

## ACTIVE CONTROL OF AN IMPROVED BOUSSINESQ SYSTEM

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**Abstract.** In this paper, optimal control of excessive water waves in a canal system, modeled by a nonlinear improved Boussinesq equation, is considered. For this aim, well-posedness and controllability properties of the system is investigated. Suppressing of the waves in the canal system is successfully obtained by means of optimally determining of canal depth control function *via* maximum principle, which transforms to optimal control problem to solving an nonlinear initial-boundary-terminal value problem. The beauty of the present paper than other studies existing in the literature is that optimal canal depth control function is analytically obtained without linearization of nonlinear term. In order to show effectiveness and robustness of the control actuation, several numerical examples are given by MATLAB in tables and graphical forms.

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### 1. INTRODUCTION

In recent years, nonlinear evolution equations have acquired great attentions due to their properties on modeling of real world problems and explaining some nonlinear phenomena. One of these evolution equations is called Boussinesq equation, which is introduced by Joseph Boussinesq for modeling of long waves of the surface of water with a small amplitude. The classical Boussinesq equation is given as follows;

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx} \quad (1.1)$$

in which  $u(t, x)$  is the elevation of the free surface of the water,  $\alpha, \beta$  are some constants. This equation was introduced in 1872 [5] and especially, over the last two decades, Boussinesq equation is studied in various aspects by different researchers. These studies can be summarized as follows but not limited to: In [28], Wazwaz investigated the logarithmic-Boussinesq equation for Gaussian solitary waves and derived the Gaussian solitary wave solutions for the logarithmic-regularized Boussinesq equation. In [23], Shakhmurov obtained the existence and uniqueness of solution of the integral boundary value problem for abstract Boussinesq equations. Global well-posedness and long time decay of the 3D Boussinesq equations presented in [16]. Also, small global solutions to the damped two-dimensional Boussinesq equations obtained in [1]. In [14], Li considered the maximum principle for an optimal control problem governed by Boussinesq equations including integral type state constraints. Analysis and approximation of linear feedback control problems for the Boussinesq equations are studied

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in [13]. In [15], authors proposed a high-order local discontinuous Galerkin method to solve improved Boussinesq equation, coupled with both explicit leap-frog and implicit midpoint energy-conserving time discretization. In [11], a finite time blow up result for the Cauchy problem of two-dimensional generalized Boussinesq-type equation with exponential type source term is considered. In [18], the limit behavior of solutions to the Cauchy problem for damped Boussinesq equation in the regime of small viscosity in  $\mathcal{R}^1$  is taken into account. In [9], improved global well-posedness for defocusing sixth-order Boussinesq equations is studied for the case when nonlinear term is cubic. In [8], Cauchy problem and vanishing dispersion limit for Schrodinger-improved Boussinesq equations is taken into account in  $\mathcal{R}^n$ . In [27], an edge numerical model was developed based on extended Boussinesq equations with the internal wave-generation method. In [17], oblique boundary feedback controller for stabilizing the equilibrium solutions to Boussinesq equations on a bounded and open domain in  $\mathcal{R}^2$  is designed. In [3], propagation of polymerization fronts with liquid monomer and liquid polymer is considered and the influence of vibrations on critical conditions of convective instability is studied. The model includes the heat equation, the equation for the concentration and the Navier-Stokes equations considered under the Boussinesq approximation. In [2], a derivation of approximate local conservation equations associated to the Kaup-Boussinesq system is studied and the spectral method is used to confirm the exact conservation of the total momentum and energy. The results on local well-posedness for the sixth-order Boussinesq equation are derived in [7]. In [6], damped infinite energy solutions of the 3D Euler and Boussinesq equations are presented. In [26], extended Boussinesq model to predict the propagation of waves in porous media is developed. The inertial and drag resistances are taken into account in the developed model. In [31], Fourier spectral approximation for the time fractional Boussinesq equation with periodic boundary condition is considered. In [29], longtime dynamics of a damped Boussinesq equation is investigated. In [25], following damped Boussinesq equation is introduced by Varlamov,

$$u_{tt} - 2\gamma u_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx} \quad (1.2)$$

in which  $\alpha, \beta, \gamma, t, x, u \in \mathcal{R}$  and dissipation is presented by second term on the left hand-side. By assuming some conditions on initial data, Varlamov achieved the classical solution of equation (1.2) and obtained long-time asymptotic of a damped Boussinesq equation by means of eigenfunction method. In [22], Eugene and Schneider took into account a water wave problem with surface tension and they derived following equation, in a class of Boussinesq equation, named improved Boussinesq equation;

$$u_{tt} = u_{xx} + u_{xxtt} + \mu u_{xxxx} - u_{ttxxxx} + (u^2)_{xx} \quad (1.3)$$

