RATIONAL WAVE SOLUTIONS TO A GENERALIZED (2+1)-DIMENSIONAL HIROTA BILINEAR EQUATION

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Abstract. A generalized form of (2+1)-dimensional Hirota bilinear (2D-HB) equation is considered herein in order to study nonlinear waves in fluids and oceans. The present goal is carried out through adopting the simplified Hirota's method as well as ansatz approaches to retrieve a bunch of rational wave structures from multiple soliton solutions to breather, rational, and complexiton solutions. Some figures corresponding to a series of rational wave structures are provided, illustrating the dynamics of the obtained solutions. The results of the present paper help to reveal the existence of rational wave structures of different types for the 2D-HB equation.

1. Introduction

One of the attractive subjects in the areas of mathematical physics is to look for rational wave solutions of differential equations. There are different rational wave solutions to differential equations; for example, multiple soliton solutions, breather solutions, rogue solutions, and so on. During the last decades, many effective methods such as inverse scattering method [4], Hirota's direct method [11], multiple exp-function method [1, 15, 21, 22], simplified Hirota's method [9, 10, 12, 27, 28, 30–33], and ansatz methods [2, 13, 25, 26, 29] have been exerted to find rational wave solutions of differential equations. More articles can be found in [3, 5–8, 16–19, 23, 24, 34, 35].

The basic goal of this paper is to study a generalized 2D-HB equation describing nonlinear waves in fluids and oceans as follows [14]

$$\phi_{yt} + c_1 \left(\phi_{xxxy} + 6\phi_x\phi_y + 3\phi_{xy}\phi + 3\phi_{xx} \int \phi_y \mathrm{d}x \right) + c_2\phi_{yy} = 0, \tag{1.1}$$

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or

$$\phi_{yt} + c_1 \left(\phi_{xxxy} + 6\phi_x \phi_y + 3\phi_{xy}\phi + 3\phi_{xx}\psi \right) + c_2\phi_{yy} = 0, \qquad \phi_y = \psi_x, \tag{1.2}$$

which is a generalization of the Hirota bilinear equation studied in [20]. The Hirota bilinear form corresponding to the above nonlinear model (1.1) is

$$B_{HBE}\ (\varphi) := \left(D_y D_t + c_1 D_x^3 D_y + c_2 D_y^2 \right) \varphi.\varphi = 2 \left(\varphi\varphi_{yt} - \varphi_y\varphi_t + c_1 \left(\varphi\varphi_{xxxy} - 3 \left(\varphi_{xxy}\varphi_x - \varphi_{xy}\varphi_{xx} \right) - \varphi_y\varphi_{xxx} \right) \right.$$
$$\left. +c_2 \left(\varphi\varphi_{yy} - \varphi_y^2 \right) \right) = 0, \tag{1.3}$$

under the specific transformation

$$\phi = 2(\ln \varphi\ )_{xx}.$$

Recently, Hua *et al.* investigated the interaction behavior associated with the above generalized 2D-HB equation in [14]. They constructed two types of interaction solutions through adopting different test functions as below

$$\varphi = g^2 + h^2 + ke^l + a_{12}, \qquad \text{and} \qquad \varphi = g^2 + h^2 + \cosh(l) + a_{12}, \tag{1.4}$$

where

$$g = a_1 x + a_2 y + a_3 t + a_4, \qquad h = a_5 x + a_6 y + a_7 t + a_8, \qquad l = a_9 x + a_{10} y + a_{11} t. \tag{1.5}$$

In order to advance the studies on the generalized 2D-HB equation (1.1); in this paper, the simplified Hirota's method as well as different ansatz approaches are utilized formally to retrieve a bunch of rational wave structures from multiple soliton solutions to breather, rational, and complexiton solutions.

2. The generalized 2D-HB equation and its rational waves solutions

In this section, a number of rational wave structures from multiple soliton solutions to breather, rational, and complexiton solutions for the generalized 2D-HB equation are derived using the simplified Hirota's method as well as ansatz approaches.

2.1. Multiple soliton solutions

In order to procure multiple soliton solutions of the generalized 2D-HB equation, first the expression

$$\phi = e^{\theta_i}, \quad \theta_i = \kappa_i x + \tau_i y - \varsigma_i t, \tag{2.1}$$

is substituted into the linear terms of the generalized 2D-HB equation; and then, the resulting equation is solved for obtaining the dispersion relation ς_i. This leads to

$$\varsigma_i = c_1 \kappa_i^3 + c_2 \tau_i, \tag{2.2}$$

and therefore, the phase variables $\theta_i, \ \ 1 \le i \le N$ can be written as

$$\theta_i = \kappa_i x + \tau_i y - \left(c_1 \kappa_i^3 + c_2 \tau_i \right) t, \quad 1 \le i \le N. \tag{2.3}$$

Now, by inserting the logarithmic transformations

$$\phi = R(\ln \varphi)_{xx}, \quad \psi = R(\ln \varphi)_{xy}, \tag{2.4}$$

into (1.2) where the auxiliary function $\varphi(x, y, t)$ is defined as

$$\varphi = 1 + e^{\kappa_1 x + \tau_1 y - \left(c_1 \kappa_1^3 + c_2 \tau_1\right)t}, \tag{2.5}$$

we find that $R = 2$.

