^2 - \frac{\beta xS}{\alpha'+x} - q_1E_1x - n_1xy - \gamma xI \\ (r_2 - \sigma_2)y + \sigma_1x - vy^2 - n_2xy \\ \frac{\sigma\beta xS}{\alpha'+x} - \delta SI - \mu S - q_2E_2S \\ \delta SI + \sigma\gamma xI - q_3E_3I - \eta I \end{pmatrix} = \begin{pmatrix} F_1(X) \\ F_2(X) \\ F_3(X) \\ F_4(X) \end{pmatrix}$. To prove the global existence and uniqueness of system (1.1), consider the region $\Omega \times [t_0, T]$,
where $\Omega = \left\{(x, y, S, I) \in \mathbb{R}^4 : max\left\{|x|, |y|, |S|, |I|\right\} \le M, M > 0\right\}$.
For any $X, X' \in \Omega$:

$$\begin{aligned} \left\|F(X) - F(X^{'})\right\|_1 &= \sum_{i=1}^{4} |F_i(X) - F_i(X')|, \\ &= |(r_1 - \sigma_1 - q_1E_1)(x - x') - (u + \tfrac{r_1}{K})(x^2 - x'^2) + \sigma_2(y - y') \\ &\quad -\beta(\tfrac{\alpha'(xS - x'S') + xx'(S - S')}{(\alpha'+x)(\alpha'+x')}) - n_1(xy - x'y') - \gamma(xI - x'I')| \\ &\quad + \left|(r_2 - \sigma_2)(y - y') + \sigma_1(x - x') - v(y^2 - y'^2) - n_2(xy - x'y')\right| \\ &\quad +|\sigma\beta(\tfrac{\alpha'(xS - x'S') + xx'(S - S')}{(\alpha'+x)(\alpha'+x')}) - \delta(SI - S'I') - (\mu + q_2E_2)(S - S')| \\ &\quad +|\delta(SI - S'I') + \sigma\gamma(xI - x'I') - (\eta + q_3E_3)(I - I')|, \\ &\le L\left\|X - X'\right\|, \end{aligned}$$

where $L = \max((r_1 - q_1E_1 + M(n_1 + \gamma + \beta(1 + \sigma) + 2(\frac{r_1}{K} + u))), (r_2 + M(n_2 + 2v)),$
$(\beta(1 + \sigma) + \mu + q_2E_2 + 2\delta M), (\sigma\gamma M + \eta + q_3E_3))$, proving the Lemma. □

Thus, $F(X)$ satisfies the Lipschitz's condition [13] with respect to X.
Now, we describe the uniform boundedness of the solutions of the system (1.1).

Lemma 2.2. *The set* $\Omega^{'} = \left\{(x,y,S,I) \in \mathbb{R}^4_+ : x+y+\frac{S}{\sigma}+\frac{I}{\sigma} \leq \frac{H}{\eta+q_3E_3}\right\}$ *is a region of attraction for all solutions initiating in the interior of the positive octant, where*

$$H = \frac{K(r_1 - q_1E_1 + \eta + q_3E_3)^2}{4(r_1+Ku)} + \frac{(r_2+\eta+q_3E_3)^2}{4v}.$$

Proof. We pose $w = x+y+\frac{S}{\sigma}+\frac{I}{\sigma}$,

$$\begin{aligned} D^\alpha w + (\eta+q_3E_3)w &= (r_1 - q_1E_1+\eta+q_3E_3)x - (\tfrac{r_1}{K}+u)x^2 - vy^2 \\ &\quad +(r_2+\eta+q_3E_3)y - (n_1+n_2)xy + \tfrac{(\eta+q_3E_3-\mu-q_2E_2)S}{\sigma}, \end{aligned}$$

Taking $\eta + q_3E_3 < \mu + q_2E_2$ we get:
$D^\alpha w + (\eta+q_3E_3)w \leq \frac{K(r_1-q_1E_1+\eta+q_3E_3)^2}{4(r_1+Ku)} + \frac{(r_2+\eta+q_3E_3)^2}{4v} = H.$
Applying the theory of fractional inequality [22] we get:

$$w(t) \leq w(0)E_\alpha(-(\eta+q_3E_3)t^\alpha) + \frac{H}{\eta+q_3E_3}\left(1 - E_\alpha(-(\eta+q_3E_3)t^\alpha)\right),$$

where $E_\alpha(z) = \sum\limits_{k=0}^{\infty} \dfrac{z^k}{\Gamma(\alpha k+1)}$ is Mittag-Leffler function [22], $\Gamma(z) = \displaystyle\int_0^\infty x^{z-1}e^{-x}\,\mathrm{d}x$ is Euler's Gamma function, and $0 < E_\alpha(-(\eta+q_3E_3)t^\alpha) \leq 1$, if $t \to \infty$, we have $0 < w(t) \leq w(0) + \frac{H}{\eta+q_3E_3}$, proving this Lemma. □

Now, we find the positive equilibria, then we study their local stability. We denote the function on the right hand side of the system (1.1) by $F_i(x,y,S,I)$, for $i = 1,\ldots,4$.

Equilibria of model (1.1) is obtained by solving $F_i(x,y,S,I) = 0$, for $i = 1,\ldots,4$. Then, we find that our model (1.1) admits five positive equilibria:

1. $P_0(0,0,0,0)$ there is a trivial equilibrium.
2. $P_1(x_1,y_1,0,0)$, where (x_1,y_1) is the positive solution of the following equations:

$$\begin{aligned} (r_1-\sigma_1-q_1E_1)x - (\tfrac{r_1+Ku}{K})x^2 + \sigma_2 y - n_1xy &= 0, \\ (r_2-\sigma_2)y + \sigma_1 x - vy^2 - n_2xy &= 0. \end{aligned} \tag{2.2}$$

Using system (2.2), x is satisfied by the following equation,

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \tag{2.3}$$

where:

$$\begin{aligned} a_3 &= (u+\tfrac{r_1}{K})\left(n_1n_2 - v(u+\tfrac{r_1}{K})\right), \\ a_2 &= \tfrac{2v(r_1+Ku)(r_1-\sigma_1-q_1E_1)}{K} - n_2\sigma_2(\tfrac{r_1}{K}+u) - n_1n_2(r_1-\sigma_1-q_1E_1) \\ &\quad -n_1(r_2-\sigma_2)(u+\tfrac{r_1}{K}) + \sigma_1 n_1^2, \\ a_1 &= -v(r_1-\sigma_1-q_1E_1)^2 + (r_1-\sigma_1-q_1E_1)\left(n_2\sigma_2 + n_1(r_2-\sigma_2)\right) \\ &\quad -2\sigma_1\sigma_2n_1 + (r_2-\sigma_2)\sigma_2(u+\tfrac{r_1}{K}), \\ a_0 &= -\sigma_2(r_2-\sigma_2)(r_1-\sigma_1-q_1E_1) + \sigma_1\sigma_2^2. \end{aligned} \tag{2.4}$$

Using criteria of Descartes [5], the above equation (2.3) had a unique positive solution if the following inequalities hold:

$$\begin{array}{lll} a_0 > 0 & \text{if} & (r_2-\sigma_2)(r_1-\sigma_1-q_1E_1) < \sigma_1\sigma_2, \\ a_1 > 0 & \text{if} & (r_1-\sigma_1-q_1E_1)\left(n_2\sigma_2+n_1(r_2-\sigma_2)\right) \\ & & +(r_2-\sigma_2)\sigma_2(u+\frac{r_1}{K}) > 2\sigma_1\sigma_2 n_1 + v(r_1-\sigma_1-q_1E_1)^2. \end{array}$$

