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SPECTRUM OF THE M^5-TRAVELING WAVES*

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Abstract. In this paper, we study the essential spectrum of the operator obtained by linearizing at traveling waves that occur in the one-dimensional version of the M^5-model for mesenchymal cell movement inside a directed tissue made up of highly aligned fibers. We show that traveling waves are spectrally unstable in $L^2(\mathbb{R};\mathbb{C}^3)$ as the essential spectrum includes the imaginary axis. Tools in the proof include exponential dichotomies and Fredholm properties. We prove that a weighted space $L^2_w(\mathbb{R};\mathbb{C}^3)$ with the same function for the tree variables of the linearized operator is no suitable to shift the essential spectrum to the left of the imaginary axis. We find a pair of appropriate weight functions whereby on the weighted space $L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})$ the essential spectrum lies on $\{\Re e\,\lambda < 0\}$, outside the imaginary axis.

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1. Introduction

Hillen [9] proposed the M^5-model for mesenchymal cell movement through directed and undirected tissues. Mesenchymal migration is a strategy of individual cell motion characterized by the action of proteases that execute degradation and remodeling of the local extracellular matrix (abbreviated ECM) — a 3D fiber network composed primarily of collagen, which provides physical support to tissues —, creating tube-like matrix defects along the path of migration [4, 6, 22]. The orientation of the matrix fibers influences cell migration, through a process termed contact guidance, the ECM induces movement in either direction parallel to the fibers [14]. The n-dimensional model was formulated at the mesoscopic level as a transport model, in which all cells are assumed to follow a velocity jump process. The model incorporates the joint actions of proteolytic degradation of the ECM substrate and alignment of cells along matrix fiber strands induced by the ECM via contact guidance.

In this paper, we are concerned with the M^5-model for 1D directed tissues. The model in one space-dimension consists of a system of transport equations for the total cell population p and the population flux j, coupled to a dynamic equation for the probability distribution q^+ for a cell to move to the right along the 1D ECM formed

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by highly aligned fibers. The dynamics of the variables is described by the strictly hyperbolic system

$$
\begin{aligned}
p_t + j_x &= 0, \\
j_t + s^2 p_x &= -\mu j + \mu s \left(2q^+ - 1\right) p, \\
q_t^+ &= \frac{2\kappa}{s} j q^+ \left(1 - q^+\right),
\end{aligned} \tag{1.1}
$$

with $x \in \mathbb{R}$ and $t \geq 0$. Cells change the direction of movement with a constant turning rate $\mu > 0$. The parameter $\kappa > 0$ describes the efficiency of matrix degradation caused by proteolytic enzymes.

1.1. M^5-Traveling waves

Wang *et al.* [21] proved that for any speed c such that $0 \leq c \leq s$, the system (1.1) possesses a continuum of traveling wave solutions of the form

$$
p(x,t) = \bar{p}(z), \quad j(x,t) = \bar{j}(z), \quad q^+(x,t) = \bar{q}^+(z), \quad z = x - ct,
$$

satisfying

$$
\begin{aligned}
&\lim_{z \to \pm\infty} \bar{p}(z) = \lim_{z \to \pm\infty} \bar{j}(z) = 0, \\
&\lim_{z \to +\infty} \bar{q}^+(z) = q_r^+, \\
&\lim_{z \to -\infty} \bar{q}^+(z) = q_l^+ \text{ with } 0 \leq q_r^+ < q_l^+ \leq 1.
\end{aligned} \tag{1.2}
$$

In the moving coordinate $z = x - ct$, the system (1.1) becomes

$$
\begin{aligned}
-c\bar{p}_z + \bar{j}_z &= 0, \\
-c\bar{j}_z + s^2 \bar{p}_z &= -\mu \bar{j} + \mu s \left(2\bar{q}^+ - 1\right) \bar{p}, \\
-c\bar{q}_z^+ &= \frac{2\kappa}{s} \bar{j} \bar{q}^+ \left(1 - \bar{q}^+\right).
\end{aligned} \tag{1.3}
$$

Under the boundary conditions (1.2),

$$
\bar{j} = c\bar{p} \tag{1.4}
$$

is an invariant of motion to the system (1.1). Then, upon substituting (1.4) into (1.3), the system (1.3) reduces to

$$
\begin{aligned}
\bar{p}_z &= \frac{\mu}{c^2 - s^2} \bar{p} \left[c - s\left(2\bar{q}^+ - 1\right)\right], \\
\bar{q}_z^+ &= -\frac{2\kappa}{s} \bar{p} \left(1 - \bar{q}^+\right) \bar{q}^+.
\end{aligned} \tag{1.5}
$$

This system has infinitely many equilibria. Indeed, note that for $0 \leq \theta \leq 1$ any point $(\bar{p}, \bar{q}^+) = (0, \theta)$ in the phase space is a steady state. With the help of LaSalle's invariant principle, Wang *et al.* [21] showed that there corresponds to each $c \in [0, s)$ a critical value $\theta^* = \frac{c+s}{2s}$ such that for any left end state q_l^+ with $\theta^* < q_l^+ < 1$ there is a heteroclinic orbit starting from $(0, q_l^+)$ and finishing at $(0, q_r^+)$ for some right end state q_r^+ with $0 < q_r^+ < \theta^*$. In regard to $\bar{q}^+ = 0$ and $\bar{q}^+ = 1$, by solving (1.5) it is seen that if $\bar{q}^+ = 0$ then $\bar{p} \to +\infty$ as

$z \to -\infty$, and that when $\bar{q}^+ = 1$, $\bar{p} \to +\infty$ as $z \to +\infty$. This implies that neither a heteroclinic connection to $(0,0)$, nor a heteroclinic connection from $(0,1)$, has the chance to exist.

The existence of traveling wave solutions of (1.1) is established by Wang *et al.* [21] as follows:

Theorem 1.1. [21] *Let us consider the system (1.5) given traveling speed c with $0 \le c < s$ and $\theta^* = \frac{c+s}{2s}$. Then for any equilibrium $(0, c_1)$ with $\theta^* < c_1 < 1$ there exists another equilibrium $(0, c_2)$ with $0 < c_2 < \theta^*$ such that there is a bounded, nonnegative, heteroclinic orbit connecting $(0, c_1)$ to $(0, c_2)$. That is, there exists a traveling solution $(\bar{p}, \bar{q}^+)$ of the system (1.5) connecting two equilibria. Particularly, the system (1.5) admits a standing wave for $c = 0$.*

The heteroclinic orbit of (1.5) corresponds to a traveling pulse solution for $\bar{p}$, and consequently for $\bar{j}$, coexisting with a decreasing traveling wave front for $\bar{q}^+$.

In [21], Wang *et al.* derived a relation between the left end state q_l^+ and the right end state q_r^+ by proving the following

Lemma 1.2. [21] *Given a speed c satisfying $0 \le c < s$, the left and right equilibria $(0, q_l^+)$ and $(0, q_r^+)$ are related as*

$$\left(\frac{1-q_r^+}{1-q_l^+}\right)^{s-c} = \left(\frac{q_l^+}{q_r^+}\right)^{s+c}, \qquad 0 \le c < s. \tag{1.6}$$

More recently, Cruz-García *et al.* [2] completed this result by settling that q_r^+ and the wave speed are related.

Proposition 1.3. [2] *Fix q_l^+. Then the right end q_r^+ is an increasing function of the wave speed.*

Wang *et al.* stress out that inside a continuum of traveling waves propagating at a fixed speed c, the amplitude $\bar{p}_{\max}$ of the continuum of pulses is an increasing function of q_l^+. Concerning the fronts spreading at the same speed, we have found that the amplitude $q_l^+ - q_r^+$ grows as q_l^+ moves up away from θ^* because q_r^+, in turn, goes down to zero. More concretely, we have the following result.

Proposition 1.4. *Let c be a given wave speed. Then the right end q_r^+ is a decreasing function of the left end.*

Proof. We use (1.6) to obtain

$$\frac{\partial q_r^+}{\partial q_l^+} = \frac{\left[2sq_l^+ - (s+c)\right] q_r^+ \left(1-q_r^+\right)}{\left[2sq_r^+ - (s+c)\right] q_l^+ \left(1-q_l^+\right)}.$$

By Theorem 1.1, $2sq_l^+ - (s+c) > 0$ and $2sq_r^+ - (s+c) < 0$, hence, since $0 < q_r^+, q_l^+ < 1$, we conclude that $\partial q_r^+ / \partial q_l^+ < 0$. □

Recently, M^5-traveling wave solutions for system (1.1) have been investigated by Cruz-García *et al.* [3]. Using Lagrange's interpolation method, the authors derived an exactly solvable approximate equation which yields analytical approximations for the standing and traveling waves. In [3], upper and lower error bounds are computed for the approximate standing wave solutions. Furthermore, error bounds for the approximate traveling wave solutions were provided under necessary conditions on the right and left endpoints q_r^+ and q_l^+.

In earlier work, Cruz-García *et al.* [2] investigated the spectral stability of the standing wave solutions of (1.1). Spectral stability means that for the operator obtained by linearizing around a wave, there is no spectrum in the closed right-half complex plane except for the origin. The spectrum of the linearization is the disjoint union of the *essential spectrum* and the *point spectrum*, the latter being a discrete set comprising all isolated eigenvalues. In [2], each standing wave was found to be spectrally stable. To handle the stability problem, the spectrum was proven to be purely essential. Goodman's *integrated variable* technique [7] permitted to establish, via energy methods, that the point spectrum is empty. Then, by Fredholm properties and Palmer's Theorem [15, 16, 19], the essential spectrum is demonstrated to be contained in the open left-half complex plane except for the eigenvalue $\lambda = 0$, which has infinite multiplicity. This deduction relies on the fact showed by Cruz-García *et al.* [2] that the traveling wave profiles converge at exponential rates to their endpoints as $z \to \pm\infty$.

The present paper addresses the spectrum of the operator obtained by linearizing at the traveling waves. We place a major emphasis on resolving the spectral instability related to the location of the essential spectrum. We should mention that the work conducted to date on the stability of M^5-traveling wave solutions has not led us to determine the precise location of the point spectrum.

Since some results in this work rely on the exponential convergence of the waves towards their endpoints, we provide a precise statement of this result.

Lemma 1.5. [2] *Traveling wave solutions $\bar{p}$ and $\bar{q}^+$ satisfy*

$$\begin{aligned}
\left|d^i/\mathrm{d}z^i\left(\bar{q}^+(z) - q_r^+\right)\right| &\le C\exp\left(-\frac{(c+s-2sq_r^+)\mu}{s^2-c^2}z\right), \quad \textit{as} \quad z\to+\infty,\\
\left|d^i/\mathrm{d}z^i\left(\bar{q}^+(z) - q_l^+\right)\right| &\le C\exp\left(-\frac{(c+s-2sq_l^+)\mu}{s^2-c^2}z\right), \quad \textit{as} \quad z\to-\infty,\\
\left|d^i/\mathrm{d}z^i\left(\bar{p}(z)\right)\right| &\le C\exp\left(-\frac{(c+s-2sq_r^+)\mu}{s^2-c^2}z\right), \quad \textit{as} \quad z\to+\infty,\\
\left|d^i/\mathrm{d}z^i\left(\bar{p}(z)\right)\right| &\le C\exp\left(-\frac{(c+s-2sq_l^+)\mu}{s^2-c^2}z\right), \quad \textit{as} \quad z\to-\infty,
\end{aligned}$$

for $i = 0, 1$, and some uniform $C > 0$.