Based on this result, a single soliton solution can be obtained as

$$\phi(x, y, t) = \frac{2\kappa_1^2 e^{\kappa_1 x + \tau_1 y - \left(c_1 \kappa_1^3 + c_2 \tau_1\right)t}}{1 + e^{\kappa_1 x + \tau_1 y - \left(c_1 \kappa_1^3 + c_2 \tau_1\right)t}} - \frac{2\kappa_1^2 e^{2\left(\kappa_1 x + \tau_1 y - \left(c_1 \kappa_1^3 + c_2 \tau_1\right)t\right)}}{\left(1 + e^{\kappa_1 x + \tau_1 y - \left(c_1 \kappa_1^3 + c_2 \tau_1\right)t}\right)^2}, \tag{2.6}$$

$$\psi(x, y, t) = \frac{2\kappa_1 \tau_1 e^{\kappa_1 x + \tau_1 y - \left(c_1 \kappa_1^3 + c_2 \tau_1\right)t}}{1 + e^{\kappa_1 x + \tau_1 y - \left(c_1 \kappa_1^3 + c_2 \tau_1\right)t}} - \frac{2\kappa_1 \tau_1 e^{2\left(\kappa_1 x + \tau_1 y - \left(c_1 \kappa_1^3 + c_2 \tau_1\right)t\right)}}{\left(1 + e^{\kappa_1 x + \tau_1 y - \left(c_1 \kappa_1^3 + c_2 \tau_1\right)t}\right)^2}. \tag{2.7}$$

For the double soliton solution, the following auxiliary function is considered

$$\varphi = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{2.8}$$

in which the phase variables θ_i, $i = 1, 2$ are as before and the phase shift a_{12} is an unknown. After performing some calculations, a double soliton solution is gained as

$$\phi = 2(\ln \varphi)_{xx}, \quad \psi = 2(\ln \varphi)_{xy}, \tag{2.9}$$

where the phase shift a_{12} is

$$a_{12} = \frac{\kappa_1 \tau_1 - \kappa_1 \tau_2 - \kappa_2 \tau_1 + \kappa_2 \tau_2}{\kappa_1 \tau_1 + \kappa_1 \tau_2 + \kappa_2 \tau_1 + \kappa_2 \tau_2}. \tag{2.10}$$

Finally, a triple soliton solution to the generalized 2D-HB equation is derived as

$$\phi = 2(\ln \varphi)_{xx}, \quad \psi = 2(\ln \varphi)_{xy}, \tag{2.11}$$

in which the auxiliary function, the phase shifts, and the phase variables are defined as

$$\varphi = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1 + \theta_2} + a_{13} e^{\theta_1 + \theta_3} + a_{23} e^{\theta_2 + \theta_3} + a_{12} a_{13} a_{23} e^{\theta_1 + \theta_2 + \theta_3}, \tag{2.12}$$

$$a_{ij} = \frac{\kappa_i \tau_i - \kappa_i \tau_j - \kappa_j \tau_i + \kappa_j \tau_j}{\kappa_i \tau_i + \kappa_i \tau_j + \kappa_j \tau_i + \kappa_j \tau_j}, \quad 1 \le i, j \le 3, \tag{2.13}$$

$$\theta_i = \kappa_i x + \tau_i y - \left(c_1 \kappa_i^3 + c_2 \tau_i\right) t, \quad 1 \le i \le 3. \tag{2.14}$$

It is easy to show that the following multiple complex soliton solutions to the generalized 2D-HB equation can be constructed

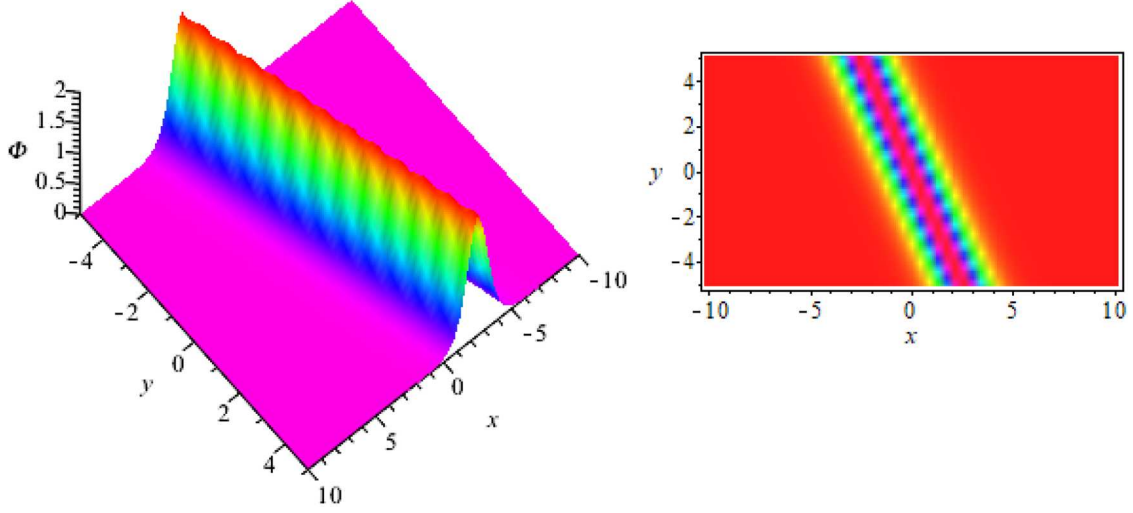$$\phi_{1,2,3} = 2(\ln \varphi_{1,2,3})_{xx}, \quad \psi_{1,2,3} = 2(\ln \varphi_{1,2,3})_{xy}, \tag{2.15}$$