Then,

$$E_1 > \frac{1}{q_1}\max\left(r_1-\sigma_1-\frac{\sigma_1\sigma_2}{r_2-\sigma_2}, r_1-\sigma_1-\frac{\sqrt{\Delta_1+n_2\sigma_2+n_1(r_2-\sigma_2)}}{2v}\right), \tag{2.5}$$

where $\Delta_1 = (n_2\sigma_2+n_1(r_2-\sigma_2))^2 + 4v(u+\frac{r_1}{K})(r_2-\sigma_2)\sigma_2$

$$\begin{array}{llll} a_2 & > & 0 & \text{if} \quad v(r_1-\sigma_1-q_1E_1) > n_2\sigma_2+n_1(r_2-\sigma_2), \\ a_3 & < & 0 & \text{if} \quad n_1n_2 < v\left(u+\frac{r_1}{K}\right). \end{array} \tag{2.6}$$

Then

$$y_1 = \frac{x_1}{\sigma_2-n_1x_1}\left(\left(\frac{r_1+Ku}{K}\right)x_1-(r_1-\sigma_1-q_1E_1)\right) > 0,$$

if

$$\frac{(r_1-\sigma_1-q_1E_1)K}{r_1+Ku} < x_1 < \frac{\sigma_2}{n_1} \quad \textit{or} \quad \frac{\sigma_2}{n_1} < x_1 < \frac{(r_1-\sigma_1-q_1E_1)K}{r_1+Ku}. \tag{2.7}$$

3. In the interior of the equilibrium $P_2(x_2, y_2, 0, I_2)$, *i.e.* $F_i(x_2, y_2, 0, I_2) = 0$, $i = 1, 2, 4$, we get a positive solution:

$$\begin{array}{lll} x_2 & = & \dfrac{\eta+q_3E_3}{\sigma\gamma}, \\ y_2 & = & \dfrac{r_2-\sigma_2-n_2x_2+\sqrt{(r_2-\sigma_2-n_2x_2)^2+4\sigma_1x_2v}}{2v}, \\ I_2 & = & \dfrac{1}{\gamma x_2}\left((r_1-\sigma_1-n_1y_2-q_1E_1)x_2-\left(\dfrac{r_1}{K}+u\right)x_2^2+\sigma_2y_2\right) > 0. \end{array} \tag{2.8}$$

if

$$0 < x_2 < \frac{r_1-\sigma_1-q_1E_1-n_1y_2+\sqrt{(r_1-\sigma_1-n_1y_2-q_1E_1)^2+4\sigma_2(u+\frac{r_1}{K})y_2}}{2(\frac{r_1}{K}+u)}. \tag{2.9}$$

4. In the interior of the equilibrium $P_3(x_3, y_3, S_3, 0)$, *i.e.* $F_i(x_3, y_3, S_3, 0) = 0$, $i = 1, 2, 3$, we get a positive solution:

$$\begin{aligned} x_3 &= \frac{\alpha'(\mu + q_2E_2)}{\sigma\beta - (\mu + q_2E_2)} > 0, \quad \text{if} \quad \sigma\beta > \mu + q_2E_2, \\ y_3 &= \frac{(r_2 - \sigma_2 - n_2x_3) + \sqrt{(r_2 - \sigma_2 - n_2x_3)^2 + 4v\sigma_1x_3}}{2v}, \\ S_3 &= \tfrac{\alpha' + x_3}{\beta x_3}\left((r_1 - \sigma_1 - q_1E_1 - n_1y_3)x_3 + \sigma_2y_3 - (u + \tfrac{r_1}{K})x_3^2\right). \end{aligned} \tag{2.10}$$

$S_3 > 0 \quad \text{if} \quad 0 < x_3 < \frac{r_1 - \sigma_1 - q_1E_1 - n_1y_3 + \sqrt{(r_1 - \sigma_1 - n_1y_3 - q_1E_1)^2 + 4\sigma_2(u + \frac{r_1}{K})y_3}}{2(\frac{r_1}{K} + u)}$.

5. Using $F_i(x_4, y_4, S_4, I_4) = 0$, $i = 1, \ldots, 4$, the equilibrium point $P_4(x_4, y_4, S_4, I_4)$ is given by:

$$\begin{aligned} y_4 &= \tfrac{r_2 - \sigma_2 - n_2x_4 + \sqrt{(r_2 - \sigma_2 - n_2x_4)^2 + 4\sigma_1x_4v}}{2v}, \\ S_4 &= \frac{\eta + q_3E_3 - \sigma\gamma x_4}{\delta} > 0, \quad \text{if} \quad x_4 < \frac{\eta + q_3E_3}{\sigma\gamma}, \\ I_4 &= \tfrac{1}{\delta}\left(\tfrac{\sigma\beta x_4}{\alpha' + x_4} - (\mu + q_2E_2)\right) > 0, \quad \text{if} \quad x_4 > \tfrac{\alpha'(\mu + q_2E_2)}{\sigma\beta - \alpha'(\mu + q_2E_2)}. \end{aligned} \tag{2.11}$$

where (x_4, y_4) is the positive solution of the following equations:

$$\begin{aligned} r_1x\left(1 - \tfrac{x}{K}\right) - \sigma_1x + \sigma_2y - ux^2 - \tfrac{\beta xS}{\alpha' + x} - q_1E_1x - n_1xy - \gamma xI &= 0, \\ (r_2 - \sigma_2)y + \sigma_1x - vy^2 - n_2xy &= 0. \end{aligned} \tag{2.12}$$

After the calculations, x is satisfied by the following equation,

$$b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0 = 0, \tag{2.13}$$

where

$$\begin{aligned} b_5 &= bn_2n_1 - b^2v, \\ b_4 &= 2abv - an_2n_1 - 2b^2\alpha'v - bdn_1 + 2b\alpha'n_2n_1 - b\sigma_2n_2 + \sigma_1n_1^2, \\ b_3 &= -a^2v + 4ab\alpha'v + adn_1 - 2a\alpha'n_2n_1 + a\sigma_2n_2 - b^2\alpha'^2v - 2bc\beta v - 2bd\alpha'v \\ &\quad + bd\sigma_2 + b\alpha'^2n_2n_1 - 2b\alpha'\sigma_2n_2 + c\beta n_2n_1 + 2\alpha'\sigma_1n_1^2 - 2\sigma_1\sigma_2n_1, \\ b_2 &= -2a^2\alpha'v + 2ab\alpha'^2v + 2ac\beta v + 2ad\alpha'n_1 - ad\sigma_2 - a\alpha'^2n_2n_1 + 2a\alpha'\sigma_2n_2 \\ &\quad - 2bc\alpha'\beta v - bd\alpha'^2n_1 + 2bd\alpha'\sigma_2 - b\alpha'^2\sigma_2n_2 - cd\beta n_1 + c\alpha'vn_2n_1 - c\beta\sigma_2n_2 \\ &\quad + \alpha'^2\sigma_1n_1^2 - 4\alpha'\sigma_1\sigma_2n_1 + \sigma_1\sigma_2^2, \\ b_1 &= -a^2\alpha'^2v + 2ac\alpha'\beta v + ad\alpha'^2n_1 - 2ad\alpha'\sigma_2 + a\alpha'^2\sigma_2n_2 + bd\alpha'^2\sigma_2 - c^2\beta^2v \\ &\quad - cd\alpha'\beta n_1 + cd\beta\sigma_2 - c\alpha'\beta\sigma_2n_2 - 2\alpha'^2\sigma_1\sigma_2n_1 + 2\alpha'\sigma_1\sigma_2^2, \\ b_0 &= -ad\alpha'^2\sigma_2 + cd\alpha'\beta\sigma_2 + \alpha'^2\sigma_2^2, \\ a &= r_1 - \sigma_1 - q_1E_1 + \tfrac{\gamma}{\delta}(\mu + q_2E_2), \\ b &= \tfrac{r_1}{K} + u, \\ c &= \tfrac{\eta + q_3E_3}{\delta}, \\ d &= r_2 - \sigma_2. \end{aligned} \tag{2.14}$$

Using the criteria of Descartes [5] it is necessary to impose that:

$\frac{\alpha'(\mu+q_2E_2)}{\sigma\beta-\alpha'(\mu+q_2E_2)} < x_4 < \frac{\eta+q_3E_3}{\sigma\gamma}$, $b_5 < 0$ and $b_i > 0$, for $i = 0, \ldots, 4$.