1.2. The spectral problem

In the moving coordinate frame $(z,t) = (x - ct, t)$, where $0 < c < s$, traveling waves $\bar{p}$, $\bar{j}$ and $\bar{q}^+$ are time-independent solutions of the system

$$\begin{aligned}
p_t &= cp_z - j_z,\\
j_t &= -s^2 p_z + cj_z - \mu j + \mu s\left(2q^+ - 1\right)p,\\
q_t^+ &= cq_z^+ + \frac{2\kappa}{s} jq^+\left(1 - q^+\right).
\end{aligned} \tag{1.7}$$

We formulate the spectral stability problem following the procedure outlined by Kapitula and Promislow [13]. It begins with the linearization of (1.7) about the traveling wave solutions. The result is the system of linear PDEs

$$\begin{aligned}
p_t &= cp_z - j_z,\\
j_t &= -s^2 p_z + cj_z - \mu j + \mu s\left[\left(2\bar{q}^+ - 1\right)p + 2\bar{p}q^+\right],\\
q_t^+ &= cq_z^+ + \frac{2\kappa}{s}\left[\bar{q}^+\left(1 - \bar{q}^+\right)j + \bar{j}\left(1 - 2\bar{q}^+\right)q^+\right].
\end{aligned}$$

Let us consider solutions of the form $e^{\lambda t}p(z)$, $e^{\lambda t}j(z)$ and $e^{\lambda t}q^+(z)$, where $\lambda \in \mathbb{C}$ and p, j and q^+ belong to the Hilbert space $L^2(\mathbb{R};\mathbb{C})$. This gives rise to the spectral system

$$\begin{aligned}
\lambda p &= cp_z - j_z,\\
\lambda j &= -s^2 p_z + cj_z - \mu j + \mu s\left(\left(2\bar{q}^+ - 1\right)p + 2\bar{p}q^+\right),\\
\lambda q^+ &= cq_z^+ + \frac{2\kappa}{s}\left(\bar{q}^+\left(1 - \bar{q}^+\right)j + \bar{j}\left(1 - 2\bar{q}^+\right)q^+\right).
\end{aligned} \tag{1.8}$$

The natural domain of solutions of (1.8) is $H^1(\mathbb{R};\mathbb{C})$.

Alternatively, system (1.8) can be rewritten as the spectral problem

$$\mathcal{L}^c \begin{pmatrix} p \\ j \\ q^+ \end{pmatrix} = \lambda \begin{pmatrix} p \\ j \\ q^+ \end{pmatrix}, \qquad \begin{pmatrix} p \\ j \\ q^+ \end{pmatrix} \in \mathcal{D}(\mathcal{L}^c), \tag{1.9}$$

where $\mathcal{L}^c : \mathcal{D}(\mathcal{L}^c) \subset L^2(\mathbb{R};\mathbb{C}^3) \to L^2(\mathbb{R};\mathbb{C}^3)$ denotes the linear operator

$$\mathcal{L}^c = \begin{pmatrix} c\partial_z & -\partial_z & 0 \\ -s^2\partial_z + \mu s(2\bar{q}^+ - 1) & c\partial_z - \mu & 2\mu s\bar{p} \\ 0 & \frac{2\kappa}{s}\bar{q}^+(1-\bar{q}^+) & c\partial_z + \frac{2\kappa}{s}\bar{j}\,(1-2\bar{q}^+) \end{pmatrix}, \quad 0 < c < s,$$

acting on $L^2(\mathbb{R};\mathbb{C}^3)$ with dense domain $\mathcal{D}(\mathcal{L}^c) = H^1(\mathbb{R};\mathbb{C}^3) \subset L^2(\mathbb{R};\mathbb{C}^3)$.

Throughout the paper we shall denote by $\sigma(\mathcal{L})$ the *spectrum* of a closed, densely defined linear operator $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{X} \to \mathcal{X}$ with dense domain $D(\mathcal{L}) \subset \mathcal{X}$ on a Banach space $\mathcal{X}$. The spectrum of $\mathcal{L}$ is the complement of the so called *resolvent set* $\rho(\mathcal{L})$ of $\mathcal{L}$ that consists of all numbers $\lambda \in \mathbb{C}$ such that the operator $\mathcal{L} - \lambda\mathcal{I}$ has a bounded inverse. We say that a complex number $\lambda \in \sigma(\mathcal{L})$ is an eigenvalue of $\mathcal{L}$ if the kernel of $\mathcal{L} - \lambda\mathcal{I}$ is nontrivial. The spectrum of $\mathcal{L}$ breaks up into two disjoint sets, the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$, which is the set of all eigenvalues λ for which $\mathcal{L} - \lambda\mathcal{I}$ is a Fredholm operator with index zero, and the essential spectrum, defined by $\sigma_{\mathrm{ess}}(\mathcal{L}) = \sigma(\mathcal{L}) \setminus \sigma_{\mathrm{pt}}(\mathcal{L})$.

Definition 1.6. Let $c \in (0, s)$ be fixed. A traveling wave solution $(\bar{p}(z), \bar{j}(z), \bar{q}^+(z))$ of system (1.1) is *spectrally stable* if no element of the spectrum of $\mathcal{L}^c$ has strictly positive real part; that is

$$\sigma(\mathcal{L}^c) \cap \{\lambda \in \mathbb{C} \mid \Re e\,\lambda \geq 0\} = \{0\}\,.$$

Otherwise, we say that the traveling wave is *spectrally unstable*.

Recently, in a study on the stability of traveling waves in nonstrictly hyperbolic systems, Rottmann-Matthes [17] proved that spectral stability implies *linear stability*— the decay of the solutions of the linearized PDE-system around the wave profiles —. In a more recent work, restricting to linear first-order operators with constant coefficients, Rottmann-Matthes showed in [18] *orbital stability*— nonlinear asymptotic stability with asymptotic phase of a traveling wave —using linear estimates from [17]. The result holds under spectral assumptions like that zero is a simple eigenvalue and the existence of a *spectral gap*, that is, $\sigma(\mathcal{L}) \cap \{\Re e\,\lambda > -\delta\} = \{0\}$ for some $\delta > 0$.

The results of Rottmann-Matthes encourage us to believe that the present paper is the first step towards the orbital stability of traveling wave solutions to (1.1).

The paper is organized as follows: In Section 2 we establish that zero is an eigenvalue with associated one-dimensional eigenspace. The result is a consequence of the exponential convergence of the wave profiles to their equilibrium states and the Gap Lemma [23]. In Section 3 we find out that the continuum of traveling wave solutions is spectrally unstable. This instability is (at least) due to the intersection of the essential spectrum with the imaginary axis. The proof is based upon exponential dichotomies and Fredholm properties. In Section 4 we prove that a weighted space of the form $L^2_w(\mathbb{R};\mathbb{C}^3)$ with the same weight w for the three variables is not suitable to shift the essential spectrum to the stable half-plane $\{\Re e\,\lambda < 0\}$. We overcome this issue in Section 5 by finding a pair of weights $(w_\alpha, w_\varepsilon)$ such that on the two-weighted space $L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})$ the essential spectrum lies to the left of the imaginary axis. Furthermore, in Section 6 we prove that in such a weighted space, zero is a simple eigenvalue that belongs to the point spectrum. We conclude with a brief discussion in Section 7.

2. The eigenvalue $\lambda = 0$

In this section, we show that unlike the standing case, in which the dimension of the eigenspace associated to the zero eigenvalue is infinity (see [2]), in the case of traveling waves, zero is an eigenvalue whose eigenspace has dimension one for all wave speed $c \in (0, s)$.

Lemma 2.1. *For each wave speed $0 < c < s$, $\lambda = 0$ is an eigenvalue of $\mathcal{L}^c$ with a 1-dimensional eigenspace generated by $(\bar{p}_z, \bar{j}_z, \bar{q}_z^+)$.*

Proof. Set $\lambda = 0$ in (1.9), integration in $(-\infty, z)$ of the equation for p, together with the condition $(p, j, q^+) \in H^1(\mathbb{R}; \mathbb{C}^3)$, yields

$$j = cp. \tag{2.1}$$

By substitution of (2.1) into (1.9), and considering $\lambda = 0$, the problem (1.9) reduces to system

$$\begin{aligned} p_z &= \frac{\mu}{s^2 - c^2}\left((2s\bar{q}^+ - (c+s))p + 2s\bar{p}q^+\right), \\ q_z^+ &= -\frac{2\kappa}{s}(\bar{q}^+\left(1 - \bar{q}^+\right)p + \bar{p}\left(1 - 2\bar{q}^+\right)q^+); \end{aligned} \tag{2.2}$$

in the equation for q^+ we have used the invariant of motion (1.4). As a result of this reduction, the spectral problem (1.9) may have at most two linearly independent eigenfunctions associated to $\lambda = 0$. Thus, in order to find the solutions of (1.9) for such λ, all we need is to solve system (2.2). Next we show that $(\bar{p}_z, \bar{q}_z^+)$ and $(\bar{p}_{q_l^+}, \bar{q}^+_{q_l^+})$ are linearly independent solutions of (2.2).

We differentiate (1.5) with respect to z and rearrange the terms to obtain that $(\bar{p}_z, \bar{q}_z^+)$ is a solution of (2.2), that is,

$$\begin{aligned} \bar{p}_{zz} &= \frac{\mu}{s^2 - c^2}\left((2s\bar{q}^+ - (c+s))\bar{p}_z + 2s\bar{p}\bar{q}_z^+\right), \\ \bar{q}_{zz}^+ &= -\frac{2\kappa}{s}(\bar{q}^+\left(1 - \bar{q}^+\right)\bar{p}_z + \bar{p}\left(1 - 2\bar{q}^+\right)\bar{q}_z^+). \end{aligned} \tag{2.3}$$

Similarly, differentiating (1.5) partially with respect to the left state q_l^+, we find that $(\bar{p}_{q_l^+}, \bar{q}^+_{q_l^+})$ is a solution to system (2.2). Hence

$$\begin{aligned} \bar{p}_{q_l^+ z} &= \frac{\mu}{s^2 - c^2}\left((2s\bar{q}^+ - (c+s))\bar{p}_{q_l^+}, + 2s\bar{p}\bar{q}^+_{q_l^+}\right), \\ \bar{q}^+_{q_l^+ z} &= -\frac{2\kappa}{s}(\bar{q}^+\left(1 - \bar{q}^+\right)\bar{p}_{q_l^+} + \bar{p}\left(1 - 2\bar{q}^+\right)\bar{q}^+_{q_l^+}). \end{aligned}$$

To be sure that $(\bar{p}_z, \bar{q}_z^+)$ and $(\bar{p}_{q_l^+}, \bar{q}^+_{q_l^+})$ are linearly independent we have to check that their Wronskian is different from zero at least at one point.

We thus proceed as follows. Wang *et al.* [21] observed that for all left state $\theta^* < q_l^+ < 1$, the pulse $\bar{p}$ attains its maximum when $\bar{q}^+ = \theta^*$ (recall $\theta^* = \frac{c+s}{2s}$); since the traveling wave solutions are invariant under translations, we can assume that such values of $\bar{p}$ and $\bar{q}^+$ are reached at $z = 0$. We then have that $\bar{p}_z(0) = 0$ and $\bar{q}^+_{q_l^+}(0) = \partial_{q_l^+}\theta^* = 0$. On the other hand, Wang and coworkers further noted that the maximum of $\bar{p}$ is a increasing function of q_l^+, this means that $\bar{p}_{q_l^+}(0) = \partial_{q_l^+}\bar{p}_{\max} > 0$. Hence, from all these results and the fact that

$\bar{q}_z^+ < 0$ for all $z \in \mathbb{R}$, we have that the Wronskian at $z = 0$ is positive:

$$w(0) = \left[\bar{p}_z \bar{q}^+_{q_l^+} - \bar{p}_{q_l^+} \bar{q}_z^+\right]_{z=0} = -\bar{p}_{q_l^+}(0)\bar{q}_z^+(0) > 0.$$

Actually, $(\bar{p}_{q_r^+}, \bar{q}^+_{q_r^+})$ is also a solution of (2.2), but in view of Proposition 1.4 we can write

$$\frac{\partial \bar{p}}{\partial q_l^+} = \frac{\partial \bar{p}}{\partial q_r^+}\frac{\partial q_r^+}{\partial q_l^+} \quad \text{and} \quad \frac{\partial \bar{q}^+}{\partial q_l^+} = \frac{\partial \bar{q}^+}{\partial q_r^+}\frac{\partial q_r^+}{\partial q_l^+},$$

which we use to compute the Wronskian of $\{(\bar{p}_{q_l^+}, \bar{q}^+_{q_l^+}), (\bar{p}_{q_r^+}, \bar{q}^+_{q_r^+})\}$, obtaining that it is zero and consequently that this pair of solutions is linearly dependent.

