

## HOPF BIFURCATION FOR AN SIR MODEL WITH AGE STRUCTURE

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**Abstract.** This paper deals with an SIR model with age structure of infected individuals. We formulate the model as an abstract non-densely defined Cauchy problem and derive the conditions for the existence of all the feasible equilibrium points of the system. The criteria for both stability and instability involving system parameters are obtained. Bifurcation analysis indicates that the system with age structure exhibits Hopf bifurcation which is the main result of this paper. Finally, some numerical examples are provided to illustrate our obtained results.

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### 1. INTRODUCTION

Age is one of the most influential factors for a population growth since individuals with different ages may have different reproduction and survival capacities. Age also plays an important role in the dynamics of infectious diseases, if the transmission rate, the mortality and other vital parameters depend on age [1]. It is natural to introduce and study age-structured epidemic models to offer better understanding and additional insights on transmission mechanism. Age-structured epidemic models have been formulated and their dynamical behavior has been studied extensively [3, 7, 9, 14].

In fact, infection age is also an important factor to the dynamics of infectious diseases, especially, those infectious diseases with longer latency and variable infectivity, such as HIV/AIDS, tuberculosis (TB), and Influenza. Taking HIV/AIDS as an example, the infectivity are different during the latent period. It is estimated that the infectivity at the acute infection period is 30 to 50 times of that at the asymptomatic latent period due to the very high viral load. TB is another example. Latently infectious individuals do not have symptoms and cannot spread the infection to others. Once the latent infection individuals becomes active, the active TB individuals will spread the bacteria in population. Therefore, the infectivity of infected individuals depend on their infection age.

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For the classic compartment epidemic model with age structure, the outflow of the infected individuals is often described by a partial differential equation

$$\frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -d(a)i(t, a) - \gamma(a)i(t, a), \quad t \geq 0, \quad a \geq 0, \quad (1.1)$$

where  $i(t, a)$  denotes the number of the infected individuals with infection age  $a$  at time  $t$ ;  $d(a)$  is the death rate of the infected individuals with infection age  $a$ , which includes the natural death and the disease-induced death; and  $\gamma(a)$  is the recovery rate of the infected individuals with infection age  $a$ . If we take the bilinear incidence, the boundary condition of model (1.1) is

$$i(t, 0) = S(t) \int_0^{+\infty} \beta(a)i(t, a)da. \quad (1.2)$$

Here,  $S(t)$  represents the number of the susceptible individuals at time  $t$ , and  $\beta(a)$  is the infection rate by an infected individual of infection age  $a$ . In fact, after the susceptible individuals are infected successfully by the infectious individuals, they have to experience the latent period, and the length of the latent period varies from disease to disease, and from person to person. In particular, these infected individuals are just infected but not yet infectious in the latent period. For most infectious diseases, it is difficult to determine the exact time when an infected individual becomes infectious. Therefore, we introduce the parameter  $\tau$  to characterize it, namely, the infected individual will become infectious only when infection age  $a$  is greater than  $\tau$ . It implies that the transmission rate  $\beta(a)$  should be expressed by the following piecewise function

$$\beta(a) := \begin{cases} \beta_*, & a \geq \tau, \\ 0, & a < \tau, \end{cases} \quad (1.3)$$

and satisfies  $\beta(a) \in L_+^\infty((0, +\infty), \mathbb{R})$ .

Let  $R(t)$  be the number of the removed individuals at time  $t$ . Following works in [4, 13], we assume that the population obeys the logistic growth. Based on the classic SIR model framework, an SIR model with age structure is established and discussed in the following:

$$\begin{cases} \frac{dS(t)}{dt} = bS(t) \left(1 - \frac{S(t)}{K}\right) - S(t) \int_0^{+\infty} \beta(a)i(t, a)da, & t \geq 0, \\ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -(d(a) + \gamma(a))i(t, a), & t \geq 0, \quad a \geq 0, \\ \frac{dR(t)}{dt} = \int_0^{+\infty} \gamma(a)i(t, a)da - \mu R(t), & t \geq 0, \\ i(t, 0) = S(t) \int_0^{+\infty} \beta(a)i(t, a)da, & t \geq 0, \end{cases} \quad (1.4)$$

where  $b$  is the intrinsic growth rate of the human population,  $K$  is the carrying capacity for the human population of a given region/area, and  $\mu$  is natural death rate of the removed individuals.

Since the first two equations of system (1.4) are independent of the third one, it suffices to consider the first two equations of system (1.4). Thus, we restrict our attention to the following reduced model:

$$\begin{cases} \frac{dS(t)}{dt} = bS(t) \left(1 - \frac{S(t)}{K}\right) - S(t) \int_0^{+\infty} \beta(a)i(t, a)da, & t \geq 0, \\ \frac{\partial i(t, a)}{\partial t} + \frac{\partial i(t, a)}{\partial a} = -\delta(a)i(t, a), & t \geq 0, \quad a \geq 0, \\ i(t, 0) = S(t) \int_0^{+\infty} \beta(a)i(t, a)da, & t \geq 0, \end{cases} \quad (1.5)$$

with the initial condition  $S(0) = S_0 \geq 0$ ,  $i(0, \cdot) = i_0(\cdot) \in L^1_+(0, +\infty)$ , and  $\delta(a) = d(a) + \gamma(a)$ , which is a continuous bounded nonnegative function of  $a$ , and represents the rate at which infected individuals move out this class due to death and treatment.

Model (1.5) involves the interaction of both the population dynamics of logistic type and the transmission dynamics of disease epidemiology. Hence the dynamic behaviors of the system (1.5) may be very complicated. In this paper, our interest is to investigate the existence of non-trivial periodic solutions for system (1.5) by using the bifurcation theory. We formulate the model as an abstract non-densely defined Cauchy problem and derive the conditions for the existence of all the feasible equilibrium points of the system. Through analysing the location of eigenvalues of the associated characteristic equation, we investigate the stability of disease free equilibrium  $E_0$  and the endemic equilibrium  $E_*$  of system (1.5) under some conditions. In addition, the criterion for existence of Hopf bifurcation around the endemic equilibrium  $E_*$  is obtained.

This paper is organized as follows. Some preliminaries results and the well-posedness of the system (1.5) are presented in Section 2. In Section 3, we prove the existence of the equilibria, especially the existence and uniqueness of endemic equilibrium, and linearize the system (1.5) around the equilibria. Then in Section 4, we discuss the stability of the disease free equilibrium  $E_0$ . Following that, in Section 5 we consider the stability of the endemic equilibrium  $E_*$  and we explore the problem of Hopf bifurcations occurring from  $E_*$ . In Section 6, we present some numerical examples to illustrate our theoretical results.

## 2. PRELIMINARIES AND WELL-POSEDNESS

In this section, we will rewrite system (1.5) into an abstract equation on a suitable Banach lattice, establish the well-posedness result for system (1.5). To this end, we first collect some preliminaries on linear operators and  $C_0$ -semigroup theory and some notations to be used in our discussion.

Let  $L : D(L) \subset X \rightarrow X$  be a linear operator on a Banach space  $X$ . Denote by  $\rho(L)$  the resolvent set of  $L$ . The spectrum of  $L$  is  $\sigma(L) = \mathbb{C} \setminus \rho(L)$ . The point spectrum of  $L$  is the set

$$\sigma_p(L) := \{\lambda \in \mathbb{C} : N(\lambda I - L) \neq \{0\}\}.$$

**Definition 2.1.** Let  $L : D(L) \subseteq X \rightarrow X$  be a linear operator. If there exist real constants  $M \geq 1$ , and  $\omega \in \mathbb{R}$ , such that  $(\omega, +\infty) \subseteq \rho(L)$ , and

$$\|(\lambda - L)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}, \quad \text{for } n \in \mathbb{N}_+, \text{ and all } \lambda > \omega.$$

Then the linear operator  $(L, D(L))$  is called a Hille-Yosida operator.

For a Hille-Yosida operator, one has the following perturbation result.