To study the local stability of equilibria, the eigenvalues of the Jacobian matrix of system (1.1) are computed by: $J(x,y,S,I)=\begin{pmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & 0 & 0 \\ J_{31} & 0 & J_{33} & J_{34} \\ J_{41} & 0 & J_{43} & J_{44} \end{pmatrix}$,

where:

$$\begin{array}{rcl} J_{11} & = & r_1 - \sigma_1 - q_1E_1 - 2(\frac{r_1}{K} + u)x - n_1y - \gamma I - \frac{\beta S\alpha'}{(\alpha'+x)^2}, \\ J_{12} & = & \sigma_2 - n_1x, \\ J_{13} & = & \frac{-\beta x}{\alpha'+x}, \\ J_{14} & = & -\gamma x, \\ J_{21} & = & \sigma_1 - n_2y, \\ J_{22} & = & r_2 - \sigma_2 - 2vy - n_2x, \\ J_{31} & = & \frac{\sigma\beta\alpha' S}{(\alpha'+x)^2}, \\ J_{33} & = & \frac{\sigma\beta x}{\alpha'+x} - \delta I - \mu - q_2E_2, \\ J_{34} & = & -\delta S, \\ J_{41} & = & \sigma\gamma I, \\ J_{43} & = & \delta I, \\ J_{44} & = & \delta S + \sigma\gamma x - q_3E_3 - \eta. \end{array} \tag{2.15}$$

Theorem 2.3. *The equilibrium $P_0(0,0,0,0)$ of the system (1.1) is unstable.*

Proof. The characteristic equation of $P_0(0,0,0,0)$ is

$$[\lambda^2 - (r_1 - \sigma_1 + r_2 - \sigma_2 - q_1E_1)\lambda + (r_2 - \sigma_2)(r_1 - \sigma_1 - q_1E_1) - \sigma_2\sigma_1]$$
$$\times(\lambda + \mu + q_2E_2)(\lambda + \eta + q_3E_3) = 0.$$

Then, the eigenvalues of matrix (2.15) to the equilibrium point P_0:
$\lambda_1 = -(\eta + q_3E_3) < 0, \lambda_2 = -(\mu + q_2E_2) < 0$, and $\lambda_3 + \lambda_4 = r_1 - \sigma_1 - q_1E_1 + r_2 - \sigma_2 > 0$. Therefore one of the eigenvalues λ_3 and λ_4 not satisfy Matignon's condition [17]. Hence, $P_0(0,0,0,0)$ is unstable. □

Theorem 2.4. *The equilibrium point $P_1(x_1,y_1,0,0)$ of the system (1.1) is locally asymptotically stable if $x_1 < \min\left(\frac{\eta+q_3E_3}{\sigma\gamma}, \frac{\alpha'(\mu+q_2E_2)}{\sigma\beta-(\mu+q_2E_2)}\right)$ and (2.16) satisfied.*

$$\begin{array}{ll} (1) & \text{If } \Delta > 0,\ P_1 \text{ is locally asymptically stable.} \\ (2) & \text{If } \Delta < 0, \text{ and } |\arg(\lambda_{3,4})| > \frac{\alpha\pi}{2},\ \ P_1 \text{ is locally asymptically stable.} \end{array} \tag{2.16}$$

Proof. From (2.15) evaluated at equilibrium point P_1, the characteristic equation is:
$(\lambda^2 + s\lambda + p)(\lambda - \sigma\gamma x_1 + \eta + q_3E_3)(\lambda - \frac{\sigma\beta x_1}{\alpha'+x_1} + \mu + q_2E_2) = 0.$
Where

$$\begin{array}{rcl} s & = & \left(\sigma_2\frac{y_1}{x_1} + (u + \frac{r_1}{K})x_1 + \sigma_1\frac{x_1}{y_1} + vy_1\right), \\ p & = & \left(\sigma_2\frac{y_1}{x_1} + (u + \frac{r_1}{K})x_1\right)\left(\sigma_1\frac{x_1}{y_1} + vy_1\right) - (\sigma_2 - n_1x_1)(\sigma_1 - n_2y_1). \end{array}$$

Therefore, the first and second eigenvalues are:

$$\begin{array}{rcl} \lambda_1 & = & \sigma\gamma x_1 - (\eta + q_3E_3) < 0,\ \text{if } x_1 < \frac{\eta+q_3E_3}{\sigma\gamma},\ \text{then } |\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}. \\ \lambda_2 & = & \frac{\sigma\beta x_1}{\alpha'+x_1} - (\mu + q_2E_2) < 0,\ \text{if } x_1 < \frac{\alpha'(\mu+q_2E_2)}{\sigma\beta-(\mu+q_2E_2)},\ \text{then } |\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}. \end{array}$$

We pose $\Delta = s^2 - 4p$. If $\Delta > 0$, λ_3 and λ_4 are purely real and negative.
If $\Delta < 0$, λ_3 and λ_4 are complex number and if $|\arg(\lambda_{3,4})| = \tan^{-1}\left(\frac{\sqrt{-\Delta}}{s}\right) > \frac{\alpha\pi}{2}$. Hence, P_1 is locally asymptotically stable. □

Theorem 2.5. *The equilibrium point $P_2(x_2, y_2, 0, I_2)$ is locally asymptotically stable if $x_2 < \frac{\alpha'(\delta I_2+\mu+q_2E_2)}{\sigma\beta-\alpha'(\delta I_2+\mu+q_2E_2)}$ and (2.17) satisfied.*

Proof. From the Jacobian matrix $J(x_2, y_2, 0, I_2)$, the characteristic equation at P_2:
$\left(\lambda - (\frac{\sigma\beta x_2}{\alpha'+x_2} - (\delta I_2 + \mu + q_2E_2))\right)(\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0) = 0$, where:

$$
\begin{aligned}
c_2 &= -(A_1 + B_1),\\
c_1 &= A_1B_1 - (\sigma_1 - n_2y_2)(\sigma_2 - n_1x_2) + \sigma\gamma^2 I_2x_2,\\
c_0 &= -B_1\sigma\gamma^2 I_2x_2,\\
A_1 &= r_1 - \sigma_1 - q_1E_1 - 2\left(\frac{r_1+Ku}{K}\right)x_2 - n_1y_2 - \gamma I_2,\\
B_1 &= r_2 - \sigma_2 - 2vy_2 - n_2x_2.
\end{aligned}
$$

$\lambda_1 = \frac{\sigma\beta x_2}{\alpha'+x_2} - (\delta I_2 + \mu + q_2E_2) < 0$ if $x_2 < \frac{\alpha'(\delta I_2+\mu+q_2E_2)}{\sigma\beta-\alpha'(\delta I_2+\mu+q_2E_2)}$,
then $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$.
We define the discriminant of polynomial $P(\lambda) = \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0$ in this form [2]:
$D(P) = 18c_1c_2c_0 + (c_2c_1)^2 - 4c_1^3 - 4c_2^3c_0 - 27c_0^2$.

