Math. Model. Nat. Phenom. 16 (2021) 51
https://doi.org/10.1051/mmnp/2021041

Mathematical Modelling of Natural Phenomena
www.mmnp-journal.org

INVARIANT MEASURE OF STOCHASTIC HIGHER ORDER KDV EQUATION DRIVEN BY POISSON PROCESSES*

Pengfei Xu[1], Jianhua Huang[1,**] and Wei Yan[2]

Abstract. The current paper is devoted to stochastic damped higher order KdV equation driven by Poisson process. We establish the well-posedness of stochastic damped higher-order KdV equation, and prove that there exists an unique invariant measure for non-random initial conditions. Some discussion on the general pure jump noise case are also provided. Some numerical simulations of the invariant measure are provided to support the theoretical results.

Mathematics Subject Classification. 60H15, 37L55.

Received March 27, 2020. Accepted July 13, 2021.

1. Introduction

In this paper,we investigate the long time behavior of the following stochastic damped $(2n+1)$-th order KdV equation with pure jump processes

$$\begin{cases} \mathrm{d}u + (\partial_x^{2n+1}u + u\partial_x u + \lambda u)\mathrm{d}t = f\mathrm{d}t + \int_Z g(u(t-),z)\eta(\mathrm{d}t,\mathrm{d}z) \\ u(0,x) = u_0. \end{cases} \tag{1.1}$$

where $\lambda > 0$,$f \in H^{2n+1}(\mathbb{R})$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete probability space,$(Z, \mathcal{B}(Z))$ be a measurable space and $\nu(\mathrm{d}z)$ be a $\sigma-$finite measure on it. Let $P = (p(t), t \in D_p)$ be a stationary $\mathcal{F}_t-$Poisson point process on Z,with characteristic measure $\nu(\mathrm{d}z)$,where D_p is a countable subset of $[0, +\infty)$ dependeing on random parameter $\omega \in \Omega$. Denote by $\eta(\mathrm{d}t, \mathrm{d}z)$ the Poisson counting measure associated with P, that is $\eta(\mathrm{d}t, \mathrm{d}z) = \Sigma_{s\in D_p, s\le t} 1_A(p(s))$, where $1_A(\cdot)$ is the indicator function with respect to A. Let $\tilde{\eta}(\mathrm{d}t, \mathrm{d}z) = \eta(\mathrm{d}t, \mathrm{d}z) - \mathrm{d}t\nu(\mathrm{d}z)$ be the compensated poisson measure, when $\nu(z) < \infty$, $\tilde{\eta}$ is a martingale with $\langle\tilde{\eta}(Z)\rangle_t = t\nu(z)$,and ν is the intensity measure of η.

The Korteweg-de Vries equation(KdV) equation is used to describe the motion of long,unidirec-tional weakly water waves on a chanel [27], which has been investigated widely by some authors, for instance, see [3–5, 12, 13, 19–23, 34, 35]. By introducing the Fourier restriction norm method, Bourgain [3, 4] proved that the Cauchy problem for the KdV equation is globally well-posed in L^2 for the periodic case and nonperiodic case. By using the Fourier restriction norm method, Kenig *et al.* [22, 23] proved that the Cauchy problem for the KdV equation is locally well-posed in $H^s(\mathbf{R})$, $s > -\frac{3}{4}$ and ill-posed in $H^s(\mathbf{R})$ with $s < -\frac{3}{4}$ in the sense that the data-to-solution

*Supported by the NSF of China (No.11771449).

Keywords and phrases: Invariant measure, damped KdV equation, pure jump noises.

[1] College of Liberal Arts and Science, National University of Defense Technology, Changsha 410073, PR China.
[2] College of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan 453007, PR China.

** Corresponding author: jhhuang32@nudt.edu.cn

map fails to be uniformly continuous as map from $H^s(\mathbf{R})$ to $C_t^0H^s(\mathbf{R})$. This means that $s=-\frac{3}{4}$ is the critical regularity index in Sobolev spaces for KdV equation. By using the I-method and the Fourier restriction norm method, Colliander *et al.* [12] showed that the Cauchy problem for the KdV equation is globally well-posed in $H^s(\mathbf{R})$ with $s>-\frac{3}{4}$. Guo [16] and Kishimoto [25] independently proved that the Cauchy problem for the KdV equation is globally well-posed in $H^{-\frac{3}{4}}(\mathbf{R})$ with the aid of the I-method and the modified Besov spaces. Kappeler and Topalov [18] proved that the flow map extends continuously to H^{-1} in the periodic case with the aid of inverse scattering transformation. Molinet [31] proved that no well-posedness result can possibly hold below $s=-1$ in the periodic case in the sense that the solution map of KdV equation does not extend to a continuous map from $H^s(\mathbf{R})$ for $s<-1$ to distribution. Molinet [30] proved that the solution-map associated with the KdV equation cannot be continuously extended in $H^s(\mathbf{R})$ for $s<-1$. Liu [28] established a priori bounds for KdV equation in $H^{-\frac{3}{4}}(\mathbf{R})$. Buckmaster and Koch [11] proved the existence of weak solutions to the KdV initial value problem on the real line with $H^{-1}(\mathbf{R})$ initial data and studied the problem of orbital and asymptotic $H^s(\mathbf{R})$ stability of solitons for integers $s=-1$; and established new a priori $H^{-1}(\mathbf{R})$ bound for solutions to the KdV equation. Visan [24] proved global well-posedness of the KdV equation for initial data in $H^{-1}(\mathbf{R})$ and showed completely parallel arguments and give a new proof of global well-posedness for KdV with periodic $H^{-1}(\mathbf{R})$ data, showed previously by Kappeler and Topalov, as well as global well-posedness for the fifth order KdV equation in $L^2(\mathbf{R})$. Li and Yan [29] established the global well-posedness of higher-order KDV equations with Gaussian noise in $L^2(\mathbf{R})$.

When $n=1$, (1.1) reduces to the stochastic KdV equation which has been studied by some people, we refer the readers to [6–8]. Bouard *et al.* [7] proved that for almost surely $\omega\in\Omega$, there exist a $T_\omega>0$ and a unique solution $u(t)$ of the problem (1.1) on $[0,T_\omega]$ which satisfies $u\in C([0,T];H^s)\cap X_{s,b}^{T_\omega}\cap\dot{X}_{s,-\frac{3}{8},b}^{T_\omega}$ with the assumption that $u_0(x,\omega)\in H^s, s>-\frac{5}{8}$ for almost surely $\omega\in\Omega$ and u_0 is $\mathscr{F}_0$-measurable and $\Phi\in L_2^{0,s}\cap L_2^0(L^2;\dot{H}^{s,-\frac{3}{8}})$. Moreover, de Bouard *et al.* [7] proved that when $\Phi\in L_2^{0,0}\cap L_2^0(L^2;\dot{H}^{0,-\frac{3}{8}})$ and $u_0\in L^2(\Omega;L^2))$ and is $\mathscr{F}_0$-measurable, the solution to problem (1.1) is global and belongs to $L^2(\Omega;C([0,T];L^2)$. Li [28] proved that for $s>-\frac{3}{4}$, there exist a $T_\omega>0$(almost surely $\omega\in\Omega$), $\tilde{s}=\tilde{s}(s), b=b(s,\tilde{s})$ which satisfies $\tilde{s}<0$ and $\tilde{s}$ can sufficiently approach zero and $0<b<\frac{1}{2}$, when $\Phi\in L_2^{0,s}\cap L_2^0(L^2;\dot{H}^{s,\tilde{s}})$ and $u_0(x,\omega)\in H^s$ for almost surely $\omega\in\Omega$ and u_0 is $\mathscr{F}_0$-measurable, the problem (1.1) possesses a unique solution on $[0,T_\omega]$ which satisfies $u\in C([0,T_\omega];H^s)\cap X_{s,b}^{T_\omega}$, thus, the result of Li improves the result of [7]. Recently, [9] proved the existence of invariant measure for stochastic Schrödinger equation with jump process, but they also did not provide with the uniqueness of the invariant measure. To the best of our knowledge, there is no result on the ergodicity on stochastic damped higher-order KdV equation with pure jump noise.