where  $t, x, \mu, u \in \mathcal{R}$ . Regarding Boussinesq equations and related studies, the books [10, 19, 24] provide a general overview. In this paper, a more general form of equation (1.3) is considered by taking the nonlinear term as  $\mathcal{N}(u)$  in stead of  $u^2$  and the system is taken into account as an optimal control problem by considering the optimally controlling of excessive water waves in a canal system, modeled by a nonlinear improved Boussinesq equation. For this aim, under some assumptions on nonlinear term and solution, wellposedness and controllability results of the system is discussed and uniqueness of the solution is given by a lemma. By means of Hilbert uniqueness method, the controllability of the study is also expressed. The performance index functional of the system to be minimized at predetermined terminal time by using minimum canal depth is considered as a sum of weighted dynamics response of the system and a functional of modified total canal depth control function. Suppressing of the waves in the canal is successfully obtained by means of optimally determining of canal depth control function *via* maximum principle, which transforms to optimal control problem to solving a nonlinear initial-boundary-terminal value problem. Specifically, The original contribution of this paper to literature is that optimal canal depth control function of the system is obtained without linearization of nonlinear term in the improved Boussinesq equation and several numerical examples are presented. Obtained results are simulated by means of MATLAB and given by tables and graphical forms for indicating the effectiveness of the introduced control actuation.

## 2. MATHEMATICAL FORMULATION OF THE PROBLEM

Consider that there are two lakes/two separate seas in a region. Due to several reasons, designers need to open a canal between two lakes/two separate seas. As estimated, this canal will have the some effects, simply, such as economically due to digging cost and physically due to excessive waves. Canal system, in Figure 1, which is fully filled with water and subjects to wind as an external excitation. In order to prevent excessive water waves and unnecessary cost, we need to optimally determine the depth of the canal. The main aim of the present control problem is to damp out the excessive water waves *via* optimal control of the canal depth. Let us consider the system of equation in general form as follows;

$$u_{tt} + \alpha(u_{ttxx} - u_{ttxxx}) + \beta u_{txx} + \gamma u_{xxxx} + u_{xx} + [\mathcal{N}(u)]_{xx} = f(t, x) + \mathcal{H}(t, x) \quad (2.1)$$

where state variable  $u$  is the elevation of the free surface of the water at  $(t, x) \in \Omega = \{t \in (0, t_f), x \in (0, \ell)\}$ ,  $t$  is time variable,  $t_f$  is predetermined terminal time,  $x$  is space variable,  $\ell$  is the length of the canal,  $\alpha$  is a constant in  $\mathcal{R}^+$ ,  $\beta > 0$  is internal damping constant,  $\gamma > 0$  is a constant depending on the depth of water,  $\mathcal{N}$  is a nonlinear function of  $u$ ,  $f$  is the external excitation function,  $\mathcal{H}(t, x) = \hbar(t)\theta(x)$  in which  $\hbar(t)$  is the optimal canal depth control function and  $\theta(x)$  is a function, affecting on canal depth function. Equation (2.1) is subject to the following boundary conditions

$$u(t, x) = 0, \quad u_{xx}(t, x) = 0 \quad \text{at} \quad x = 0, \ell \quad (2.2)$$

and initial conditions

$$u(t, x) = u_0(x), \quad u_t(t, x) = u_1(x) \quad \text{at} \quad t = 0. \quad (2.3)$$

Let us assume following on the solution;

$$\mathcal{N}, f, \mathcal{H} \quad \text{are continuous and bounded functions on} \quad \Omega, \quad (2.4a)$$

$$\mathcal{N} \quad \text{is a nonlinear nonincreasing function,} \quad (2.4b)$$

$$u, \frac{\partial^{i+j} u}{\partial t^j \partial x^i} \in L^2(\Omega), \quad j = 0, 1, 2, \quad i = 0, 1, \dots, 4 \quad (2.4c)$$

$$u_0(x) \in H^1(0, \ell) = \{u_0(x) \in L^2(0, \ell) : \frac{\partial u_0(x)}{\partial x} \in L^2(0, \ell)\}, \quad u_1(x) \in L^2(0, \ell), \quad (2.4d)$$

where  $L^2(\Omega)$  denote the Hilbert space of real-valued square-integrable functions defined in the domain  $\Omega$  in the Lebesgue sense with usual inner product and norm defined by

$$\|\rho\|^2 = \langle \rho, \rho \rangle, \quad \langle \rho, \eta \rangle_\Omega = \int_\Omega \rho \eta d\Omega.$$

With these assumptions, the system under consideration has a solution [30]. For obtaining the uniqueness of the solution, we need to consider two cases;  $\mathcal{N}$  is a nonlinear or linear function of  $u$ . For sake of shortness, let us assume the system has a unique solution in case  $\mathcal{N}$  is nonlinear and let us introduce the following lemma for obtaining the uniqueness of the solution in case  $\mathcal{N}$  is linear function of  $u$ .

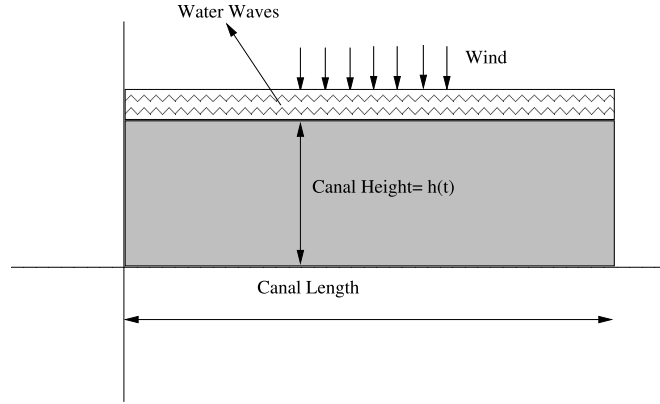


FIGURE 1. A Canal system.