Below we deduce that the associated eigenspace has a dimension equal to one. From (2.3) and the results of Lemma 1.5 on the exponential decay of $\bar{p}_z$ and $\bar{q}_z^+$, we have that

$$(\bar{p}_z, \bar{j}_z, \bar{q}_z^+) = (\bar{p}_z, c\bar{p}_z, \bar{q}_z^+) \in H^1(\mathbb{R}; \mathbb{C}^3).$$

Therefore, $\lambda = 0$ is an eigenvalue and $(\bar{p}_z, \bar{j}_z, \bar{q}_z^+)$ is an associated eigenfunction. To prove that this is the only eigenfunction we will show that $(\bar{p}_{q_l^+}, \bar{q}^+_{q_l^+})$ tends to a nonzero limit as $z \to \pm\infty$, which automatically leaves the function $(\bar{p}_{q_l^+}, \bar{j}_{q_l^+}, \bar{q}^+_{q_l^+}) = (\bar{p}_{q_l^+}, c\bar{p}_{q_l^+}, \bar{q}^+_{q_l^+})$ out of $L^2(\mathbb{R}; \mathbb{C}^3)$ and therefore out of $H^1(\mathbb{R}; \mathbb{C}^3)$.

Let us write (2.2) in the vector form

$$\mathbf{W}_z = \mathbb{A}(z)\,\mathbf{W}, \tag{2.4}$$

where $\mathbf{W} = (p, q^+)^t$ and

$$\mathbb{A}(z) = \begin{pmatrix} \frac{(2s\bar{q}^+ - (c+s))\mu}{s^2 - c^2} & \frac{2\mu s\bar{p}}{s^2-c^2} \\ -\frac{2\kappa\bar{q}^+(1-\bar{q}^+)}{s} & -\frac{2\kappa\bar{p}(1-2\bar{q}^+)}{s} \end{pmatrix}.$$

Define the parameters

$$\alpha_r := \lim_{z\to+\infty} \frac{(c+s-2s\bar{q}^+)\mu}{2(s^2-c^2)} = \frac{(c+s-2sq_r^+)\mu}{2(s^2-c^2)} > 0,$$
$$\alpha_l := \lim_{z\to-\infty} \frac{(c+s-2s\bar{q}^+)\mu}{2(s^2-c^2)} = \frac{(c+s-2sq_l^+)\mu}{2(s^2-c^2)} < 0.$$

Letting $z \to \pm\infty$, we obtain

$$\begin{aligned}\mathbb{A}_\pm &:= \lim_{z\to\pm\infty} \mathbb{A}(z) \\ &= \begin{pmatrix} -2\alpha_m & 0 \\ -\frac{2\kappa q_m^+(1-q_m^+)}{s} & 0 \end{pmatrix},\end{aligned}$$

where $m = r_q l$ at $\pm\infty$, respectively.

It turns out that $\mathbb{A}(z)$ approaches exponentially to its limits $\mathbb{A}_\pm$ as $z \to \pm\infty$, which is a direct consequence of the exponential convergence of the waves to their steady states. Hence, from the results in Lemma 1.5, we can obtain the exponential decay estimates

$$|\mathbb{A}(z) - \mathbb{A}_{\pm}| \leq Ce^{-2\alpha_m z}, \quad \text{for} \quad z \to \pm\infty, \tag{2.5}$$

with $\alpha_m = \alpha_{r,l}$ at $\pm\infty$, respectively.

This makes it possible to use the Gap Lemma [23] to establish a relation between the solutions of (2.4) and the solutions of the constant-coefficient system

$$\mathbf{Z}_z = \mathbb{A}_{\pm}\mathbf{Z}. \tag{2.6}$$

The idea consists of using such a relationship to deduce that the limit of $\bar{q}^+_{q_l^+}$ as $z \to \pm\infty$ must be nonzero.

The eigenvalues of $\mathbb{A}_{\pm}$ and their respective associated eigenvectors are

$$\begin{aligned} \nu_1^{\pm} &= 0, & \nu_2^{\pm} &= -2\alpha_m, \\ v_1^{\pm} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & v_2^{\pm} &= \begin{pmatrix} 1 \\ -\frac{2\kappa q_m^+(1-q_m^+)}{s\nu_2^{\pm}} \end{pmatrix}, \end{aligned}$$

with $m = r_q l$ at $z = \pm\infty$, respectively.

Then the solutions of (2.6) are given by

$$\mathbf{Z}_j^{\pm} = v_j^{\pm} e^{\nu_j^{\pm} z}, \quad j = 1, 2.$$

We now apply the Gap Lemma [23], according to which the existence of the uniform decay rates (2.5) implies that the system (2.4) has a set of solutions $\mathbf{W}_j^{\pm}(z)$, $j = 1, 2$, satisfying

$$\begin{aligned} \mathbf{W}_j^{-}(z) &= \left(v_j^- + \mathcal{O}\left(e^{-\alpha|x|}\left|v_j^-\right|\right)\right) e^{\nu_j^- z}, \quad (j = 1, 2) \quad z < 0, \\ \mathbf{W}_j^{+}(z) &= \left(v_j^+ + \mathcal{O}\left(e^{-\alpha|x|}\left|v_j^+\right|\right)\right) e^{\nu_j^+ z}, \quad (j = 1, 2) \quad z > 0, \end{aligned}$$

for all $\alpha < \min\{-2\alpha_l, 2\alpha_r\}$.

In other words, the Gap Lemma says that there exist solutions $\mathbf{W}_j^{\pm}(z)$, $j = 1, 2$, with the asymptotic limits

$$\lim_{z \to \pm\infty} \mathbf{W}_1^{\pm}(z) = (0, 1)^t \quad \text{and} \quad \lim_{z \to \pm\infty} \mathbf{W}_2^{\pm}(z) = (0, 0)^t. \tag{2.7}$$

Since $\bar{p}_z, \bar{q}_z^+ \to 0$ as $z \to \pm\infty$, we can conclude that $(\bar{p}_z, \bar{q}_z^+)^t$ in $z > 0$ is spanned by $\mathbf{W}_2^+(z)$, and is spanned by $\mathbf{W}_2^-(z)$ in $z < 0$. Thus, $(\bar{p}_z, \bar{q}_z^+)^t = \alpha_0 \mathbf{W}_2^+(z) = \beta_0 \mathbf{W}_2^-(z)$ for some nonzero constants $\alpha_0, \beta_0 \in \mathbb{C}$.

Regarding $(\bar{p}_{q_l^+}, \bar{q}^+_{q_l^+})^t$, it is spanned by $\left\{\mathbf{W}_1^+(z), \mathbf{W}_2^+(z)\right\}$ in $z > 0$, meanwhile it is spanned by $\left\{\mathbf{W}_1^-(z), \mathbf{W}_2^-(z)\right\}$ in $z < 0$. Then

$$(\bar{p}_{q_l^+}, \bar{q}^+_{q_l^+})^t = \begin{cases} \alpha_0^- \mathbf{W}_1^-(z) + \beta_0^- \mathbf{W}_2^-(z), & \text{for } z < 0, \\ \alpha_0^+ \mathbf{W}_1^+(z) + \beta_0^+ \mathbf{W}_2^+(z), & \text{for } z > 0, \end{cases} \tag{2.8}$$

for nonzero $\alpha_0^{\pm} \in \mathbb{C}$ and some $\beta_0^{\pm} \in \mathbb{C}$. From this and (2.7) we have that

$$\lim_{z \to \pm\infty} (\bar{p}_{q_l^+}, \bar{q}^+_{q_l^+})^t = \alpha_0^{\pm}(0, 1)^t, \quad \alpha_0^{\pm} \neq 0.$$

Thus $(\bar{p}_{q_l^+}, \bar{j}_{q_l^+}, \bar{q}^+_{q_l^+}) \notin H^1(\mathbb{R}, \mathbb{C}^3)$, and consequently $(\bar{p}_z, \bar{j}_z, \bar{q}^+_z)$ is the only eigenfunction associated to $\lambda = 0$. This proves Lemma 2.1. □

3. Spectrally unstable waves

We state the first main result of the paper.

Theorem 3.1. *For each fixed c satisfying* $0 < c < s$*, the corresponding continuum of traveling waves is spectrally unstable, that is,* $\sigma(\mathcal{L}^c) \cap \{\lambda \in \mathbb{C}\backslash\{0\} \mid \Re e\lambda \geq 0\} \neq \emptyset$.

To prove the theorem, we shall show that the essential spectrum touches the closed right-half complex plane.

3.1. The essential spectrum falls inside the unstable half plane

Following the classical approach by Alexander, Gardner, and Jones [1, 13, 19], let us recast (1.9) as a first-order ODE system

$$\mathbf{Y}_z = \mathbf{A}(z, \lambda)\mathbf{Y}, \tag{3.1}$$

with $\mathbf{Y} = (p, j, q^+)^t \in H^1(\mathbb{R}; \mathbb{C}^3)$ and

$$\mathbf{A}(z, \lambda) = \begin{pmatrix} \frac{1}{c^2-s^2}\left(-\mu s\left(2\bar{q}^+ - 1\right) + c\lambda\right) & \frac{1}{c^2-s^2}\left(\mu + \lambda\right) & -\frac{2\mu s \bar{p}}{c^2-s^2} \\ \frac{1}{c^2-s^2}\left(-c\mu s\left(2\bar{q}^+ - 1\right) + s^2\lambda\right) & \frac{c}{c^2-s^2}\left(\mu + \lambda\right) & -\frac{2c\mu s \bar{p}}{c^2-s^2} \\ 0 & -\frac{2\kappa\bar{q}^+\left(1-\bar{q}^+\right)}{cs} & -\frac{2\kappa\bar{j}\left(1-2\bar{q}^+\right)}{cs} + \frac{\lambda}{c} \end{pmatrix}, \quad 0 < c < s, \tag{3.2}$$

and consider the family of linear, closed, and densely defined operators

$$\begin{aligned} \mathcal{T}^c(\lambda) : \mathcal{D}(\mathcal{T}^c) = H^1(\mathbb{R}; \mathbb{C}^3) \subset L^2(\mathbb{R}; \mathbb{C}^3) &\to L^2(\mathbb{R}; \mathbb{C}^3), \\ \mathbf{Y} &\mapsto \mathbf{Y}_z - \mathbf{A}(z, \lambda)\mathbf{Y}, \quad \lambda \in \mathbb{C}. \end{aligned}$$

It is well documented (see [19] and the references therein) that $\mathcal{L}^c - \lambda\mathcal{I}$ and $\mathcal{T}^c(\lambda)$ are endued with the same Fredholm properties, and that consequently, their Fredholm indices match.

In order to determine the location of the essential spectrum, we begin by taking the limit as $|z| \to +\infty$ in (3.1). Thus we obtain the constant-coefficient limiting system

$$\mathbf{Y}_z = \mathbf{A}_\pm(\lambda)\mathbf{Y},$$

where

$$\mathbf{A}_\pm(\lambda) = \lim_{z \to \pm\infty} \mathbf{A}(z, \lambda).$$

These asymptotic matrices are

$$\mathbf{A}_\pm(\lambda) = \begin{pmatrix} (-\mu s(2q_m^+ - 1) + c\lambda)/(c^2 - s^2) & (\mu + \lambda)/(c^2 - s^2) & 0 \\ (-c\mu s(2q_m^+ - 1) + s^2\lambda)/(c^2 - s^2) & c\left(\mu + \lambda\right)/(c^2 - s^2) & 0 \\ 0 & -2\kappa q_m^+(1 - q_m^+)/cs & \lambda/c \end{pmatrix},$$

with $q_m^+ = q_{r,l}^+$ at $z = \pm\infty$, respectively.

It is a well-known fact that the location of the essential spectrum and the hyperbolicity of the asymptotic matrices $\mathbf{A}_\pm$ are linked.

Let $\mathbb{C}^+$ denote the open right-half complex plane $\{\lambda \in \mathbb{C} \mid \Re e\,\lambda > 0\}$, and let $\mathbb{E}^s_\pm(\lambda)$ and $\mathbb{E}^u_\pm(\lambda)$ denote the stable and unstable eigenspaces of $\mathbf{A}_\pm(\lambda)$, respectively.

Lemma 3.2. *For all $\lambda \in \mathbb{C}^+$, the asymptotic matrices $\mathbf{A}_\pm(\lambda)$ are hyperbolic, and* $\dim[\mathbb{E}^s_\pm(\lambda)] = 1$ *and* $\dim[\mathbb{E}^u_\pm(\lambda)] = 2$.

Proof. We follow closely Jin's ideas [12]. We start by proving that the matrices $\mathbf{A}_\pm(\lambda)$ have purely imaginary eigenvalues only if $\lambda \in \mathbb{C}\backslash\mathbb{C}^+$.

