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ASYMPTOTIC BEHAVIOR OF A BAM NEURAL NETWORK WITH DELAYS OF DISTRIBUTED TYPE

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Abstract. In this paper, we examine a Bidirectional Associative Memory neural network model with distributed delays. Using a result due to Cid [*J. Math. Anal. Appl.* **281** (2003) 264–275], we were able to prove an exponential stability result in the case when the standard Lipschitz continuity condition is violated. Indeed, we deal with activation functions which may not be Lipschitz continuous. Therefore, the standard Halanay inequality is not applicable. We will use a nonlinear version of this inequality. At the end, the obtained differential inequality which should imply the exponential stability appears 'state dependent'. That is the usual constant depends in this case on the state itself. This adds some difficulties which we overcome by a suitable argument.

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1. Introduction

Recently, artificial neural networks have received a lot of attention in sciences and engineering, more specifically, in economics, biology, medicine, gas and petroleum industry [1, 25, 40] and in many other disciplines. They are derived in the context of Artificial Intelligence. Their standard design is based on three neural layers known as the input layer, output layer, and some hidden layers. In each layer, the artificial neurons are connected to all neurons in the surrounding layers *via* artificial synapses in a similar manner to biological neurons. The main operations of neural networks are summed up to help predicting, classifying, recognizing models and solving optimization problems. The Recurrent neural networks encompass Hopfield neural networks, Cohen-Grossberg neural networks, Bidirectional Associative Memory neural networks, Cellular networks and many others.

In 1983, Hopfield suggested a single-layer self-associative Hebbian correlator model as follows

$$x_i'(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)), \quad t > 0, \quad i = 1, 2, \ldots, n.$$

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After a few years, Kosko introduced an expansion of Hopfield neural networks named bidirectional associative memory (BAM) neural network [13–15]. It is used in pattern recognition, signal and image processing, optimization problems, and automatic control [21]. Its architecture relies on two interconnected hidden layers of neurons that have no links in a single layer. It is described by the system

$$\begin{cases} x_i'(t) = -c_i x_i(t) + \sum\limits_{j=1}^{m} a_{ji} f_j(y_j(t)) + I_i, \quad i = 1, 2, \dots, n, \\ y_j'(t) = -\bar{c}_j y_j(t) + \sum\limits_{i=1}^{n} \bar{a}_{ij} g_i(x_i(t)) + J_j, \quad j = 1, 2, \dots, m. \end{cases}$$

In applications, it is frequent that, oscillations, divergences, chaos, and bifurcations affect negatively structures. One of the causes may be the occurrence of delays. Owing to the existence of a multitude of parallel paths with axons of different sizes and lengths, neural networks typically have a spatial extent, and therefore a distribution of propagation delays over a time span occurs [19, 26, 41].

Due to the increasing interest in the asymptotic behavior of solutions for designing neural networks, researchers have recently addressed the stability of delayed neural networks (see for instance [5, 8, 10, 12, 16–18, 20, 23, 24, 38, 39] and references therein). In [18, 38, 39], a set of sufficient conditions based on the system parameters guaranteeing the exponential stability of various retarded BAM neural network models was derived by analytical techniques and Lyapunov functionals. In addition, the authors in [5, 10, 17, 20, 23] obtained some LMI-dependent sufficient conditions ensuring either the exponential or asymptotic stability of BAM neural networks involving delays, *via* Karasovski Lyapunov functionals and analytical inequalities. In [8, 16, 24], the asymptotic stability of a class of delayed BAM neural networks was investigated utilizing the LMI approach, Lyapunov functionals, and analytical inequalities. Through a combination of the graph-theoretic approach, degree theory, and Lyapunov functionals, the exponential stability of a BAM neural network with delays was established in [12].

On the other hand, one of the basic components of artificial neural networks involves activation functions, linking the inputs to the outputs of the networks. Generally, the activation function of hidden neurons introduces a degree of nonlinearity that is of significant value in most applications of artificial neural networks. At first, such functions were assumed bounded, smooth, and monotonic [6, 36, 37]. For example, we can cite the threshold function, the piecewise linear function, and the sigmoid function. Thereafter, these conditions were eased somewhat to be of Lipschitz type, which is commonly considered in the existing literature [12, 18, 20, 23, 24, 39]. In view of the importance of non-Lipschitz activation functions in implementations [15], a relaxation of the Lipschitz condition is necessary. This has motivated some researchers to consider discontinuous functions and Hölder-type functions, one can refer to [2, 3, 7, 11, 27–33, 35].