**Lemma 2.1.** *Let  $u$  and  $u^\circ$  satisfy the system defined by equations (2.1)–(2.3) and corresponding control functions are  $\bar{h}(t)$  and  $\bar{h}^\circ(t)$ , respectively. Difference function is given by  $\Delta u(t, x) = u(t, x) - u^\circ(t, x)$  and it is clear that  $\Delta u(t, x)$  satisfies the following equation*

$$\Delta u_{tt} + \alpha(\Delta u_{ttxx} - \Delta u_{ttxxx}) + \beta\Delta u_{txx} + \gamma\Delta u_{xxx} + \Delta u_{xx} + \Delta[\mathcal{N}(u)]_{xx} = \Delta\mathcal{H}(t, x), \quad \Delta\mathcal{H}(t, x) = \theta(x)\Delta\bar{h}(t) \quad (2.5)$$

and following boundary conditions

$$\Delta u(t, x) = \Delta u_{xx}(t, x) = 0 \quad \text{at } x = 0, \ell \quad (2.6)$$

and initial conditions

$$\Delta u(t, x) = \Delta u_t(t, x) = 0 \quad \text{at } t = 0. \quad (2.7)$$

Then,

$$\int_0^\ell \Delta u_t^2(t_f, x) dx = \int_0^\ell \Delta u^2(t_f, x) dx = o(\varepsilon) \quad (2.8)$$

and

$$\iint_{\Omega} \Delta u^2(t, x) d\Omega = o(\varepsilon) \quad (2.9)$$

in which  $o(\varepsilon)$  is a quantity such that  $\lim_{\varepsilon \rightarrow 0} (o(\varepsilon)/|\varepsilon|) = 0$ .

*Proof.* Let  $(t_1, x_1), \dots, (t_p, x_p)$  be  $P$  arbitrary points in the open region  $\Omega$  and  $\varepsilon_j$  are the coefficients that the rectangles,  $R_j = [t_j, t_j + \sqrt{\varepsilon_j}] \times [x_j, x_j + \sqrt{\varepsilon_j}]$  are disjoint for  $1 \leq j \leq p$ . Let us define the following energy

integral [12],

$$\mathcal{E}(t) = \frac{1}{2} \int_0^\ell \left\{ (\Delta u_t^2) + \frac{\alpha}{2} \frac{\partial^2}{\partial x^2} (\Delta u_t^2) - \frac{\alpha}{2} \frac{\partial^4}{\partial x^4} (\Delta u_t^2) + \gamma \frac{\partial^4}{\partial x^4} (\Delta u^2) + \kappa \frac{\partial^2}{\partial x^2} (\Delta u) \right\} dx \quad (2.10)$$

Applying integration by parts and using boundary conditions given by equation (2.6), one observes equation (2.10) as follows;

$$\begin{aligned} \mathcal{E}(t) &= \int_0^t \frac{dE(\tau)}{d\tau} d\tau = \iint_{\Omega} \left\{ \Delta u_{tt} + \alpha [\Delta u_{ttxx} - \Delta u_{ttxxxx}] + \gamma \Delta u_{xxxx} + \kappa \Delta u_{xx} \right\} \Delta u_t d\Omega \\ &= \iint_{\Omega} \left\{ \Delta \mathcal{H}(t, x) - \beta \Delta u_{txx} \right\} \Delta u_t d\Omega \leq \iint_{\Omega} \Delta \mathcal{H}(t, x) \Delta u_t d\Omega \end{aligned}$$

Applying the Cauchy-Schwartz inequality to the space integral [4], following equality is obtained;

$$\mathcal{E}(t) = \int_0^t \left[ \int_0^\ell \Delta u_t^2 dx \right]^{1/2} \left[ \int_0^\ell \Delta \mathcal{H}^2(t, x) dx \right]^{1/2} d\tau \leq \int_0^t \mathcal{E}^{1/2}(\tau) \left[ \int_0^\ell \Delta \mathcal{H}^2(t, x) dx \right]^{1/2} d\tau.$$

Taking the sup of the both sides of the last integral leads to

$$\sup \mathcal{E}(t) \leq \sup \mathcal{E}^{1/2}(t) \left[ \int_0^\ell \Delta \mathcal{H}^2(t, x) dx \right]^{1/2} d\tau = \sup \mathcal{E}^{1/2}(t) \sum_{i=1}^P O(\varepsilon_i^{5/4}) \quad (2.11)$$

where  $O(r)$  is a quantity such that

$$\lim_{r \rightarrow 0^+} (O(r)/r) = \text{constant}.$$

By means of equation (2.11), the following inequality is observed for each  $t \in [0, t_f]$

$$0 \leq \mathcal{E}^{1/2}(t) \leq O(\varepsilon^{5/4}).$$

Because  $5/4 > 0$  [21], the following equality is obtained

$$\mathcal{E}(t) = o(\varepsilon). \quad (2.12)$$

Because the coefficients of equation (2.5) are bounded away from zero, the conclusion of the Lemma 2.1 is obtained from equation (2.12). By considering the result of Lemma 2.1 that

$$\lim_{\Delta \mathcal{H}(t, x) \rightarrow 0} \Delta u(t, x) = 0$$

. Hence, it is concluded that the system equations (2.1)–(2.3) has a unique solution.  $\square$