$$
\begin{array}{ll}
(1) & \text{If } D(P) > 0,\ P_2 \text{ is asymptotically stable if } c_0, c_1, c_2 > 0 \text{ and } c_2c_1 - c_0 > 0\\
& \text{for all } \alpha \in [0,1[.\\
(2) & \text{If } D(P) < 0 \text{ and } c_0, c_1, c_2 > 0\ \ P_2 \text{ is asymptotically stable if } \alpha < \frac{2}{3} \text{ and}\\
& c_2c_1 - c_0 > 0.\\
(3) & \text{If } D(P) < 0,\ c_0, c_1, c_2 > 0 \ \text{ and } \ c_2c_1 = c_0,\ P_2 \text{ is asymptotically stable}\\
& \text{for all } \alpha \in [0,1[.
\end{array}
\tag{2.17}
$$

□

Theorem 2.6. *The equilibrium point $P_3(x_3, y_3, S_3, 0)$ is locally asymptotically stable if $\delta S_3 + \sigma\gamma x_3 < \eta + q_3E_3$ and (2.18) are satisfied.*

Proof. The characteristic equation of matrix $J(x_3, y_3, S_3, 0)$:
$(\lambda - (\delta S_3 + \sigma\gamma x_3 - \eta - q_3E_3))(\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0) = 0$, where:

$$
\begin{aligned}
d_2 &= -(A_2 + B_2),\\
d_1 &= A_2B_2 - (\sigma_1 - n_2y_3)(\sigma_2 - n_1x_3) + \frac{\sigma\beta^2\alpha' x_3S_3}{(\alpha'+x_3)^3},\\
d_0 &= -B_2\frac{\sigma\beta^2\alpha' x_3S_3}{(\alpha'+x_3)^3},\\
A_2 &= r_1 - \sigma_1 - q_1E_1 - 2\frac{r_1+Ku}{K}x_3 - n_1y_3 - \frac{\beta\alpha' S_3}{(\alpha'+x_3)^2},\\
B_2 &= r_2 - \sigma_2 - 2vy_3 - n_2x_3.
\end{aligned}
$$

The first eigenvalues $\lambda_1 = \delta S_3 + \sigma\gamma x_3 - \eta - q_3E_3 < 0$ if $\delta S_3 + \sigma\gamma x_3 < \eta + q_3E$,
then $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$.
We define the discriminant of polynomial $Q(\lambda) = \lambda^3 + d_2\lambda^2 + d_1\lambda + d_0$ in this form [2]:
$D(Q) = 18d_1d_2d_0 + (d_2d_1)^2 - 4d_1^3 - 4d_2^3d_0 - 27d_0^2$.

$$
\begin{array}{ll}
(1) & \text{If } D(Q) > 0,\ P_3 \text{ is asymptotically stable if } d_0, d_1, d_2 > 0 \text{ and } d_2d_1 - d_0 > 0\\
& \text{for all } \alpha \in [0,1[.\\
(2) & \text{If } D(Q) < 0 \text{ and } d_0, d_1, d_2 > 0\ \ P_3 \text{ is asymptotically stable if } \alpha < \frac{2}{3} \text{ and } c_2c_1 - c_0 > 0.\\
(3) & \text{If } D(Q) < 0,\ d_0, d_1, d_2 > 0 \ \text{ and } \ d_2d_1 = d_0,\ P_3 \text{ is asymptotically stable}\\
& \text{for all } \alpha \in [0,1[.
\end{array}
\tag{2.18}
$$

□

Theorem 2.7. *The equilibrium point $P_4(x_4, y_4, S_4, I_4)$ is locally asymptotically stable if $\phi_0, \phi_1, \phi_2, \phi_3 > 0$, and $\phi_2\phi_3 - \phi_1 > \frac{\phi_0\phi_3^2}{\phi_1}$.*

Proof. From the Jacobian $J(x_4, y_4, S_4, I_4)$ matrix, the characteristic equation of P_4 is:

$$R(\lambda) = \lambda^4 + \phi_3\lambda^3 + \phi_2\lambda^2 + \phi_1\lambda + \phi_0,$$

where:

$$\begin{array}{rcl}
\phi_3 &=& -(a_6 + f_6), \\
\phi_2 &=& a_6 f_6 - b_6 e_6 - c_6 g_6 - d_6 i_6 - h_6 j_6, \\
\phi_1 &=& a_6 h_6 j_6 + c_6 f_6 g_6 + d_6 f_6 i_6 + f_6 h_6 j_6 - i_6 c_6 h_6 - d_6 g_6 j_6, \\
\phi_0 &=& j_6 b_6 e_6 h_6 + i_6 c_6 f_6 h_6 + d_6 f_6 g_6 j_6 - j_6 a_6 f_6 h_6, \\
a_6 &=& r_1 - \sigma_1 - q_1 E_1 - 2(\frac{r_1}{K} + u)x_4 - n_1 y_4 - \gamma I_4 - \frac{\beta\alpha' S_4}{(\alpha'+x_4)^2} - \gamma I_4, \\
b_6 &=& \sigma_2 - n_1 x_4, \\
c_6 &=& \frac{-\beta x_4}{\alpha'+x_4}, \\
d_6 &=& -\gamma x_4, \\
e_6 &=& \sigma_1 - n_2 y_4, \\
f_6 &=& r_2 - \sigma_2 - 2v y_4 - n_2 x_4, \\
g_6 &=& \frac{\sigma\beta\alpha' S_4}{(\alpha'+x_4)^2}, \\
h_6 &=& -\delta S_4, \\
i_6 &=& \sigma\gamma I_4, \\
j_6 &=& \delta I_4.
\end{array} \tag{2.19}$$

We pose the discriminant $D(R)$ in following form:

$$\begin{array}{rcl}
D(R) &=& 256\phi_0^3 - 192\phi_3\phi_1\phi_0^2 - 128\phi_2^2\phi_0 + 144\phi_2\phi_1^2\phi_0 - 27\phi_1^4 + 144\phi_3^2\phi_2\phi_0^2 \\
&& -6\phi_3^2\phi_1^2\phi_0 - 80\phi_3\phi_2^2\phi_1\phi_0 + 18\phi_3\phi_2\phi_1^3 + 16\phi_2^4\phi_0 - 4\phi_2^3\phi_1^2 - 27\phi_3^4\phi_0^2 \\
&& +18\phi_3^3\phi_2\phi_1\phi_0 - 4\phi_3^3\phi_1^3 - 4\phi_3^2\phi_2^3\phi_0 + (\phi_3\phi_2\phi_1)^2.
\end{array} \tag{2.20}$$

Using the results of [2],

(2.21)

(1) If $D(R) > 0$, $\phi_0, \phi_1, \phi_2, \phi_3 > 0$ and $\phi_2\phi_3 - \phi_1 > \frac{\phi_0\phi_3^2}{\phi_1}$, P_4 is locally asymptotically stable for all $\alpha \in [0, 1[$.

(2) If $D(R) < 0$, $\phi_0, \phi_1, \phi_2, \phi_3 > 0$ and $\alpha < \frac{1}{3}$, P_4 is locally asymptotically stable.

(3) If $D(R) < 0$, $\phi_0, \phi_1, \phi_2, \phi_3 > 0$ and $\phi_2 = \frac{\phi_3\phi_0}{\phi_1} + \frac{\phi_1}{\phi_3}$, P_4 is locally asymptotically stable for all $\alpha \in [0, 1[$.