As is well known, Cauchy problems described by

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(t_0) = x_0 \end{cases} \tag{1.1}$$

have local solutions under the continuity of the function f in a neighborhood of (t_0, x_0). This can shown by the theorem of Peano, whilst the uniqueness of the solution is guaranteed under the Lipschitz continuity with respect to the second argument. Weaker conditions have been considered in several papers by Nagumo, Perron, Osgood, Kamke, Tonelli, and many others (see for instance [9]). In particular, Nagumo and Osgood proved the uniqueness of the solution under the condition

$$|f(t, x) - f(t, y)| \le \phi(|x - y|), \quad t > 0,$$

where $\phi(u)$ is a nondecreasing function of 'continuity-modulus' type satisfying the Osgood criterion: $\phi : [0,\infty) \to [0,\infty)$, with $\phi(0) = 0$ and $\phi(u) > 0$ for $u > 0$ and

$$\int_0^\delta \frac{\mathrm{d}u}{\phi(u)} = \infty, \quad \delta > 0.$$

In case one of the components of the function f fails to be Lipschitz continuous and f is Lipschitz continuous with respect to all the other remaining components including the first one t, Cid [4] succeeded in proving the uniqueness of the solution for a system of differential equations.

Motivated by the discussions above, in this paper, we examine the exponential stability of a BAM neural network model with distributed delays. Our first result is based on a nonlinear Halanay inequality, whilst the second one is proved in a direct fashion. Compared to previous literature, we deal here with less restrictive assumptions on the activation functions, which allows a larger class than the ones considered so far. More precisely, we treat non-linear and non-Lipschitz continuous activation functions, satisfying

$$|\phi(t,x) - \phi(t,y)| \le \psi(|x-y|) = |x-y|\tilde{\phi}(|x-y|),\ t > 0 \tag{1.2}$$

for some non-decreasing function $\tilde{\phi}$.

This paper is arranged as follows: In Section 2, we introduce some notation, definitions, and technical lemmas, while Section 3 contains our exponential stability results proven with the help of a non-linear Halanay inequality. Numerical illustrations to confirm the obtained results are given in Section 4.

2. Model description and preliminaries

In this paper, we consider the following BAM neural network with continuously distributed delays, for $i = 1, 2, \dots, n,\ j = 1, 2, \dots, m$

$$\begin{cases} x_i'(t) = -c_i(x_i(t)) + \sum\limits_{j=1}^m a_{ji} f_{1j}(y_j(t)) + \sum\limits_{j=1}^m d_{ji} \int\limits_0^\infty k_{ji}(s) f_{2j}(y_j(t-s))\mathrm{d}s + I_i, \quad t > 0, \\ y_j'(t) = -r_j(y_j(t)) + \sum\limits_{i=1}^n \bar{a}_{ij} g_{1i}(x_i(t)) + \sum\limits_{i=1}^n \bar{d}_{ij} \int\limits_0^\infty h_{ij}(s) g_{2i}(x_i(t-s))\mathrm{d}s + J_j, \quad t > 0, \\ x_i(t) = \phi_i(t), \quad t \le 0, \\ y_j(t) = \varphi_j(t), \quad t \le 0, \end{cases} \tag{2.1}$$

where n and m denote the number of neurons in the first layer L_x and in the second layer L_y, respectively; x_i is the state of the ith neuron in L_x and y_j is the state of the jth neuron in L_y; c_i and r_j refer to the charging time functions of the ith and the jth neuron, respectively; $a_{ji}, \bar{a}_{ij}, d_{ji}$ and $\bar{d}_{ij}$ correspond to the constants accounting for the synaptic connection strengths between neurons; f_{lj} and g_{li} for $l = 1, 2$ represent the activation functions of the ith state and the jth state, respectively; k_{ji} and h_{ij} account for the delay kernel functions; ϕ_i and φ_j are the history functions of the ith state and the jth state, respectively; I_i and J_j stand for the external inputs of the ith and the jth neuron.

Throughout this paper, we assume that

(A1) The functions c_i and r_j are monotone increasing continuous functions and there exist constants $\beta_i > 0$ and $\bar{\beta}_j > 0$ such that

$$\frac{c_i(x) - c_i(y)}{x - y} \ge \beta_i,\ i = 1, 2, ..., n, \text{ for all } x,\ y \in \mathbb{R} \text{ with } x \ne y,$$

$$\frac{r_j(x) - r_j(y)}{x - y} \geq \bar{\beta}_j, \ j = 1, 2, ..., m, \text{ for all } x, \ y \in \mathbb{R} \text{ with } x \neq y.$$

We denote by $\beta = \min_{1 \leq i \leq n} \beta_i, \ \bar{\beta} = \min_{1 \leq j \leq m} \bar{\beta}_j$.