### 3. OPTIMAL CONTROL PROBLEM

The main aim of this study is optimally to determine the canal depth control function  $h(t)$  and minimize the dynamic response of the water waves system at a predetermined terminal time  $t_f$ . Before defining the performance index functional of the system, let us define the admissible canal depth control function set as follows;

$$\mathring{h}_{ad} = \{\mathring{h}(t) | \mathring{h} \in L^2(\Omega), \quad |\mathring{h}(t)| \leq \mathring{h}_0 < \infty\}. \quad (3.1)$$

Then, the performance index functional of the system is given by as follows;

$$\mathcal{J}(\mathring{h}(t)) = \int_0^\ell [\vartheta_1 u^2(t_f, x) + \vartheta_2 u_t^2(t_f, x)] dx + \vartheta_3 \int_0^{t_f} \mathring{h}^2(t) dt \quad (3.2)$$

in which  $\vartheta_1, \vartheta_2 \geq 0$ ,  $\vartheta_1 + \vartheta_2 \neq 0$  and  $\vartheta_3 > 0$  are weighting constants. First integral on the right-hand side in equation (3.2) represents the modified dynamics response of the water waves system. First and second terms in this integral are quadratic functional of the displacement and velocity of the water wave, respectively. Second term on the right-hand side in equation (3.2) is the measure of the total canal depth on the  $(0, t_f)$ . Then, optimal canal depth control problem is stated as follows;

$$\mathcal{J}(\mathring{h}^\circ(t)) = \min_{\mathring{h}(t) \in \mathring{h}_{ad}} \mathcal{J}(\mathring{h}(t)) \quad (3.3)$$

subject to the equation (2.1)–(2.3). In order to achieve the Maximum principle for obtaining optimal canal depth control function, let us introduce an adjoint variable  $w \in L^*$ ,  $L^*$  is the dual to  $L^2(\Omega)$  and has the same norm and inner product like in  $L^2(\Omega)$ . Adjoint system corresponding to equation (2.1)–(2.3) is expressed as follows;

$$w_{tt} + \alpha(w_{ttxx} - w_{ttxxxx}) - \beta w_{txx} + \gamma w_{xxxx} + w_{xx} = 0 \quad (3.4)$$

subjects to following boundary and terminal conditions, respectively;

$$w(t, x) = 0, \quad w_{xx}(t, x) = 0, \quad \text{at } x = 0, \ell \quad (3.5)$$

$$-2\vartheta_1 u(t, x) = w_t(t, x) + \alpha[w_{ttxx}(t, x) - w_{ttxxxx}(t, x)] - \beta w_{xx}(t, x), \quad (3.6a)$$

$$2\vartheta_2 u_t(t, x) = w(t, x) + \alpha[w_{xx}(t, x) - w_{xxxx}(t, x)] \quad \text{at } t = t_f. \quad (3.6b)$$

The existence and uniqueness of the solutions to adjoint system defined by equations (3.4)–(3.6) can be obtained similar to equations (2.1)–(2.3). Note that  $u$  is the unique solution to system given by equations (2.1)–(2.3). By considering the conclusion of Lemma 1, it is concluded that when  $u$  is unique solution to system given by equations (2.1)–(2.3), corresponding canal depth control function  $\mathring{h}(t)$  also must be unique for preserving the uniqueness of the solution to equations (2.1)–(2.3). Then, the system is called as observable. By keeping Hilbert Uniqueness method [20] in mind, it is concluded that observable is equal to controllable. Namely, the system equation (2.1)–(2.3) is controllable. Maximum principle is derived a necessary condition for the optimal control function in terms of Hamiltonian functional. In case of some convexity assumptions, satisfied by equation (3.2), on performance index functional of the system, maximum principle is also sufficient condition for optimal control function. Maximum principle is elegant tool for obtaining the optimal control function and maximum

principle sets up a direct relation between state variable and optimal control function *via* terminal conditions of adjoint system. Hence, maximum principle converts an optimal control problem to solving a system of equations, including state and adjoint variables, subjects to initial-boundary-terminal conditions. Let us derive the maximum principle as follows;

**Theorem 3.1.** [*Maximum principle*] *The maximization problem is stated as follows;*

$$\text{If } \mathcal{H}[t; u^\circ, w^\circ, \bar{h}^\circ(t)] = \max_{\bar{h}(t) \in \bar{h}_{ad}} \mathcal{H}[t; u, w, \bar{h}(t)] \quad (3.7)$$

in which, Hamiltonian function is defined by

$$\mathcal{H}[t; u, w, \bar{h}(t)] = \Psi(t) - \bar{h}(t)\Phi(t) - \vartheta_3 \bar{h}^2(t), \quad (3.8)$$

$$\Phi(t) = \int_0^\ell w(t, x)\theta(x)dx, \quad \Psi(t) = \int_0^\ell w(t, x)[\mathcal{N}(u)]_{xx}dx$$

then

$$\mathcal{J}[\bar{h}^\circ(t)] \leq \mathcal{J}[\bar{h}(t)], \quad \forall \bar{h}(t) \in \bar{h}_{ad}. \quad (3.9)$$

where  $\bar{h}^\circ(t)$  is the optimal canal depth control function.