If ai, with $a \in \mathbb{R}$, is an eigenvalue of $\mathbf{A}_\pm(\lambda)$, it solves the equation

$$\det\left(\mathbf{A}_\pm(\lambda) - aiI\right) = 0.$$

It is straightforward to compute that the characteristic equation is

$$\left(\frac{\lambda}{c} - ai\right)\left(\lambda^2 + B(a)\lambda + C(a)\right) = 0, \tag{3.3}$$

where

$$B(a) = \mu - 2aci, \quad C(a) = a^2\left(s^2 - c^2\right) + a\mu\left(2sq_m^+ - (c+s)\right)i,$$

with $q_m^+ = q_{r,l}^+$ at $z = \pm\infty$, respectively.

Equation (3.3) has three roots

$$\lambda_1(a) = aci \quad \text{and} \quad \lambda_{2,3}(a) = \frac{-B(a) \pm \sqrt{D(a)}}{2},$$

where

$$D(a) = B^2(a) - 4C(a) = \mu^2 - 4a^2s^2 - 4a\mu s(2q_m^+ - 1)i.$$

It is clear that $\Re e\,\lambda_1(a) = 0$ for all $a \in \mathbb{R}$. This shows that $\mathbf{A}_\pm(\lambda)$ are nonhyperbolic matrices when λ is an element of the imaginary axis.

Now, taking the real part of $\lambda_{2,3}(a)$ yields

$$\Re e\,\lambda_{2,3}(a) = \left(-\Re e\,B(a) \pm \Re e\sqrt{D(a)}\right)/2.$$

Since $\Re e\,B(a) > 0$, then clearly $\Re e\,\lambda_3(a) < 0$ for all $a \in \mathbb{R}$. On the other hand, we observe that $\Re e\,\lambda_2(a) \le 0$ if and only if

$$(\Re e\,B(a))^2 \ge \left(\Re e\sqrt{D(a)}\right)^2 = \frac{1}{2}\left(\Re e\,D(a) + \sqrt{(\Re e\,D(a))^2 + (\Im m\,D(a))^2}\right).$$

After some calculation, we obtain that the above inequality is satisfied if and only if

$$(\Im m\,D(a))^2 + 4\,(\Re e\,B(a))^2\,\Re e\,D(a) - 4\,(\Re e\,B(a))^4 \le 0. \tag{3.4}$$

One can get that

$$\begin{aligned}&(\Im m\, D(a))^2 + 4\left(\Re e\, B(a)\right)^2 \Re e\, D(a) - 4\left(\Re e\, B(a)\right)^4 \\ &\qquad = (4a\mu s(2q_m^+ - 1))^2 + 4\mu^2\left(\mu^2 - 4a^2s^2\right) - 4\mu^4 = -64a^2\mu^2s^2q_m^+(1-q_m^+),\end{aligned}$$

since $0 < q_m^+ < 1$ for $q_m^+ = q_{r,l}^+$, it follows that inequality (3.4) is true for all $a \in \mathbb{R}$, which implies that $\Re e\, \lambda_2(a) \leq 0$ for all $a \in \mathbb{R}$. This, together with the fact that $\Re e\, \lambda_1(a) = 0$ for all $a \in \mathbb{R}$, lead us to conclude that whenever the asymptotic matrices $\mathbf{A}_\pm(\lambda)$ are nonhyperbolic, it holds that λ belongs to the closed left-half complex plane. Therefore the matrices $\mathbf{A}_\pm(\lambda)$ are hyperbolic in $\mathbb{C}^+$.

The next concern is to calculate $\dim[\mathbb{E}^s_\pm(\lambda)]$ and $\dim[\mathbb{E}^u_\pm(\lambda)]$ in $\mathbb{C}^+$, for this purpose we find the eigenvalues of $\mathbf{A}_\pm(\lambda)$, which are:

$$\begin{aligned}\eta_1^\pm(\lambda) &= \frac{\lambda}{c}, \\ \eta_{2,3}^\pm(\lambda) &= \frac{c(\mu+2\lambda) - \mu s(2q_m^+ - 1) \pm \sqrt{(c(\mu+2\lambda) - \mu s(2q_m^+ - 1))^2 + 4(s^2 - c^2)\lambda(\mu+\lambda)}}{2(c^2 - s^2)}.\end{aligned}$$

Clearly $\Re e \eta_1^\pm(\lambda) > 0$ for all $\lambda \in \mathbb{C}^+$. Set now $\lambda = \beta \in \mathbb{R}^+ = (0, +\infty)$, we have that $\eta_2^\pm(\beta) < 0$ and $\eta_3^\pm(\beta) > 0$, because $c < s$. Since $\eta_{2,3}^\pm(\lambda)$ are continuous functions and $\mathbb{C}^+$ is connected, their real part does not change sign on $\mathbb{C}^+$, and consequently $\Re e \eta_2^\pm(\lambda) < 0$ and $\Re e \eta_3^\pm(\lambda) > 0$ for all $\lambda \in \mathbb{C}^+$. Hence, $\dim[\mathbb{E}^s_\pm(\lambda)] = 1$ and $\dim[\mathbb{E}^u_\pm(\lambda)] = 2$ for all $\lambda \in \mathbb{C}^+$. □

Lemma 3.3. *The essential spectrum is a subset of the left-half complex plane and contains the imaginary axis.*

Proof. Let $\lambda \in \mathbb{C}^+$, from Lemma 3.2 we know that $\mathbf{A}_+(\lambda)$ and $\mathbf{A}_-(\lambda)$ are hyperbolic matrices, which, by Theorem 3.3 in [19] implies that equation (3.1) has exponential dichotomies in $\mathbb{R}^+$ and $\mathbb{R}^-$ with Morse indices $i_+(\lambda) = \dim[\mathbb{E}^u_+(\lambda)] = 2$ and $i_-(\lambda) = \dim[\mathbb{E}^u_-(\lambda)] = 2$, respectively. Hence, as a consequence of Lemma 4.2 in [15], this signifies that $\mathcal{T}^c(\lambda)$ is Fredholm with index zero, given by

$$\operatorname{ind} \mathcal{T}^c(\lambda) = i_-(\lambda) - i_+(\lambda) = 0.$$

Therefore, according to the definition of essential spectrum we have that the essential spectrum lies outside $\mathbb{C}^+$, that is, $\sigma_{\text{ess}}(\mathcal{T}^c) \subset \mathbb{C} \setminus \mathbb{C}^+$.

From the proof of Lemma 3.2 we have that $\mathbf{A}_\pm(\lambda)$ are nonhyperbolic for all $\lambda \in i\mathbb{R}$, thus, in view of Theorem 3.3 in [19], the equation (3.1) has no exponetial dichotomies nethier on $\mathbb{R}^+$ nor $\mathbb{R}^-$. Thus, due to Palmer's Theorem [16], $\mathcal{T}^c(\lambda)$ is not Fredholm, which lead us to conclude that $i\mathbb{R} \subset \sigma_{\text{ess}}(\mathcal{T}^c)$. □

Corollary 3.4. *The eigenvalue $\lambda = 0$ is embedded in the essential spectrum.*

In light of Theorem 3.1, none of the results of Rottmann-Matthes [17, 18] (see Sect. 1) can be used. Nevertheless, there is still the possibility of achieving spectral stability; the idea is to find a suitable weighted space in which the essential spectrum can be shifted to the left so that none of the elements of the imaginary axis belong to the essential spectrum. However, we emphasize that the existence of such space does not assure spectral stability since the point spectrum might reach the unstable half-plane $\{\Re e\, \lambda > 0\}$.

4. The unsuitable weighted space

The technique of weighted spaces introduced by Sattinger [20] involves choosing a weight function in such a way that the essential spectrum on the weighted space is contained in the stable half-plane $\{\Re e\, \lambda < 0\}$. Thus the natural idea would be to attempt to shift the essential spectrum to the open left-half of the complex plane in a weighted space

$$L^2_w(\mathbb{R};\mathbb{C}^3) = \left\{(p,j,q^+) \,|\, w(p,j,q^+) \in L^2(\mathbb{R};\mathbb{C}^3)\right\}, \tag{4.1}$$

wherein the operator $\mathcal{L}^c : L^2_w(\mathbb{R};\mathbb{C}^3) \to L^2_w(\mathbb{R};\mathbb{C}^3)$ has the domain

$$\mathcal{D}(\mathcal{L}^c) = H^1_w(\mathbb{R};\mathbb{C}^3) = \left\{(p,j,q^+) \,|\, w(\partial^i_z p, \partial^i_z j, \partial^i_z q^+) \in L^2(\mathbb{R};\mathbb{C}^3) \text{ for } i=0,1\right\} \subset L^2_w(\mathbb{R};\mathbb{C}^3),$$

with a weight function w satisfying the conditions

$$\frac{w_z}{w}(+\infty) = w_+ \quad \text{and} \quad \frac{w_z}{w}(-\infty) = w_-, \quad w_\pm \in \mathbb{R}.$$

But as a matter of fact, this is not achievable. We will show below that one single weight function is not capable of pushing the essential spectrum to the left of the imaginary axis. That means that a weighted space of the form (4.1), in which the same weight is assigned to the three variables, is not suitable to overcome the spectral instability due to the essential spectrum.

We proceed as follows. Suppose that $(p,j,q^+) \in H^1_w(\mathbb{R};\mathbb{C}^3)$ for some weight function w, then there exists a vector function $(\tilde{p},\tilde{j},\tilde{q}^+) \in H^1(\mathbb{R};\mathbb{C}^3)$ such that $(p,j,q^+) = w^{-1}(\tilde{p},\tilde{j},\tilde{q}^+)$. Upon substituting the latter into (1.9), we reach the eigenvalue problem

$$\begin{aligned}
\lambda\tilde{p} &= \tilde{p}_z - c\tilde{p}\frac{w_z}{w} - \tilde{j}_z + \tilde{j}\frac{w_z}{w},\\
\lambda\tilde{j} &= -s^2\tilde{p}_z + s^2\tilde{p}\frac{w_z}{w} + c\tilde{j}_z - c\tilde{j}\frac{w_z}{w} - \mu\tilde{j} + \mu s\left[\left(2\bar{q}^+ - 1\right)\tilde{p} + 2\bar{p}\tilde{q}^+\right],\\
\lambda\tilde{q}^+ &= c\tilde{q}^+_z - c\tilde{q}^+\frac{w_z}{w} + \frac{2\kappa}{s}\left[\bar{q}^+\left(1-\bar{q}^+\right)\tilde{j} + \bar{j}\left(1-2\bar{q}^+\right)\tilde{q}^+\right].
\end{aligned} \tag{4.2}$$

For notational convenience, the superscript tilde notation will be omitted on $\tilde{p}$, $\tilde{j}$ and $\tilde{q}^+$.

We recast (4.2) as the first order system

$$\mathbf{Y}_z = \mathbf{A}^w(z,\lambda)\mathbf{Y},$$

where $\mathbf{Y} = (p,j,q^+)^t \in H^1(\mathbb{R};\mathbb{C}^3)$ and

$$\mathbf{A}^w(z,\lambda) = \begin{pmatrix} \frac{w_z}{w} + \frac{1}{c^2-s^2}\left(-\mu s\left(2\bar{q}^+-1\right)+c\lambda\right) & \frac{1}{c^2-s^2}(\mu+\lambda) & -\frac{2\mu s\bar{p}}{c^2-s^2} \\ \frac{1}{c^2-s^2}\left(-c\mu s\left(2\bar{q}^+-1\right)+s^2\lambda\right) & \frac{w_z}{w} + \frac{c}{c^2-s^2}(\mu+\lambda) & -\frac{2c\mu s\bar{p}}{c^2-s^2} \\ 0 & -\frac{2\kappa\bar{q}^+\left(1-\bar{q}^+\right)}{cs} & \frac{w_z}{w} - \frac{2\kappa\bar{j}\left(1-2\bar{q}^+\right)}{cs} + \frac{\lambda}{c} \end{pmatrix}.$$

The asymptotic matrices are given by

$$\mathbf{A}^w_\pm(\lambda) = \begin{pmatrix} w_\pm + \frac{1}{c^2-s^2}\left(-\mu s\left(2q^+_m-1\right)+c\lambda\right) & \frac{1}{c^2-s^2}(\mu+\lambda) & 0 \\ \frac{1}{c^2-s^2}\left(-c\mu s\left(2q^+_m-1\right)+s^2\lambda\right) & w_\pm + \frac{c}{c^2-s^2}(\mu+\lambda) & 0 \\ 0 & -\frac{2\kappa q^+_m\left(1-q^+_m\right)}{cs} & w_\pm + \frac{\lambda}{c} \end{pmatrix},$$

with $q_m^+ = q_{r,l}^+$ at $\pm\infty$, respectively.