This means that the charging time functions c_i and r_j are "superlinear". With these assumptions, the first two terms in the right hand sides of (2.1) are indeed dissipative terms (see the first two terms in the right hand sides of (3.4) and (3.5) below).

(A2) The initial data ϕ_i and $\varphi_j, \ i = 1, 2, ..., n, \ j = 1, 2, ..., m, \ t \leq 0$ are real-valued continuous functions.

(A3) The delay kernels k_{ji} and h_{ij} are nonnegative continuous functions such that for $i = 1, 2, ..., n, \ j = 1, 2, ..., m$,

$$\int_0^\infty e^{\alpha_0 s} k_{ji}(s) \mathrm{d}s < \infty, \quad \int_0^\infty e^{\alpha_0 s} h_{ij}(s) \mathrm{d}s < \infty, \quad \alpha_0 > 0.$$

The significance of these conditions is that the kernels are "subexponential". Roughly, they assert that the densities in the cells are decreasing in time. This is in line with the fading memory principle. Simple examples are the exponentially decaying kernels.

Next, we present some definitions and necessary lemmas for our main results.

Definition 2.1. [4] The function $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$, U open set, is said to be Lipschitz continuous when fixing a component $i_0 \in \{0, 1, ..., n\}$ if there exists a constant $L > 0$ such that

$$\begin{aligned} &\|f(y_0, \ldots, z_{i_0}, \ldots, y_n) - f(\bar{y}_0, \ldots, z_{i_0}, \ldots, \bar{y}_n)\|_\infty \\ &\leq L\|(y_0, \ldots, y_{i_0-1}, y_{i_0+1}, \ldots, y_n) - (\bar{y}_0, \ldots, \bar{y}_{i_0-1}, \bar{y}_{i_0+1}, \ldots, \bar{y}_n)\|_\infty, \end{aligned}$$

for all $(y_0, \ldots, z_{i_0}, \ldots, y_n), (\bar{y}_0, \ldots, z_{i_0}, \ldots, \bar{y}_n) \in U$.

Definition 2.2. We say that the system (1) is globally exponentially stable, if for any two solutions $(x_i(t), y_j(t))$ and $(\bar{x}_i(t), \bar{y}_j(t))$ (with $(\phi_i(t), \varphi_j(t))$ and $(\bar{\phi}_i(t), \bar{\varphi}_j(t))$ as initial data), there exist two positive constants M and ν such that

$$\|x(t) - \bar{x}(t)\| + \|y(t) - \bar{y}(t)\| \leq M(\|\phi(t) - \bar{\phi}(t)\| + \|\varphi(t) - \bar{\varphi}(t)\|)e^{-\nu t}, \quad t \geq 0.$$

If there exists a unique equilibrium $(x_i^*, y_j^*), \ i = 1, \ldots, n, \ j = 1, \ldots, m$, that is a solution of the system

$$\begin{cases} c_i(x_i^*) = \sum\limits_{j=1}^m a_{ji} f_{1j}(y_j^*) + \sum\limits_{j=1}^m d_{ji} \int\limits_0^\infty k_{ji}(s) f_{2j}(y_j^*) \mathrm{d}s + I_i, \\ r_j(y_j^*) = \sum\limits_{i=1}^n \bar{a}_{ij} g_{1i}(x_i^*) + \sum\limits_{i=1}^n \bar{d}_{ij} \int\limits_0^\infty h_{ij}(s) g_{2i}(x_i^*) \mathrm{d}s + J_j, \end{cases}$$

then we get the usual exponential stability of this equilibrium.

The above properties are local if they hold only nearby the initial data.

Definition 2.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. The upper right Dini derivative $D^+ f(t)$ is defined as

$$D^+ f(t) = \overline{\lim}_{h \to 0^+} \frac{1}{h}(f(t+h) - f(t)).$$

Let $f := (f_1, f_2, \ldots, f_n)$ and f_{i_0} is one of the components of f, where $i_0 \in \{1, 2, \ldots, n\}$.

The main result of Cid [4] is given below.