FIGURE 1. Single soliton solution when $\kappa_1 = 1$, $\tau_1 = 2$, and $t = 0$.

in which

$$\varphi_1 = I + e^{\theta_1}, \qquad I = \sqrt{-1}, \tag{2.16}$$

$$\varphi_2 = I + e^{\theta_1} + e^{\theta_2} - Ia_{12}e^{\theta_1+\theta_2}, \qquad I = \sqrt{-1}, \tag{2.17}$$

$$\varphi_3 = I + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} - Ia_{12}e^{\theta_1+\theta_2} - Ia_{13}e^{\theta_1+\theta_3} - Ia_{23}e^{\theta_2+\theta_3} - a_{12}a_{13}a_{23}e^{\theta_1+\theta_2+\theta_3}, \qquad I = \sqrt{-1}, \tag{2.18}$$

and

$$a_{ij} = \frac{\kappa_i\tau_i - \kappa_i\tau_j - \kappa_j\tau_i + \kappa_j\tau_j}{\kappa_i\tau_i + \kappa_i\tau_j + \kappa_j\tau_i + \kappa_j\tau_j}, \qquad 1 \le i, j \le 3, \tag{2.19}$$

$$\theta_i = \kappa_i x + \tau_i y - \left(c_1\kappa_i^3 + c_2\tau_i\right)t, \qquad 1 \le i \le 3. \tag{2.20}$$

The plots of single, double, and triple soliton solutions have been provided in Figures 1–3, illustrating the dynamics of the multiple solutions.

Figure 1 shows a bright soliton wave whereas Figure 2 demonstrates the interaction of two bright soliton waves. Furthermore, the interaction of three bright soliton waves including two strong waves and one weak wave has been illustrated in Figure 3.

2.2. Breather and rational solutions

In the present subsection, first, the breather solution of the generalized 2D-HB equation is acquired by adopting a specific ansatz approach which is a combination of exponential functions and a trigonometric function as follows

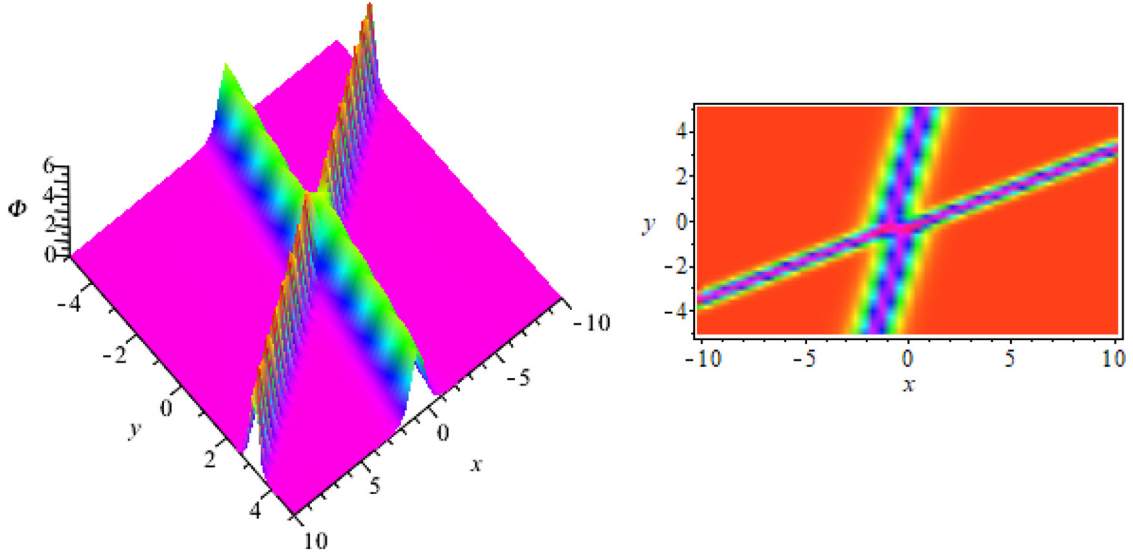$$\varphi = e^{-kX} + b_0 \cos(hY) + b_1 e^{kX}, \tag{2.21}$$

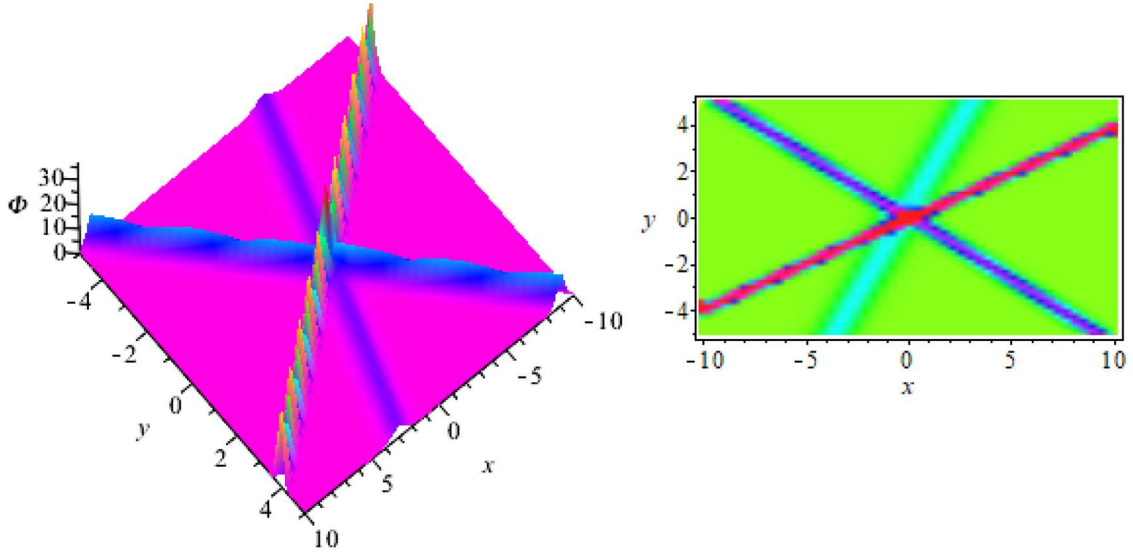FIGURE 2. Double soliton solution when $\kappa_1 = -3.5$, $\tau_1 = 10$, $\kappa_2 = 3$, $\tau_2 = -1$, and $t = 0$.