Observe that if $\bar{u}(t)$ solves

$$\mathrm{d}u+\partial_x^{2n+1}u\mathrm{d}t=0,$$

Let $u(t):=e^{-\lambda t}\bar{u}(t)$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t)=-\lambda e^{-\lambda t}\bar{u}(t)+e^{-\lambda t}\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}(t)=-\lambda u(t)-e^{-\lambda t}\partial_x^{2n+1}\bar{u}(t)=-\lambda u(t)-\partial_x^{2n+1}u(t).$$

Thus, $u(t)$ solves the equation $\mathrm{d}u+(\partial_x^{2n+1}u+\lambda u)\mathrm{d}t=0$. Denote $S_\lambda(t)$ by the solution operator of $\mathrm{d}u+(\partial_x^{2n+1}u+\lambda u)\mathrm{d}t=0$, then we have $S_\lambda(t)=e^{-\lambda t}S_0(t)$. Then the mild solution of equation (1.1) can be written as

$$u(t)=S_\lambda(t)u_0-\int_0^t S_\lambda(t-s)u\partial_x u\mathrm{d}s+\int_0^t S_\lambda(t-s)f\mathrm{d}s+\int_0^t\int_Z S_\lambda(t-s)g(u(s-),z)\eta(\mathrm{d}s,\mathrm{d}z). \tag{1.2}$$

Let $u_0\in H^n(\mathbb{R})$ be a deterministic condition,and $u(t)$ be the solution of equation (1.1). For all $B\in\mathfrak{B}(H^n(\mathbb{R}))$, the transition probabilities of the equation can be defined by $P_t(u_0,B)=P(u(t)\in B)$. For any function $\xi\in$

$C_b(H^n, \mathbb{R})$, we denote

$$P_t\xi(u_0) = \mathbb{E}[\xi(u(t))] = \int_{H^n(\mathbb{R})} \xi(u) P_t(u_0, \mathrm{d}u).$$

In this paper, we will investigated the ergodicity of stochastic damped higher order KdV equation with pure jump noise. We prove the global well-posedness and the existence of invariant measure for stochastic damped higher order KdV equation with random initial value. Moreover, we obtain the ergodicity of stochastic damped higher order KdV equation with non-random initial conditions.

Comparing with stochastic damped KdV equation with Gaussian noise, the non-Gaussian noise driven higher Order KdV equation leads to the trajectories of the solutions are Càdlàg. To overtake the difficulty caused by dispersive term $\partial_x^{2n+1}u$ instead of the dissipative term u_{xx}, motivated by the idea from [15], we establish the uniform estimates for L^2 norm and H^n norm respectively, which are critical to obtain the existence of invariant measure. Moreover, we give some numerical simulations of the invariant measure to support the theoretical results.

The rest of paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we prove that the solutions in H^n space are uniformly bounded and establish the global well-posedness. In Section 4, we establish the existence of invariant measure for stochastic damped stochastic damped higher order KdV equation. In Section 5, we prove that the invariant measure is unique for deterministic initial value. In Section 6, we discuss the general pure jump noise case,some numerical simulations of the invariant measure are provided to support the theoretical results.

2. Preliminaries

In this section, some basic concepts and some inequalities are provided, which plays the crucial role in establishing the main theorems.

Let Y be a separable and complete metric space and $T > 0$. The space $\mathbb{D}([0,T];Y)$ denotes the space of all right continuous functions $x : [0,T] \to Y$ with left limits,$P(\mathbb{D}([0,T];Y))$ the space of Borel probability measures on $\mathbb{D}([0,T];Y)$. We equip $\mathbb{D}([0,T];Y)$ with the Skorohod topology such that $\mathbb{D}([0,T];Y)$ is both separable and complete.

Definition 2.1. Assume that (X,T) is a polish space, Σ is a σ-algebra on X, M is a set of measures on Σ. M is said to be tight if for any $\varepsilon > 0$, there exists a compact set $K_\epsilon \subset X$ such that

$$|\mu(K_\varepsilon^c)| < \varepsilon, \text{for any } \mu \in M.$$

Theorem 2.2 (Cor. A.1, [9])**.** *Let $\{x_n : n \in \mathbb{N}\}$ be a sequence of càdlàg processes, each of the process defined on a probability space $(\Omega_n, \mathscr{F}_n, \mathbb{P}n)$. Then the sequence of laws of $\{x_n : n \in \mathbb{N}\}$ is tight on $\mathbb{D}([0,T];Y)$ if*

(a) there exists a space Y_1, with $Y_\hookrightarrow Y$ compactly and some $r > 0$, such that

$$\mathbb{E}^n|x_n(t)|_{Y_1}^r \le C, \quad \forall n \in \mathbb{N};$$

(b) there exist two constants $c > 0$ and $\gamma > 0$ and a real number $r > 0$ such that for all $\theta > 0, t \in [0, T-\theta]$, and $n \ge 0$

$$\mathbb{E}_n \sup_{t\le s\le t+\theta} |x_n(t) - x_n(s)|_Y^r \le c\theta^\gamma.$$

In the sequel, we impose the following hypothesis on the Poisson noise as

(H1) For any $u \in X(T)$, there exists a constant $C < \infty$ such that

$$\int \|g(u,z)\|^2_{X(T)} \nu(\mathrm{d}z) \leq C(1 + \|u\|^2_{X(T)});$$

(H2) $\nu(0) = 0, \int_Z (\|z\|^2_Z \nu(\mathrm{d}z) < \infty$ and $\nu(Z) = \rho < \infty$;
(H3) Z is continuously embedded in $H^n(\mathbb{R})$.

Lemma 2.3 ([19])**.** *There exist some monotontic increasingly continuous functions $\tilde{C_1}(t)$ and a constant $\alpha > 0, \forall h, g \in X(T)$ such that*

$$\| \int_0^t S_0(t-s)h(s)\partial_x g(s)\mathrm{d}s \|_{X(T)} \leq \tilde{C_1}(T) T^\alpha \| h \|_{X(T)} \| g \|_{X(T)} .$$

Lemma 2.4 ([19])**.** *There exist some monotonely increasingly continous functions $\tilde{C_2}(t)$ such that*

$$\| S_0(t)u_0 \|_{X(T)} \leq \tilde{C_2}(T) \| u_0 \|_{X(T)} .$$

Lemma 2.5. *If $u(t) \in X(T)$ solves the equation (1.2), then there exists a constant $C > 0$ such that*

$$\mathbb{E}[\sup_{0 \leq t \leq T} \|u\|^2_{L^2}] \leq C(1 + E[\|u_0\|^2_{L^2}]),$$
$$\mathbb{E}[\sup_{0 \leq t \leq T} \|u\|^2_{H^n}] \leq C(1 + E[\|u_0\|^2_{H^n}]).$$

Proof. Define a stopping time by $\tau_N = \inf\{t : \|u(t)\|^2_{L^2}) > N\}$. By using It$\hat{o}$ formula, we have

$$\begin{aligned}
\|u(t)\|^2_{L^2} &= \|u_0\|^2_{L^2} + 2\int_0^t \langle u(s), -(\partial_x^{2n+1} u + u\partial_x u + \lambda u) + f \rangle \mathrm{d}s \\
&\quad + \int_0^t \int_Z (\|g(u(s-), z)\|^2_{L^2} + 2\langle u(s-), g(u(s-), z) \rangle) \tilde{\eta}(\mathrm{d}s, \mathrm{d}z) \\
&\quad + \int_0^t \int_Z (\|g(u(s), z)\|^2_{L^2} + 2\langle u(s), g(u(s), z) \rangle) \mathrm{d}s \nu(\mathrm{d}z) \\
&= \|u_0\|^2_{L^2} + I_1(t) + M_t + I_2(t).
\end{aligned}$$

Next, we will estimate $\|u(t \wedge \tau_N)\|^2_{L^2}$.