We find that λ is a root of the characteristic polynomial

$$\det\left(\mathbf{A}_{\pm}^{w}(\lambda) - aiI\right), \quad a \in \mathbb{R},$$

if and only if

$$\left(\frac{\lambda}{c} + w_{\pm} - ai\right)\left(\lambda^2 + \mathsf{b}(a)\lambda + \mathsf{c}(a)\right) = 0,$$

where

$$\mathsf{b}(a) = \mu + 2c(w_{\pm} - ai)$$

and

$$\mathsf{c}(a) = -w_{\pm}\mu(2sq_m^+ - (s+c)) + (c^2 - s^2)(w_{\pm}^2 - a^2) + (\mu(2sq_m^+ - (s+c)) - 2w_{\pm}(c^2 - s^2))ai.$$

As before, the characteristic equation has three roots

$$\lambda_1(a) = -w_{\pm}c + aci \quad \text{and} \quad \lambda_{2,3}(a) = \frac{-\mathsf{b}(a) \pm \sqrt{\mathsf{d}(a)}}{2},$$

where $\mathsf{d}(a) = \mathsf{b}^2(a) - 4\mathsf{c}(a)$.

Notice that, in order to shift the essential spectrum to the left, it is necessary that $w_{\pm} > 0$ and $\mathfrak{Re}\lambda_{2,3}(a) < 0$ for all $a \in \mathbb{R}$. Upon assuming the former, we have that $\mathfrak{Re}\lambda_{2,3}(a) < 0$ if and only if

$$\left(\mathfrak{Im}\,\mathsf{d}(a)\right)^2 + 4\left(\mathfrak{Re}\,\mathsf{b}(a)\right)^2 \mathfrak{Re}\,\mathsf{d}(a) - 4\left(\mathfrak{Re}\,\mathsf{b}(a)\right)^4 < 0, \quad \forall a \in \mathbb{R}. \tag{4.3}$$

One can derive that $\mathfrak{Im}\,\mathsf{d}(a) = -4as(\mu(2q_m^+ - 1) + 2sw_{\pm})$. By plugging this, together with $\mathfrak{Re}\,\mathsf{d}(a) = (\mathfrak{Re}\,\mathsf{b}(a))^2 - (\mathfrak{Im}\,\mathsf{b}(a))^2 - 4\mathfrak{Re}\,\mathsf{c}(a)$, into (4.3), we find that

$$\begin{aligned}
&\left(\mathfrak{Im}\,\mathsf{d}(a)\right)^2 - 4\left(\mathfrak{Re}\,\mathsf{b}(a)\right)^2\left((\mathfrak{Im}\,\mathsf{b}(a))^2 + 4\mathfrak{Re}\,\mathsf{c}(a)\right)\\
&\quad = 16a^2s^2(\mu(2q_m^+ - 1) + 2sw_{\pm})^2\\
&\qquad - 4(\mu + 2cw_{\pm})^2(4a^2c^2 + 4(-w_{\pm}\mu(2sq_m^+ - (s+c)) + (c^2 - s^2)(w_{\pm}^2 - a^2)))\\
&\quad = 16a^2s^2((\mu(2q_m^+ - 1) + 2sw_{\pm})^2 - (\mu + 2cw_{\pm})^2)\\
&\qquad + 16w_{\pm}(\mu + 2cw_{\pm})^2(\mu(2sq_m^+ - (s+c)) + w_{\pm}(s^2 - c^2)).
\end{aligned}$$

We observe that this quantity is negative for all $a \in \mathbb{R}$ if and only if

$$(\mu(2q_m^+ - 1) + 2sw_{\pm})^2 - (\mu + 2cw_{\pm})^2 < 0 \quad \text{and} \quad \mu(2sq_m^+ - (s+c)) + w_{\pm}(s^2 - c^2) < 0. \tag{4.4}$$

Now, see that the second inequality in (4.4) follows if and only if

$$w_{\pm}(s^2 - c^2) < -\mu(2sq_m^+ - (s+c)).$$

In the case $q_m^+ = q_l^+$ we have that $2sq_l^+ - (s+c) > 0$, since from Theorem 1.1 one has that $q_l^+ > (c+s)/2s$. Thus, it is impossible to satisfy the latter inequality since $w_-(s^2 - c^2)$ is positive and $-\mu(2sq_l^+ - (s+c))$ is negative. The analysis shows that one single weight function cannot satisfy all the necessary conditions so that the essential spectrum belongs to the stable half-plane.

In light of the preceding discussion, endeavor to stabilize the essential spectrum on a weighted space of the type of (4.1) is hopeless. Our proposal to fulfill the requirements to accomplish the desired stabilization is to employ simultaneously two different weights within a weighted space.

5. The stable essential spectrum

5.1. An appropriate weighted space

In this section, we move the essential spectrum to the left of the imaginary axis. To resolve the issue that the essential spectrum reaches the imaginary axis, we introduce the space

$$L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2)\times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})=\left\{(p,j,q^+)\,|\,w_\alpha(p,j)\in L^2(\mathbb{R};\mathbb{C}^2)\,\text{and}\,w_\varepsilon q^+\in L^2(\mathbb{R};\mathbb{C})\right\},$$

where the weight functions are $w_\alpha(z)=e^{\alpha_l z}+e^{\alpha_r z}$ and $w_\varepsilon(z)=e^{-\alpha_l\varepsilon z}$, with $\varepsilon>0$, $\varepsilon\ll 1$ sufficiently small.

Recall that

$$\alpha_l=\frac{(c+s-2sq_l^+)\mu}{2(s^2-c^2)}<0\quad\text{and}\quad\alpha_r=\frac{(c+s-2sq_r^+)\mu}{2(s^2-c^2)}>0.$$

Consider the operator $\mathcal{L}^c$ acting on $L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2)\times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})$, where

$$\mathcal{L}^c: L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2)\times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})\to L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2)\times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C}),$$

with domain

$$H^1_{w_\alpha}(\mathbb{R};\mathbb{C}^2)\times H^1_{w_\varepsilon}(\mathbb{R};\mathbb{C})=\left\{(p,j,q^+)\,|\,w_\alpha(\partial_z^i p,\partial_z^i j)\in L^2(\mathbb{R};\mathbb{C}^2)\,\text{and}\,w_\varepsilon\partial_z^i q^+\in L^2(\mathbb{R};\mathbb{C})\text{ for } i=0,1\right\}.$$

Let $(p,j,q^+)\in H^1_{w_\alpha}(\mathbb{R};\mathbb{C}^2)\times H^1_{w_\varepsilon}(\mathbb{R};\mathbb{C})$, since w_α'/w_α and $w_\varepsilon'/w_\varepsilon$ tend to finite limits as $z\to\pm\infty$, then $(p,j,q^+)=(w_\alpha^{-1}\tilde{p},w_\alpha^{-1}\tilde{j},w_\varepsilon^{-1}\tilde{q}^+)$ for some $(\tilde{p},\tilde{j},\tilde{q}^+)\in H^1(\mathbb{R};\mathbb{C}^3)$.

In the variables $\tilde{p}$, $\tilde{j}$ and $\tilde{q}^+$, the spectral problem (1.9) becomes

$$\begin{aligned}
\lambda\tilde{p}&=c\tilde{p}_z-c\tilde{p}\frac{w_\alpha'}{w_\alpha}-\tilde{j}_z+\tilde{j}\frac{w_\alpha'}{w_\alpha},\\
\lambda\tilde{j}&=-s^2\tilde{p}_z+s^2\tilde{p}\frac{w_\alpha'}{w_\alpha}+c\tilde{j}_z-c\tilde{j}\frac{w_\alpha'}{w_\alpha}-\mu\tilde{j}+\mu s\left[\left(2\overline{q}^+-1\right)\tilde{p}+2\overline{p}\tilde{q}^+\frac{w_\alpha}{w_\varepsilon}\right],\\
\lambda\tilde{q}^+&=c\tilde{q}^+_z+c\alpha_l\varepsilon\tilde{q}^++\frac{2\kappa}{s}\left[\overline{q}^+\left(1-\overline{q}^+\right)\tilde{j}\frac{w_\varepsilon}{w_\alpha}+\overline{j}\left(1-2\overline{q}^+\right)\tilde{q}^+\right].
\end{aligned}\tag{5.1}$$

We drop the tilde notation and rewrite system (5.1) as

$$\mathbf{Y}_z=\mathbf{A}^{\alpha,\varepsilon}(z,\lambda)\mathbf{Y},$$

where $\mathbf{Y} = (p, j, q^+)^t \in H^1(\mathbb{R}; \mathbb{C}^3)$ and

$$\mathbf{A}^{\alpha,\varepsilon}(z,\lambda) = \begin{pmatrix} \frac{w_\alpha'}{w_\alpha} + \frac{1}{c^2-s^2}\left(-\mu s\left(2\bar{q}^+ - 1\right) + c\lambda\right) & \frac{1}{c^2-s^2}(\mu+\lambda) & -\frac{2\mu s\bar{p}w_\alpha}{(c^2-s^2)w_\varepsilon} \\ \frac{1}{c^2-s^2}\left(-c\mu s\left(2\bar{q}^+ - 1\right) + s^2\lambda\right) & \frac{w_\alpha'}{w_\alpha} + \frac{c}{c^2-s^2}(\mu+\lambda) & -\frac{2c\mu s\bar{p}w_\alpha}{(c^2-s^2)w_\varepsilon} \\ 0 & -\frac{2\kappa\bar{q}^+\left(1-\bar{q}^+\right)w_\varepsilon}{csw_\alpha} & \frac{w_\varepsilon'}{w_\varepsilon} - \frac{2\kappa\bar{j}\left(1-2\bar{q}^+\right)}{cs} + \frac{\lambda}{c} \end{pmatrix}.$$

To calculate the limit of $\mathbf{A}^{\alpha,\varepsilon}(z,\lambda)$ as $z \to \pm\infty$, we will first show that

$$\lim_{z\to\pm\infty} \frac{w_\alpha}{w_\varepsilon}\bar{p} = 0 \quad \text{and} \quad \lim_{z\to\pm\infty} \frac{w_\varepsilon}{w_\alpha} = 0. \tag{5.2}$$

In fact, by Lemma 1.5 we have that

$$\bar{p} = \mathcal{O}(e^{-2\alpha_l z}), \quad z<0, \quad \text{and} \quad \bar{p} = \mathcal{O}(e^{-2\alpha_r z}), \quad z>0.$$

We therefore obtain

$$\frac{w_\alpha}{w_\varepsilon}\bar{p} = \begin{cases} \mathcal{O}(e^{-\alpha_l(1-\varepsilon)z} + e^{(-2\alpha_l(1-\varepsilon/2)+\alpha_r)z}), & z<0, \\ \mathcal{O}(e^{(\alpha_l(1+\varepsilon)-2\alpha_r)z} + e^{(\alpha_l\varepsilon-\alpha_r)z}), & z>0. \end{cases}$$

Hence, the first limit in (5.2) is true, because $-\alpha_l > 0$, $-\alpha_r < 0$ and $\varepsilon \ll 1$ is a small positive number.

Finally, through a direct computation we have

$$\frac{w_\varepsilon}{w_\alpha} = \frac{1}{e^{\alpha_l(1+\varepsilon)z} + e^{(\alpha_r+\alpha_l\varepsilon)z}}.$$

The second limit in (5.2) holds since $\alpha_r + \alpha_l\varepsilon > 0$ if $\varepsilon \ll 1$ is chosen small enough.

In view of (5.2), when $z \to \pm\infty$, $\mathbf{A}^{\alpha,\varepsilon}(z,\lambda)$ approaches to

$$\mathbf{A}^{\alpha,\varepsilon}_\pm(\lambda) = \begin{pmatrix} \alpha_m + \frac{1}{c^2-s^2}\left(-\mu s\left(2q_m^+ - 1\right) + c\lambda\right) & \frac{1}{c^2-s^2}(\mu+\lambda) & 0 \\ \frac{1}{c^2-s^2}\left(-c\mu s\left(2q_m^+ - 1\right) + s^2\lambda\right) & \alpha_m + \frac{c}{c^2-s^2}(\mu+\lambda) & 0 \\ 0 & 0 & -\alpha_l\varepsilon + \frac{\lambda}{c} \end{pmatrix},$$

where $m = r, l$ at $\pm\infty$, respectively.