FIGURE 3. Triple soliton solution when $\kappa_1 = 3$, $\tau_1 = -2$, $\kappa_2 = -5$, $\tau_2 = -9$, $\kappa_3 = -8$, $\tau_3 = 20$, and $t = 0$.

in which

$$X = a_1 x + a_2 y + a_3 t + a_4, \qquad Y = a_5 x + a_6 y + a_7 t + a_8, \tag{2.22}$$

and k, b_0, h, b_1 and a_i, $1 \le i \le 8$ are constants to be computed. Inserting equation (2.21) into equation (1.3) and performing some calculations, yields the following nonlinear algebraic system

$$2h^4 a_5^3 a_6 b_0 c_1 - 6h^2 k^2 a_1^2 a_5 a_6 b_0 c_1 - 6h^2 k^2 a_1 a_2 a_5^2 b_0 c_1 + 2k^4 a_1^3 a_2 b_0 c_1 - 2h^2 a_6^2 b_0 c_2 + 2k^2 a_2^2 b_0 c_2 - 2h^2 a_6 a_7 b_0$$
$$+2k^2 a_2 a_3 b_0 = 0, \tag{2.23}$$

$$2h^4 a_5^3 a_6 b_0 b_1 c_1 - 6h^2 k^2 a_1^2 a_5 a_6 b_0 b_1 c_1 - 6h^2 k^2 a_1 a_2 a_5^2 b_0 b_1 c_1 + 2k^4 a_1^3 a_2 b_0 b_1 c_1 - 2h^2 a_6^2 b_0 b_1 c_2 + 2k^2 a_2^2 b_0 b_1 c_2$$
$$-2h^2 a_6 a_7 b_0 b_1 + 2k^2 a_2 a_3 b_0 b_1 = 0, \tag{2.24}$$

$$-6h^3 k a_1 a_5^2 a_6 b_0 b_1 c_1 - 2h^3 k a_2 a_5^3 b_0 b_1 c_1 + 2hk^3 a_1^3 a_6 b_0 b_1 c_1 + 6hk^3 a_1^2 a_2 a_5 b_0 b_1 c_1 + 4hk a_2 a_6 b_0 b_1 c_2 + 2hk a_2 a_7 b_0 b_1$$
$$+2hk a_3 a_6 b_0 b_1 = 0, \tag{2.25}$$

$$6h^3 k a_1 a_5^2 a_6 b_0 c_1 + 2h^3 k a_2 a_5^3 b_0 c_1 - 2hk^3 a_1^3 a_6 b_0 c_1 - 6hk^3 a_1^2 a_2 a_5 b_0 c_1 - 4hk a_2 a_6 b_0 c_2 - 2hk a_2 a_7 b_0 - 2hk a_3 a_6 b_0 = 0, \tag{2.26}$$

$$8h^4 a_5^3 a_6 b_0^2 c_1 + 32k^4 a_1^3 a_2 b_1 c_1 - 2h^2 a_6^2 b_0^2 c_2 - 2h^2 a_6 a_7 b_0^2 + 8k^2 a_2^2 b_1 c_2 + 8k^2 a_2 a_3 b_1 = 0, \tag{2.27}$$

whose solution is

$$a_3 = 3h^2 a_1 a_5^2 c_1 - k^2 a_1^3 c_1 - a_2 c_2, \quad a_7 = h^2 a_5^3 c_1 - 3k^2 a_1^2 a_5 c_1 - a_6 c_2, \quad b_1 = -\frac{a_5 h^2 a_6 b_0^2}{4a_1 k^2 a_2}. \tag{2.28}$$

Now, a breather solution is obtained as

$$\phi = 2(\ln \varphi)_{xx}, \quad \psi = 2(\ln \varphi)_{xy}, \tag{2.29}$$

where

$$\varphi = e^{-k(a_1 x + a_2 y + a_3 t + a_4)} + b_0 \cos(h(a_5 x + a_6 y + a_7 t + a_8)) + b_1 e^{k(a_1 x + a_2 y + a_3 t + a_4)}, \tag{2.30}$$

and a_3, a_7, and b_1 have been defined in (2.28).

The plots of the breather solution (2.29) have been depicted in Figure 4, demonstrating its dynamical behavior. It is evident that Figure 4 presents the interaction of bright and dark waves with a periodic property.

Now, the rational solution is obtained from the breather solution (2.29) in a specific way.