For $M_{t \wedge \tau_N}$, direct computation shows that

$$\begin{aligned}
&[M_{\tau_N}, M_{\tau_N}]_t^{\frac{1}{2}} = [M, M]^{\frac{1}{2}}_{\tau_N \wedge t} \\
&= (\sum_{s \in D_p, s \leq \tau_N \wedge t} (\|g(u(s-), z)\|^2_{L^2} + 2\langle u(s-), g(u(s-), p(s)) \rangle)^2)^{\frac{1}{2}} \\
&\leq (2 \sum_{s \in D_p, s \leq \tau_N \wedge t} (\|g(u(s-), z)\|^4_{L^2} + 8 \sum_{s \in D_p, s \leq \tau_N \wedge t} \|u(s-)\|^2_{L^2} \|g(u(s-), p(s))\|^2_{L^2})^{\frac{1}{2}} \\
&\leq C(\sum_{s \in D_p, s \leq \tau_N \wedge t} \|g(u(s-), z)\|^4_{L^2})^{\frac{1}{2}} + C(\sum_{s \in D_p, s \leq \tau_N \wedge t} \|u(s-)\|^2_{L^2} \|g(u(s-), p(s))\|^2_{L^2})^{\frac{1}{2}} \\
&\leq C \sum_{s \in D_p, s \leq \tau_N \wedge t} \|g(u(s-), z)\|^2_{L^2} + C(\sum_{s \in D_p, s \leq \tau_N \wedge t} \|u(s-)\|^2_{L^2} \|g(u(s-), p(s))\|^2_{L^2})^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq C\sum_{s\in D_p,s\leq\tau_N\wedge t}\|g(u(s-),z)\|_{L^2}^2+C\sup_{s\leq t\wedge\tau_N}\|u(s-)\|_{L^2}(\sum_{s\in D_p,s\leq\tau_N\wedge t}\|g(u(s-),p(s))\|_{L^2}^2)^{\frac{1}{2}}\\
&\leq C\sum_{s\in D_p,s\leq\tau_N\wedge t}\|g(u(s-),z)\|_{L^2}^2+\frac{1}{2}\sup_{s\leq t\wedge\tau_N}\|u(s-)\|_{L^2}^2.
\end{aligned}$$

Taking the expectation and using Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
&\mathbb{E}[\sup_{0\leq s\leq t}|M_s^{\tau_N}|]\leq C\mathbb{E}([M_{\tau_N},M_{\tau_N}]_t^{\frac{1}{2}})\\
&\leq C\mathbb{E}[\sum_{s\in D_p,s\leq\tau_N\wedge t}\|g(u(s-),z)\|_{L^2}^2]+\frac{1}{2}\mathbb{E}[\sup_{s\leq t\wedge\tau_N}\|u(s-)\|_{L^2}^2]\\
&=C\mathbb{E}[\int_0^{t\wedge\tau_N}\int_Z\|g(u(s-),z)\|_{L^2}^2\mathrm{d}s\nu(\mathrm{d}z)]+\frac{1}{2}\mathbb{E}[\sup_{s\leq t\wedge\tau_N}\|u(s-)\|_{L^2}^2]\\
&\leq C\mathbb{E}[\int_0^{t\wedge\tau_N}(1+\|u(s-)\|_{L^2}^2)\mathrm{d}s]+\frac{1}{2}\mathbb{E}[\sup_{s\leq t\wedge\tau_N}\|u(s-)\|_{L^2}^2]\\
&\leq C(1+N)t+\frac{1}{2}N<\infty.
\end{aligned}$$

For $I_1(t\wedge\tau_N)$, we can deduce

$$\begin{aligned}
\mathbb{E}[I_1(t\wedge\tau_N)]&=-2\lambda\mathbb{E}[\int_0^{t\wedge\tau_N}\|u(s)\|_{L^2}^2\mathrm{d}s]+2\mathbb{E}[\int_0^{t\wedge\tau_N}\langle u(s),f\rangle\mathrm{d}s]\\
&\leq-2\lambda\mathbb{E}[\int_0^{t\wedge\tau_N}\|u(s)\|_{L^2}^2\mathrm{d}s]+2\mathbb{E}[\int_0^{t\wedge\tau_N}\|u(s)\|_{L^2}\|f\|_{L^2}\mathrm{d}s]\\
&\leq\frac{1}{2\lambda}\mathbb{E}[\int_0^{t\wedge\tau_N}\|f\|_{L^2}^2\mathrm{d}s]\leq\frac{1}{2\lambda}t\|f\|_{L^2}^2\leq Ct<\infty.
\end{aligned}$$

For $I_2(t\wedge\tau_N)$, by using a direct calculation, we have

$$\begin{aligned}
&\mathbb{E}[I_2(t\wedge\tau_N)]=\mathbb{E}[\int_0^{t\wedge\tau_N}\int_Z(\|g(u(s),z)\|_{L^2}^2+2\langle u(s),g(u(s),z)\rangle)\mathrm{d}s\nu(\mathrm{d}z)]\\
&\leq\mathbb{E}[\int_0^{t\wedge\tau_N}\int_Z(\|g(u(s),z)\|_{L^2}^2+2\|u(s)\|_{L^2}^2\|g(u(s),z)\|_{L^2}^2)\mathrm{d}s\nu(\mathrm{d}z)]\\
&\leq 2\mathbb{E}[\int_0^{t\wedge\tau_N}\int_Z\|g(u(s),z)\|_{L^2}^2\mathrm{d}s\nu(\mathrm{d}z)]+\mathbb{E}[\int_0^{t\wedge\tau_N}\int_Z\|u(s)\|_{L^2}^2\mathrm{d}s\nu(\mathrm{d}z)]\\
&=2\mathbb{E}[\int_0^{t\wedge\tau_N}\int_Z\|g(u(s),z)\|_{L^2}^2\mathrm{d}s\nu(\mathrm{d}z)]+\mathbb{E}[\int_0^{t\wedge\tau_N}\int_Z\|u(s-)\|_{L^2}^2\mathrm{d}s\nu(\mathrm{d}z)]\\
&\leq C\mathbb{E}[\int_0^{t\wedge\tau_N}(1+\|u(s-)\|_{L^2}^2)\mathrm{d}s]+\rho\mathbb{E}[\int_0^{t\wedge\tau_N}\|u(s-)\|_{L^2}^2\mathrm{d}s]\\
&\leq Ct+(C+\rho)Nt<\infty.
\end{aligned}$$

From the above estimates, we have

$$\mathbb{E}[\sup_{0\leq s\leq t\wedge\tau_N}\|u(s)\|_{L^2}^2]=\mathbb{E}[\sup_{0\leq s\leq t}\|u(s\wedge\tau_N)\|_{L^2}^2]$$

$$\le \mathbb{E}[\|u_0\|_{L^2}^2] + \mathbb{E}[\sup_{0\le s\le t} I_1(s\wedge\tau_N)] + \mathbb{E}[\sup_{0\le s\le t} |M_s^{\tau_N}|] + \mathbb{E}[\sup_{0\le s\le t} I_2(s\wedge\tau_N)]$$
$$\le \mathbb{E}[\|u_0\|_{L^2}^2] + Ct + C\mathbb{E}[\int_0^{t\wedge\tau_N}(1+\|u(s-)\|_{L^2}^2)\mathrm{d}s] + \frac{1}{2}E[\sup_{s\le t\wedge\tau_N}\|u(s-)\|_{L^2}^2]$$
$$+C\mathbb{E}[\int_0^{t\wedge\tau_N}(1+\|u(s-)\|_{L^2}^2)\mathrm{d}s] + \rho E[\int_0^{t\wedge\tau_N}\|u(s-)\|_{L^2}^2\mathrm{d}s]$$
$$\le \mathbb{E}[\|u_0\|_{L^2}^2] + Ct + CE[\int_0^{t\wedge\tau_N}(\sup_{s\le t\wedge\tau_N}\|u(s)\|_{L^2}^2)\mathrm{d}s] + \frac{1}{2}E[\sup_{s\le t\wedge\tau_N}\|u(s)\|_{L^2}^2].$$

Therefore, we deduce

$$\mathbb{E}[\sup_{0\le s\le t\wedge\tau_N}\|u(s)\|_{L^2}^2] \le 2\mathbb{E}[\|u_0\|_{L^2}^2] + Ct + C\mathbb{E}[\int_0^{t\wedge\tau_N}(\sup_{s\le t\wedge\tau_N}\|u(s)\|_{L^2}^2)\mathrm{d}s].$$

The Gronwall's inequality leads to

$$\mathbb{E}[\sup_{0\le s\le t\wedge\tau_N}\|u(s)\|_{L^2}^2] \le C(Ct + \mathbb{E}[\|u_0\|_{L^2}^2])e^{Ct}.$$

Since $\tau_N \wedge T \to T$ as $N\to\infty$, P.$a.s.$, we have

$$\mathbb{E}[\sup_{0\le t\le T}\|u\|_{L^2}^2] \le C(1+\mathbb{E}[\|u_0\|_{L^2}^2]).$$

Similarly, applying Itô formula and Burkholder-Davis-Gundy inequality as well as Sobolev embeddings Lemmas to

$$\frac{(-1)^n}{2}\int_{\mathbf{R}}(\partial_x^n u)^2\mathrm{d}x + \frac{1}{6}\int_{\mathbf{R}} u^3\mathrm{d}x,$$

we also obtain

$$\mathbb{E}[\sup_{0\le t\le T}\|u\|_{H^n}^2] \le C(1+\mathbb{E}[\|u_0\|_{H^n}^2]).$$

We have completed the proof of Lemma 2.5. □

3. Existence of mild solution of equation (1.1)

In this section, we will establish the well-posedness of (1.1) in the space $X(T) = \mathbb{D}(0,T;H^n(R))$.