**Lemma 2.2** (see [5, 10]). *Let  $(A, D(A))$  be a Hille-Yosida operator on a Banach space  $X$  and  $B \in \mathcal{L}(X)$ ,  $\mathcal{L}(X)$  denotes the set of all bounded linear operators on  $X$ , then the sum  $C = A + B$  is a Hille-Yosida operator as well.*

If  $(L, D(L))$  is a Hille-Yosida operator on the Banach space  $X$  and set

$$\begin{aligned} X_0 &:= (\overline{D(L)}, \|\cdot\|), \\ D(L_0) &:= \{x \in D(L) : Lx \in X_0\}, \\ L_0x &:= Lx, \quad \text{for } x \in D(L_0). \end{aligned}$$

Then the operator  $(L_0, D(L_0))$  is called the part of  $L$  in  $X_0$ . Further, there is the following result for the operator  $(L_0, D(L_0))$ .

**Lemma 2.3** (see [5, 10]). *If  $(L, D(L))$  is a Hille-Yosida operator, then its part  $(L_0, D(L_0))$  generates a  $C_0$ -semigroup  $(T_0(t))_{t \geq 0}$  on  $X_0$ .*

Now we set about to rewrite system (1.5) into an abstract evolution equation. Let

$$X = \mathbb{R}^2 \times L^1((0, +\infty), \mathbb{R}).$$

Define the linear operator  $\mathcal{A} : D(\mathcal{A}) \subseteq X \rightarrow X$  by

$$\mathcal{A} \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} -\mu x \\ -y(0) \\ -y' - \delta(a)y \end{pmatrix},$$

with  $D(\mathcal{A}) = \mathbb{R} \times \{0\} \times W^{1,1}((0, +\infty), \mathbb{R})$ . Then  $\overline{D(\mathcal{A})} = \mathbb{R} \times \{0\} \times L^1((0, +\infty), \mathbb{R})$  which shows  $D(\mathcal{A})$  is not dense in  $X$ . We also introduce a nonlinear map  $\mathcal{F} : D(\mathcal{A}) \rightarrow X$  given by

$$\mathcal{F} \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} rx(1 - \frac{x}{K}) + \frac{\mu}{K}x^2 - x \int_0^{+\infty} \beta(a)y(a)da \\ x \int_0^{+\infty} \beta(a)y(a)da \\ 0 \end{pmatrix},$$

here, let  $b = r - \mu$ ,  $r$  and  $\mu$  are the birth rate and natural death rate of susceptible individuals, respectively, and let

$$u(t) = (S(t), 0, i(t, \cdot))^T.$$

Then we can reformulate system (1.5) as the following abstract Cauchy problem:

$$\begin{cases} \frac{d}{dt} (u(t)) = \mathcal{A}u(t) + \mathcal{F}(u(t)), & t \geq 0, \\ u(0) = u_0, \end{cases} \quad (2.1)$$

where  $u_0 = (S_0, 0, i_0(a))^T$ .

In general, it is difficult to find a strong solution for an abstract differential equation like (2.1). So, we solve (2.1) in integrated form

$$u(t) = u_0 + \mathcal{A} \int_0^t u(s)ds + \int_0^t \mathcal{F}(u(s))ds. \quad (2.2)$$

Set

$$X_0 = \overline{D(\mathcal{A})} = \mathbb{R} \times \{0\} \times L^1((0, +\infty), \mathbb{R}),$$

$$X_{0+} = \mathbb{R}_+ \times \{0\} \times L_+^1((0, +\infty), \mathbb{R}).$$

We will show that  $(\mathcal{A}, D(\mathcal{A}))$  is a Hille-Yosida operator in Theorem 3.2 in the next section, and hence from Lemma 2.3 it generates a  $C_0$ -semigroup on the closure of its domain. As a result, we have the following well-posedness theorem for the system (2.1).

**Theorem 2.4.** *For any  $u_0 \in X_{0+}$ , the system of equations (1.5) represented by the integral equation (2.2) has a unique continuous solution with values in  $X_{0+}$ . Moreover, the map  $\Phi : [0, +\infty) \times X_{0+} \mapsto X_{0+}$  defined*

by  $\Phi(t, u_0) = u(t, u_0)$  is a continuous semi-flow, i.e. the map is continuous and satisfies that  $\Phi(0, \cdot) = I$  and  $\Phi(t, \Phi(s, \cdot)) = \Phi(t + s, \cdot)$ .

Due to the biological interpretation of model (1.5), only non-negative solutions are meaningful to be considered. The following result reveals that  $(S(t), i(t, a))$  associated with non-negative initial condition remains non-negative and bounded ultimately.

**Theorem 2.5.** *All solutions of system (1.5) with non-negative initial condition remain non-negative for all  $t \geq 0$  and are ultimately bounded.*

*Proof.* First of all, we show that  $S(t)$  is non-negative for all  $t \geq 0$ . From the first equation of (1.5), we have  $S(t) = S_0 e^{\int_0^t [b(1 - \frac{S(\theta)}{K}) - \int_0^\infty \beta(a) i(\theta, a) da] d\theta} > 0$ . Integrating the second equation in the system (1.5) along the characteristic line yields that

$$i(t, a) = \begin{cases} i(t-a, 0) e^{-\int_0^a \delta(\theta) d\theta}, & a \leq t, \\ i_0(a-t) e^{-\int_{a-t}^a \delta(\theta) d\theta}, & a > t. \end{cases} \quad (2.3)$$

It is clear that  $i(t, a)$  remains nonnegative for any nonnegative initial values.

Next, we will show that the solutions of (1.5) are ultimately bounded. Because the positivity of  $S(t)$  and  $i(t, a)$ , we have

$$\frac{dS(t)}{dt} \leq bS(t) \left(1 - \frac{S(t)}{K}\right).$$

Hence, we have  $\limsup_{t \rightarrow +\infty} S(t) \leq K$  and by the biological significance of the parameter  $K$ , we can obtain  $S(t) \leq K$ . Let  $I(t) = \int_0^{+\infty} i(t, a) da$ , which represents the total number of infected individuals at time  $t$ , and  $\delta^* = \min_{a \geq 0} \delta(a)$ . It is reasonable to assume that the age is finite, which leads to  $\lim_{a \rightarrow +\infty} i(t, a) = 0$ . Then from system (1.5), we have

$$\begin{aligned} (S(t) + I(t))' &= bS(t) \left(1 - \frac{S(t)}{K}\right) - i(t, 0) + \int_0^{+\infty} \left(-\frac{\partial i(t, a)}{\partial a} - \delta(a) i(t, a)\right) da \\ &= bS(t) \left(1 - \frac{S(t)}{K}\right) - \lim_{a \rightarrow +\infty} i(t, a) - \int_0^{+\infty} \delta(a) i(t, a) da \\ &\leq bS(t) - \delta^* I(t) \\ &\leq rK - \zeta (S(t) + I(t)), \end{aligned}$$

where  $\zeta := \min\{\mu, \delta^*\}$ . Therefore,

$$\limsup_{t \rightarrow +\infty} (S(t) + I(t)) \leq \frac{rK}{\zeta}.$$

It follows that the omega limit set of system (1.5) is contained in the following bounded feasible region:

$$\Gamma = \left\{ (S, i(\cdot)) : 0 \leq S \leq K, i(\cdot) \geq 0, S + \int_0^{+\infty} i(t, a) da \leq \frac{rK}{\zeta} \right\}.$$

Obviously, this region is positively invariant with respect to system (1.5).  $\square$

### 3. EQUILIBRIA AND LINEARIZED EQUATIONS

In this section, we devote to linearizing the nonlinear system (1.5) around the equilibrium solutions. To this end, we first discuss the existence of equilibria of system (1.5).