Remark 2.1. It should be noted that by setting $a_1 = a_5$, $a_2 = -a_6$, $b_0 = -2$, and $h = k$ in (2.29) and letting $k \to 0$, a rational solution to the generalized 2D-HB equation can be constructed as

$$\phi(x, y, t) = 8a_5^2 \left(\frac{1}{\mathcal{X}^2 + \mathcal{Y}^2} - \frac{(\mathcal{X} + \mathcal{Y})^2}{(\mathcal{X}^2 + \mathcal{Y}^2)^2} \right), \tag{2.31}$$

$$\psi(x, y, t) = \frac{8a_5 a_6 (\mathcal{X}^2 - \mathcal{Y}^2)}{(\mathcal{X}^2 + \mathcal{Y}^2)^2}, \tag{2.32}$$

in which $\mathcal{X} = a_5 x - a_6 y + c_2 a_6 t + a_4$ and $\mathcal{Y} = a_5 x + a_6 y - c_2 a_6 t + a_8$.

To show it, by considering $a_1 = a_5$, $a_2 = -a_6$, $b_0 = -2$, and $h = k$; it is obvious that the breather solution (2.29) can be written as

$$\phi(x, y, t) = \frac{2a_5^2 k^2 (\cosh(kX) + \cos(kY))}{\cosh(kX) - \cos(kY)} - \frac{2(a_5 k (\sinh(kX) + \sin(kY)))^2}{(\cosh(kX) - \cos(kY))^2}, \tag{2.33}$$

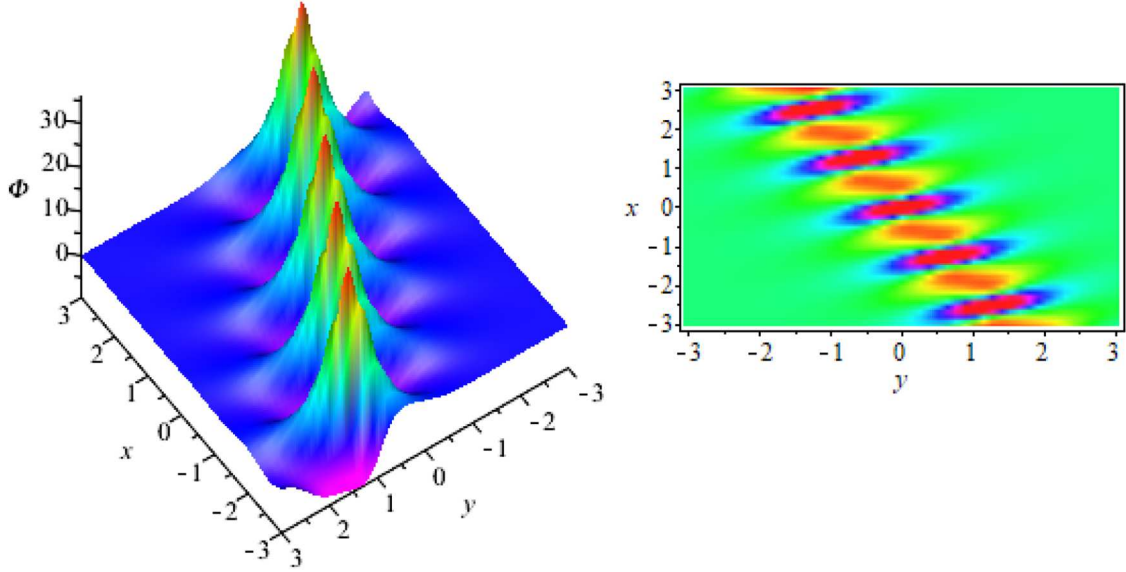$$\psi(x, y, t) = -2a_5 a_6 k^2 + \frac{2a_5 a_6 k^2 (\sinh^2(kX) - \sin^2(kY))}{(\cosh(kX) - \cos(kY))^2}. \tag{2.34}$$

FIGURE 4. Breather solution when $a_1 = 1$, $a_2 = 2$, $a_4 = 0$, $a_5 = -2$, $a_6 = 1$, $a_8 = 0$, $k = 1$, $h = -2$, $b_0 = -1$, and $t = 0$.

Applying the Taylor expansion results in

$$
\phi(x, y, t) = \frac{2a_5^2 k^2 \left(\left(1 + \frac{k^2 X^2}{2!} + \ldots\right) + \left(1 - \frac{k^2 Y^2}{2!} + \ldots\right)\right)}{\left(1 + \frac{k^2 X^2}{2!} + \ldots\right) - \left(1 - \frac{k^2 Y^2}{2!} + \ldots\right)}
$$
$$
- \frac{2\left(a_5 k\left(\left(kX + \frac{k^3 X^3}{3!} + \ldots\right) + \left(kY - \frac{k^3 Y^3}{3!} + \ldots\right)\right)\right)^2}{\left(\left(1 + \frac{k^2 X^2}{2!} + \ldots\right) - \left(1 - \frac{k^2 Y^2}{2!} + \ldots\right)\right)^2}, \tag{2.35}
$$

$$
\psi(x, y, t) = -2a_5 a_6 k^2 + \frac{2a_5 a_6 k^2 \left(\left(kX + \frac{k^3 X^3}{3!} + \ldots\right)^2 - \left(kY - \frac{k^3 Y^3}{3!} + \ldots\right)^2\right)}{\left(\left(1 + \frac{k^2 X^2}{2!} + \ldots\right) - \left(1 - \frac{k^2 Y^2}{2!} + \ldots\right)\right)^2}. \tag{2.36}
$$

Now, when k tends to zero; we find a rational solution as

$$
\phi(x, y, t) = 8a_5^2 \left(\frac{1}{\mathcal{X}^2 + \mathcal{Y}^2} - \frac{(\mathcal{X} + \mathcal{Y})^2}{(\mathcal{X}^2 + \mathcal{Y}^2)^2}\right), \tag{2.37}
$$

$$
\psi(x, y, t) = \frac{8a_5 a_6 (\mathcal{X}^2 - \mathcal{Y}^2)}{(\mathcal{X}^2 + \mathcal{Y}^2)^2}, \tag{2.38}
$$

in which $\mathcal{X} = a_5 x - a_6 y + c_2 a_6 t + a_4$ and $\mathcal{Y} = a_5 x + a_6 y - c_2 a_6 t + a_8$.