Theorem 3.1. *Assume the conditions* **(H1), (H2)** *and* **(H3)** *hold. Then the equation* (1.1) *has a unique global mild solution in $X(T)$, which is cádlag.*

Proof. Let $\{\tau_n : n\in\mathbb{N}\}$ be a family of independent exponential distributed random variables with parameter ρ, and set

$$T_n = \sum_{j=1}^n \tau_j, n\in\mathbb{N}.$$

Define $\{N(t) : t \geqslant 0\}$ as the counting process

$$N(t) = \Sigma_{j=1}^{\infty} 1_{[T_j,\infty)}(t), t \geqslant 0.$$

Then $N(t)$ is a Poisson distributed random variable with parameter ρt for any fixed $t > 0$. Let $\{Y_n : n \in \mathbb{N}\}$ be a family of independent, ν/ρ distributed random variables. Then

$$\int_0^t \int_Z g(u(s-), z)\tilde{\eta}(\mathrm{d}s, \mathrm{d}z) = \begin{cases} -\int_0^t \int_Z g(u(s-), z)\nu(\mathrm{d}z)\mathrm{d}s, & \text{if } N(t) = 0, \\ \Sigma_{j=1}^{N(t)} g(u(t-), Y_n) - \int_0^t \int_Z g(u(s-), z)\nu(\mathrm{d}z)\mathrm{d}s, & \text{if } N(t) > 0. \end{cases}$$

Notice that $N(t) = 0$ on the interval $[0, T_1)$, then the stochastic equation (1.1) can be rewritten as

$$\begin{cases} \mathrm{d}u + (\partial_x^{2n+1} u + u\partial_x u + \lambda u)\mathrm{d}t = f\mathrm{d}t, \\ u(0, x) = u_0. \end{cases} \tag{3.1}$$

The mild solution of equation (3.1) can be represented by

$$u(t) = S_\lambda(t)u_0 - \int_0^t S_\lambda(t-s)u\partial_x u\mathrm{d}s + \int_0^t S_\lambda(t-s)f\mathrm{d}s.$$

Define the operator F by

$$Fu = S_\lambda(t)u_0 - \int_0^t S_\lambda(t-s)u\partial_x u\mathrm{d}s + \int_0^t S_\lambda(t-s)f\mathrm{d}s.$$

Then it follows that

$$\|Fu\|_{X(T_1)} \leq \|S_\lambda(t)u_0\|_{X(T_1)} + \|\int_0^t S_\lambda(t-s)u\partial_x u\mathrm{d}s\|_{X(T_1)} + \|\int_0^t S_\lambda(t-s)f\mathrm{d}s\|_{X(T_1)}$$

Due to Lemma 2.3, Lemma 2.4, we obtain that there exists some constant $C > 0$ such that

$$\|Fu\|_{X(T_1)} \leq C(\|u_0\|_{X(T_1)} + T_1^\alpha \|u\|_{X(T_1)}^2) < \infty, \quad \text{whenever} \quad \|u\|_{X(T_1)} < \infty.$$

Thus, F maps $X(T_1)$ into itself, and

$$\begin{aligned} &\|Fu_1 - Fu_2\|_{X(T_1)} = \|\int_0^t S_\lambda(t-s)(u_1\partial_x u_1 - u_2\partial_x u_2)\mathrm{d}s\|_{X(T_1)} \\ &\leq \|\int_0^t S_\lambda(t-s)u_1\partial_x(u_1 - u_2)\mathrm{d}s\|_{X(T_1)} + \|\int_0^t S_\lambda(t-s)(u_1 - u_2)\partial_x u_2\mathrm{d}s\|_{X(T_1)} \\ &\leq C_1(T_1)T_1^\alpha(\|u_1\|_{X(T_1)} + \|u_2\|_{X(T_1)})\|u_1 - u_2\|_{X(T_1)}, \end{aligned}$$

for any $u_1, u_2 \in X(T_1)$. Hence, the Banach Fixed point theorem guarantees that there exists a sufficient small $T_1 > 0$ such that the stochastic equation (1.1) admits a unique local solution. Based on the estimate in Lemma 2.5, we can extend this local solution to the global one. Denote the global solution on $[0, T_1)$ by u_1.

Next, we will consider the solution on $[T_1, T_2)$. Since a jump with size $g(u(T_1-), Y_n)$ occurs at time T_1, we denote $u_1^0 = u_1(T_1-) + g(u_1(T_1-), Y_n)$, and consider a second process on $[T_1, T_2)$ follows as

$$\begin{cases} \mathrm{d}u + (\partial_x^{2n+1}u + u\partial_x u + \lambda u)\mathrm{d}t = f\mathrm{d}t \\ u(T_1, x) = u_1^0. \end{cases} \tag{3.2}$$

Similarly, equation (3.2) posses a unique golobal mild solution u_2 on $[T_1, T_2)$. Repeating the above arguments, we can deduce that equation (1.1) has a unque global mild solution in $X(T)$.

Since $P(N(t) < \infty) = 1$, then the solution u is a.s. defined on $[0, T]$, and the jumps take place at each T_j with $\lim_{t\downarrow T_j} u(t) = u_j^0$, Hence, $\lim_{t\uparrow T_j} u(t)$ exists, and the solution u is *cádlag*.

We have completed the proof of Theorem 3.1. $\square$

4. Existence of invariant measure

In this section, we will prove the existence of the invariant measure for equation (1.1).

Let P_t is a semigroup defined by the global solution of equation (1.1) on a Banach space X. It is well known that

$$\lim_{k\to\infty} P_t\xi(u_0^k) = P_t\xi(u_0).$$

for any sequence $u_0^k \in X, k \in \mathbb{N}\}$ with $\|u_0^k - u_0\|_B \to 0, k \to \infty$, for any $t > 0$ and $\xi \in B_b(X, \mathbb{R})$. Then P_t is a strong Feller semigroup on B.

Theorem 4.1. *Assume the conditions* **(H1), (H2)** *and* **(H3)** *hold. Then P_t is a strong Feller semigroup on $H^n(\mathbb{R})$.*

Proof. We only need to prove that $\mathbb{E}[|\xi(u(t)) - \xi(v(t))|] \to 0$ for any solution u, v, where u and v are solution of equation (1.1) with initial value $u_0, v_0 \in H^n(\mathbb{R})$ respectively, and v_0 converges to u_0 for $t > 0, \xi \in B_b(H^n, \mathbb{R})$. Let $R_0 = \|u_0\|_{H^n} + 1$ and $M = \sup_{v\in H^n} |\xi(v)|$.

By using Chebychev's inequality, we obtain that there exists some constant $C(R_0) > 0$ for $\|v_0 - u_0\| \leq 1$,

$$\begin{aligned} &\mathbb{P}\{\max\{\sup_{s\in[0,t]} \|u(s)\|_{H^n}^2, \sup_{s\in[0,t]} \|v(s)\|_{H^n}^2\} \geqslant R\} \\ &\leq \mathbb{P}\{\sup_{s\in[0,t]} \|u(s)\|_{H^n}^2 + \sup_{s\in[0,t]} \|v(s)\|_{H^n}^2 \geqslant R\} \\ &\leq \frac{1}{R}\mathbb{E}[\sup_{s\in[0,t]} \|u(s)\|_{H^n}^2 + \sup_{s\in[0,t]} \|v(s)\|_{H^n}^2] \leq \frac{C(R_0)}{R}. \end{aligned}$$

Choosing a sufficient large $R > 0$ such that $\frac{C(R_0)}{R} \leq \frac{\varepsilon}{6M}$, we have

$$\mathbb{P}\{\max\{\sup_{s\in[0,t]} \|u(s)\|_{H^n}^2, \sup_{s\in[0,t]} \|v(s)\|_{H^n}^2\} \geqslant R\} \leq \frac{\varepsilon}{6M}.$$

Denote $\bar{u}(t) = \int_0^t \int_Z S_\lambda(t-s)g(u(s-), z)\eta(\mathrm{d}s, \mathrm{d}z)$, and set

$$T_h(g)(s) = S_\lambda(s)h - \int_0^s S_\lambda(s-r)g(r)\partial_x g(r)\mathrm{d}s + \int_0^s S_\lambda(s-r)f\mathrm{d}r + \bar{u}(s), \quad s \leq t.$$

Then $u(s) = T_{u_0}(u)(s)$ and $v(s) = T_{v_0}(v)(s)$.