*Proof.* Before giving the proof, let us introduce an operator as follows;

$$\varphi(u) = \varphi_1(u) + \varphi_2(u) + \varphi_3(u), \quad (3.10a)$$

$$\varphi_1(u) = u_{tt} + \alpha(u_{ttxx} - u_{ttxxx}), \quad \varphi_2(u) = \beta u_{txx}, \quad (3.10b)$$

$$\varphi_3(u) = \gamma u_{xxxx} + u_{xx}, \quad (3.10c)$$

and its adjoint operator as follows;

$$\varphi^*(w) = \varphi_1(w) - \varphi_2(w) + \varphi_3(w) \quad (3.11)$$

The deviations in the state variable and its derivatives with respect to the time variable are defined by

$$\Delta u = u - u^\circ, \quad \Delta u_t = u_t - u_t^\circ.$$

The operator

$$\varphi(\Delta u) = \Delta \mathcal{H}(t, x) - \Delta \varpi, \quad \Delta \varpi = [N(u)]_{xx} - [N(u^\circ)]_{xx}$$

is subject to the following boundary and initial conditions, respectively;

$$\Delta u(t, x) = \Delta u_{xx}(t, x) = 0 \quad \text{at } x = 0, \ell \quad (3.12)$$

and initial conditions

$$\Delta u(t, x) = \Delta u_t(t, x) = 0 \quad \text{at} \quad t = 0. \quad (3.13)$$

Let us take into account the following functional

$$\iint_{\Omega} \left\{ w\varphi(\Delta u) - \Delta u\varphi^*(w) \right\} d\Omega = \iint_{\Omega} \left\{ w[\Delta \mathcal{H}(t, x) - \Delta \varpi] \right\} d\Omega \quad (3.14)$$

Focus on the integral on the left side of equation (3.14);

$$\begin{aligned} & \iint_{\Omega} \{w\varphi(\Delta u) - \Delta u\varphi^*(w)\} d\Omega \\ &= \iint_{\Omega} \{w[\varphi_1(\Delta u) + \varphi_2(\Delta u) + \varphi_3(\Delta u) + \varphi_4(\Delta u)] - \Delta u[\varphi_1^*(w) - \varphi_2^*(w) + \varphi_3^*(w)]\} d\Omega \end{aligned} \quad (3.15)$$

$$= \iint_{\Omega} \left\{ \underbrace{[w\varphi_1(\Delta u) - \Delta u\varphi_1^*(w)]}_{I_1} + \underbrace{[w\varphi_2(\Delta u) + \Delta u\varphi_2^*(w)]}_{I_2} + \underbrace{[w\varphi_3(\Delta u) - \Delta u\varphi_3^*(w)]}_{I_3} \right\} d\Omega. \quad (3.16)$$

Applying the integration by parts to each term in the integral in equation (3.16) and using boundary conditions given by equations (3.5), (3.12) and (3.13), following equalities are obtained;

$$\begin{aligned} I_1 &= \iint_{\Omega} \left\{ [w\varphi_1(\Delta u) - \Delta u\varphi_1^*(w)] \right\} d\Omega = \int_0^{\ell} \left\{ [w(t_f, x)\Delta u_t(t_f, x) - \Delta u(t_f, x)w_t(t_f, x)] \right. \\ &\quad \left. + \alpha[w_{xx}(t_f, x)\Delta u_t(t_f, x) - w_{txx}(t_f, x)\Delta u(t_f, x) + w_{txxx}(t_f, x)\Delta u(t_f, x) - w_{xxxx}(t_f, x)\Delta u_t(t_f, x)] \right\} dx, \end{aligned} \quad (3.17a)$$

$$I_2 = \iint_{\Omega} \left\{ [w\varphi_2(\Delta u) - \Delta u\varphi_2^*(w)] \right\} d\Omega = \beta \int_0^{\ell} [w_{xx}(t_f, x)\Delta u(t_f, x)] dx, \quad (3.17b)$$

$$I_3 = \iint_{\Omega} \left\{ [w\varphi_3(\Delta u) - \Delta u\varphi_3^*(w)] \right\} d\Omega = 0, \quad (3.17c)$$

by means of equation (3.6),

$$\{I_1 + I_2 + I_3\} = 2 \int_0^1 \{\vartheta_1 w(t_f, x)\Delta w(t_f, x) + \vartheta_2 w_t(t_f, x)\Delta w_t(t_f, x)\} dx = \iint_{\Omega} \{w[\Delta \mathcal{H}(t, x) - \Delta \varpi]\} d\Omega. \quad (3.18)$$



Consider the difference of the performance index

$$\begin{aligned}\Delta\mathcal{J}[\bar{h}(t)] &= \mathcal{J}[\bar{h}(t)] - \mathcal{J}[\bar{h}^\circ(t)] \\ &= \int_0^{\ell} \left\{ \vartheta_1[u^2(t_f, x) - u^{\circ 2}(t_f, x)] + \vartheta_2[u_t^2(t_f, x) - u_t^{\circ 2}(t_f, x)] \right\} dx \\ &\quad + \int_0^{t_f} \vartheta_3[\bar{h}^2(t) - \bar{h}^{\circ 2}(t)] dt.\end{aligned}\tag{3.19}$$