It is clear that system (1.5) always has the trivial equilibrium  $O = (0, 0)$  and the disease free equilibrium  $E_0 = (S^{(0)}, 0) = (K, 0)$ . Next, we discuss the existence of the endemic equilibrium  $E_* = (S_*, i_*(a))$ . Put

$$\begin{cases} bS_* \left(1 - \frac{S_*}{K}\right) - S_* \int_0^{+\infty} \beta(a) i_*(a) da = 0, \\ \frac{di_*(a)}{da} = -\delta(a) i_*(a), \\ i_*(0) = S_* \int_0^{+\infty} \beta(a) i_*(a) da. \end{cases} \quad (3.1)$$

Solving the second equation of (3.1), we get

$$i_*(a) = i_*(0) e^{-\int_0^a \delta(\theta) d\theta}. \quad (3.2)$$

Then, from the last equation of (3.1), we have  $i_*(0) = S_* i_*(0) \int_0^{+\infty} \beta(a) e^{-\int_0^a \delta(\theta) d\theta} da$ , namely,  $S_* = \frac{1}{\int_0^{+\infty} \beta(a) e^{-\int_0^a \delta(\theta) d\theta} da}$ . Substituting (3.2) into the first equation of (3.1) we obtain

$$i_*(0) = \frac{b \left(1 - \frac{S_*}{K}\right)}{\int_0^{+\infty} \beta(a) e^{-\int_0^a \delta(\theta) d\theta} da}. \quad (3.3)$$

In the following, we define the basic reproduction number as

$$R_0 := \frac{K}{S_*} = K \int_0^{+\infty} \beta(a) e^{-\int_0^a \delta(\theta) d\theta} da. \quad (3.4)$$

Here,  $\beta(a)$  is the infection rate by an infected individual of infection age  $a$  and  $K = S^{(0)}$  denotes the total number of susceptible individuals. As  $e^{-\int_0^a \delta(\theta) d\theta}$  is the probability that an infected individual survives to age  $a$ ,  $\int_0^{+\infty} \beta(a) e^{-\int_0^a \delta(\theta) d\theta} da$  describes the average number of secondary infections resulting from one primary infection into an otherwise susceptible population. Therefore,  $R_0$  represents the total number of the newly infected cases by an infectious individual in the entire infection period.

Furthermore, (3.3) turns out to be

$$i_*(0) = \frac{b}{\int_0^{+\infty} \beta(a) e^{-\int_0^a \delta(\theta) d\theta} da} \left(1 - \frac{1}{R_0}\right).$$

Therefore, if  $R_0 > 1$ , the system (1.5) has a unique endemic equilibrium  $E_*$ , specifically, we infer that the following result:

**Theorem 3.1.** *The system (1.5) has always the trivial equilibrium  $O = (0, 0)$  and the disease free equilibrium  $E_0 = (S^{(0)}, 0)$ . If  $R_0 > 1$ , then there exists a unique endemic equilibrium  $E_* = (S_*, i_*(a))$  for system (1.5).*

Next, let  $S(t) = x(t) + \bar{S}$ ,  $i(t, a) = y(t, a) + \bar{i}(a)$ , where  $\bar{E} = (\bar{S}, \bar{i}(a))$  is any steady state of system (1.5). Let  $\tilde{u}(t) = (x(t), 0, y(t, a))$ ,  $\tilde{u} = (\bar{S}, 0, \bar{i}(a))$ . Then the system (2.1) is equivalent to the following Cauchy problem

$$\begin{cases} \frac{d}{dt} \tilde{u}(t) = \mathcal{A} \tilde{u}(t) + \mathcal{F}(\tilde{u}(t) + \tilde{u}) - \mathcal{F}(\tilde{u}(t)), & t \geq 0, \\ \tilde{u}(0) = u(0) - \tilde{u}. \end{cases}$$

By the direct computations, we can obtain the linearized system of (2.1) around  $\bar{u}$  as the following form

$$\begin{cases} \frac{d}{dt}\tilde{u}(t) = \mathcal{A}\tilde{u}(t) + D\mathcal{F}(\bar{u})(\tilde{u}(t)), & t \geq 0, \\ \tilde{u}(0) = u(0) - \bar{u}, \end{cases} \quad (3.5)$$

in which

$$D\mathcal{F}(\bar{u}) \begin{pmatrix} x(t) \\ 0 \\ y(t, a) \end{pmatrix} = \begin{pmatrix} \left( r - \frac{2b}{K}\bar{S} - \int_0^{+\infty} \beta(a)\bar{i}(a)da \right) x(t) - \bar{S} \int_0^{+\infty} \beta(a)y(t, a)da \\ x(t) \int_0^{+\infty} \beta(a)\bar{i}(a)da + \bar{S} \int_0^{+\infty} \beta(a)y(t, a)da \\ 0 \end{pmatrix}$$

Obviously,  $D\mathcal{F}(\bar{u})$  is a compact bounded linear operator on  $X$ .

Denote  $\Omega := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > -\zeta\}$ , where  $\zeta$  is defined in Theorem 2.5, then we may verify that

**Theorem 3.2.** *The operator  $(\mathcal{A}, D(\mathcal{A}))$  is a Hille-Yosida operator.*

*Proof.* For  $(\varphi, \omega, \psi) \in X$ ,  $(\alpha, 0, \eta) \in D(\mathcal{A})$ ,  $\lambda \in \Omega$ , we have

$$(\lambda - \mathcal{A})^{-1} \begin{pmatrix} \varphi \\ \omega \\ \psi \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \eta \end{pmatrix} \Leftrightarrow \begin{cases} (\lambda + \mu)\alpha = \varphi, \\ \eta(0) = \omega, \\ \eta' + (\lambda + \delta(a))\eta = \psi. \end{cases}$$

It then follows that

$$\begin{cases} \alpha = \frac{\varphi}{\lambda + \mu}, \\ \eta = \omega e^{-\int_0^a (\lambda + \delta(\theta))d\theta} + \int_0^a \psi(s) e^{-\int_s^a (\lambda + \delta(\theta))d\theta} ds. \end{cases} \quad (3.6)$$

Integrating the last equation of (3.6) with regard to the age variable  $a$  and adding all the equations, we have

$$|\alpha| + \|\eta\|_{L^1} \leq \frac{1}{\lambda + \zeta} (|\varphi| + |\omega| + \|\psi\|_{L^1}).$$

Thus, we have

$$\|(\lambda - \mathcal{A})^{-1}\| \leq \frac{1}{\lambda + \zeta}, \quad \text{for all } \lambda \in \Omega,$$

which shows  $(\mathcal{A}, D(\mathcal{A}))$  is a Hille-Yosida operator.  $\square$

Combining Lemma 2.2 and Theorem 3.2, and then, we can obtain immediately that

**Theorem 3.3.** *The operator  $\mathcal{A} + D\mathcal{F}(\bar{u})$  is a Hille-Yosida operator.*

**Theorem 3.4.** *The part of  $(\mathcal{A}, D(\mathcal{A}))$  and  $(\mathcal{A} + D\mathcal{F}(\bar{u}), D(\mathcal{A} + D\mathcal{F}(\bar{u})))$  generate  $C_0$ -semigroups  $(\mathcal{S}(t))_{t \geq 0}$  and  $(\mathcal{T}(t))_{t \geq 0}$ , respectively, on space  $X_0$ .*

In order to establish the stability results for model (1.5), we will analyze the compactness of the generated  $C_0$ -semigroups. For this, we introduce the definition of quasi-compactness for a semigroup below.

**Definition 3.5** (cf. [8]). A  $C_0$ -semigroup  $(\mathcal{T}(t))_{t \geq 0}$  is called quasi-compact if  $\mathcal{T}(t) = \mathcal{T}_1(t) + \mathcal{T}_2(t)$  with the operator families  $\mathcal{T}_1(t)$  and  $\mathcal{T}_2(t)$  satisfying that

- (i)  $\mathcal{T}_1(t) \rightarrow 0$ , as  $t \rightarrow +\infty$ ,

(ii)  $\mathcal{T}_2(t)$  is eventually compact, that is, there is  $t_0 > 0$ , such that  $\mathcal{T}_2(t)$  is compact for all  $t > t_0$ .