Define the stopping time τ by

$$\tau = \inf\{s \geqslant 0 : 8\tilde{C}(t)s^{\alpha}(\tilde{C}(t)R + \|\bar{u}\|_{X(s)} + \|\int_0^s S_\lambda(s-r)f\mathrm{d}r\|_{X(s)}) > 1\}.$$

Then we have

$$\begin{aligned}\mathbb{P}(\tau < s) &\leq \mathbb{P}(8\tilde{C}(t)s^{\alpha}(\tilde{C}(t)R + \|\bar{u}\|_{X(s)} + \|\int_0^s S_\lambda(s-r)f\mathrm{d}r\|_{X(s)}) > 1)\\ &\leq \mathbb{E}[8\tilde{C}(t)s^{\alpha}(\tilde{C}(t)R + \|\bar{u}\|_{X(s)} + \|\int_0^s S_\lambda(s-r)f\mathrm{d}r\|_{X(s)})]\\ &\leq 8\tilde{C}^2(t)s^{\alpha}(R+1) \leq C(R)s^{\alpha},\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[\tau] &= \int_0^\infty \mathbb{P}(\tau > s)\mathrm{d}s = \int_0^\infty (1 - \mathbb{P}(\tau < s))\mathrm{d}s\\ &\geq \int_0^t (1 - \mathbb{P}(\tau < s))\mathrm{d}s \geq \int_0^t (1 - 1 \wedge (C(R)s^{\alpha}))\mathrm{d}s \geq \frac{1}{C_0(R)}.\end{aligned}$$

Let $\tau_0 = \tau$, and define inductively a sequence of stopping times by

$$\tau_{k+1} = \inf\{s \geqslant \tau_k : 8\tilde{C}(t)(s-\tau_k)^{\alpha}(\tilde{C}(t)R + \|\bar{u_{\tau_k}}\|_{X(\tau_k,s)} + \|\int_{\tau_k}^s S_\lambda(s-r)f\mathrm{d}r\|_{X(\tau_k,s)}) > 1\},$$

where $\bar{u}_{\tau_k}(s) = \int_{\tau_k}^s \int_Z S_\lambda(t-s)g(u(s-),z)\eta(\mathrm{d}s,\mathrm{d}z)$, and $X(\tau_k, s)$ is defined on $[\tau_k, s]$. Then $\tau_{k+1} - \tau_k$ and τ are independent with the same distribution. The law of large number gives that

$$\frac{\tau_n}{n} = \frac{1}{n}\sum_{0\leq i\leq n}(\tau_i - \tau_{i-1}) \to \mathbb{E}[\tau] \geqslant \frac{1}{C_0(R)}.$$

Therefore, $\mathbb{P}(\tau_n \leq t) \to 0$ as $n \to \infty$. Hence, there exists $n > 0, n \in \mathbb{N}$ such that

$$\mathbb{P}(\tau_n \leq t) \leq \frac{\varepsilon}{6M}.$$

It follows that for any v_0 with $\|v_0 - u_0\|_{H^n} \leq 1$,

$$\begin{aligned}\mathbb{E}[|\xi(u(t) - \xi(v(t)|] &\leq \mathbb{E}[|\xi(u(t) - \xi(v(t)|]I_{\max\{\sup\limits_{s\in[0,t]}\|u(s)\|^2_{H^n},\ \sup\limits_{s\in[0,t]}\|v(s)\|^2_{H^n}\}\geq R\}}\\ &\quad + \mathbb{E}[|\xi(u(t) - \xi(v(t)|]I_{\max\{\sup\limits_{s\in[0,t]}\|u(s)\|^2_{H^n},\ \sup\limits_{s\in[0,t]}\|v(s)\|^2_{H^n}\}\leq R\}}I_{\tau_n\leq t}\\ &\quad + \mathbb{E}[|\xi(u(t) - \xi(v(t)|]I_{\max\{\sup\limits_{s\in[0,t]}\|u(s)\|^2_{H^n},\ \sup\limits_{s\in[0,t]}\|v(s)\|^2_{H^n}\}\leq R\}}I_{\tau_n\geqslant t}\\ &\leq \frac{2}{3}\varepsilon + \mathbb{E}[|\xi(u(t) - \xi(v(t)|]I_{\max\{\sup\limits_{s\in[0,t]}\|u(s)\|^2_{H^n},\ \sup\limits_{s\in[0,t]}\|v(s)\|^2_{H^n}\}\leq R\}}I_{\tau_n\geqslant t}.\end{aligned}$$

Since v_0 converges to u_0, then

$$\|u(t) - v(t)\|_{H^n} \leq (2\tilde{C}(t))^{k+1}\|u(0) - v(0)\|_{H^n} \to 0.$$

Thus, it holds that $\mathbb{E}[|\xi(u(t))-\xi(v(t))|]\to 0$. the proof of Theorem 4.1 is complete. □

Definition 4.2. Let (X,T) be a topological space, and let $\{X_n, n\in\mathbb{N}\}X_0$ be a (X,T)-valued random variable, X_n is said to converge to X_0 in distribution, if for any bounded continuous function $F: X\to\mathbb{R}$,

$$\lim_{n\to\infty}\mathbb{E}[F(X_n)]=\mathbb{E}[F(X_0)].$$

Theorem 4.3. *Assume the conditions* **(H1), (H2)** *and* **(H3)** *hold, and for any sequence of deterministic initial conditions* $\{u_0^n\}$ *with* $R:=\sup_{n\in\mathbb{N}}\{\| u_0^n \|_{H^n}\}<\infty$, *it holds that* $\{P_{t_n}(u_0^n,\cdot): n\in\mathbb{N}\}$ *is tight on* H^n *for any* $\{t_n>0: \lim_{n\to\infty} t_n=\infty\}$.

Proof. Without loss of generality, we assume that t_n is an increasing sequence, and denote by $u_n(t)$ the solution with $u_n(0)$ be the initial condition. Next, we will prove that $u_n(t)$ converges to ξ in distribution in $L^2_{loc}(\mathbb{R})$. In fact, since

$$\mathbb{E}[\sup_{0\le t\le T}\|u\|^2_{H^n}]\le C(1+\mathbb{E}[\|u_0\|^2_{H^n}]),$$

then we have

$$\sup_{t\geqslant 0}\mathbb{E}[\| u_n(t_n) \|^2_{H^n}]\le C(\sup_n(\| u_n(0) \|^2_{H^n})+\sup_n(\| u_0 \|^2_{L^2})^2+1):=C(R).$$

Since the bounded sets in $H^n(\mathbb{R})$ is relatively compact in $L^2_{loc}(\mathbb{R})$. Then there exists $L^2_{loc}(\mathbb{R})$-valued variable ξ and a subsequence $u_n(t_n)$ (we also denote it as $u_n(t_n)$) such that $u_n(t_n)$ converges to ξ in distribution in $L^2_{loc}(\mathbb{R})$. Assume that $\{f_i\}$ is a set of smooth and compactly supported orthonormal basis of $H^n(\mathbb{R})$, then we have

$$\mathbb{E}[\sum_i(u_n(t_n,f_i)^2_{H^n}\wedge M^2]\to\mathbb{E}[\sum_i(\xi,f_i)^2_{H^n}\wedge M^2].$$

Therefore,

$$\mathbb{E}[\sum_i(\xi,f_i)^2_{H^n}\wedge M^2]\le\mathbb{E}[\|\xi\|^2_{H^n}\wedge M^2]\le C(R),$$

and $\mathbb{E}[\sum_i(\xi,f_i)^2_{H^n}]\le C(R)$ as M increases. Thus, it holds that ξ takes value in H^n.