The above rational solution is plotted graphically in Figure 5 for $a_4 = 0$, $a_5 = -2$, $a_6 = 1$, $a_8 = 0$, and $t = 0$.
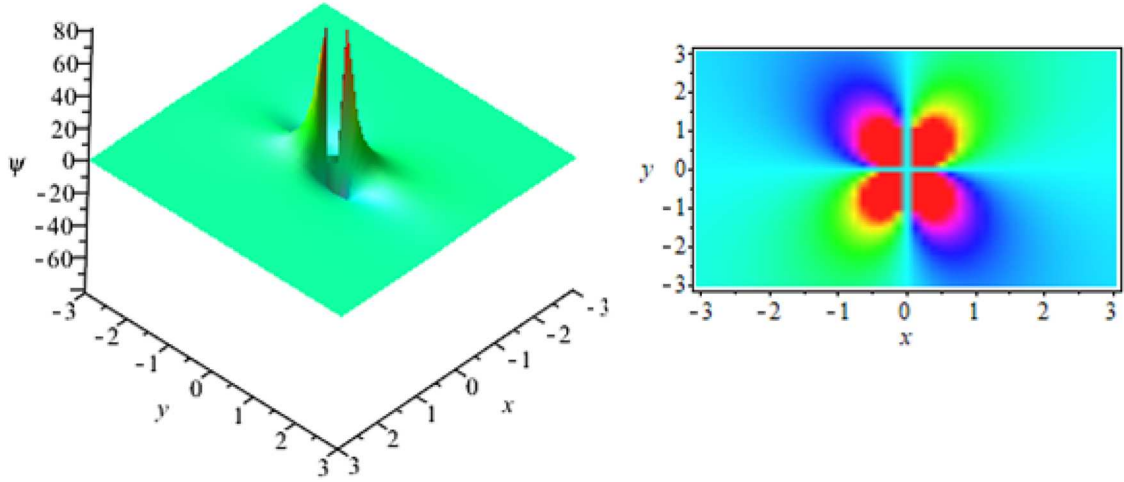
FIGURE 5. Rational solution when $a_4 = 0$, $a_5 = -2$, $a_6 = 1$, $a_8 = 0$, and $t = 0$.

2.3. Complexiton solutions

This subsection deals with complexiton solutions of the generalized 2D-HB equation. For this purpose, a test function is exerted as

$$\varphi = 1 + 2e^{\theta_1}\cos\left(\theta_2\right) + a_{12}e^{2\theta_1}, \tag{2.39}$$

in which

$$\theta_1 = k_1 x + r_1 y + w_1 t, \qquad \theta_2 = k_2 x + r_2 y + w_2 t, \tag{2.40}$$

and w_1 and w_2 satisfy

$$P\left(k, r, w\right) = P\left(\overline{k}, \overline{r}, \overline{w}\right) = 0, \quad k = k_1 + Ik_2, \quad r = r_1 + Ir_2, \quad w = w_1 + Iw_2, \quad P = yt + c_1 x^3 y + c_2 y^2, \quad I = \sqrt{-1}. \tag{2.41}$$

Here, the polynomial P has been defined owing to the Hirota operator (1.3). It is easy to prove that

$$w_1 = -c_1 k_1^3 + 3c_1 k_1 k_2^2 - c_2 r_1, \tag{2.42}$$

$$w_2 = c_1 k_2^3 - 3c_1 k_1^2 k_2 - c_2 r_2. \tag{2.43}$$

The unknown a_{12} also can be gained as

$$a_{12} = -\frac{P\left(2Ik_2, 2Ir_2, 2Iw_2\right)}{P\left(2k_1, 2r_1, 2w_1\right)} = -\frac{16c_1 k_2^3 r_2 - 4c_2 r_2^2 - 4r_2\left(-3c_1 k_1^2 k_2 + c_1 k_2^3 - c_2 r_2\right)}{16c_1 k_1^3 r_1 + 4c_2 r_1^2 + 2r_1\left(-2c_1 k_1^3 + 6c_1 k_1 k_2^2 - 2c_2 r_1\right)}. \tag{2.44}$$