□

3. Global stability of equilibira

In this section, we prove the global stability of each equilibrium point of system (1.1) using Lyapunov functions.

Theorem 3.1. *The equilibrium $P_1(x_1, y_1, 0, 0)$ is globally asymptotically stable if (2.5), (2.6), (2.7) and $n_1 + \frac{\sigma_2 n_2 y_1}{\sigma_1 x_1} < 2\min\left(\frac{\sigma_2 y_1 v}{\sigma_1 x_1}, \frac{r_1}{K} + u\right)$ are realized.*

Proof. Consider the following positive definite Lyapunov function about $P_1(x_1, y_1, 0, 0)$:
$V_1(x,y) = \left(x - x_1 - x_1 \ln\left(\frac{x}{x_1}\right)\right) + \frac{\sigma_2 y_1}{\sigma_1 x_1}\left(y - y_1 - y_1 \ln\left(\frac{y}{y_1}\right)\right)$.
Using [28], we get:

$$\begin{aligned} D^\alpha V_1 &\leq \tfrac{x-x_1}{x} D^\alpha x + \tfrac{\sigma_2 y_1}{\sigma_1 x_1}\left(\tfrac{y-y_1}{y}\right) D^\alpha y, \\ &\leq (x-x_1)\left(-\left(\tfrac{r_1+Ku}{K}\right)(x-x_1) + \sigma_2\left(\tfrac{y}{x} - \tfrac{y_1}{x_1} - n_1(y-y_1)\right)\right) \\ &\quad + \tfrac{\sigma_2 y_1}{\sigma_1 x_1}(y-y_1)\left(-v(y-y_1) + \sigma_1\left(\tfrac{x}{y} - \tfrac{x_1}{y_1}\right) - n_2(x-x_1)\right). \end{aligned}$$

Using $-(x-x_1)(y-y_1) < \frac{1}{2}\left((x-x_1)^2 + (y-y_1)^2\right)$ we find,

$$\begin{aligned} D^\alpha V_1 &\leq -\left(\tfrac{r_1+Ku}{K}\right)(x-x_1)^2 - \tfrac{v\sigma_2 y_1}{\sigma_1 x_1}(y-y_1)^2 - \tfrac{\sigma_2}{x x_1 y}(y x_1 - y_1 x)^2 \\ &\quad -\left(n_1 + \tfrac{\sigma_2 y_1 n_2}{\sigma_1 x_1}\right)(x-x_1)(y-y_1), \\ &\leq \left(\tfrac{1}{2}\left(n_1 + \tfrac{\sigma_2 y_1 n_2}{\sigma_1 x_1}\right) - \tfrac{r_1+Ku}{K}\right)(x-x_1)^2 + \left(\tfrac{1}{2}\left(n_1 + \tfrac{\sigma_2 y_1 n_2}{\sigma_1 x_1}\right) - \tfrac{v\sigma_2 y_1}{\sigma_1 x_1}\right)(y-y_1)^2 \\ &\quad - \tfrac{\sigma_2}{x x_1 y}(y x_1 - y_1 x)^2. \end{aligned}$$

Therefore, $D^\alpha V_1 < 0$ if $n_1 + \frac{\sigma_2 n_2 y_1}{\sigma_1 x_1} < 2\min\left(\frac{\sigma_2 y_1 v}{\sigma_1 x_1}, \frac{r_1}{K} + u\right)$.

□

Theorem 3.2. *The equilibrium $P_2(x_2, y_2, 0, I_2)$ is globally asymptotically stable if (2.9) and $n_1 + \frac{\sigma_2 n_2 y_2}{\sigma_1 x_2} < 2\min\left(\frac{\sigma_2 y_2 v}{\sigma_1 x_2}, \frac{r_1}{K} + u\right)$ are realized.*

Proof. Consider the following positive definite Lyapunov function about $P_2(x_2, y_2, 0, I_2)$:
$V_2(x,y,I) = \left(x - x_2 - x_2 \ln\left(\frac{x}{x_2}\right)\right) + \frac{\sigma_2 y_2}{\sigma_1 x_2}\left(y - y_2 - y_2 \ln\left(\frac{y}{y_2}\right)\right) + \frac{1}{\sigma}\left(I - I_2 - I_2 \ln\left(\frac{I}{I_2}\right)\right)$.
Using [28] we get:

$$\begin{aligned} D^\alpha V_2 &\leq (x-x_2)\left(-\left(\tfrac{r_1+Ku}{K}\right)(x-x_2) + \sigma_2\left(\tfrac{y}{x} - \tfrac{y_2}{x_2}\right) - n_1(y-y_2) - \gamma(I-I_2)\right) \\ &\quad + \tfrac{\sigma_2 y_2}{\sigma_1 x_2}(y-y_2)\left(-v(y-y_2) + \sigma_1\left(\tfrac{x}{y} - \tfrac{x_2}{y_2}\right) - n_2(x-x_2)\right) + (I-I_2)\gamma(x-x_2), \\ &\leq -\tfrac{v\sigma_2 y_2}{\sigma_1 x_2}(y-y_2)^2 - \tfrac{\sigma_2}{x x_2 y}(y x_2 - y_2 x)^2 - \left(n_1 + \tfrac{\sigma_2 y_2 n_2}{\sigma_1 x_2}\right)(x-x_2)(y-y_2), \\ &\leq \left(\tfrac{1}{2}\left(n_1 + \tfrac{\sigma_2 y_2 n_2}{\sigma_1 x_2}\right) - \tfrac{r_1+Ku}{K}\right)(x-x_2)^2 - \left(\tfrac{1}{2}\left(n_1 + \tfrac{n_2 \sigma_2 y_2}{\sigma_1 x_2}\right) - \tfrac{v\sigma_2 y_2}{\sigma_1 x_2}\right)(y-y_2)^2 \\ &\quad - \tfrac{\sigma_2}{x x_2 y}(y x_2 - y_2 x)^2. \end{aligned}$$

Therefore, $D^\alpha V_2 < 0$ if $n_1 + \frac{\sigma_2 n_2 y_2}{\sigma_1 x_2} < 2\min\left(\frac{\sigma_2 y_2 v}{\sigma_1 x_2}, \frac{r_1}{K} + u\right)$.

□

Theorem 3.3. *The equilibrium $P_3(x_3, y_3, S_3, 0)$ is globally asymptotically stable if (2.10) and $n_1 + \frac{\sigma_2 n_2 y_3}{\sigma_1 x_3} < 2\min\left(\frac{\sigma_2 y_3 v}{\sigma_1 x_3}, \frac{r_1}{K} + u - \frac{\beta S_3}{\alpha'(\alpha' + x_3)}\right)$ are realized.*

Proof. Consider the following positive definite Lyapunov function about $P_3(x_3, y_3, S_3, 0)$:

$$\begin{aligned} V_3(x,y,S) &= \left(x - x_3 - x_3 \ln\left(\tfrac{x}{x_3}\right)\right) + \tfrac{\sigma_2 y_3}{\sigma_1 x_3}\left(y - y_3 - y_3 \ln\left(\tfrac{y}{y_3}\right)\right) \\ &\quad + \tfrac{\alpha' + x_3}{\alpha' \sigma}\left(S - S_3 - S_3 \ln\left(\tfrac{S}{S_3}\right)\right). \end{aligned}$$