Next we shall show the convergence in $L^2(\mathbb{R})$. Assume that $\{g_i\}$ is a set of smooth orthonormal basis with compactly support of $H^n(\mathbb{R})$, then we have

$$\mathbb{E}[\sum_i(u_n(t_n,g_i)^2_{L^2}\wedge M^2]\to\mathbb{E}[\sum_i(\xi,g_i)^2_{L^2}\wedge M^2].$$

Since $\mathbb{E}[\| u_n(t_n) \|^2_{L^2}]\le C(\mathbb{E}[\| u_n(0) \|^2_{L^2}]+1)$, we have

$$\mathbb{E}[\sum_i(u_n(t_n,g_i)^2_{L^2}]\to\mathbb{E}[\sum_i(\xi,g_i)^2_{L^2}]$$

as M increases. Therefore, it follows that

$$\mathbb{E}[\sum_{i=N+1}^{\infty}(u_n(t_n,g_i)_{L^2}^2]\to\mathbb{E}[\sum_{i=N+1}^{\infty}(\xi,g_i)_{L^2}^2].$$

Since $\mathbb{E}[\sum_{i=N+1}^{\infty}(\xi,g_i)_{L^2}^2]\to 0$, we have

$$\lim_{N\to\infty}\sup_n\mathbb{E}[\sum_{i=N+1}^{\infty}(u_n(t_n,g_i)_{L^2}^2]=0.$$

Therefore, $u_n(t_n)$ converges to ξ in distribution in $L^2(\mathbb{R})$. From the above arguments, we have

$$\mathbb{E}[\|\partial_x^n\xi\|_{L^2}^2]=\lim_{n\to\infty}\mathbb{E}[\|\partial_x^n u_n(t_n))\|_{L^2}^2].$$

Therefore $u_n(t_n)$ converges to ξ in distribution in$H^n(\mathbb{R})$, and $\{P_{t_n}(u_0^n,\cdot):n\in\mathbb{N}\}$ is tight on $\{H^n\}$.

We have completed the proof of Theorem (4.3). □

Theorem 4.4. *Assume the conditions* **(H1), (H2)** *and* **(H3)** *hold. If K is a compact set of $\{H^n\}$, then the sequence of measure $\{P_s(v,\cdot):s\in[0,1],v\in K\}$ is tight on $\{H^n\}$.*

Proof. It suffices to prove that $\{P_s(v,\cdot):s\in[0,1],v\in K\}$ posses a convergent subsequence. Assume that $\{(s_n,v_n)\}\in[0,1]\times K$, then it posses a convergent subsequence, we also denote it as $\{(s_n,v_n)\}$. Without loss of generality, we assume that $(s_n,v_n)\in[0,1]\times K$ converges to $(s,v)\in[0,1]\times K$. Let $u_n(t)$ be the solution with initial data v_n, and $u(t)$ be the solution with the initial data v. Since $u\in\mathbb{D}([0,1];H^n)$, we have

$$\lim_{n\to\infty}\|u(s_n)-u(s)\|_{H^n}=0,\ \text{P.a.s.}$$

Let $R_0=\sup_{v\in K}\|v\|_{H^n}+1$, we can choose a suitable $R>0$ such that for any $\varepsilon>0,\delta>0$

$$\begin{aligned}&P\{\max\{\sup_{s\in[0,t]}\|u(s)\|_{H^n}^2,\sup_{s\in[0,t]}\|u_n(s)\|_{H^n}^2\}\geqslant R\}\\&\leq P\{\sup_{s\in[0,t]}\|u(s)\|_{H^n}^2+\sup_{s\in[0,t]}\|v(s)\|_{H^n}^2\geqslant R\}\\&\leq\frac{1}{R}\mathbb{E}[\sup_{s\in[0,t]}\|u(s)\|_{H^n}^2+\sup_{s\in[0,t]}\|u_n(s)\|_{H^n}^2]\leq\frac{C(R_0)}{R}\leq\frac{\varepsilon}{2}.\end{aligned}$$

Define a stopping time τ as

$$\tau=\inf\{s\geqslant 0:8\tilde{C}(t)s^{\alpha}(\tilde{C}(t)R+\|\bar{u}\|_{X(s)}+\|\int_0^s S_\lambda(s-r)f\mathrm{d}r\|_{X(s)})>1\}.$$

Let $\tau_0=\tau$,

$$\tau_{k+1}=\inf\{s\geqslant\tau_k:8\tilde{C}(t)(s-\tau_k)^{\alpha}(\tilde{C}(t)R+\|\bar{u_{\tau_k}}\|_{X(\tau_k,s)}+\|\int_{\tau_k}^s S_\lambda(s-r)f\mathrm{d}r\|_{X(\tau_k,s)})>1\}.$$

Choose some proper N such that $P(\tau_N \le 1) \le \frac{\varepsilon}{2}$. Let

$$A_n = \{\max\{\sup_{s\in[0,t]} \|u(s)\|^2_{H^n}, \sup_{s\in[0,t]} \|u_n(s)\|^2_{H^n}\} \le R\}.$$

Then, we have

$$\sup_{s\in[0,1]} \|u(s) - v(s)\|_{H^n} \le (2\tilde{C}(t))^{N+1}\|v - v_n\|_{H^n} \to 0.$$

on the interval $A_n \bigcap \{\tau_N \geqslant 1\}$, and $(2\tilde{C}(t))^{N+1}\|v - v_n\|_{H^n} < \delta$ provided with n being sufficiently large. Therefore, have

$$P(\sup_{s\in[0,1]} \|u(s) - u_n(s)\|_{H^n} \le \delta) \geqslant P(A_n \bigcap \{\tau_N \geqslant 1\}) \geqslant 1 - \varepsilon.$$

Therefore, it holds that $\lim\limits_{n\to\infty} \sup\limits_{s\in[0,1]} \|u(s) - u_n(s)\|_{H^n} \to 0,$ P.a.s., which implies that there exists a sequence n_k such that

$$\lim_{k\to\infty} \|u(s) - u_{n_k}(s)\|_{H^n} \to 0, \text{ P.a.s.}$$

and

$$|P_{s_{n_k}}\xi(v_{n_k} - P_s\xi(v)| \le \mathbb{E}[|\xi(u_{n_k}(s_{n_k}) - \xi(u(s_{n_k})|] + \mathbb{E}[|\xi(u(s_{n_k}) - \xi(u(s)|] \to 0$$

for any real valued uniformly continuous function ξ on $H^n(\mathbb{R})$. Therefore, $\{P_s(v,\cdot) : s \in [0,1], v \in K\}$ has a convergent subsequence.

We have completed the proof of Theorem 4.4. □

Theorem 4.5. *Assume the conditions* **(H1), (H2)** *and* **(H3)** *hold. Then* $\mu_n(\cdot) = \frac{1}{n}\int_0^n P_t(0,\cdot)\mathrm{d}t, n = 1, 2, ...$ *is tight on* $H^n(\mathbb{R})$.

Proof. For any $\varepsilon > 0$, since $\{P_n(0,\cdot) : n \geqslant 0\}$ is tight, then we can choose a compact set $K_\varepsilon \subset H^n(\mathbb{R})$ such that $\sup\limits_n\{P_n(0, K^c_\varepsilon) \le \frac{\varepsilon}{2}$. Since $\{P_s(v,\cdot) : s \in [0,1], v \in K\}$ is tight on$\{H^n\}$, then by choosing a compact set $A_\varepsilon \subset H^m(\mathbb{R})$ such that

$$\sup_{s\in[0,1],v\in K_\varepsilon} \{P_n(0, A^c_\varepsilon) \le \frac{\varepsilon}{2}.$$

We can deduce

$$\begin{aligned}
\mu_n(A^c_\varepsilon) &= \frac{1}{n}\int_0^n P_t(0, A^c_\varepsilon)\mathrm{d}t \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\int_i^{i+1}\int_{H^n} P_i(0,\mathrm{d}y)P_{t-i}(y, A^c_\varepsilon)\mathrm{d}t \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\int_i^{i+1}[\int_{K_\varepsilon} P_i(0,\mathrm{d}y)P_{t-i}(y, A^c_\varepsilon) + \int_{K^c_\varepsilon} P_i(0,\mathrm{d}y)P_{t-i}(y, A^c_\varepsilon)]\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{n}\sum_{i=0}^{n-1}\int_i^{i+1}[\frac{\varepsilon}{2}\int_{K_\varepsilon} P_i(0,\mathrm{d}y)+\int_{K_\varepsilon^c} P_i(0,\mathrm{d}y)]\mathrm{d}t \\
&\leq \frac{1}{n}\sum_{i=0}^{n-1}\int_i^{i+1}\varepsilon \mathrm{d}t=\varepsilon,
\end{aligned}
$$

which implies that $\mu_n(\cdot)=\frac{1}{n}\int_0^n P_t(0,\cdot)\mathrm{d}t, n=1,2,...$ is tight on $H^n(\mathbb{R})$.