Now, a complexiton solution to the generalized 2D-HB equation is obtained as

$$\phi = 2(\ln\varphi)_{xx}, \qquad \psi = 2(\ln\varphi)_{xy}, \tag{2.45}$$

in which

$$\varphi = 1 + 2e^{\theta_1}\cos\left(\theta_2\right) + a_{12}e^{2\theta_1}, \tag{2.46}$$

$$\theta_1 = k_1 x + r_1 y + \left(-c_1 k_1^3 + 3c_1 k_1 k_2^2 - c_2 r_1\right)t, \qquad \theta_2 = k_2 x + r_2 y + \left(c_1 k_2^3 - 3c_1 k_1^2 k_2 - c_2 r_2\right)t, \tag{2.47}$$

$$a_{12} = -\frac{16c_1 k_2^3 r_2 - 4c_2 r_2^2 - 4r_2\left(-3c_1 k_1^2 k_2 + c_1 k_2^3 - c_2 r_2\right)}{16c_1 k_1^3 r_1 + 4c_2 r_1^2 + 2r_1\left(-2c_1 k_1^3 + 6c_1 k_1 k_2^2 - 2c_2 r_1\right)}. \tag{2.48}$$

It is worth mentioning that the following complex complexiton solution to the generalized 2D-HB equation can be extracted

$$\phi = 2(\ln\varphi)_{xx}, \qquad \psi = 2(\ln\varphi)_{xy}, \tag{2.49}$$

in which

$$\varphi = I + 2e^{\theta_1}\cos\left(\theta_2\right) - Ia_{12}e^{2\theta_1}, \qquad I = \sqrt{-1}, \tag{2.50}$$

$$\theta_1 = k_1 x + r_1 y + \left(-c_1 k_1^3 + 3c_1 k_1 k_2^2 - c_2 r_1\right)t, \qquad \theta_2 = k_2 x + r_2 y + \left(c_1 k_2^3 - 3c_1 k_1^2 k_2 - c_2 r_2\right)t, \tag{2.51}$$

$$a_{12} = -\frac{16c_1 k_2^3 r_2 - 4c_2 r_2^2 - 4r_2\left(-3c_1 k_1^2 k_2 + c_1 k_2^3 - c_2 r_2\right)}{16c_1 k_1^3 r_1 + 4c_2 r_1^2 + 2r_1\left(-2c_1 k_1^3 + 6c_1 k_1 k_2^2 - 2c_2 r_1\right)}. \tag{2.52}$$

The above complexiton solution has been portrayed graphically in Figure 6 for suitable parameters.

2.4. Hyperbolic solutions

In this subsection, hyperbolic solutions of the generalized 2D-HB equation are obtained with the use of ansatz methods. To start, let's consider the solution of the generalized 2D-HB equation as

$$\phi = A_0 + A_1 \tanh\left(\kappa x + \tau y + \omega t\right)^2, \tag{2.53}$$

$$\psi = \frac{\tau}{\kappa}\left(A_0 + A_1 \tanh\left(\kappa x + \tau y + \omega t\right)^2\right). \tag{2.54}$$

By setting equations (2.53) and (2.54) in the system (1.2) and performing some calculations, we find

$$A_1 = -2\kappa^2, \quad \omega = 8c_1\kappa^3 - 6A_0 c_1 \kappa - c_2\tau. \tag{2.55}$$

Therefore, the following hyperbolic solution to the generalized 2D-HB equation is obtained

$$\phi = A_0 - 2\kappa^2 \tanh\left(\kappa x + \tau y + \left(8c_1\kappa^3 - 6A_0 c_1 \kappa - c_2\tau\right)t\right)^2, \tag{2.56}$$

$$\psi = \frac{\tau}{\kappa}\left(A_0 - 2\kappa^2 \tanh\left(\kappa x + \tau y + \left(8c_1\kappa^3 - 6A_0 c_1 \kappa - c_2\tau\right)t\right)^2\right). \tag{2.57}$$
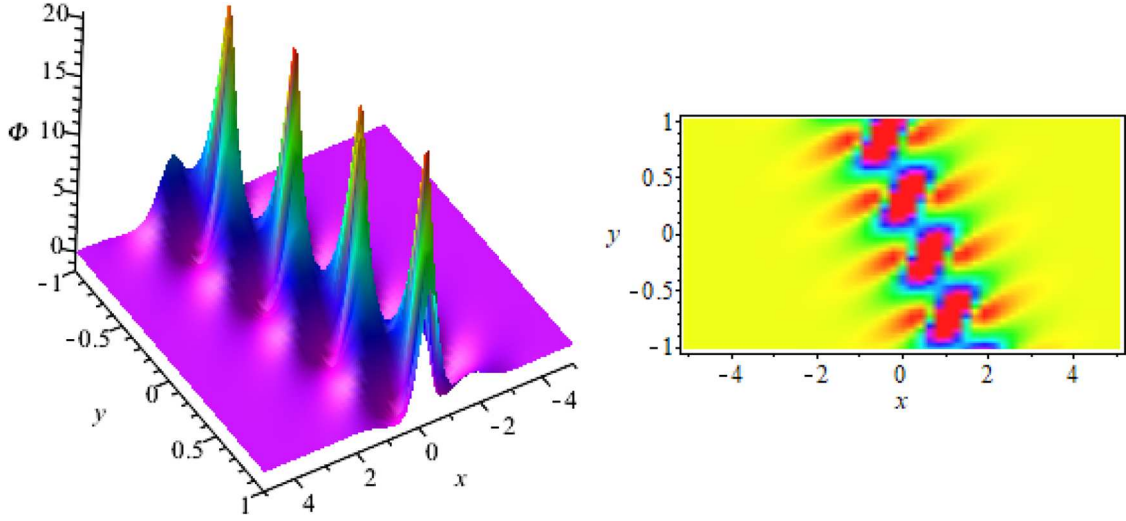
FIGURE 6. Complexiton solution when $k_1 = -2$, $k_2 = -2$, $r_1 = -2$, $r_2 = 10$, and $t = 0$.