For a quasi-compact  $C_0$ -semigroup, one has that

**Lemma 3.6** (cf. [8]). *Let  $(\mathcal{T}(t))_{t \geq 0}$  be a quasi-compact  $C_0$ -semigroup and  $(B, D(B))$  its infinitesimal generator. Then  $e^{\epsilon t} \|\mathcal{T}(t)\| \rightarrow 0$  as  $t \rightarrow +\infty$  for  $\epsilon > 0$  if and only if all eigenvalues of  $B$  have strictly negative real part.*

By the Hille-Yosida estimate in the proof of Theorem 3.2, we have  $\|\mathcal{S}(t)\| \leq e^{-\xi t}$ . Furthermore,  $D\mathcal{F}(\bar{u})\mathcal{S}(t) : X_0 \rightarrow X$  is compact for every  $t > 0$ . Since

$$\mathcal{S}(t) = e^{D\mathcal{F}(\bar{u})t} \mathcal{S}(t) = \mathcal{S}(t) + \sum_{k=1}^{+\infty} \frac{(D\mathcal{F}(\bar{u})t)^k}{k!} \mathcal{S}(t),$$

it is seen that  $(\mathcal{T}(t))_{t \geq 0}$  is quasi-compact. Then by Lemma 3.6 we deduce that, for some  $\epsilon > 0$ ,  $e^{\epsilon t} \|\mathcal{S}(t)\| \rightarrow 0$  as  $t \rightarrow +\infty$  whenever all the eigenvalues of  $(\mathcal{A} + D\mathcal{F}(\bar{u}))$  have negative real part.

From the above arguments we can now make the following conclusion.

**Theorem 3.7.** *The solution semi-flow  $\Phi(t, u_0)$  of system (1.5), defined as in Theorem 2.4, satisfies the following properties.*

- (i) *If all the eigenvalues of  $(\mathcal{A} + D\mathcal{F}(\bar{u}))$  have strictly negative real part, then the steady state  $\bar{u}$  is locally asymptotically stable.*
- (ii) *If, however, at least one eigenvalue of  $(\mathcal{A} + D\mathcal{F}(\bar{u}))$  has strictly positive part, then the steady state  $\bar{u}$  is unstable.*

It is easy to prove that the trivial equilibrium  $O(0, 0)$  of system (1.5) is always unstable. Therefore, we will mainly discuss the stability of the disease free equilibrium  $E_0$  and the endemic equilibrium  $E_*$  in the following sections.

#### 4. STABILITY OF DISEASE FREE EQUILIBRIUM

Based on the preceding discussion, in this section we first discuss the stability of the disease free equilibrium  $E_0 = (S^{(0)}, 0) = (K, 0)$ .

**Theorem 4.1.** *If  $R_0 < 1$ , then the disease free equilibrium  $E_0 = (S^{(0)}, 0)$  of system (1.5) is locally asymptotically stable. If  $R_0 > 1$ ,  $E_0 = (S^{(0)}, 0)$  is unstable.*

*Proof.* Let  $x(t) = S(t) - S^{(0)} = S(t) - K$ ,  $y(t, a) = i(t, a)$ , and the linearizing system (3.5) at  $E_0$  turns out to the following system

$$\begin{cases} x'(t) = -bx(t) - K \int_0^{+\infty} \beta(a) i(t, a) da, \\ \frac{\partial y(t, a)}{\partial t} + \frac{\partial y(t, a)}{\partial a} = -\delta(a) y(t, a), \\ y(t, 0) = K \int_0^{+\infty} \beta(a) i(t, a) da. \end{cases} \quad (4.1)$$

We substitute  $x(t) = x_0 e^{\lambda t}$ ,  $y(t, a) = y_0(a) e^{\lambda t}$  into the equations of (4.1), giving the equations

$$\begin{cases} (\lambda + b) x_0 = -K \int_0^{+\infty} \beta(a) y_0(a) da, \\ \frac{dy_0(a)}{da} = -(\lambda + \delta(a)) y_0(a), \\ y_0(0) = K \int_0^{+\infty} \beta(a) y_0(a) da. \end{cases} \quad (4.2)$$



By the second and third equation in (4.2), we have

$$y_0(a) = y_0(0)e^{-\int_0^a (\lambda + \delta(\theta))d\theta}. \quad (4.3)$$

By some computation, we get the characteristic equation of the system (4.1) as below:

$$\Delta_0(\lambda) = (\lambda + b) \left( K \int_0^{+\infty} \beta(a) e^{-\int_0^a (\lambda + \delta(\theta))d\theta} da - 1 \right) = 0. \quad (4.4)$$

Obviously,  $\Delta_0(\lambda) = 0$  has one negative real root  $\lambda_1 = -b$ . For

$$\Lambda(\lambda) := K \int_0^{+\infty} \beta(a) e^{-\int_0^a (\lambda + \delta(\theta))d\theta} da - 1,$$

if  $\lambda$  is restricted to  $\mathbb{R}$ , then  $\Lambda(\lambda)$  is a continuous real function strictly decreasing and satisfies that

$$\lim_{\lambda \rightarrow +\infty} \Lambda(\lambda) = -1, \quad \Lambda(0) = R_0 - 1.$$

Thus, when  $R_0 > 1$ ,  $\Lambda(\lambda) = 0$  has at least one positive real root, thus the disease free equilibrium  $E_0$  is unstable. If, however,  $R_0 < 1$ , then  $\Lambda(\lambda) = 0$  has no complex solutions with nonnegative real part. In fact, suppose that  $\lambda = \alpha + \omega i$  is an arbitrary complex root with  $\alpha \geq 0$ , then

$$\begin{aligned} 1 = |\Lambda(\lambda) + 1| &= \left| K \int_0^{+\infty} \beta(a) e^{-\int_0^a (\alpha + \omega i + \delta(\theta))d\theta} da \right| \\ &= K \int_0^{+\infty} \beta(a) \left| e^{-\int_0^a (\alpha + \omega i + \delta(\theta))d\theta} \right| da \\ &= K \int_0^{+\infty} \beta(a) e^{-\int_0^a (\alpha + \delta(\theta))d\theta} da \\ &= \Lambda(\alpha) + 1 \\ &\leq \Lambda(0) + 1 \\ &= R_0 < 1, \end{aligned}$$

which is a contradiction. Thus, the solutions of  $\Lambda(\lambda) = 0$  must have negative real part. Therefore, the disease free equilibrium  $E_0$  is locally asymptotically stable when  $R_0 < 1$ .  $\square$

**Theorem 4.2.** *If  $R_0 < 1$ , then the disease free equilibrium  $E_0 = (S^{(0)}, 0)$  of system (1.5) is globally asymptotically stable in  $X_{0+}$ .*

*Proof.* Theorem 4.1 implies that the disease free equilibrium  $E_0 = (S^{(0)}, 0)$  of system (1.5) is locally asymptotically stable. Therefore, here, we just prove that  $E_0$  is globally attractive when  $R_0 < 1$ . That is, for any non-negative solutions  $(S(t), i(t, a))$  of system (1.5), it should be only proved that  $\lim_{t \rightarrow +\infty} (S(t), i(t, a)) = (S^{(0)}, 0)$ .

By the first equation of (1.5), we can obtain

$$\frac{dS(t)}{dt} \leq bS(t) \left( 1 - \frac{S(t)}{K} \right).$$

For  $\frac{dS_1(t)}{dt} = bS_1(t) \left( 1 - \frac{S_1(t)}{K} \right)$ , we have  $\lim_{t \rightarrow \infty} S_1(t) = K$ . Namely, for any  $\varepsilon > 0$ ,  $\exists t_1 > 0$ , such that

$$S_1(t) \leq K + \varepsilon, \quad t \geq t_1.$$

The comparison principle implies that,

$$S(t) \leq K + \varepsilon, \quad t \geq t_1.$$

Furthermore,  $i(t, a) \leq \bar{i}(t, a)$ , where  $\bar{i}(t, a)$  is the solution of the following equations

$$\begin{cases} \frac{\partial \bar{i}(t, a)}{\partial t} + \frac{\partial \bar{i}(t, a)}{\partial a} = -\delta(a)\bar{i}(t, a), \\ \bar{i}(t, 0) = (K + \varepsilon) \int_0^{+\infty} \beta(a)\bar{i}(t, a)da. \end{cases} \quad (4.5)$$