The λ-roots of the characteristic polynomial

$$\det\left(\mathbf{A}^{\alpha,\varepsilon}_\pm(\lambda) - aiI\right), \quad a \in \mathbb{R},$$

satisfy

$$\left(\frac{\lambda}{c} - \alpha_l\varepsilon - ai\right)\left(\lambda^2 + b(a)\lambda + c(a)\right) = 0,$$

where $b(a) = \mu + 2c\alpha_m - 2aci$ and $c(a) = (s^2 - c^2)(\alpha_m^2 + a^2)$.

Such roots are

$$\lambda_1(a) = \alpha_l \varepsilon c + aci \quad \text{and} \quad \lambda_{2,3}(a) = \frac{-b(a) \pm \sqrt{d(a)}}{2},$$

where

$$d(a) = b^2(a) - 4c(a) = (\mu + 2c\alpha_m)^2 - 4\alpha_m^2(s^2 - c^2) - 4a^2s^2 - 4ac(\mu + 2c\alpha_m)i.$$

Notice that $\mathfrak{Re}\,\lambda_1(a)$ is negative for all $a \in \mathbb{R}$. Our concern is now to ensure that, for all $a \in \mathbb{R}$, $\mathfrak{Re}\,\lambda_{2,3}(a)$ is negative as well. To this end, we begin by asserting that $\mathfrak{Re}\,b(a) = \mu + 2c\alpha_m$ is positive. For $\alpha_m = \alpha_r$ this is obvious, since $\alpha_r > 0$. Regarding $\alpha_m = \alpha_l$, from the facts that $c < s$ and $q_l^+ < 1$, there holds

$$\mu + 2c\alpha_l = \mu + \frac{c(c + s - 2sq_l^+)\mu}{s^2 - c^2} > \mu + \frac{c(c + s - 2s)\mu}{s^2 - c^2} = \mu - \frac{c\mu}{s + c} > \mu - \frac{c\mu}{2c} = \frac{\mu}{2} > 0.$$

Thanks to the positivity of $\mathfrak{Re}\,b(a)$ we have that

$$\mathfrak{Re}\,\lambda_3(a) \leq \mathfrak{Re}\,\lambda_2(a), \quad \forall a \in \mathbb{R}.$$

Thus, it all reduces to show that $\mathfrak{Re}\,\lambda_2(a) < 0$ for all $a \in \mathbb{R}$. The real part of $\lambda_2(a)$ is determined by

$$\mathfrak{Re}\,\lambda_2(a) = \frac{1}{2}\left(-\mathfrak{Re}\,b(a) + \frac{1}{\sqrt{2}}\sqrt{\mathfrak{Re}\,d(a) + \sqrt{\left(\mathfrak{Re}\,d(a)\right)^2 + \left(\mathfrak{Im}\,d(a)\right)^2}}\right). \tag{5.3}$$

In what follows, we first prove that the quantity $\mathfrak{Re}\,d(a) + 4a^2s^2 = (\mu + 2c\mu\alpha_m)^2 - 4\alpha_m^2(s^2 - c^2)$ is positive. Secondly, from this fact we check that the inequality

$$\sqrt{\left(\mathfrak{Re}\,d(a)\right)^2 + \left(\mathfrak{Im}\,d(a)\right)^2} \leq \mathfrak{Re}\,d(a) + 8a^2s^2, \quad \forall a \in \mathbb{R}, \tag{5.4}$$

holds.

To show that $(\mu + 2c\alpha_m)^2 - 4\alpha_m^2(s^2 - c^2)$ is positive, we write

$$(\mu + 2c\alpha_m)^2 - 4\alpha_m^2(s^2 - c^2) = \left(\mu + 2c\alpha_m - 2\alpha_m\sqrt{s^2 - c^2}\right)\left(\mu + 2c\alpha_m + 2\alpha_m\sqrt{s^2 - c^2}\right).$$

It turns out that both factors are positive. Certainly, in the case $\alpha_m = \alpha_l$, the first factor is positive because $\mu + 2c\alpha_l > 0$ and $-\alpha_l > 0$. Concerning the second factor, we can see that

$$\begin{aligned} \mu + 2c\alpha_l + 2\alpha_l\sqrt{s^2 - c^2} &> \mu + 2c\alpha_l + 2\alpha_l s \\ &= \mu + 2\alpha_l(s + c) = \mu + \frac{(c + s - 2sq_l^+)\mu}{s - c} \\ &> \mu + \frac{(c + s - 2s)\mu}{s - c} = \mu - \frac{(s - c)\mu}{s - c} = 0. \end{aligned} \tag{5.5}$$

When $\alpha_m = \alpha_r$, the second factor is the sum of positive numbers. As regards the first factor, we derive

$$\begin{aligned} \mu + 2c\alpha_r - 2\alpha_r\sqrt{s^2-c^2} &> \mu + 2c\alpha_r - 2\alpha_r s \\ &= \mu - 2\alpha_r(s-c) = \mu - \frac{(c+s-2sq_r^+)\mu}{s+c} \\ &> \mu + \frac{2s\mu q_r^+}{s+c} - \mu = \frac{2s\mu q_r^+}{s+c} > 0. \end{aligned} \tag{5.6}$$

We now proceed to verify (5.4). The inequality is true if and only if

$$\begin{aligned} &(\mathfrak{Re}\, \boldsymbol{d}(a))^2 + (\mathfrak{Im}\, \boldsymbol{d}(a))^2 \le \left(\mathfrak{Re}\, \boldsymbol{d}(a) + 8a^2s^2\right)^2, \\ \iff \quad & (\mathfrak{Im}\, \boldsymbol{d}(a))^2 - 16a^2s^2\mathfrak{Re}\, \boldsymbol{d}(a) - 64a^4s^4 \le 0, \quad \forall a \in \mathbb{R}. \end{aligned} \tag{5.7}$$

Substitution of $\mathfrak{Re}\, \boldsymbol{d}(a)$ and $\mathfrak{Im}\, \boldsymbol{d}(a)$ yields after a straightforward calculation,

$$(\mathfrak{Im}\, \boldsymbol{d}(a))^2 - 16a^2s^2\mathfrak{Re}\, \boldsymbol{d}(a) - 64a^4s^4 = -16a^2(s^2-c^2)\left((\mu+2c\alpha_m)^2 - 4s^2\alpha_m^2\right).$$

Inequality (5.7) will follow as soon as we establish that $(\mu+2c\alpha_m)^2 - 4s^2\alpha_m^2 > 0$. This can be factored as

$$(\mu + 2\alpha_m(s+c))(\mu - 2\alpha_m(s-c)).$$

For $\alpha_m = \alpha_r$ the first factor is positive, and for $\alpha_m = \alpha_l$ the second factor is positive too. That $\mu + 2\alpha_l(s+c)$ and $\mu - 2\alpha_r(s-c)$ are positive has already been proved in (5.5) and (5.6), respectively. Then inequality (5.7) is true, therefore so is inequality (5.4).

We will use (5.4) in order to find a negative upper bound for $\mathfrak{Re}\,\lambda_2(a)$. Applying that inequality in (5.3) we get

$$\begin{aligned} \mathfrak{Re}\,\lambda_2(a) &< \frac{1}{2}\left(-\mathfrak{Re}\, \boldsymbol{b}(a) + \sqrt{\mathfrak{Re}\, \boldsymbol{d}(a) + 4a^2s^2}\ \right) \\ &= \frac{1}{2}\left(-(\mu+2c\mu\alpha_m) + \sqrt{(\mu+2c\mu\alpha_m)^2 - 4\alpha_m^2(s^2-c^2)}\ \right), \quad \forall a \in \mathbb{R}. \end{aligned}$$

The resulting bound is negative because $(\mu+2c\mu\alpha_m)^2 - 4\alpha_m^2(s^2-c^2) > 0$ is lower that $\mu + 2c\mu\alpha_m > 0$ for $\alpha_m = \alpha_{r,l}$. So we have that $\mathfrak{Re}\,\lambda_3(a) < \mathfrak{Re}\,\lambda_2(a) < 0$ for all $a \in \mathbb{R}$. Finally, we can choose $\varepsilon \ll 1$ sufficiently small in such a way that $\alpha_l\varepsilon c$, *i.e.* the real part of $\lambda_1(a)$, is larger than

$$\frac{1}{2}\left(-(\mu+2c\mu\alpha_m) + \sqrt{(\mu+2c\mu\alpha_m)^2 - 4\alpha_m^2(s^2-c^2)}\ \right).$$

We denote by Ω the open connected region in $\mathbb{C}$ bounded on the left by $\lambda_1(a)$, namely

$$\Omega = \{\lambda \in \mathbb{C} \,|\, \mathfrak{Re}\,\lambda > \alpha_l\varepsilon c\}.$$

To summarize, we have found that the parameter λ belongs to the complement of Ω whenever the matrices $\mathbf{A}_\pm^{\alpha,\varepsilon}(\lambda)$ are nonhyperbolic. That shows the first statement of the following lemma.

Lemma 5.1. *Let λ be an element of Ω. Then, the asymptotic matrices $\mathbf{A}_\pm^{\alpha,\varepsilon}(\lambda)$ are hyperbolic. Furthermore, the stable and unstable eigenspaces $\mathbb{E}_\pm^{\alpha,\varepsilon,s}(\lambda)$ and $\mathbb{E}_\pm^{\alpha,\varepsilon,u}(\lambda)$ have dimension 1 and 2, respectively.*

Proof. To prove the second statement, we need to compute the eigenvalues of the matrices $\mathbf{A}_{\pm}^{\alpha,\varepsilon}(\lambda)$. A direct calculation yields

$$\begin{aligned} \eta_1^{\alpha,\varepsilon\pm}(\lambda) &= -\alpha_l \varepsilon + \frac{\lambda}{c}, \\ \eta_{2,3}^{\alpha,\varepsilon\pm}(\lambda) &= -\frac{c\lambda}{s^2-c^2} \mp \sqrt{\left(\frac{c\lambda}{s^2-c^2} + \alpha_m\right)^2 + \frac{\lambda(\mu+\lambda)}{s^2-c^2}}, \end{aligned} \tag{5.8}$$

with $\alpha_m = \alpha_{r,l}$ at $\pm\infty$, respectively.

Since $-\alpha_l > 0$, we have that $\Re e \eta_1^{\alpha,\varepsilon\pm}(\lambda) > 0$ for all $\lambda \in \Omega$. Set now $\lambda = 0$ in (5.8), we get $\eta_{2,3}^{\alpha,\varepsilon-}(0) = \pm\alpha_l$ and $\eta_{2,3}^{\alpha,\varepsilon+}(0) = \mp\alpha_r$, thus we have that $\eta_2^{\alpha,\varepsilon\pm}(0) < 0$ and $\eta_3^{\alpha,\varepsilon\pm}(0) > 0$. Due to the fact that Ω is a connected set, the above is enough to conclude that $\Re e \eta_2^{\alpha,\varepsilon\pm}(\lambda) < 0$ and $\Re e \eta_3^{\alpha,\varepsilon\pm}(\lambda) > 0$ for all $\lambda \in \Omega$, so the proof of the lemma is complete. ☐

5.2. The essential spectrum on the two-weighted space

The question of finding the essential spectrum of the operator $\mathcal{L}^c : L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C}) \to L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})$ corresponds to the problem of determining the essential spectrum of the family of linear, closed and densely defined operators

$$\begin{aligned} \mathcal{T}^c(\lambda) : \mathcal{D}(\mathcal{T}^c) \subset L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C}) &\to L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C}) \\ \mathbf{Y} &\mapsto \mathbf{Y}_z - \mathbf{A}(z,\lambda)\mathbf{Y}, \quad \lambda \in \mathbb{C}, \end{aligned} \tag{5.9}$$

with domain $\mathcal{D}(\mathcal{T}^c) = H^1_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times H^1_{w_\varepsilon}(\mathbb{R};\mathbb{C})$.