Using [28], we obtain:

$$
\begin{aligned}
D^\alpha V_3 \leq\ & (x-x_3)\Big(-\big(\tfrac{r_1+Ku}{K}\big)(x-x_3)+\sigma_2\left(\tfrac{y}{x}-\tfrac{y_3}{x_3}\right)-n_1(y-y_3)-\beta(\tfrac{S}{\alpha'+x}-\tfrac{S_3}{\alpha'+x_3})\Big)\\
&+\tfrac{\sigma_2 y_3}{\sigma_1 x_3}(y-y_1)\Big(-v(y-y_1)+\sigma_1(\tfrac{x}{y}-\tfrac{x_1}{y_1})-n_2(x-x_2)\Big)\\
&+\tfrac{\alpha'+x_3}{\alpha'\sigma}(S-S_3)\sigma\beta\left(\tfrac{x}{x+\alpha'}-\tfrac{x_3}{\alpha'+x_3}\right),\\
\leq\ & -\big(\tfrac{r_1+Ku}{K}\big)(x-x_3)^2-\tfrac{v\sigma_2 y_3}{\sigma_1 x_3}(y-y_3)^2-\tfrac{\sigma_2}{xx_3y}(yx_3-y_3x)^2\\
&-(n_1+\tfrac{\sigma_2 y_3 n_2}{\sigma_1 x_3})(x-x_3)(y-y_3)+\tfrac{\beta S_3(x-x_3)^2}{(\alpha'+x)(\alpha'+x_3)},\\
\leq\ & \left(\tfrac{1}{2}(n_1+\tfrac{\sigma_2 y_3 n_2}{\sigma_1 x_3})-\tfrac{r_1+Ku}{K}+\tfrac{\beta S_3}{\alpha'(\alpha'+x_3)}\right)(x-x_3)^2\\
&-\left(\tfrac{1}{2}(n_1+\tfrac{\sigma_2 y_3 n_2}{\sigma_1 x_3})-\tfrac{v\sigma_2 y_3}{\sigma_1 x_3}\right)(y-y_3)^2-\tfrac{\sigma_2}{xx_3y}(yx_3-y_3x)^2.
\end{aligned}
$$

Therefore, $D^\alpha V_3 < 0$ if $n_1+\frac{\sigma_2 n_2 y_3}{\sigma_1 x_3} < 2\min\left(\frac{\sigma_2 y_3 v}{\sigma_1 x_3}, \frac{r_1}{K}+u-\frac{\beta S_3}{\alpha'(\alpha'+x_3)}\right)$. □

Theorem 3.4. *The equilibrium $P_4(x_4, y_4, S_3, I_4)$ is globally asymptotically stable if (2.11) and $n_1+\frac{\sigma_2 n_2 y_4}{\sigma_1 x_4} < 2\min\left(\frac{\sigma_2 y_4 v}{\sigma_1 x_4}, \frac{r_1}{K}+u+\frac{\beta(x_4-2S_4)}{2(\alpha'+x_4)}\right)$ are realized.*

Proof. Consider the following positive definite Lyapunov function about $P_4(x_4, y_4, S_4, I_4)$:

$$
\begin{aligned}
V_4(x,y,S,I) =\ & \left(x-x_4-x_4\ln\left(\tfrac{x}{x_4}\right)\right)+\tfrac{\sigma_2 y_4}{\sigma_1 x_4}\left(y-y_4-y_4\ln\left(\tfrac{y}{y_4}\right)\right)\\
&+\tfrac{1}{\sigma}\left(S-S_4-S_4\ln\left(\tfrac{S}{S_4}\right)\right)+\tfrac{1}{\sigma}\left(I-I_4-I_4\ln\left(\tfrac{I}{I_4}\right)\right).
\end{aligned}
$$

Using [28], we obtain:

$$
\begin{aligned}
D^\alpha V_4 \leq\ & (x-x_4)\Big(-\big(\tfrac{r_1+Ku}{K}\big)(x-x_4)+\sigma_2\left(\tfrac{y}{x}-\tfrac{y_4}{x_4}\right)-n_1(y-y_4)-\beta(\tfrac{S}{\alpha'+x}-\tfrac{S_4}{\alpha'+x_4})-\gamma(I-I_4)\Big)\\
&+\tfrac{\sigma_2 y_4}{\sigma_1 x_4}(y-y_4)\Big(-v(y-y_4)+\sigma_1(\tfrac{x}{y}-\tfrac{x_4}{y_4})-n_2(x-x_4)\Big)\\
&+\tfrac{1}{\sigma}(S-S_4)\left(\sigma\beta\left(\tfrac{x}{x+\alpha'}-\tfrac{x_4}{\alpha'+x_4}\right)-\delta(I-I_4)\right)+\tfrac{(I-I_4)(\sigma\gamma(x-x_4)+\delta(S-S_4))}{\sigma},\\
\leq\ & -\big(\tfrac{r_1+Ku}{K}\big)(x-x_4)^2-\tfrac{v\sigma_2 y_4}{\sigma_1 x_4}(y-y_4)^2-\tfrac{\sigma_2}{xx_4y}(yx_4-y_4x)^2\\
&-(n_1+\tfrac{\sigma_2 y_4 n_2}{\sigma_1 x_4})(x-x_4)(y-y_4)-\tfrac{\beta(x-x_4)}{(\alpha'+x)(\alpha'+x_4)}(Sx_4-xS_4),\\
\leq\ & \left(\tfrac{1}{2}(n_1+\tfrac{\sigma_2 y_4 n_2}{\sigma_1 x_4})-\tfrac{r_1+Ku}{K}+\tfrac{\beta S_4}{\alpha'(\alpha'+x_4)}-\tfrac{\beta x_4}{2(\alpha'+x_4)}\right)(x-x_4)^2-\tfrac{\sigma_2}{xx_4y}(yx_4-y_4x)^2\\
&+\left(\tfrac{1}{2}(n_1+\tfrac{\sigma_2 y_4 n_2}{\sigma_1 x_4})-\tfrac{v\sigma_2 y_4}{\sigma_1 x_4}\right)(y-y_4)^2-\tfrac{\beta x_4}{(\alpha'+x)(\alpha'+x_4)}(S-S_4)^2.
\end{aligned}
$$

Therefore, $D^\alpha V_4 < 0$ if $n_1+\frac{\sigma_2 n_2 y_4}{\sigma_1 x_4} < 2\min\left(\frac{\sigma_2 y_4 v}{\sigma_1 x_4}; \frac{r_1}{K}+u+\frac{\beta(x_4-2S_4)}{2(\alpha'+x_4)}\right)$.

□

4. Numerical simulations

To show the influence of the parameter α on our fractional order model, we take the different values of α in numerical simulations of the curves $x(t), y(t), S(t)$ and $I(t)$ that are shown in Figures 2–5. These figures show that the system (1.1) reaches the equilibrium state for the different values of α. These results show the effectiveness of Theorems 2.4–2.7. As we can see, numerical solutions are permanently dependent on the fractional order derivative α and the model reaches the equilibrium point more rapidly by reducing α. In other words, the model approaches the steady state more quickly when the memory factor effect is increased.

To demonstrate the theoretical results obtained in this paper, we give some numerical simulations. We consider the parameters values as given by [14]:
$r_1 = 5, r_2 = 1, n_1 = 0.5, n_2 = 0.3, \sigma_1 = 1, \sigma_2 = 0.9, q_1 = 0.1, q_2 = 0.2, q_3 = 0.4, E_1 = 5, E_2 = 5.2; E_3 = 4.8,$
$u = 0.0001, v = 0.333, K = 4, \alpha' = 0.7, \beta = 0.94, \sigma = 0.998, \delta = 10, \mu = 1, \gamma = 5.5, \eta = 60$, with initial conditions $(x(0), y(0), S(0), I(0)) = (1,1,1,1)$, so P_1 is locally asymptotically stable.