We have completed the proof of Theorem 4.5. □

Theorem 4.6. *Assume the conditions* **(H1), (H2)** *and* **(H3)** *hold. Then there exists an invariant measures of equation* (1.1).

Proof. By using the Krylov-Bogoliubov Theorem and combing Theorem 4.1 with Theorem 4.5, we can obtain the existence of the invariant measures for the semigroup P_t.

We have completed the proof of Theorem 4.6. □

5. Ergodicity of invariant measure

Theorem 5.1. *Assume the conditions* **(H1), (H2)** *and* **(H3)** *hold. There exists an ergodic invariant measure of equation* (1.1) *with deterministic initial conditions.*

Proof. Let K be the set of all the invariant measures. It is easy to check that K is convex. Assume that $\{\mu_n\}_{n\in\mathbb{N}^+}$ is a sequence of invariant measures in K. Then there exists some constant C such that

$$
\sup_n\int \parallel u\parallel_{H^n}^2\ \mu_n(\mathrm{d}u)\leq C, \quad \text{and} \quad \sup_n\int \parallel u\parallel_{H^n}^4\ \mu_n(\mathrm{d}u)\leq C,
$$

and $\{(\mu_n P_{t_n})(\cdot):n\in\mathbb{N}^+\}=\{\mu_n(\cdot):n\in\mathbb{N}^+\}$ is tight for any deterministic initial condition since K is a close set,then K is compact. The Krein-Milman theorem yields that a convex compact set possesses extremal point. By using Theorem 3.2.7 in [32], we deduce that this extremal point is ergodic. Therefore, the equation (1.1) has an ergodic invariant measure.

We have completed the proof of Theorem 4.6. □

6. Discussion

In this section, we consider the following stochastic weakly damped higher-order KdV equation driven by pure jump noise

$$
\begin{cases}
\mathrm{d}u+(\partial_x^{2n+1}u+u\partial_x u+\lambda u)=f\mathrm{d}t+\int_Z u(t,x)g(z(x))\tilde{\eta}(\mathrm{d}z,\mathrm{d}t)+\int_Z u(t,x)h(z(x))\nu(\mathrm{d}x),\\
u(0,x)=u_0(x),
\end{cases}
\tag{6.1}
$$

where $\lambda,\tilde{\eta},\nu$ are the same with that in the first section. To prove the existence of invariant measure for Markov semigroup generated by equation (6.1), we impose some assumption on noise:

Hypothesis 6.1. (a) For any $u\in X(T)$, there exists a constant $C<\infty$ such that

$$
\int_Z(\|u(t,x)g(z(x))\|_{X(T)}^2+\|u(t,x)h(z(x))\|_{X(T)}^2)\nu(\mathrm{d}z)\leq C(1+\|u\|_{X(T)}^2);
$$

(b) $\nu(0)=0$, $\int_Z(\|z\|_Z^2\nu(\mathrm{d}z)<\infty$ and $\nu(Z)=\rho<\infty$;

(c) Z is continuously embedded in $H^n(\mathbb{R})$.

The mild solution of equation (6.1) can be written as

$$
\begin{aligned}
u(t) &= S_\lambda(t)u_0 - \int_0^t S_\lambda(t-s)u\partial_x u \mathrm{d}s + \int_0^t S_\lambda(t-s)f\mathrm{d}s \\
&+ \int_0^t \int_Z S_\lambda(t-s)u(s,x)g(z(x))\tilde{\eta}(\mathrm{d}z,\mathrm{d}s) + \int_0^t \int_Z S_\lambda(t-s)u(s,x)h(z(x))\nu(\mathrm{d}z)\mathrm{d}s. \qquad (6.2)
\end{aligned}
$$

Lemma 6.2. *If $u(t) \in X(T)$ solves the equation (6.2), under the condition of hypothesis 6.1, there exists a constant $C > 0$,such that:*

$$
\begin{aligned}
&\mathbb{E}[\sup_{0\le t\le T} \|u\|_{L^2}^2] \le C(1 + \mathbb{E}[\|u_0\|_{L^2}^2]), \\
&\mathbb{E}[\sup_{0\le t\le T} \|u\|_{H^n}^2] \le C(1 + \mathbb{E}[\|u_0\|_{H^n}^2]).
\end{aligned}
$$

Theorem 6.3. *Under the conditions of hypothesis 6.1, the equation (6.1) has a unque global mild solution in $X(T)$, which is cádlag.*

With the similar scheme, we obtain the following result.

Theorem 6.4. *Under the conditions of theorem 6.3, P_t is a strong Feller semigroup on $H^n(\mathbb{R})$,and $\mu_n(\cdot) = \frac{1}{n}\int_0^n P_t(0,\cdot)\mathrm{d}t, n = 1,2,...$ is tight on $H^n(\mathbb{R})$, hence, there exists invariant measures of equation (6.1). Furthermore, there exists an ergodic invariant measure with deterministic initial conditions.*

Remark 6.5. If $\nu(Z) = +\infty$, then $u(t)$ is not square integrable, and the uniqueness of solution can not obtain. Hence, we could not prove the existence of invariant measure in this case.

Then we will give some numerical simulation of the invariant measure. To the end, we give the distribution of the solution $\frac{1}{T_M}\sum_{m=0}^{T_M} \mathbb{E}\left[\Phi\left(u(t_m)\right)\right]$ by using the so-called the Monte Carl method as following, one can prove theoretically that it does have a unique invariant measure, which derive to the ergodicity [33].

We firstly use the the norm conservative finite difference scheme introduced by [2] to simulate the equation (1.1) to the

$$
\begin{aligned}
&\frac{U_j^{(n+1)} - U_j^{(n)}}{\Delta t} + \beta\delta^{(5)}\left(\frac{U_j^{(n)} + U_j^{(n+1)}}{2}\right) \\
&+\frac{1}{3}\left(\delta^{(1)}\left(\frac{U_j^{(n)} + U_j^{(n+1)}}{2}\right)^2 + \frac{U_j^{(n)} + U_j^{(n+1)}}{2}\delta^{(2)}\left(\frac{U_j^{(n)} + U_j^{(n+1)}}{2}\right)\right) \\
&= \lambda\left(\frac{U_j^{(n)} + U_j^{(n+1)}}{2}\right) + \sum_{t_i\le s\le t_{i+1}} g\left(\Delta L_s\right) 1_A\left(\Delta L_s(\omega)\right) \qquad (6.3)
\end{aligned}
$$

where

$$
\begin{aligned}
\delta^{(1)}U_j^{(n)} &= \frac{U_{j+1}^{(n)} - U_{j-1}^{(n)}}{2\Delta x}, \\
\delta^{(5)}U_j^{(n)} &= \frac{U_{j+4}^{(n)} - 2U_{j+3}^{(n)} - 2U_{j+2}^{(n)} + 6U_{j+1}^{(n)} - 6U_{j-1}^{(n)} + 2U_{j-2}^{(n)} + 2U_{j+3}^{(n)} - U_{j+4}^{(n)}}{4(\Delta x)^5},
\end{aligned}
$$

and $\sum_{s\le t} 1_A\left(\Delta L_s(\omega)\right)$ is a possion process with the parameter $P\Delta t$.

Now we set $\beta = 0.01, \lambda = 0.5$, and $u_0(x) = sin(x)$. The simulation of (1.1) driven by Poisson noise with $g(u(t-),z) = 0.2u(t-)z$ is given in Figure 1. Figure 2 gives time changes of $\|u(t,\cdot)\|_{L_x^2}$ using different sample trajectories.

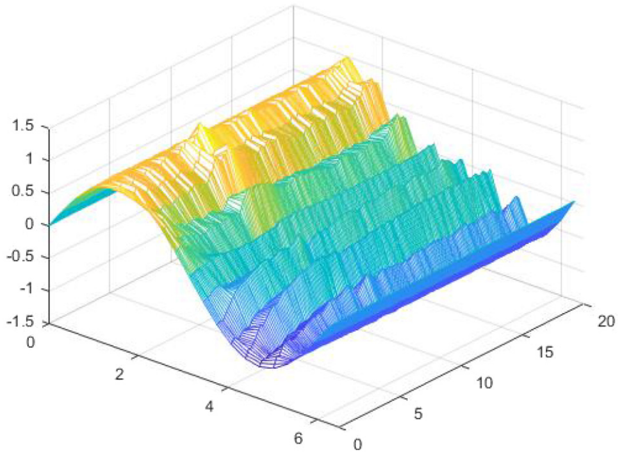

FIGURE 1. Solution of u with Poisson Process.