Using the similar arguments to the proof of Theorem 4.1, we can infer that system (4.5) admits a solution of the form  $\bar{i}(t, a) = \bar{i}_0(a)e^{\lambda_0 t}$ , where  $\bar{i}_0(a)$  is positive and  $\lambda_0$  is a root of the corresponding characteristic equation of system (4.5). The comparison principle yields that  $i(t, a) \leq \bar{i}(t, a) = \bar{i}_0(a)e^{\lambda_0 t}$ . For  $\varepsilon$  sufficiently small, since  $R_0 < 1$ , all the eigenvalues of system (4.5) have negative real part, which implies that

$$\lim_{t \rightarrow +\infty} i(t, a) = 0.$$

That is, the first equation in (1.5) are asymptotic to equation

$$\frac{d\check{S}(t)}{dt} = b\check{S}(t) \left( 1 - \frac{\check{S}(t)}{K} \right),$$

and

$$\lim_{t \rightarrow +\infty} \check{S}(t) = K = S^{(0)}.$$

Finally, by the asymptotic autonomous semi-flow theory (see Corollary 4.3 in [12]), we conclude that

$$\lim_{t \rightarrow +\infty} S(t) = K = S^{(0)}.$$

Therefore, if  $R_0 < 1$ , then

$$\lim_{t \rightarrow +\infty} (S(t), i(t, a)) = (S^{(0)}, 0).$$

This completes the proof. □

## 5. HOPF BIFURCATION AROUND ENDEMIC EQUILIBRIUM $E_*$

In this section, we are interested in the stability of the endemic equilibrium  $E_* = (S_*, i_*(a))$  and the existence of Hopf bifurcations around  $E_*$  in the case where  $\delta(a) = \delta$ .

Based on (1.3), we can rewritten  $R_0$  in (3.4) as

$$R_0(\tau) = \frac{K\beta_*e^{-\delta\tau}}{\delta},$$

and, accordingly, we rewritten the endemic equilibrium as

$$E_* = (S_*, i_*(a)) = \left( \frac{\delta}{\beta_* e^{-\delta\tau}}, \frac{b(1 - \frac{1}{R_0(\tau)})\delta e^{-\delta a}}{\beta_* e^{-\delta\tau}} \right).$$

To show the local stability of  $E_*$ , we linearize the system (1.5) around  $E_*$ . Namely, we introduce the perturbation variables  $x(t) = S(t) - S_*$ ,  $y(t, a) = i(t, a) - i_*(a)$ , which leads to

$$\begin{cases} x'(t) = \left( b - \frac{2b}{K}S_* - \int_0^{+\infty} \beta(a)i_*(a)da \right) x(t) - S_* \int_0^{+\infty} \beta(a)y(t, a)da, \\ \frac{\partial y(t, a)}{\partial t} + \frac{\partial y(t, a)}{\partial a} = -\delta y(t, a), \\ y(t, 0) = x(t) \int_0^{+\infty} \beta(a)i_*(a)da + S_* \int_0^{+\infty} \beta(a)y(t, a)da. \end{cases}$$

To analyze the asymptotic behavior of  $E_*$ , we look for solutions of the form  $x(t) = x_0 e^{\lambda t}$ ,  $y(t, a) = y_0(a) e^{\lambda t}$ . Thus, we can consider the following eigenvalue problem:

$$\begin{cases} \lambda x_0 = \left( b - \frac{2b}{K}S_* - \int_0^{+\infty} \beta(a)i_*(a)da \right) x_0 - S_* \int_0^{+\infty} \beta(a)y_0(a)da, \\ \frac{dy_0(a)}{da} = -(\lambda + \delta)y_0(a), \\ y_0(0) = x_0 \int_0^{+\infty} \beta(a)i_*(a)da + S_* \int_0^{+\infty} \beta(a)y_0(a)da. \end{cases} \quad (5.1)$$

Solving the second and the third equation in (5.1) we then get that

$$y_0(a) = y_0(0)e^{-(\lambda+\delta)a}, \text{ and } y_0(0) \left( 1 - S_* \int_0^{+\infty} \beta(a)e^{-(\lambda+\delta)a}da \right) = x_0 \int_0^{+\infty} \beta(a)i_*(a)da. \quad (5.2)$$

The first equation in (5.1) yields that

$$\left( \lambda - b \left( 1 - \frac{2S_*}{K} \right) \right) x_0 = -y_0(0). \quad (5.3)$$

From the second equation in (5.2) and (5.3), we find that (divide out  $x_0$ )

$$(\lambda - b(1 - \frac{2S_*}{K})) \left( 1 - S_* \int_0^{+\infty} \beta(a)e^{-(\lambda+\delta)a}da \right) + \int_0^{+\infty} \beta(a)i_*(a)da = 0.$$

By some computation, we get the characteristic equation

$$\Delta_1(\lambda, \tau) := \frac{\lambda^2 + a_1(\tau)\lambda + a_0(\tau) + (b_1\lambda + b_0(\tau))e^{-\lambda\tau}}{\lambda + \delta} =: \frac{f(\lambda, \tau)}{g(\lambda)} = 0,$$

where

$$a_1(\tau) = \delta + \frac{b}{R_0(\tau)}, \quad a_0(\tau) = \frac{\delta b}{R_0(\tau)}, \quad b_1 = -\delta, \quad b_0(\tau) = \delta b \left( 1 - \frac{2}{R_0(\tau)} \right).$$

It is easy to see that

$$\{\lambda \in \Omega : \det(\Delta_1(\lambda, \tau)) = 0\} = \{\lambda \in \Omega : f(\lambda, \tau) = 0\}.$$

In addition, if  $\tau = 0$ , then

$$f(\lambda, 0) = \lambda^2 + (a_1(0) + b_1)\lambda + a_0(0) + b_0(0) = 0. \quad (5.4)$$

It is clear that  $a_1(0) + b_1(0) = \frac{b}{R_0(0)} > 0$ . In addition, when  $R_0(0) > 1$ , we have  $a_0(0) + b_0(0) = \delta b \left(1 - \frac{1}{R_0(0)}\right) > 0$ . Hence, from the Routh-Hurwitz criterion, all the roots of (5.4) have negative real part. Then, in this case, the endemic equilibrium  $E_*$  is locally asymptotically stable. That is, the following result holds:

**Theorem 5.1.** *Suppose that  $\tau = 0$ . Then, the endemic equilibrium  $E_*$  of system (1.5) is locally asymptotically stable when  $R_0(0) > 1$ .*

In fact, the characteristic roots have continuous dependence on  $\tau$  implies that Theorem 5.1 is still true for  $\tau > 0$  sufficiently small, and such that  $R_0(\tau) > 1$ . However, some roots of  $f(\lambda, \tau) = 0$  may cross the imaginary axis to the right part as  $\tau$  increases. To analyze this case, we rewrite the characteristic equation  $f(\lambda, \tau) = 0$  as

$$P(\lambda, \tau) + Q(\lambda, \tau)e^{-\lambda\tau} = 0, \quad (5.5)$$

where  $P(\lambda, \tau) = \lambda^2 + a_1(\tau)\lambda + a_0(\tau)$ ,  $Q(\lambda, \tau) = b_1\lambda + b_0(\tau)$ . Obviously,  $P(\lambda, \tau)$  and  $Q(\lambda, \tau)$  are both analysis function respect to  $\lambda$  and differentiable respect to  $\tau$ . We will first justify the following hypotheses hold.

- (i)  $P(0, \tau) + Q(0, \tau) \neq 0$ ;
- (ii)  $P(i\omega, \tau) + Q(i\omega, \tau) \neq 0$ ;
- (iii)  $\limsup \left\{ \left| \frac{Q}{P} \right| : |\lambda| \rightarrow \infty, \operatorname{Re} \lambda \geq 0 \right\} < 1, \forall \tau \in R_+$ ;
- (iv)  $F(\omega, \tau) = |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2$ ;
- (v) Each positive roots  $\omega(\tau)$  of  $F(\omega, \tau) = 0$  is continuous and differentiable in  $\tau$  whenever it exists.