Expanding  $u^2(t_f, x)$  and  $u_t^2(t_f, x)$  in Taylor series around  $u^{\circ 2}(t_f, x)$  and  $u_t^{\circ 2}(t_f, x)$ , yields

$$u^2(t_f, x) - u^{\circ 2}(t_f, x) = 2u^\circ(t_f, x)\Delta u(t_f, x) + r,\tag{3.20a}$$

$$u_t^2(t_f, x) - u_t^{\circ 2}(t_f, x) = 2u_t^\circ(t_f, x)\Delta u_t(t_f, x) + r_t\tag{3.20b}$$

where  $r = 2(\Delta u)^2 + \text{higher order terms} > 0$  and  $r_t = 2(\Delta u_t)^2 + \text{higher order terms} > 0$ . Substituting equations (3.20) into (3.19) gives

$$\begin{aligned}\Delta\mathcal{J}[\bar{h}(t)] &= \int_0^1 \left\{ \vartheta_1[2u^\circ(t_f, x)\Delta u(t_f, x) + r] \right. \\ &\quad \left. + \vartheta_2[2u_t^\circ(t_f, x)\Delta u_t(t_f, x) + r_t] \right\} dx + \int_0^{t_f} \vartheta_3[\bar{h}(t)^2 - \bar{h}(t)^{\circ 2}] dt.\end{aligned}\tag{3.21}$$

From equation (3.18) and because of  $\vartheta_1 r + \vartheta_2 r_t > 0$ , one obtains

$$\Delta\mathcal{J}[\bar{h}(t)] \geq \iint_{\Omega} \left\{ w\theta(x)\Delta\bar{h}(t) - w\Delta\varpi \right\} d\Omega + \int_0^{t_f} \vartheta_3[\bar{h}^2(t) - \bar{h}^{\circ 2}(t)] dt \geq 0\tag{3.22}$$

which leads to

$$\int_0^{t_f} \left\{ \Delta\bar{h}(t)\Phi(t) - \Delta\Psi(t) \right\} dt + \int_0^{t_f} \vartheta_3[\bar{h}^2(t) - \bar{h}^{\circ 2}(t)] dt \geq 0\tag{3.23}$$

that is,

$$\mathcal{H}[t; u^\circ, w^\circ, \bar{h}^\circ] \geq \mathcal{H}[t; u, w, \bar{h}].$$

Hence, we obtain

$$\mathcal{J}[\bar{h}] \geq \mathcal{J}[\bar{h}^\circ], \quad \forall \bar{h} \in \bar{h}_{ad}.$$

Therefore, the optimal control function is given by

$$\bar{h}(t) = \frac{-\Phi(t)}{2\vartheta_3}.\tag{3.24}$$

□

## 4. NUMERICAL RESULTS AND DISCUSSIONS

In this section, obtained theoretical results are simulated by solving following system of equations linked by initial-boundary-terminal conditions *via* MATLAB for indicating the effectiveness and robustness of the introduced control algorithm for damping excessive water waves in a canal system by optimally determined canal depth control function.

$$u_{tt} + \alpha(u_{ttxx} - u_{ttxxx}) + \beta u_{txx} + \gamma u_{xxxx} + u_{xx} + [\mathcal{N}(u)]_{xx} = f(t, x) + \mathcal{H}(t, x), \quad (4.1a)$$

$$\mathcal{H}(t, x) = \bar{h}(t)\theta(x), \quad \bar{h}(t) = \frac{-\Phi(t)}{2\vartheta_3}, \quad \Phi(t) = \int_0^\ell w(t, x)\theta(x)dx, \quad (4.1b)$$

$$u(t, x) = 0, \quad u_{xx}(t, x) = 0, \quad \text{at } x = 0, \ell, \quad (4.1c)$$

$$u(t, x) = u_0(x), \quad u_t(t, x) = u_1(x) \quad \text{at } t = 0. \quad (4.1d)$$

$$w_{tt} + \alpha(w_{ttxx} - w_{ttxxx}) - \beta w_{txx} + \gamma w_{xxxx} + w_{xx} = 0, \quad (4.2a)$$

$$w(t, x) = 0, \quad w_{xx}(t, x) = 0, \quad \text{at } x = 0, \ell, \quad (4.2b)$$

$$-2\vartheta_1 u(t, x) = w_t(t, x) + \alpha[w_{txx}(t, x) - w_{txxx}(t, x)] - \beta w_{xx}(t, x) \quad \text{at } t = t_f, \quad (4.2c)$$

$$2\vartheta_2 u_t(t, x) = w(t, x) + \alpha[w_{xx}(t, x) - w_{xxx}(t, x)] \quad \text{at } t = t_f. \quad (4.2d)$$