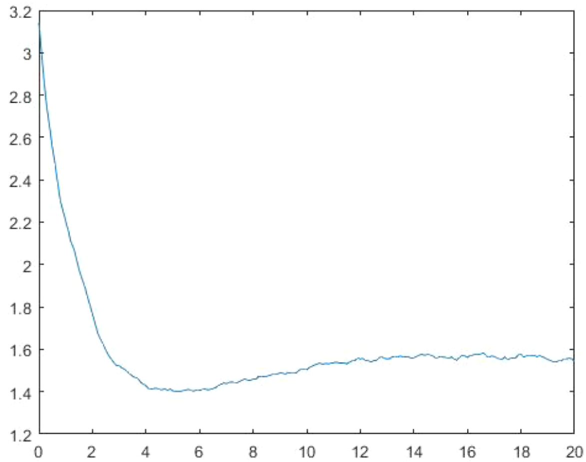

FIGURE 2. Changes of $\|u(t,\cdot)\|_{L^2_x}$.

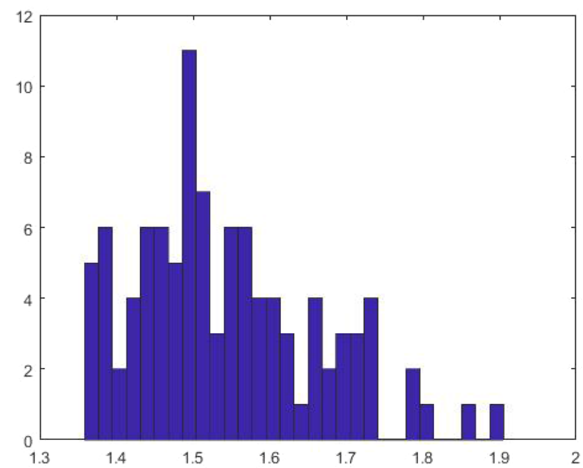

FIGURE 3. Distribution of $\frac{1}{N+1}\sum_{n=0}^{N}\mathbb{E}\left[\Phi\left(U^{(n)}\right)\right]$.

It can be clearly seen from Figure 2 that the decay rate of the equation is slowed down under the influence of noise. At the same time, as shown in Figure 3, it can be seen that for $\Phi(y) = exp(-|y|^2)$, the distribution of the solution $\frac{1}{N+1}\sum_{n=0}^{N}\mathbb{E}\left[\Phi\left(U^{(n)}\right)\right]$ tends to a measure μ as $T \to \infty$.

The above numerical simulation of stochastic damped KdV equation (Fig. 3) in the sense of $\mathbb{E}\|u(t,\cdot)\|_{L_x^2}$ reveals that stochastic damped KdV equation driven by Poisson noise posses an unique ergodic invariant measure.

References

[1] D. Applebaum, Lévy Process and Stochastic Calculus, *Cambridge Studies in Advance Mathematics*. Cambridge University Press (2004).

[2] U.M. Ascher and R.I. McLachlan, On symplectic and multisymplectic schemes for the KdV equation, *J. Sci. Comput.* **25** (2005) 83–104.

[3] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part I: Schrödinger equations. *Geom. Funct. Anal.* **3** (1993) 107–156.

[4] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part II: The KdV equation, *Geom. Funct. Anal.* **3** (1993) 209–262.

[5] J. Bourgain, Periodic Korteweg de vries equation with measures as initial data. *Sel. Math.* **3** (1997) 115–159.

[6] A. Bouard and A. Debussche, On the stochastic Korteweg-de Vries equation. *J. Funct. Anal.* **154** (1998) 215–251.

[7] A. Bouard, A. Debussche and Y. Tsutsumi, White noise driven Korteweg-de Vries equation. *J. Funct. Anal.* **169** (1999) 532–558.

[8] A. Bouard and A. Debussche, Random modulation of solitons for the stochastic Korteweg-de Vries equation. *Ann. Inst. Henri Poincaré (C) Analyse Non Linéaire* **24** (2007) 251–278.

[9] A. Bouard and E. Hausenblas, The nonlinear Schrödinger equation driven by jump processes. *J. Math. Anal. Appl.* **475** (2019) 215–252.

[10] Z. Brzéniak and J. Zabczyk, Regularity of Ornstein-Uhlenbeck process driven by a Lévy white noise. *Potential Anal.* **32** (2010) 153–188.

[11] T. Buckmaster and H. Koch, The Korteweg-de Vries equation at H^{-1} regularity. *Ann. Inst. Henri Poincaré-AN* **32** (2015) 1071–1098.

[12] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Sharp global well-posedness for KdV and modified KdV on R and T. *J. Am. Math. Soc.* **16** (2003) 705–749.

[13] M. Christ, J. Colliander and T. Tao, Asympotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. *Am. J. Math.* **125** (2003) 1235–1293.

[14] Z. Dong, T. Xu and T. Zhang, Invariant measures for stochastic evolution equations of pure jump type. *Stoch. Proc. Appl.* **119** (2009) 410–427.

[15] I. Ekren and I. Kukavica, Existence of invariant measures for the stochastic damped Kdv equation. *Indiana Univ. Math. J.* **3** (2018) 1221–1254.

[16] Z. Guo, Global well-posedness of the Korteweg-de Vries equation in $H^{-3/4}$, *J. Math. Pures Appl.* **91** (2009) 583–597.

[17] M. Hairer, Ergodicity for stochastic PDEs, 2008. http://www.hairer.org/notes/Imperial.pdf.

[18] T. Kappeler and P. Topalov, Global wellposedness of KdV in $H^{-1}(T,R)$. *Duke Math. J.* **135** (2006) 327–360.

[19] C. Kenig, G. Ponce and L. Vega, Well-posedness of the initial value problem for the Korteweg-de Vries equation. *J. Am. Math. Soc.* **4** (1991) 323-347.

[20] C. Kenig, G. Ponce and L. Vega, The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices. *Duke Math. J.* **71** (1993) 1–21.

[21] C. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Comm. Pure Appl. Math.* **46** (1993) 527–620.

[22] C. Kenig, G. Ponce and L. Vega, A bilinear estimate with applications to the KdV equation. *J. Am. Math. Soc.* **9** (1996) 573–603.

[23] C. Kenig, G. Ponce and L. Vega, On the ill-posedness of some canonical dispersive equations. *Duke Math. J.* **106** (2001) 617–633.

[24] R. Killip and M. Visan, KdV is well-posed in H^{-1}. *Ann. Math.* **190** (2019) 249–305.

[25] N. Kishimoto, Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity. *Diff. Int. Equ.* **22** (2009) 447–464.

[26] S. Klainerman and M. Machedon, Smoothing estimates for null forms and applications. *Internat. Math. Res. Notices* (1994) 383ff. approx. 7 pp. (electronic).

[27] D. Korteweg and G.de Vries, On the change of form of long waves advancing in a rectangular channel and on a new type of long stationary waves. *Phil. Mag.* **39** (1985) 422–443.

[28] S. Li, *Well-posedness and asymptotic behavior for some nonlinear evolution equations*. Dissertation, 2015.

[29] Y. Li and W. Yan, The Cauchy problem for higher-order KdV equations forced by white noise. *J. Henan Normal Univ.* **45** (2017) 1–8.

[30] L. Molinet, A note on ill posedness for the KdV equation. *Diff. Int. Equ.* **24** (2011) 759–765.
[31] L. Molinet, Sharp ill-posedness results for the KdV and mKdV equations on the torus. *Adv. Math.* **230** (2012) 1895–1930.
[32] G. Prato and J. Zabczyk, Ergodicity for infinite-dimensional systems. Cambridge University Press, Cambridge (1996).
[33] G. Prato and J. Zabczyk, Stochastic equations in infinite dimensions. Cambridge University Press, Cambridge (1992).
[34] T. Tao, Multilinear weighted convolution of L^2 functions, and applications to non-linear dispersive equations. *Am. J. Math.* **123** (2001) 839–908.
[35] N. Tzvetkov, Remark on the local ill-posedness for KdV equation. *C. R. Acad Sci Paris* **329** (1999) 1043–1047.