By the directly computation, we obtain

$$\begin{aligned} P(0, \tau) + Q(0, \tau) &= a_0(\tau) + b_0(\tau) = \delta b \left(1 - \frac{1}{R_0(\tau)}\right) > 0, \\ P(i\omega, \tau) + Q(i\omega, \tau) &= \frac{b}{R_0(\tau)}i + \delta b \left(1 - \frac{1}{R_0(\tau)}\right) - \omega^2 \neq 0, \\ \lim_{|\lambda| \rightarrow \infty} \left| \frac{Q(\lambda, \tau)}{P(\lambda, \tau)} \right| &= \lim_{|\lambda| \rightarrow \infty} \left| \frac{b_1\lambda + b_0(\tau)}{\lambda^2 + a_1(\tau)\lambda + a_0(\tau)} \right| = 0. \end{aligned}$$

It implies that (i), (ii), and (iii) all are satisfied.

Since  $|P(i\omega, \tau)|^2 = a_1^2(\tau)\omega^2 + (a_0(\tau) - \omega^2)^2$ , and  $|Q(i\omega, \tau)|^2 = b_1^2\omega^2 + b_0^2(\tau)$ , we have

$$F(\omega, \tau) = \omega^4 + (a_1^2(\tau) - b_1^2 - 2a_0(\tau))\omega^2 + a_0^2(\tau) - b_0^2(\tau). \quad (5.6)$$

Obviously, (iv) is satisfied, and the implicit function theorem implies that (v) is also satisfied.

Let  $\lambda = i\omega$ ,  $\omega > 0$ , be purely imaginary roots of  $f(\lambda, \tau) = 0$ . Then, we have

$$\begin{aligned} \cos \omega\tau &= \frac{(b_0(\tau) - a_1(\tau)b_1)\omega^2 - a_0(\tau)b_0(\tau)}{b_0^2(\tau) + b_1^2\omega^2}, \\ \sin \omega\tau &= \frac{b_1\omega^3 + (a_1(\tau)b_0(\tau) - a_0(\tau)b_1)\omega}{b_0^2(\tau) + b_1^2\omega^2}. \end{aligned}$$

Put  $\Theta = \omega^2$ , then (5.6) can be rewrite as

$$Q(\Theta) := \Theta^2 + (a_1^2(\tau) - b_1^2 - 2a_0(\tau))\Theta + a_0^2(\tau) - b_0^2(\tau). \quad (5.7)$$

We find that  $a_1^2(\tau) - b_1^2 - 2a_0(\tau) = \frac{b^2}{R_0(\tau)} > 0$ , and  $a_0^2(\tau) - b_0^2(\tau) = \delta^2 b^2 (1 - \frac{1}{R_0(\tau)}) (\frac{3}{R_0(\tau)} - 1)$ . Then, the following lemma holds.

**Lemma 5.2.** *Let  $\tau > 0$ .*

(i) *If  $R_0(\tau) > 3$ , then  $Q(\Theta) = 0$  has a unique positive root.*

(ii) *If  $R_0(\tau) \in (1, 3]$ , then  $Q(\Theta) = 0$  has no positive root.*

(iii) *If  $R_0(\tau) \leq 1$ , then there is no endemic equilibrium. Moreover, if  $R_0(\tau) < 1$ , then the disease free equilibrium  $E_0$  is globally asymptotically stable in  $X_{0+}$ .*

If  $Q(\Theta) = 0$  does not have the positive root, then the stability of the equilibrium  $E_*$  will not change as  $\tau$  increases. Therefore, we have the following result:

**Theorem 5.3.**

(i) *If  $\frac{K\beta_*}{\delta} \leq 1$ , then for any  $\tau \geq 0$  we have  $R_0(\tau) \leq 1$ .*

(ii) *If  $\frac{K\beta_*}{\delta} \in (1, 3]$ , then for any  $\tau \in [0, \bar{\tau}_2)$  we have  $R_0(\tau) \in (1, 3]$  and the endemic equilibrium  $E_*$  is locally asymptotically stable. While if  $\tau \geq \bar{\tau}_2$ , then we have  $R_0(\tau) \leq 1$ , where  $\bar{\tau}_2$  is defined by  $\frac{1}{\delta} \ln \frac{K\beta_*}{\delta}$ .*

(iii) *If  $\frac{K\beta_*}{\delta} > 3$ , then for each  $\tau \in [\bar{\tau}_1, \bar{\tau}_2)$  we have  $R_0(\tau) \in (1, 3]$  and the endemic equilibrium  $E_*$  is locally asymptotically stable. While if  $\tau \geq \bar{\tau}_2$ , then we have  $R_0(\tau) \leq 1$ , and if  $\tau \in [0, \bar{\tau}_1)$ , then we have  $R_0(\tau) > 3$ , where  $\bar{\tau}_1$  is defined by  $\frac{1}{\delta} \ln \frac{K\beta_*}{3\delta}$ .*

If  $Q(\Theta) = 0$  has the positive root, then the stability of the endemic equilibrium  $E_*$  may change when  $\tau$  passes through some specific values.

Obviously, when  $0 < \tau < \bar{\tau}_1$  and  $\frac{K\beta_*}{\delta} > 3$ ,  $Q(\Theta) = 0$  has the unique positive real root

$$\Theta_* = -\frac{b^2}{R_0^2(\tau)} + \sqrt{\frac{b^4}{R_0^4(\tau)} + 4\delta^2 b^2 (1 - \frac{1}{R_0(\tau)}) (1 - \frac{3}{R_0(\tau)})}.$$

Namely,  $\omega(\tau_*) = \sqrt{\Theta_*}$  is the unique positive real root of  $F(\omega, \tau) = 0$ . Furthermore, a set is defined by

$$\Sigma = \{\tau : 0 < \tau < \bar{\tau}_1\}.$$

That is, for  $\tau \in \Sigma$ , there exists  $\omega = \omega(\tau) > 0$  such that  $F(\omega, \tau) = 0$ .

Let  $\theta(\tau) \in (0, 2\pi]$  is defined for  $\tau \in \Sigma$  by the following equation

$$\begin{aligned} \cos \theta(\tau) &= \frac{(b_0(\tau) - a_1(\tau)b_1)\omega^2 - a_0(\tau)b_0(\tau)}{b_0^2(\tau) + b_1^2\omega^2}, \\ \sin \theta(\tau) &= \frac{b_1\omega^3 + (a_1(\tau)b_0(\tau) - a_0(\tau)b_1)\omega}{b_0^2(\tau) + b_1^2\omega^2}. \end{aligned} \quad (5.8)$$

Then, we have  $\omega(\tau)\tau = \theta(\tau) + 2n\pi$ . Hence,  $i\omega_* > 0$  with  $\omega_* = \omega(\tau_*)$  is a purely imaginary root of (5.5) if and only if  $\tau_*$  is a zero of  $C_n(\tau)$  for some  $n \in \mathbb{N}$ , which is defined by

$$C_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad \tau \in \Sigma, \quad n \in \mathbb{N}. \quad (5.9)$$

Theorem 2.2 in [2] implies that the following lemma is true.

**Lemma 5.4.** (see [2]): *Assume that  $\omega(\tau)$  is a positive real root of  $F(\omega, \tau) = 0$  for  $\tau \in \Sigma$ , and at some  $\tau_* \in \Sigma$ ,*

$$C_n(\tau_*) = 0, \quad \text{for some } n \in \mathbb{N}.$$

Then a pair of simple conjugate pure imaginary roots  $\lambda_+(\tau_*) = i\omega(\tau_*)$ , and  $\lambda_-(\tau_*) = -i\omega(\tau_*)$  of the characteristic equation (5.5) exists at  $\tau = \tau_*$  which crosses the imaginary axis from left to right if  $H(\tau_*) > 0$  and crosses the imaginary axis from right to left if  $H(\tau_*) < 0$ , where

$$H(\tau_*) = \text{sign}\left\{\frac{d\text{Re}(\lambda)}{d\tau}\Big|_{\lambda=i\omega(\tau_*)}\right\} = \text{sign}\{F'_\omega(\omega(\tau_*), \tau_*)\} \text{sign}\left\{\frac{dC_n(\tau)}{d\tau}\Big|_{\tau=\tau_*}\right\}.$$