Theorem 2.4. *For an open set $U \subset \mathbb{R}^{n+1}$, let $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$, and $(t_0, x_0) \in U$. Assume that f is continuous and locally Lipschitz continuous when fixing a component $i_0 \in \{0, 1, \dots, n\}$. If either $i_0 = 0$ or $f_{i_0}(t_0, x_0) \neq 0$, then there exists $a > 0$ such that (1.1) has a unique solution in $[t_0 - a, t_0 + a]$.*

Lemma 2.5. *[22] Let $u : \mathbb{R} \to [0, \infty)$ satisfy the inequality*

$$u'(t) \leq -au(t) + b \int_0^\infty k(s)u(t-s)\mathrm{d}s, \ t > t_0,$$

where $a, b > 0$ and k be a nonnegative piecewise continuous function satisfying $\int_0^\infty k(s)e^{\alpha_0 s}\mathrm{d}s < \infty$ for some $\alpha_0 > 0$. If $b\int_0^\infty k(s)\mathrm{d}s < a$, then there is a positive constant α such that $0 < \alpha < \alpha_0$ and

$$u(t) \leq \bar{u}(t_0)e^{-\alpha(t-t_0)}, \quad t > t_0,$$

where $\bar{u}(t_0) = \sup\limits_{-\infty<\sigma\leq t_0} u(\sigma)$.

In order to prove the first exponential stability result, we appeal to a nonlinear Halanay inequality.

Lemma 2.6. ***Nonlinear Halanay Inequality*** *Let $z(t)$ be a nonnegative differentiable function satisfying*

$$\begin{cases} z'(t) \leq -az(t) + f(z(t)) + \int\limits_{-\infty}^{t} k(t-s)g(z(s))\mathrm{d}s, \ t > 0, \\ z(t) = \varphi(t), \ -\infty < t \leq 0, \end{cases} \tag{2.2}$$

where

$$f(z) \leq z\tilde{f}(z), \ g(z) \leq z\tilde{g}(z) \tag{2.3}$$

for some nonnegative continuous functions $\tilde{f}$ and $\tilde{g}$ with $\tilde{f}(0) = \tilde{g}(0) = 0$. The kernel k is a nonnegative continuous function such that $\int\limits_0^t k(s)e^{\alpha_0 s}\mathrm{d}s < \infty$ for some $\alpha_0 > 0$. Then, there exist $\nu > 0$ and α such that $0 < \alpha < \alpha_0$ and

$$z(t) \leq \bar{z}(0)e^{-\alpha t}, \ t \geq 0, \tag{2.4}$$

when

$$\bar{z}(0) := \sup_{-\infty<t\leq 0} \varphi(t) \leq \nu. \tag{2.5}$$

Proof. Let ν be a real number such that $\tilde{g}(\nu)\int_0^\infty k(s)\mathrm{d}s < a - \tilde{f}(\nu)$. In view of the condition (2.3), the functions $\tilde{f}(z)$ and $\tilde{g}(z)$ can be non-decreasing. As $\tilde{f}(0) = \tilde{g}(0) = 0$, we see that $\nu > 0$. Assume that $\bar{z}(0) = \sup\limits_{-\infty<t\leq 0} \varphi(t) < \nu$ and that $z(s) \leq \nu$ for all $0 \leq s \leq T$. Then, in light of this assumption and (2.2), we find

$$\begin{aligned} z'(t) &\leq -az(t) + z(t)\tilde{f}(z(t)) + \int_{-\infty}^t k(t-s)z(s)\tilde{g}(z(s))\mathrm{d}s \\ &\leq -az(t) + z(t)\tilde{f}(\nu) + \tilde{g}(\nu)\int_{-\infty}^t k(t-s)z(s)\mathrm{d}s \end{aligned}$$

$$\leq -\Big(a - \tilde{f}(\nu)\Big)z(t) + \tilde{g}(\nu)\int_{-\infty}^{t} k(t-s)z(s)\mathrm{d}s,\ t \leq T. \tag{2.6}$$

We can apply Lemma 2.5 to (2.6) in case $\int_0^t k(s)e^{\alpha_0 s}\mathrm{d}s < \infty$ for some $\alpha_0 > 0$, to arrive at

$$z(t) \leq \bar{z}(0)e^{-\alpha t}$$

on $[0, T]$. Assume that there exists a T^* such that $0 < T^* \leq T$ and $z(T^*) = \nu$. Then

$$\nu \leq \bar{z}(0)e^{-\alpha T^*} < \nu$$

which is a contradiction. Therefore, by repeating the process, we obtain

$$z(t) \leq \bar{z}(0)e^{-\alpha t}, \quad t \geq 0.$$

The proof is complete. □

This result has been proved in [34] (submitted). For the sake of completeness, we reproduced the proof here.