Before evaluating the numerical results in tables and graphs, consider the optimal canal depth control function given by equation (3.24), in which, it is clear that as the value of  $\vartheta_3$  is decreasing, the value of the canal depth is increasing. As a conclusion of this situation, dynamic response of the excessive water waves given by first integral on the left side of the equation (3.2) is minimized by using minimum canal depth. Effectiveness of the introduced control actuation is examined in two cases. Both of two cases,  $t_f$  is taken into account as 3. Also, in numerical computations,  $\mathcal{N}(u)$  is considered as 0 due to difficulties on solving system of equations given by equations (4.1)–(4.2). Weighted coefficients are taken into account as  $\vartheta_{1,2} = 1$  and  $\vartheta_3 = 10^4$  and  $\vartheta_3 = 10^{-4}$  for uncontrolled and controlled case, respectively. Canal length is  $\ell = 1$ . All figures are plotted in the middle of the canal,  $x = 0.5$ . The introduced control algorithm is valid for all coefficients in the system but due the stability of the solutions of equations (2.1)–(2.3), following cases are imposed. In the first case, followings are taken into account;

$$\alpha = 0.01, \quad \beta = 0.001, \quad \gamma = 0.0001, \quad \theta(x) = 1,$$

$$f(t, x) = te^x, \quad u_0(x) = \cos(\pi x), \quad u_1(x) = \sqrt{2}\sin(\pi x).$$

For case a, controlled and uncontrolled displacements of the water waves at the midpoint of the canal are given by Figure 2 and it can be clearly observed that excessive waves in the canal system is damped out successfully. Same observation is also valid that the velocities of excessive water waves is also effectively suppressed by using minimum canal depth *via* introduced control actuation. In the second case, control actuation is applied by considering followings;

$$\alpha = 0.01, \quad \beta = 0.01 \quad \text{and} \quad \beta = 0.0001, \quad \gamma = 0.01, \quad \theta(x) = x,$$

$$f(t, x) = 1 + \sin(x), \quad u_0(x) = \sqrt{2}\sin(\pi x), \quad u_1(x) = \sqrt{2}\sin(\pi x).$$

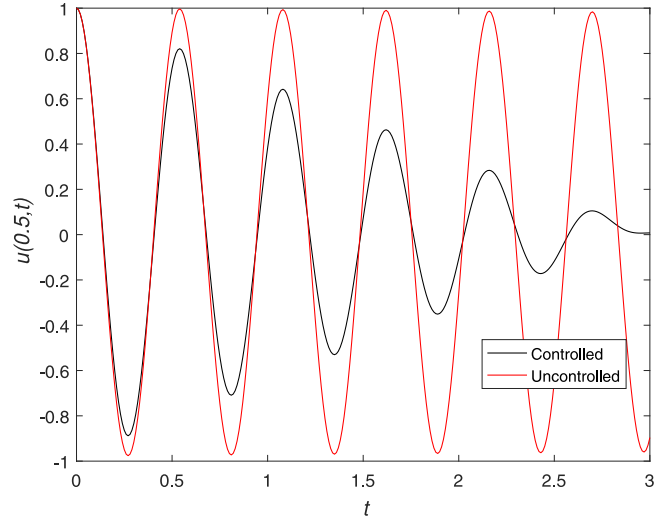


FIGURE 2. Uncontrolled and controlled displacements for case a.

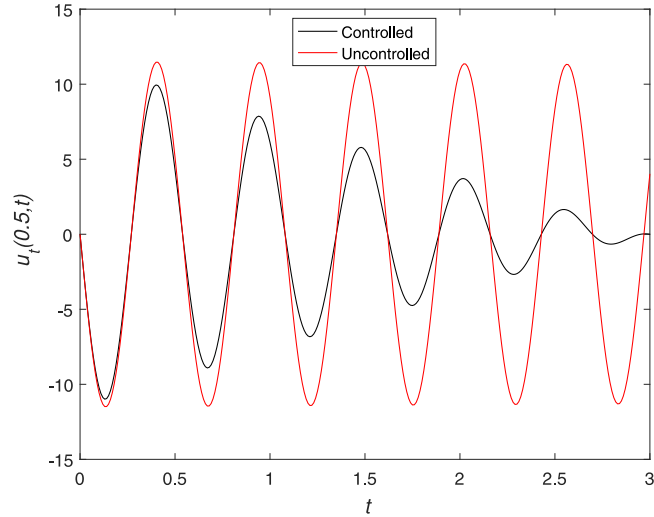


FIGURE 3. Uncontrolled and controlled velocities for case a.

For case b, controlled and uncontrolled displacement of water waves is given in Figure 4. It is easy to see that excessive waves in the canal system are effectively damped out by means of optimal canal depth function. Also, in Figure 4, note that successfully suppressed displacements are plotted for  $\beta = 0.01$  and  $\beta = 0.0001$ . In Figure 4, the uncontrolled displacement for  $\beta = 0.0001$  is greater than the displacement for plotted  $\beta = 0.01$  due to insufficient internal damping. The magnitude of controlled displacement for  $\beta = 0.01$  is less than the controlled one for  $\beta = 0.0001$  due to larger internal damping in the corresponding system and the system having larger internal damping is on trend for easily suppressing of excessive waves. In Figure 5, the velocities of the system corresponding to  $\beta = 0.01$  and  $\beta = 0.0001$  are plotted and it can be easily seen that decreasing of velocities for these systems is successfully obtained. Also, similar observation to Figure 4 can be obtained by