To locate the region where the essential spectrum is contained, we consider the matrix

$$\mathbf{Q}(z) := \begin{pmatrix} \frac{1}{w_\alpha} & 0 & 0 \\ 0 & \frac{1}{w_\alpha} & 0 \\ 0 & 0 & \frac{1}{w_\varepsilon} \end{pmatrix},$$

and the family of linear, closed and densely defined operators

$$\begin{aligned} \mathcal{T}^c_{\alpha,\varepsilon}(\lambda) : \mathcal{D}(\mathcal{T}^c_{\alpha,\varepsilon}) \subset L^2(\mathbb{R};\mathbb{C}^3) &\to L^2(\mathbb{R};\mathbb{C}^3) \\ \tilde{\mathbf{Y}} &\mapsto \tilde{\mathbf{Y}}_z - \mathbf{A}^{\alpha,\varepsilon}(z,\lambda)\tilde{\mathbf{Y}}, \quad \lambda \in \mathbb{C}, \end{aligned}$$

with domain $\mathcal{D}(\mathcal{T}^c_{\alpha,\varepsilon}) = H^1(\mathbb{R};\mathbb{C}^3)$.

For $\tilde{\mathbf{Y}} \in H^1(\mathbb{R};\mathbb{C}^3)$ we substitute $\mathbf{Y} = \mathbf{Q}(z)\tilde{\mathbf{Y}} \in H^1_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times H^1_{w_\varepsilon}(\mathbb{R};\mathbb{C})$ in (5.9) to find the relation:

$$\mathcal{T}^c(\lambda)\mathbf{Y} = \mathbf{Q}(z)\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)\tilde{\mathbf{Y}}, \qquad \tilde{\mathbf{Y}} = \mathbf{Q}^{-1}(z)\mathbf{Y} \in H^1(\mathbb{R};\mathbb{C}^3). \tag{5.10}$$

Proposition 5.2. *Fix $c \in (0,s)$. Then the operator $\mathcal{T}^c(\lambda)$ in $L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})$ is Fredholm if and only if $\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)$ is, in which case their Fredholm indices agree.*

Proof. Assume that $\mathcal{T}^c(\lambda)$ is a Fredholm operator.

1. From (5.10) we infer that

$$\begin{aligned} &\mathrm{R}(\mathcal{T}^c(\lambda)) = \left\{\mathbf{Q}(z)\mathbf{f} \,|\, \mathbf{f} \in \mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))\right\} = \mathbf{Q}(z)\mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)), \quad \text{or, equivalently} \\ &\mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)) = \mathbf{Q}^{-1}(z)\mathrm{R}(\mathcal{T}^c(\lambda)). \end{aligned} \tag{5.11}$$

This implies that $\mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))$ must be closed. In effect, suppose by contradiction that $\{\mathbf{f}_n\} \subset \mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))$ is a sequence that converges to $\mathbf{f} \notin \mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))$. Because $\mathbf{f}_n = \mathbf{Q}^{-1}(z)\mathbf{g}_n$ for some $\mathbf{g}_n \in \mathrm{R}(\mathcal{T}^c(\lambda))$ for all $n \in \mathbb{N}$, we have that for each $\epsilon > 0$ there exists $N > 0$ such that

$$\begin{aligned}\epsilon > \|\mathbf{f}_n - \mathbf{f}\|_{L^2(\mathbb{R};\mathbb{C}^3)} &= \left\|\mathbf{Q}^{-1}(z)\left(\mathbf{g}_n - \mathbf{Q}(z)\mathbf{f}\right)\right\|_{L^2(\mathbb{R};\mathbb{C}^3)} \\ &= \|\mathbf{g}_n - \mathbf{Q}(z)\mathbf{f}\|_{L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2)\times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})}, \quad \text{whenever } n > N.\end{aligned}$$

This means that $\mathbf{g}_n \to \mathbf{Q}(z)\mathbf{f}$. Note that $\mathbf{Q}(z)\mathbf{f}$ belongs to $L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})$ but not to $\mathrm{R}(\mathcal{T}^c(\lambda))$; if it did, $\mathbf{f} = \mathbf{Q}^{-1}(z)\mathbf{Q}(z)\mathbf{f}$ would belong to $\mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))$. This leads to a contradiction, since $\mathrm{R}(\mathcal{T}^c(\lambda))$ contains all of its limit points due to fact it is closed inasmuch as $\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)$ is Fredhom; therefore we conclude that $\mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))$ is closed.

2. It also follows from (5.10) that

$$\begin{aligned}\ker(\mathcal{T}^c(\lambda)) &= \mathbf{Q}(z)\ker(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)), \quad \text{or, equivalently} \\ \ker(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)) &= \mathbf{Q}^{-1}(z)\ker(\mathcal{T}^c(\lambda)).\end{aligned} \tag{5.12}$$

From the bijective relation that (5.12) represents, it results that $\dim[\ker(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))] = \dim[\ker(\mathcal{T}^c(\lambda))] < \infty$. Let $\{\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n\}$ a basis of $\ker(\mathcal{T}^c(\lambda))$ and let $\tilde{\mathbf{Y}} \in \ker(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))$ be arbitrary. By (5.12), $\tilde{\mathbf{Y}} = \mathbf{Q}^{-1}(z)\mathbf{Y}$ for some $\mathbf{Y} \in \ker(\mathcal{T}^c(\lambda))$, hence

$$\begin{aligned}\tilde{\mathbf{Y}} = \mathbf{Q}^{-1}(z)\mathbf{Y} &= \mathbf{Q}^{-1}(z)(a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2 + \cdots + a_n\mathbf{Y}_n) \\ &= a_1\mathbf{Q}^{-1}(z)\mathbf{Y}_1 + a_2\mathbf{Q}^{-1}(z)\mathbf{Y}_2 + \cdots + a_n\mathbf{Q}^{-1}(z)\mathbf{Y}_n,\end{aligned}$$

for some constants $a_1, a_2, \ldots, a_n \in \mathbb{C}$. Thus

$$\mathrm{Span}(\ker(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))) = \mathbf{Q}^{-1}(z)\mathrm{Span}(\ker(\mathcal{T}^c(\lambda))).$$

3. Note in addition that

$$\begin{aligned}L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C}) &= \mathbf{Q}(z)L^2(\mathbb{R};\mathbb{C}^3), \quad \text{or, equivalently} \\ L^2(\mathbb{R};\mathbb{C}^3) &= \mathbf{Q}^{-1}(z)L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C}).\end{aligned}$$

4. Next we show that the codimension of $\mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))$ equals the codimension of $\mathrm{R}(\mathcal{T}^c(\lambda))$. To do so, we use (5.11) and (5.12) to see that

$$\begin{aligned}L^2(\mathbb{R};\mathbb{C})/\mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)) &= L^2(\mathbb{R};\mathbb{C}) + \mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)) \\ &= \mathbf{Q}^{-1}(z)\left(L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C}) + \mathrm{R}(\mathcal{T}^c(\lambda))\right),\end{aligned}$$

from which we get

$$\mathrm{Span}\left(L^2(\mathbb{R};\mathbb{C})/\mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)\right) = \mathbf{Q}^{-1}(z)\mathrm{Span}\left(L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})/\mathrm{R}(\mathcal{T}^c(\lambda))\right);$$

the statement can be proven analogously to point 2.
Therefore

$$\begin{aligned}\mathrm{codim}[\mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))] &= \dim\left[L^2(\mathbb{R};\mathbb{C})/\mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))\right] \\ &= \dim\left[L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})/\mathrm{R}(\mathcal{T}^c(\lambda))\right] \\ &= \mathrm{codim}[\mathrm{R}(\mathcal{T}^c(\lambda))] < \infty.\end{aligned}$$

5. It follows from all the previous points that $\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)$ is a Fredholm operator with index

$$\begin{aligned}\operatorname{ind}\mathcal{T}^c_{\alpha,\varepsilon}(\lambda) &= \dim[\ker(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))] - \operatorname{codim}[\mathrm{R}(\mathcal{T}^c_{\alpha,\varepsilon}(\lambda))] \\ &= \dim[\ker(\mathcal{T}^c(\lambda))] - \operatorname{codim}[\mathrm{R}(\mathcal{T}^c(\lambda))] = \operatorname{ind}\mathcal{T}^c(\lambda).\end{aligned}$$

The converse implication, that if $\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)$ is Fredholm then so is $\mathcal{T}^c(\lambda)$, follows analogously. □

This proposition leads us to the second main result of the paper.

Theorem 5.3. *The essential spectrum of $\mathcal{L}^c$ on the space $L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2)\times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})$ is contained in the stable complex half plane.*

Proof. We know that $\sigma_{\mathrm{ess}}(\mathcal{L}^c)$ is given by $\sigma_{\mathrm{ess}}(\mathcal{T}^c)$, and in turn, from Proposition 5.2, $\sigma_{\mathrm{ess}}(\mathcal{T}^c)$ in $L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2)\times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})$ is given by $\sigma_{\mathrm{ess}}(\mathcal{T}^c_{\alpha,\varepsilon})$. Therefore, the proof consists in finding the region where the essential spectrum of $\mathcal{T}^c_{\alpha,\varepsilon}$ resides.

With the same arguments as in the proof of Lemma 3.3 we infer from Lemma 5.1 that, for all $\lambda\in\Omega$, $\mathcal{T}^c_{\alpha,\varepsilon}(\lambda)$ is Fredholm with index zero, which comes from

$$\operatorname{ind}\mathcal{T}^c_{\alpha,\varepsilon}(\lambda) = i_-^{\alpha,\varepsilon}(\lambda) - i_+^{\alpha,\varepsilon}(\lambda) = 0.$$

This allows us to conclude that $\sigma_{\mathrm{ess}}(\mathcal{T}^c_{\alpha,\varepsilon})\subset\mathbb{C}\setminus\Omega$, proving the theorem. □

6. The eigenvalue $\lambda = 0$ on the two-weighted space

Following Flores and Plaza [5], and Sandstede [19], we express the matrix (3.2) as

$$\mathbf{A}(z,\lambda) = \mathbf{A}_0(z) + \lambda\mathbf{A}_1,$$

where

$$\mathbf{A}_0(z) = \begin{pmatrix} \frac{-\mu s(2\bar{q}^+-1)}{c^2-s^2} & \frac{\mu}{c^2-s^2} & -\frac{2\mu s\bar{p}}{c^2-s^2} \\ \frac{-c\mu s(2\bar{q}^+-1)}{c^2-s^2} & \frac{c\mu}{c^2-s^2} & -\frac{2c\mu s\bar{p}}{c^2-s^2} \\ 0 & -\frac{2\kappa\bar{q}^+(1-\bar{q}^+)}{cs} & -\frac{2\kappa\bar{j}(1-2\bar{q}^+)}{cs} \end{pmatrix}$$

and

$$\mathbf{A}_1 = \begin{pmatrix} \frac{c}{c^2-s^2} & \frac{1}{c^2-s^2} & 0 \\ \frac{s^2}{c^2-s^2} & \frac{c}{c^2-s^2} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}.$$

Definition 6.1. Consider $\lambda\in\sigma_{\mathrm{pt}}(\mathcal{T}^c)$. The maximal number of linearly independent eigenfunctions of $\mathcal{T}^c(\lambda)$ is called the geometric multiplicity of λ. Suppose that $\ker(\mathcal{T}^c(\lambda)) = \operatorname{Span}\{\mathbf{Y}_1\}$, the eigenvalue λ is said to have algebraic multiplicity m if there is a solution to

$$\mathcal{T}^c(\lambda)\mathbf{Y}_j = \mathbf{A}_1\mathbf{Y}_{j-1},$$

for each $j = 2, \dots, m$, with $\mathbf{Y}_j \in H^1_{w_\alpha} \times H^1_{w_\varepsilon}$, but there does not exist a $H^1_{w_\alpha} \times H^1_{w_\varepsilon}$ solution to

$$\mathcal{T}^c(\lambda)\mathbf{Y} = \mathbf{A}_1 \mathbf{Y}_m.$$

If $\lambda \in \sigma_{\rm pt}(\mathcal{T}^c)$ has geometric multiplicity ℓ, that is to say, $\ker(\mathcal{T}^c(\lambda)) = \mathrm{Span}\{\mathbf{Y}_1 \dots \mathbf{Y}_\ell\}$, the algebraic multiplicity is the sum of the algebraic multiplicities of a maximal set of linearly independent eigenfunctions of $\mathcal{T}^c(\lambda)$.

Further, we say that λ is simple if the geometric and algebraic multiplicities are equal to 1.