It can be readily shown that another hyperbolic solution can be derived as

$$\phi = A_0 - 2\kappa^2 \coth\left(\kappa x + \tau y + \left(8c_1\kappa^3 - 6A_0 c_1 \kappa - c_2 \tau\right) t\right)^2 , \tag{2.58}$$

$$\psi = \frac{\tau}{\kappa}\left(A_0 - 2\kappa^2 \coth\left(\kappa x + \tau y + \left(8c_1\kappa^3 - 6A_0 c_1 \kappa - c_2 \tau\right) t\right)^2 \right). \tag{2.59}$$

Now, let us consider the following hyperbolic solution to the generalized 2D-HB equation

$$\phi = A_0 + A_1 \operatorname{sech}(\kappa x + \tau y + \omega t)^2 , \tag{2.60}$$

$$\psi = \frac{\tau}{\kappa}\left(A_0 + A_1 \operatorname{sech}(\kappa x + \tau y + \omega t)^2 \right). \tag{2.61}$$

Substituting equations (2.60) and (2.61) into the system (1.2) and performing some calculations, results in

$$A_1 = 2\kappa^2, \quad \omega = -4c_1\kappa^3 - 6A_0 c_1 \kappa - c_2 \tau. \tag{2.62}$$

Therefore, the following hyperbolic solution to the generalized 2D-HB equation is derived

$$\phi = A_0 + 2\kappa^2 \operatorname{sech}\left(\kappa x + \tau y + \left(-4c_1\kappa^3 - 6A_0 c_1 \kappa - c_2 \tau\right) t\right)^2 , \tag{2.63}$$

$$\psi = \frac{\tau}{\kappa}\left(A_0 + 2\kappa^2 \operatorname{sech}\left(\kappa x + \tau y + \left(-4c_1\kappa^3 - 6A_0 c_1 \kappa - c_2 \tau\right) t\right)^2 \right). \tag{2.64}$$

In a parallel manner, it can be demonstrated that

$$\phi = A_0 - 2\kappa^2 \operatorname{csch}\left(\kappa x + \tau y + \left(-4c_1\kappa^3 - 6A_0 c_1 \kappa - c_2 \tau\right) t\right)^2 , \tag{2.65}$$
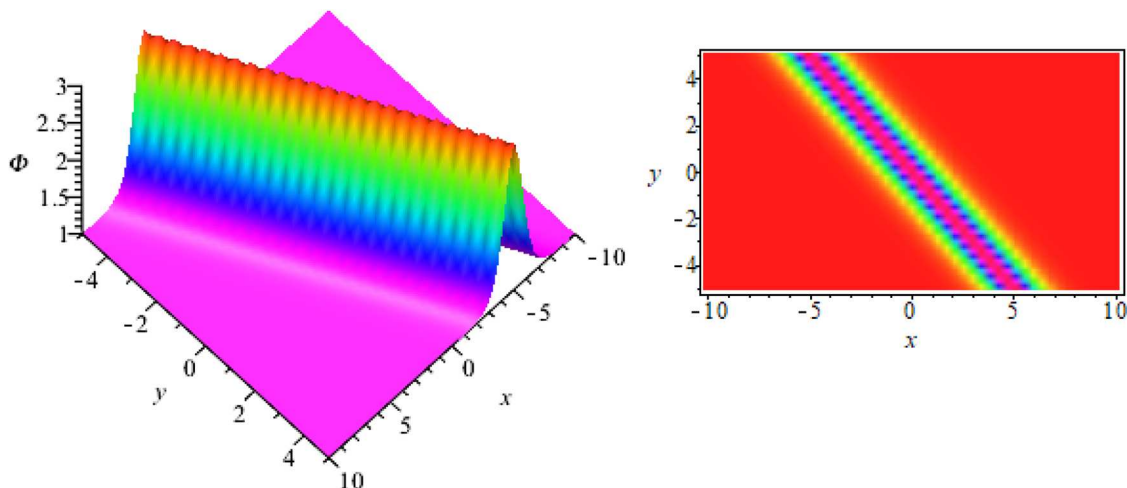
FIGURE 7. Hyperbolic solution (2.63) when $A_0 = 1$, $\kappa = 1$, $\tau = 1$, and $t = 0$.

$$\psi = \frac{\tau}{\kappa}\left(A_0 - 2\kappa^2\,\mathrm{csch}\!\left(\kappa x + \tau y + \left(-4c_1\kappa^3 - 6A_0c_1\kappa - c_2\tau\right)t\right)^2\right),\tag{2.66}$$

is the solution of the generalized 2D-HB equation.

The plots of the hyperbolic solution (2.63) have been provided in Figure 7. Obviously, Figure 7 shows a bright soliton wave solution.

Remark 2.2. Different classes of rational wave solutions have been reported in the present paper which definitely are different from the exact solutions of reference [14].

3. Concluding remarks

A generalized (2+1)-dimensional Hirota bilinear equation which describes nonlinear waves in fluids and oceans was comprehensively explored, in this paper. The current goal was performed by exerting the simplified Hirota's method and ansatz approaches as reliable techniques. Different classes of rational wave structures such as multiple soliton solutions, breather solutions, rational solutions, and complexiton solutions to the generalized 2D-HB equation were successfully obtained. Physical behaviors of a series of rational wave structures were demonstrated graphically. The results of the current paper certainly helped to reveal the existence of rational wave structures of different types for the 2D-HB equation.

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