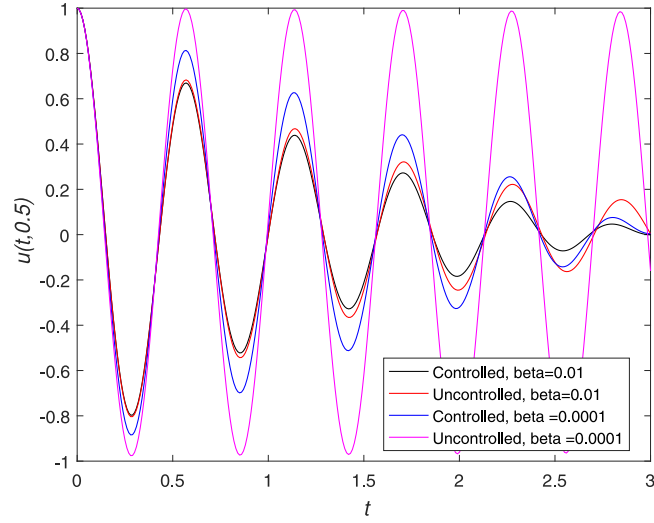


FIGURE 4. Uncontrolled and controlled displacements for case b.

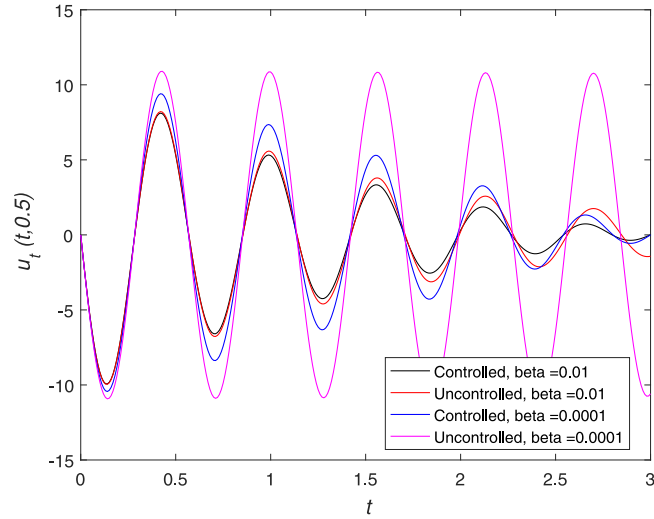


FIGURE 5. Uncontrolled and controlled velocities for case b.

seeing Figure 5 that The magnitude of controlled displacement for  $\beta = 0.0001$  is grater than the controlled one for  $\beta = 0.01$  due to smaller internal damping in the corresponding system and the system having larger internal damping is on trend for easily suppressing of velocities of excessive waves. Let us give the dynamic response of the wave in the canal system as follows;

$$\mathcal{J}(u) = \int_0^1 [u^2(t_f, x) + u_t^2(t_f, x)] dx \quad (4.3)$$

TABLE 1. The values of  $\mathcal{J}(u)$  and  $\mathcal{J}(h)$  for different values of  $\vartheta_3$  in case a.

$\vartheta_3$	$\mathcal{J}_a(u)$	$\mathcal{J}_a(h)$
$10^4$	1.8 e-3	2.7 e-9
$10^0$	5.0 e-4	2.2 e-3
$10^{-4}$	3.9 e-10	6.0 e-2

TABLE 2. The values of  $\mathcal{J}(u)$  and  $\mathcal{J}(h)$  for  $\beta = 0.01$  and different values of  $\vartheta_3$  in case b.

$\vartheta_3$	$\mathcal{J}_{0.01}(u)$	$\mathcal{J}_{0.01}(h)$
$10^4$	2.03	2.8 e-7
$10^0$	2.7 e-3	3.8 e-5
$10^{-4}$	2.2 e-8	27.6

TABLE 3. The values of  $\mathcal{J}(u)$  and  $\mathcal{J}(h)$  for  $\beta = 0.0001$  and different values of  $\vartheta_3$  in case b.

$\vartheta_3$	$\mathcal{J}_{0.0001}(u)$	$\mathcal{J}_{0.0001}(h)$
$10^4$	111.3	6.0 e-5
$10^0$	11.0	6.7
$10^{-4}$	4.8 e-7	383

and used canal depth accumulates over  $(0, t_f)$ ;

$$\mathcal{J}(\bar{h}) = \int_0^{t_f} \bar{h}^2(t) dt. \quad (4.4)$$

The dynamic response of the wave in the canal system is given by table forms and it seemed from tables that as weighted coefficient  $\vartheta_3$  in canal depth control function decreases, dynamic response of the wave decreases due to an increasing in the value of canal depth control function. Also, by comparing the Tables 2 and 3, it is obtained that dynamics response of the system corresponding to larger internal damping is less than the system having less internal damping. Besides this, totally canal depth corresponding to system having larger internal damping is more less than the system corresponding to smaller internal damping. These observations show that introduced control actuation for water waves in a canal system is successful and effective.

## 5. CONCLUSION

Damping out the excessive water waves in a canal system problem is taken into account and suppressing of the waves in the canal is successfully obtained by means of optimally determining of canal depth control function *via* maximum principle, which transforms to optimal control problem to solving an initial-boundary-terminal value problem. Optimal canal depth control function is analytically obtained without linearization of nonlinear term. Obtained theoretical results are simulated by means of MATLAB and results indicate that introduced control actuation is very effective and applicable to other waves control system including nonlinear terms.

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