Lemma 6.2. *In $L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})$, $\lambda = 0$ is an element of $\sigma_{pt}(\mathcal{L}^c)$ and it is simple.*

Proof. Lemma 2.1 establishes that in $L^2(\mathbb{R};\mathbb{C}^3)$ the eigenspace associated to $\lambda = 0$ is spanned by the eigenfunction $(\bar{p}_z, \bar{j}_z, \bar{q}^+_z) \in H^1(\mathbb{R};\mathbb{C}^3)$. In addition, from Theorem 5.3, $\mathrm{ind}\,\mathcal{L}^c = \mathrm{ind}\,\mathcal{T}^c_{\alpha,\varepsilon}(0) = 0$ in $L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})$, thus, by proving that this function belongs to $H^1_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times H^1_{w_\varepsilon}(\mathbb{R};\mathbb{C})$, it will follow that $\lambda = 0$ is an element of $\sigma_{\rm pt}(\mathcal{L}^c)$ with geometric multiplicity 1 on the space of functions $L^2_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R};\mathbb{C})$.

Verification of $(\bar{p}_z, \bar{j}_z, \bar{q}^+_z) \in H^1_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times H^1_{w_\varepsilon}(\mathbb{R};\mathbb{C})$ requires showing that $w_\alpha \partial^i_z \bar{p}$ and $w_\varepsilon \partial^i_z \bar{q}^+$ belong to $L^2(\mathbb{R};\mathbb{C})$ for $i = 1, 2$. Once achieved that, we will have

$$\begin{aligned}\left\|(\bar{p}_z, \bar{j}_z, \bar{q}^+_z)\right\|^2_{H^1_{w_\alpha}(\mathbb{R};\mathbb{C}^2)\times H^1_{w_\varepsilon}(\mathbb{R};\mathbb{C})} &= \left(\|w_\alpha \bar{p}_z\|^2_{L^2(\mathbb{R};\mathbb{C})} + \|w_\alpha \bar{p}_{zz}\|^2_{L^2(\mathbb{R};\mathbb{C})}\right)(1 + c^2) \\ &\quad + \left\|w_\varepsilon \bar{q}^+_z\right\|^2_{L^2(\mathbb{R};\mathbb{C})} + \left\|w_\varepsilon \bar{q}^+_{zz}\right\|^2_{L^2(\mathbb{R};\mathbb{C})} < \infty.\end{aligned}$$

According to Lemma 1.5,

$$\bar{p}_z, \bar{q}^+_z = \mathcal{O}(e^{-2\alpha_l z}), \quad z < 0, \quad \text{and} \quad \bar{p}_z, \bar{q}^+_z = \mathcal{O}(e^{-2\alpha_r z}), \quad z > 0.$$

Then we get

$$w_\alpha \bar{p}_z = \begin{cases} \mathcal{O}(e^{-\alpha_l z} + e^{(\alpha_r - 2\alpha_l)z}), & z < 0, \\ \mathcal{O}(e^{(\alpha_l - 2\alpha_r)z} + e^{-\alpha_r z}), & z > 0, \end{cases} \quad \text{and} \quad w_\varepsilon \bar{q}^+_z = \begin{cases} \mathcal{O}(e^{-\alpha_l(2+\varepsilon)z}), & z < 0, \\ \mathcal{O}(e^{-(2\alpha_r + \alpha_l \varepsilon)z}), & z > 0. \end{cases} \tag{6.1}$$

This suffices to conclude that $w_\alpha \bar{p}_z$ and $w_\varepsilon \bar{q}^+_z$ are elements of $L^2(\mathbb{R};\mathbb{C})$, provided $\varepsilon \ll 1$ is small enough.

From (2.3), $w_\alpha \bar{p}_{zz}$ and $w_\varepsilon \bar{q}^+_{zz}$ are given by

$$\begin{aligned} w_\alpha \bar{p}_{zz} &= \frac{\mu}{s^2 - c^2}\left((2s\bar{q}^+ - (c+s)) w_\alpha \bar{p}_z + 2s w_\alpha \bar{p} \bar{q}^+_z\right), \\ w_\varepsilon \bar{q}^+_{zz} &= -\frac{2\kappa}{s}(\bar{q}^+\left(1 - \bar{q}^+\right) w_\varepsilon \bar{p}_z + \bar{p}\left(1 - 2\bar{q}^+\right) w_\varepsilon \bar{q}^+_z). \end{aligned} \tag{6.2}$$

Thus, to check that both belong to $L^2(\mathbb{R};\mathbb{C})$ we only need to verify that all the terms in the sums on the right-hand side of (6.2) belong to $L^2(\mathbb{R};\mathbb{C})$.

We can estimate

$$\left\|(2s\bar{q}^+ - (c+s)) w_\alpha \bar{p}_z\right\|_{L^2(\mathbb{R};\mathbb{C})} < (s + c)\left\|w_\alpha \bar{p}_z\right\|_{L^2(\mathbb{R};\mathbb{C})},$$

thanks to (6.1), $\|w_\alpha \bar{p}_z\|_{L^2(\mathbb{R};\mathbb{C})} < \infty$.

From the second equation in (1.5),

$$\left\|w_\alpha \bar{p} \bar{q}^+_z\right\|_{L^2(\mathbb{R};\mathbb{C})} = \frac{2\kappa}{s}\left\|w_\alpha \bar{p}^2 (1 - \bar{q}^+) \bar{q}^+\right\|_{L^2(\mathbb{R};\mathbb{C})} < \frac{2\kappa}{s} p_{\max} q^+_l \left\|w_\alpha \bar{p}\right\|_{L^2(\mathbb{R};\mathbb{C})};$$

applying Lemma 1.5, we derive

$$w_\alpha \bar{p} = \begin{cases} \mathcal{O}(e^{-\alpha_l z} + e^{(\alpha_r - 2\alpha_l)z}), & z < 0, \\ \mathcal{O}(e^{(\alpha_l - 2\alpha_r)z} + e^{-\alpha_r z}), & z > 0, \end{cases}$$

which guarantees that $\|w_\alpha \bar{p}\|_{L^2(\mathbb{R};\mathbb{C})}$ exists.

Similarly,

$$w_\varepsilon \bar{p}_z = \begin{cases} \mathcal{O}(e^{-\alpha_l(2+\varepsilon)z}), & z < 0, \\ \mathcal{O}(e^{-(2\alpha_r + \alpha_l \varepsilon)z}), & z > 0, \end{cases}$$

hence,

$$\left\| w_\varepsilon \bar{q}^+ (1 - \bar{q}^+) \bar{p}_z \right\|_{L^2(\mathbb{R};\mathbb{C})} < \| w_\varepsilon \bar{p}_z \|_{L^2(\mathbb{R};\mathbb{C})} < \infty.$$

Finally, we have the bound

$$\left\| w_\varepsilon \bar{p}(1 - 2\bar{q}^+) \bar{q}_z^+ \right\|_{L^2(\mathbb{R};\mathbb{C})} < \bar{p}_{\max} \left\| w_\varepsilon \bar{q}_z^+ \right\|_{L^2(\mathbb{R};\mathbb{C})}.$$

As a consequence of the asymptotic behaviour of $w_\varepsilon \bar{q}_z^+$, described in (6.1), the right-hand side converges. Hence, it is true that $w_\alpha \bar{p}_{zz}$, $w_\varepsilon \bar{q}_{zz}^+ \in L^2(\mathbb{R};\mathbb{C})$.

Concerning the algebraic multiplicity, we take $\lambda = 0$ in (1.8) and derive with respect to c. Then,

$$\begin{aligned} c\bar{p}_{cz} - \bar{j}_{cz} &= -\bar{p}_z, \\ -s^2 \bar{p}_{cz} + c\bar{j}_{cz} - \mu \bar{j}_c + \mu s \left(\left(2\bar{q}^+ - 1\right) \bar{p}_c + 2\bar{p}\bar{q}_c^+ \right) &= -\bar{j}_z, \\ cq_{cz}^+ + \frac{2\kappa}{s} \left(\bar{q}^+ \left(1 - \bar{q}^+\right) \bar{j}_c + \bar{j} \left(1 - 2\bar{q}^+\right) \bar{q}_c^+ \right) &= -\bar{q}_z^+. \end{aligned} \tag{6.3}$$

We write (6.3) as the equivalent system

$$\mathcal{T}^c(0)\mathbf{Y}_2 = \partial_x \mathbf{Y}_2 - \mathbf{A}(z,0)\mathbf{Y}_2 = \mathbf{A}_1 \mathbf{Y}_1,$$

where $\mathbf{Y}_1 = (\bar{p}_z, \bar{j}_z, \bar{q}_z^+)^t$ and $\mathbf{Y}_2 = -(\bar{p}_c, \bar{j}_c, \bar{q}_c^+)^t$.

Despite the fact that $\mathbf{Y}_2$ solves equation

$$\mathcal{T}^c(0)\mathbf{Y} = \mathbf{A}_1 \mathbf{Y}_1,$$

the Jordan chain ends with algebraic multiplicity 1, the reason is that $\mathbf{Y}_2$ does not belong to $L^2(\mathbb{R};\mathbb{C}^3)$, which means that it is not a member of the space $H^1_{w_\alpha}(\mathbb{R};\mathbb{C}^2) \times H^1_{w_\varepsilon}(\mathbb{R};\mathbb{C})$.

Certainly, we compute $\bar{q}_c^+$ through

$$\frac{\partial \bar{q}^+}{\partial c} = \frac{\partial \bar{q}^+}{\partial q_l^+} \frac{\partial q_l^+}{\partial q_r^+} \frac{\partial q_r^+}{\partial c}.$$

From Propositions 1.4 and 1.3, and by (2.8), $\bar{q}_c^+$ tends to a nonzero limit whose value depends on the wave speed and the end states. This leaves $\mathbf{Y}_2$ out of $L^2(\mathbb{R};\mathbb{C}^3)$. □

Remark 6.3. While the results in Sections 5 and 6 are encouraging, unstable point spectrum for the operator $\mathcal{L}^c$ is not ruled out on the unweighted, L^2, and the weighted, $L^2_{w_\alpha} \times L^2_{w_\varepsilon}$, function spaces. So far, energy methods (see *e.g.* [2, 5, 11]), Goodman's integrated variable [7] and Goodman's weighted energy method [8] (computations not shown here) have failed to conclude that there are no/are eigenvalues in the unstable half-plane $\{\Re e\lambda > 0\}$. The main issue is to control the size of quadratic forms acting on (p_z, j), (p, j) and (q^+, j). We leave the determination of the point spectrum as an open problem, perhaps in extensions of our work it would be worthwhile to pursue numerical computation of an Evans function (see *e.g.* [10, 11])— a complex-valued function analytic to the right of the essential spectrum, whose zeros correspond to the point spectrum —to figure out whether any unstable eigenvalue exists, and therefore whether the waves are spectrally unstable not only due to the location of essential spectrum.

7. Discussion

Our contribution is a deep study of the essential spectrum of a linearized operator around traveling wave solutions for a hyperbolic system. The traveling wave solutions of the M^5-model for 1D directed tissues are spectrally unstable in $L^2(\mathbb{R}; \mathbb{C}^3)$ (at least) because the essential spectrum of the linear operator $\mathcal{L}^c$ reaches the imaginary axis. Moreover, it turns out that one single weight function w is not enough to push the essential to the left of the imaginary axis. Or in other words, a weighted space of the form $L^2_w(\mathbb{R}; \mathbb{C}^3)$ is not an appropriate space of functions to search for spectral stability. Nonetheless, we were able to find such a pair of clever weight functions $(w_\alpha, w_\varepsilon)$ so that the operator $\mathcal{L}^c$ on $L^2_{w_\alpha}(\mathbb{R}; \mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R}; \mathbb{C})$ has essential spectrum contained in the stable complex half-plane $\{\Re e\lambda < 0\}$. As far as we know, this is the first time exploring the idea of using distinct weights simultaneously to remove the essential spectrum from the unstable half-plane. We think that this technique has the potential to achieve spectral stability in traveling waves that are spectrally unstable in both function spaces $\mathcal{X}$ and $\mathcal{X}_w$ (e.g. Jin. *et al.* [12], in the case of a conservation hyperbolic–parabolic system).

The eigenvalue $\lambda = 0$ is proven to be simple in $L^2_{w_\alpha}(\mathbb{R}; \mathbb{C}^2) \times L^2_{w_\varepsilon}(\mathbb{R}; \mathbb{C})$. However, spectral stability (or instability) is not concluded. Calculation of Evans function suggest a direction for further investigation.

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