As it's shown in this example the parameter values are chosen as:
$r_1 = 6, r_2 = 8, n_1 = 0.5, n_2 = 0.3, \sigma_1 = 2, \sigma_2 = 2, q_1 = 0.1, q_2 = 0.2, u = 0.4, v = 0.4, K = 5, \alpha' = 0.45, \beta = 1,$

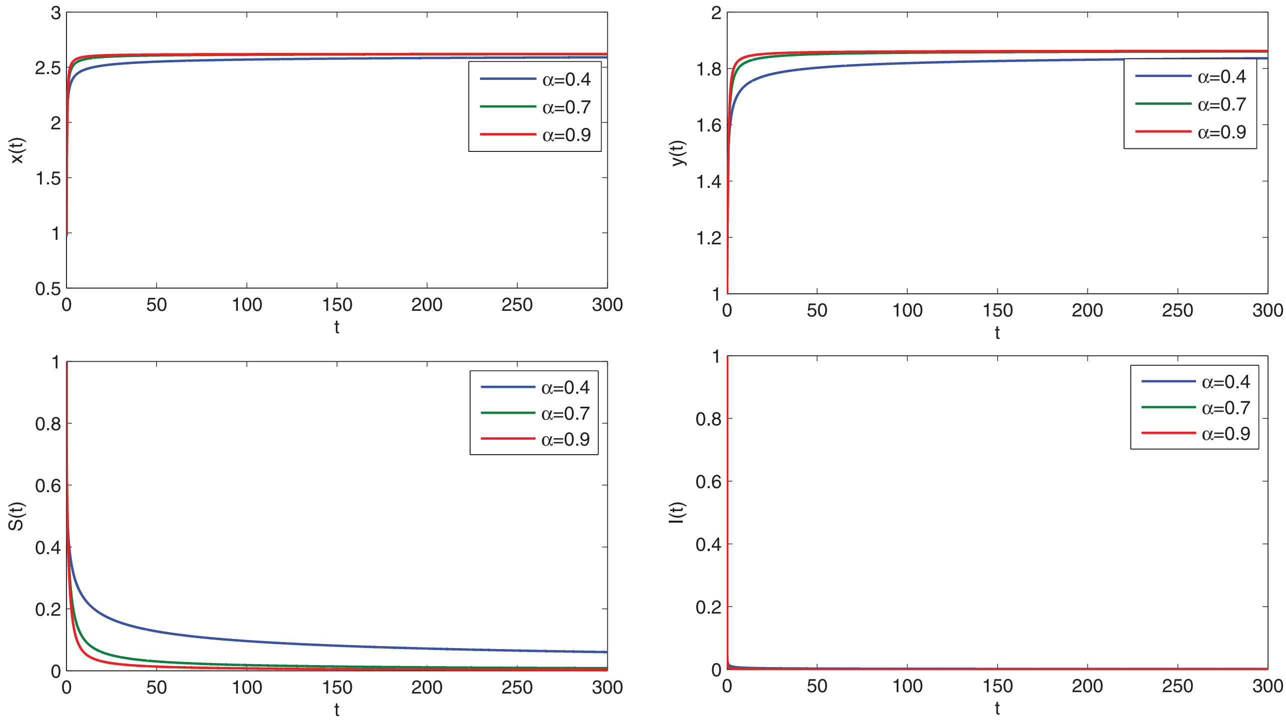

FIGURE 2. Solution curves corresponding to the set values parameters of the system (1.1) of equilibrium P_1 with different values of α.

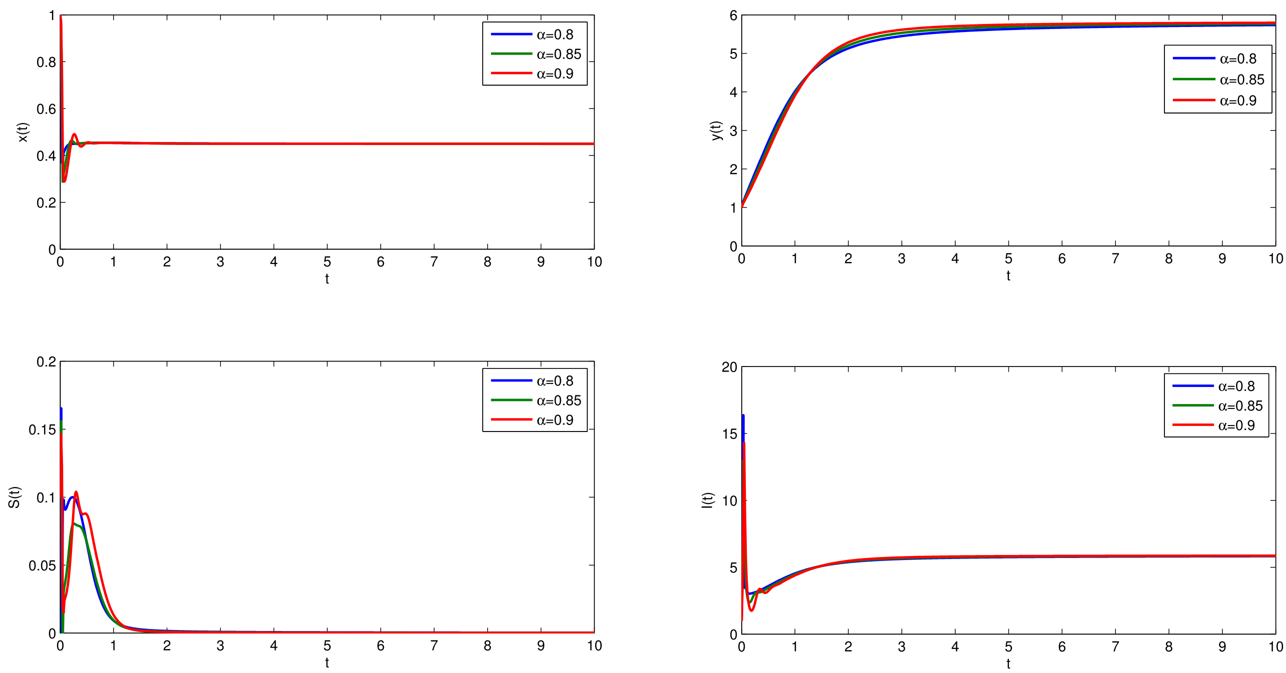

FIGURE 3. Solution curves corresponding to the set values parameters of the system (1.1) of equilibrium P_2 with different values of α.