The relationship between  $F(\omega, \lambda)$  and  $Q(\Theta)$  leads to

$$H(\tau_*) = \text{sign}\left\{\frac{d\text{Re}(\lambda)}{d\tau}\Big|_{\lambda=i\omega(\tau_*)}\right\} = \text{sign}\{Q'(\Theta_*)\} \text{sign}\left\{\frac{dC_n(\tau)}{d\tau}\Big|_{\tau=\tau_*}\right\}.$$

Since  $Q'(\Theta_*) = 2\Theta_* + \frac{b^2}{R_0(\tau_*)} > 0$ , we have

$$H(\tau_*) = \text{sign}\left\{\frac{d\text{Re}(\lambda)}{d\tau}\Big|_{\lambda=i\omega(\tau_*)}\right\} = \text{sign}\left\{\frac{dC_n(\tau)}{d\tau}\Big|_{\tau=\tau_*}\right\}.$$

It is easy to observe that  $C_n(0) < 0$ . Moreover, for all  $\tau \in \Sigma$ ,  $C_n(\tau) > C_{n+1}(\tau)$  with  $n \in \mathbb{N}$ . Then, if  $C_0(\tau)$  has no zero in  $\Sigma$ , then the function  $C_n(\tau)$  has no zero in  $\Sigma$ . And if the function  $C_n(\tau)$  has positive zeros  $\tau \in \Sigma$  for some  $n \in \mathbb{N}$ , there exists at least one zero satisfying  $\frac{dC_n(\tau)}{d\tau} > 0$ .

Since  $\omega(\tau) \rightarrow 0$  as  $\tau \rightarrow \bar{\tau}_1$ , (5.8) implies that  $\cos \theta(\tau) \rightarrow 1$ ,  $\sin \theta(\tau) \rightarrow 0$ . Therefore,  $\theta(\tau) \rightarrow 0$ . By using of (5.9), we have  $C_n(\tau) \rightarrow -\infty$  as  $\tau \rightarrow \bar{\tau}_1$ .

The above analysis implies that, for fixed  $n$ , the number of the zeros of  $C_n(\tau)$  is even when  $C'_n(\tau) \neq 0$  for each root  $\tau$  of (5.9). That is, the number of roots of (5.9) is even.

By Theorem 5.1, Lemma 5.4, and Corollary 2.4 in [11], we have the following result:

**Lemma 5.5.** *If (5.9) has roots, denoted by*

$$\{\tau_1, \tau_2, \dots, \tau_q\} \text{ with } \tau_k < \tau_{k+1}, \quad (5.10)$$

and  $C'_{n_k}(\tau_k) \neq 0$ , then  $q$  is an even number, and all roots of (5.5) have the negative real parts when  $\tau \in [0, \tau_1) \cup (\tau_q, \bar{\tau}_1)$ , and (5.5) has at least a pair of roots with positive real parts when  $\tau \in (\tau_1, \tau_q)$ , where  $n_k \in \mathbb{N}$  such that  $S_{n_k}(\tau_k) = 0$ . Furthermore, all other roots of (5.5), except a pair of purely imaginary roots, have negative real parts when  $\tau = \tau_1$  and  $\tau_q$ .

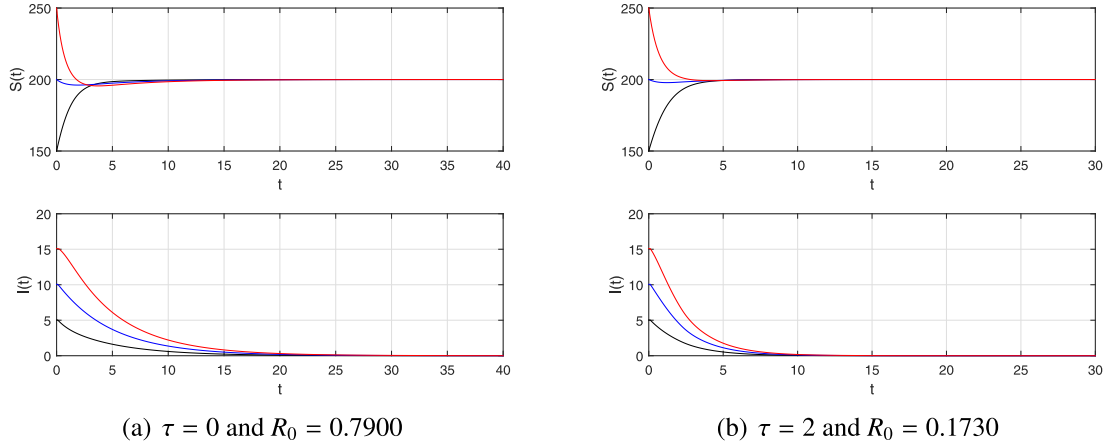
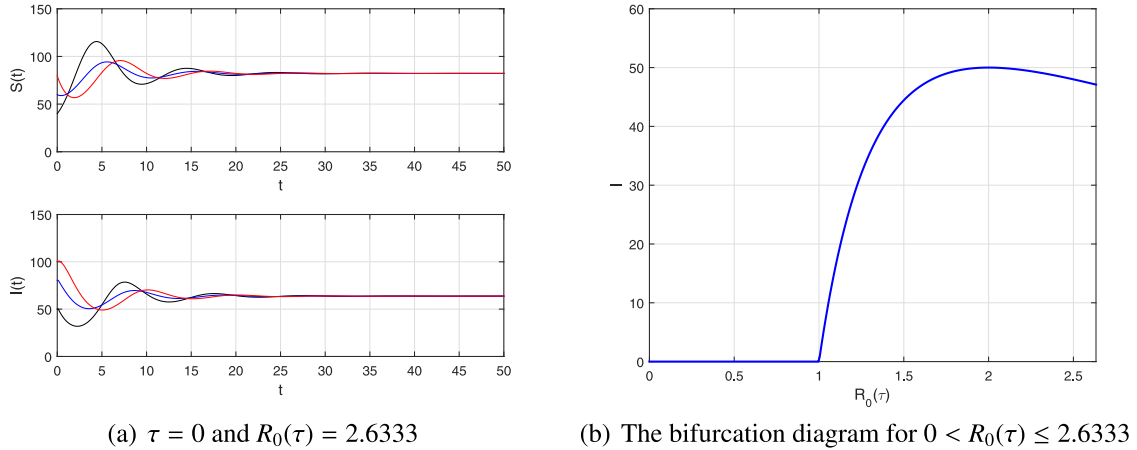
Based on Lemma 5.5 and the Hopf bifurcation theorem for functional differential equations [6], we have the following result.

**Theorem 5.6.** *If (5.9) has positive roots in  $\Sigma$  denoted by (5.10), then the endemic equilibrium  $E_*$  of system (1.5) is asymptotically stable for all  $\tau \in [0, \tau_1) \cup (\tau_q, \bar{\tau}_1)$ ; and unstable when  $\tau \in (\tau_1, \tau_q)$ , with a Hopf bifurcation occurring when  $\tau = \tau_k$ ,  $k = 1, 2, \dots, q$ .*

## 6. NUMERICAL SIMULATIONS AND CONCLUSION

In this paper, we have established and analyzed an SIR model by incorporating the infection age and time delay caused by different infection age.

Firstly, we showed that the well-posedness of the system (1.5), and solutions of the system (1.5) are positive and ultimately bounded. Then we derived the basic reproduction number  $R_0$ , and proved that it is the threshold to determine extinction or survival of disease. Precisely, the disease free equilibrium is globally asymptotically stable if  $R_0 < 1$  and is unstable if  $R_0 > 1$ . We further explored the local stability of endemic equilibrium by analyzing the distribution of roots to the related characteristic equation and the existence of Hopf bifurcation around the endemic equilibrium for the epidemic model (1.5) when  $\delta(a)$  is a constant function.


 FIGURE 1. The stability of  $E_0$  of system (1.5) when  $\frac{K\beta_*}{\delta} \leq 1$ .

 FIGURE 2. The stability of  $E_*$  and the bifurcation diagram of system (1.5) when  $1 < \frac{K\beta_*}{\delta} \leq 3$ .

In the following, we will use Matlab to demonstrate the nonlinear dynamics of system (1.5). We chose the maximum infection age is 100,  $K = 200$ ,  $b = 1$ , and  $\delta = 0.7595$ .

Firstly, we will show the stability of the disease free equilibrium  $E_0$  in the case where  $\frac{K\beta_*}{\delta} \leq 1$ . Taking  $\beta(a) = \beta_* = 0.003$ , we have  $\frac{K\beta_*}{\delta} = 0.7900 < 1$ . When  $\tau = 0$  and  $\tau = 2$ , respectively, accordingly, we can get  $R_0 = 0.7900 < 1$ , and  $R_0 = 0.1730 < 1$ . Figure 1 shows that the solutions of system (1.5) with three different initial values will trend to the disease free equilibrium  $E_0$  as  $t$  trends infinity. It implies that  $E_0 = (S^{(0)}, 0)$  is stable when  $R_0 < 1$ .

Secondly, we will display the stability of the endemic equilibrium  $E_*$  and the dynamic behaviors of system (1.5) in the case where  $1 < \frac{K\beta_*}{\delta} \leq 3$ . Let  $\beta(a) = \beta_* = 0.01$ , then  $\frac{K\beta_*}{\delta} = 2.6333$ , which satisfies  $1 < \frac{K\beta_*}{\delta} \leq 3$ . Firstly, we show that the stability of  $E_*$  in the case where  $R_0(0) > 1$ . When  $\tau = 0$ , we have  $R_0(0) = 2.6333 > 1$ , Figure 2a illustrates that  $E_*$  is stable for  $R_0(0) > 1$  and  $1 < \frac{K\beta_*}{\delta} \leq 3$ . And then, we show that the bifurcation diagram of system (1.5) in the case where  $1 < \frac{K\beta_*}{\delta} \leq 3$ . Figure 2b illustrates that system (1.5) will experience the classic threshold dynamics behaviors. That is, in the case where  $1 < \frac{K\beta_*}{\delta} \leq 3$ , as  $\tau$  changes, the disease free equilibrium  $E_0$  is stable when  $R_0(\tau) < 1$ , otherwise, the endemic equilibrium  $E_*$  is stable. The numerical

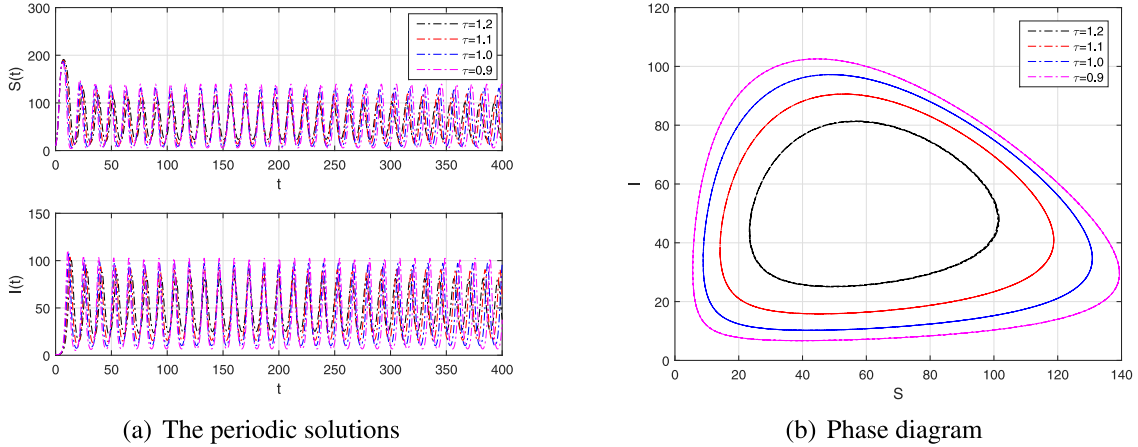


FIGURE 3. The Hopf bifurcation around the endemic equilibrium  $E_*$  when  $\frac{K\beta_*}{\delta} > 3$ .

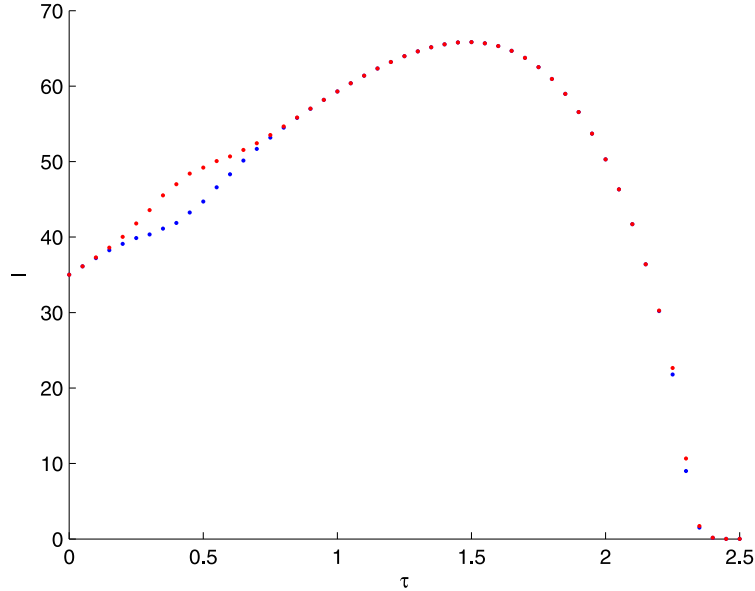


FIGURE 4. The dynamics behaviors of system (1.5) as the delay  $\tau$  changes when  $\frac{K\beta_*}{\delta} > 3$ .

simulations results are consistent with Theorem 5.1 and 5.3. It means if we can't control the basic regeneration number  $R_0(\tau) < 1$ , the disease will persist in the population.

Thirdly, we verify the Hopf bifurcation happens around the endemic equilibrium  $E_*$  in the case where  $\frac{K\beta_*}{\delta} > 3$ . Choosing  $\beta(a) = \beta_* = 0.04$ , we have  $\frac{K\beta_*}{\delta} = 10.5332 > 3$ , and  $\bar{\tau}_1 = 1.6536$ . When  $\tau = 1.2, 1.1, 1.0, 0.9 < \bar{\tau}_1$ , respectively, we can get  $R_0(0.9) > R_0(1.0) > R_0(1.1) > R_0(1.2) = 4.2340 > 3$ . The Hopf bifurcations at the endemic equilibrium  $E_*$  with different delay  $\tau$  are displayed in Figure 3. That is, Figure 3a shows that the periodic solution of system (1.5) with different delay, accordingly, Figure 3b shows that the phase of system (1.5) with different delay.

Finally, we demonstrate the dynamic behaviors of system (1.5) as the delay  $\tau$  changes in the case where  $\frac{K\beta_*}{\delta} > 3$ , see Figure 4. Here,  $\beta(a) = \beta_* = 0.025$ ,  $\frac{K\beta_*}{\delta} = 6.5832$ ,  $\bar{\tau}_1 = 1.0348$ , and  $\bar{\tau}_2 = 2.4813$ . Figure 4 demonstrates that the endemic equilibrium  $E_*$  of system (1.5) will undergoes stability to instability and then to



stability, and eventually changes to the stability of the disease free equilibrium  $E_0$ . In particular, when the endemic equilibrium  $E_*$  is unstable, system (1.5) will experience Hopf bifurcation.

In addition, when Hopf bifurcation exists, it may be global continuation. That is, Hopf bifurcation may exist for any  $\tau \in (\tau_k, \tau_{k+1})$ ,  $k = 1, 2, \dots, q$ , and  $\frac{K\beta_*}{\delta} > 3$ . Although the result has not been proved theoretically due to difficulties in theoretical analysis, it also can be demonstrated by the numerical simulations in Figure 3. Moreover, the direction of Hopf bifurcation and the stability of the bifurcation periodic solutions from  $E_*$  under conditions of Theorem 5.6 both don't been studied in this paper. In addition to these problems, we have not discussed the stability of the disease free equilibrium  $E_0$  when  $R_0(\tau) = 1$ . Whether the endemic equilibrium  $E_*$  is globally stable is also an issue worthy of attention for  $R_0(0) > 1$ . We will continue to discuss these aspects in the future.

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