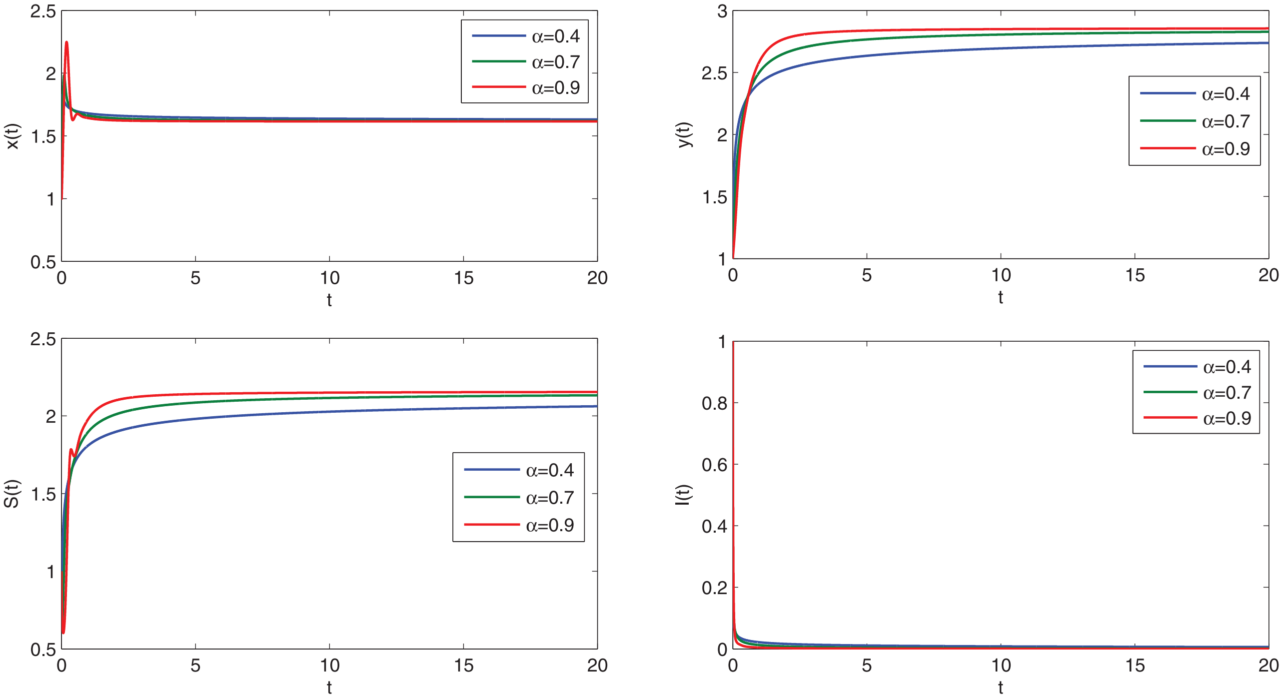

FIGURE 4. Solution curves corresponding to the set values parameters of the system (1.1) of equilibrium P_3 with different values of α.

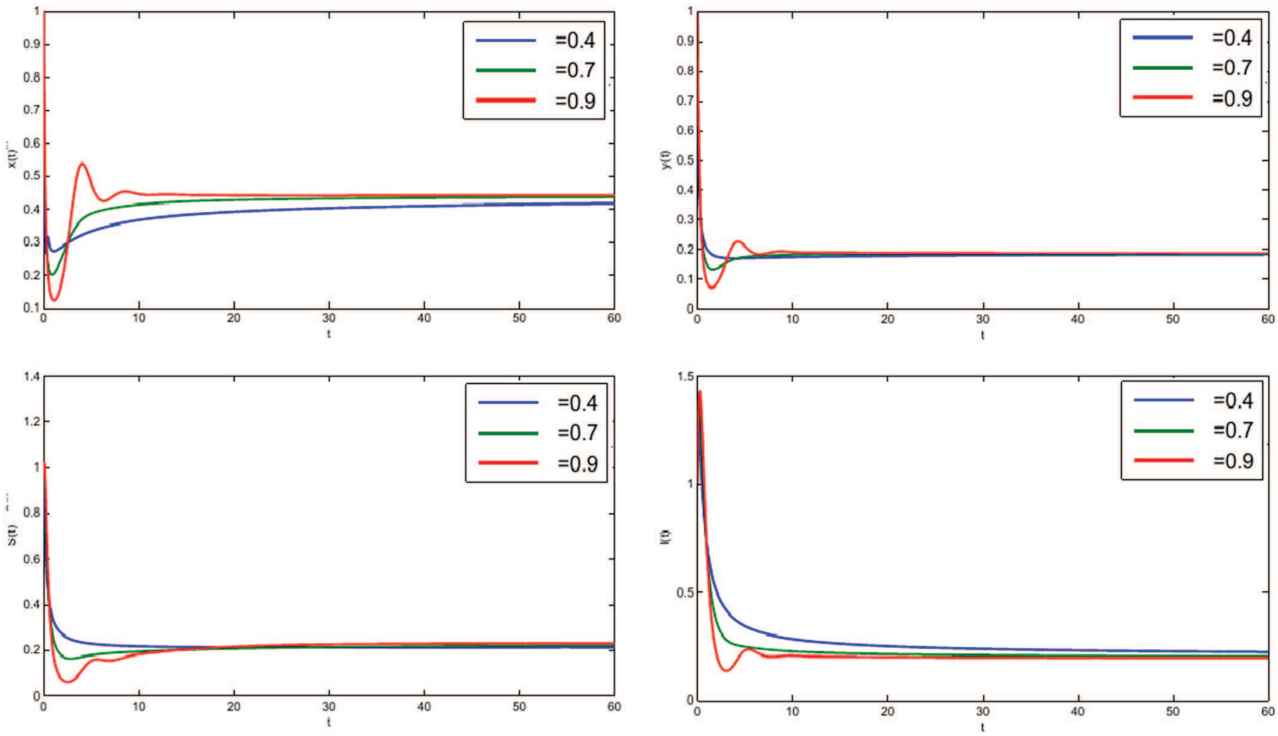

FIGURE 5. Solution curves corresponding to the set values parameters of the system (1.1) of equilibrium P_4 with different values of α.

$\sigma = 4, \delta = 10, \mu = 1, \gamma = 5.5, \eta = 60$. The system (1.1) with initial conditions $(x(0), y(0), S(0), I(0)) = (1, 1, 0.1, 1)$, so P_2 is locally asymptotically stable.

In this example, the parameter values are:
$r_1 = 10, r_2 = 8, n_1 = 0.5, n_2 = 0.3, \sigma_1 = 2, \sigma_2 = 7.5, q_1 = 0.1, q_2 = 0.2, q_3 = 0.5, E_1 = 3, E_2 = 3.2,$
$E_3 = 4, u = 0.01, v = 0.4, K = 5, \alpha' = 5, \beta = 50, \sigma = 1.5, \delta = 10, \mu = 18, \gamma = 5.33, \eta = 60$, with initial conditions $(x(0), y(0), S(0), I(0)) = (1, 1, 1, 1)$, so P_3 is locally asymptotically stable.

In this example, the parameter values are:
$r_1 = 2.34, r_2 = 3, n_1 = 0.5, n_2 = 0.3, \sigma_1 = 2, \sigma_2 = 7.5, q_1 = 0.1, q_2 = 0.2, q_3 = 0.3, E_1 = 3, E_2 = 2.6, E_3 = 2.2,$
$u = 0.01, v = 0.4, K = 0.7, \alpha' = 0.7, \beta = 0.94, \sigma = 0.998, \delta = 1.34, \mu = 2.41, \gamma = 8, \eta = 2.68$, with initial conditions $(x(0), y(0), S(0), I(0)) = (1, 1, 1, 1)$, so P_4 is locally asymptotically stable.

We observe from simulations, the effect of reducing the order of the time derivative can be observed. As the fractional order α decreases, the system (with Caputo derivative) stabilizes more quickly. It is the largest "memory" of the system of past states, the greater the damping of the oscillations in the dynamics of the system. The simulations show that, even with fairly moderate reductions in α, the amplitude of the population density oscillations is greatly delayed.

5. Conclusion

In this paper, we investigated a Dynamics of the fractional order prey–predator model in the presence of competition and toxicity using the Caputo fractional derivative. We have established the existence and boundedness of the solutions. After calculating the equilibrium of our model under certain conditions, we have analyzed the local stability using Matiginon's conditions [17]. Global stability has been studied using Lyapunov functions. From our numerical results, we can observe that the different values of α have no effect on the stability of equilibria but have an effect on the time necessary to achieve equilibrium states. These variations are verified in the numerical simulations illustrated in the Figures 2–5, as the curves x, y, S and I that converge towards the equilibrium points. Finally, we can conclude that the memory effect of the fractional order derivative affects the dynamics of our proposed system.

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