3. Exponential stability

In this section, we establish a local exponential stability result of system (2.1). As stated in Theorem 3.1, our result is valid for solutions having close enough initial data. To this end, we need the following assumption on the activation functions.

(A4) The functions f_{lj} and g_{li} are Lipschitz continuous when fixing a component $i_0 \in \{0, 1, \ldots, n\}$ and $j_0 \in \{0, 1, \ldots, m\}$ with the Lipschitz constants L_{lj} and M_{li}, $l = 1, 2,\ i = 1, 2, ..., n,\ j = 1, 2, ..., m$.

In the sense of the definition above, the existence of a local solution is guaranteed by Theorem 2.4. We can extend our solution to be global by establishing a priori estimates.

(A5) Assume that the functions f_{lj_0} and g_{li_0}, $l = 1, 2$ satisfy

$$|f_{lj_0}(y_{j_0}) - f_{lj_0}(\bar{y}_{j_0})| \leq |y_{j_0} - \bar{y}_{j_0}|\tilde{f}_{lj_0}(|y_{j_0} - \bar{y}_{j_0}|),\ l = 1, 2,\ y_{j_0}, \bar{y}_{j_0} \in \mathbb{R}$$

and

$$|g_{li_0}(x_{i_0}) - g_{li_0}(\bar{x}_{i_0})| \leq |x_{i_0} - \bar{x}_{i_0}|\tilde{g}_{li_0}(|x_{i_0} - \bar{x}_{i_0}|),\ l = 1, 2,\ x_{i_0}, \bar{x}_{i_0} \in \mathbb{R},$$

where $\tilde{f}_{lj_0}$ and $\tilde{g}_{li_0}$ are nonnegative, non-decreasing and continuous functions such that $\tilde{f}_{lj_0}(0) = \tilde{g}_{li_0}(0) = 0$.

This assumption avoids and relaxes the classical Lipschitz condition imposed on the activation functions. As a simple example, the pulse-code signal function (average of sampled pulses with an exponential weight) is not Lipschitz continuous.

(A6) The dominance property

$$\Big(\max_{i\neq i_0} M_{2i} + \max_{j\neq j_0} L_{2j}\Big)\int_0^{\infty} H(t)\mathrm{d}t < \min\Big\{\beta - \sum_{i=1}^{n}\sum_{j\neq j_0} L_{1j}|a_{ji}|, \overline{\beta} - \sum_{j=1}^{m}\sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}|\Big\},$$

where

$$H(t) = \sum_{i=1}^{n}\sum_{j=1}^{m} H_{ij}(t), \quad H_{ij}(t) = \sup_{t\geq 0}\Big\{|d_{ji}|k_{ji}(t), |\bar{d}_{ij}|h_{ij}(t)\Big\},\ 1 \leq i \leq n,\ 1 \leq j \leq m$$

holds.

This assumption is a natural one, it has been always assumed in such cases. It guarantees the dissipativity of the system as a whole. Indeed, it states that the passive decay rates dominate the synaptic connection strengths.

Notice that, denoting by $u_i(t)=x_i(t)-\bar{x}_i(t)$, $v_j(t)=y_j(t)-\bar{y}_j(t)$, $u(t)=\sum\limits_{i=1}^{n}|u_i(t)|$ and $v(t)=\sum\limits_{j=1}^{m}|v_j(t)|$, we infer

$$\begin{cases} u_i'(t)=-\tilde{c}_i(u_i(t))+\sum\limits_{j=1}^{m}a_{ji}\tilde{f}_{1j}(v_j(t))+\sum\limits_{j=1}^{m}d_{ji}\int\limits_0^\infty k_{ji}(s)\tilde{f}_{2j}(v_j(t-s))\mathrm{d}s,\ t>0,\ i=1,2,...,n,\\ v_j'(t)=-\tilde{r}_j(v_j(t))+\sum\limits_{i=1}^{n}\bar{a}_{ij}\tilde{g}_{1i}(u_i(t))+\sum\limits_{i=1}^{n}\bar{d}_{ij}\int\limits_0^\infty h_{ij}(s)\tilde{g}_{2i}(u_i(t-s))\mathrm{d}s,\ t>0,\ j=1,2,...,m,\\ u_i(t)=\tilde{\phi}_i(t):=\phi_i(t)-\bar{\phi}_i(t),\ t\le 0,\ i=1,2,...,n,\\ v_j(t)=\tilde{\varphi}_j(t):=\varphi_j(t)-\bar{\varphi}_j(t),\ t\le 0,\ j=1,2,...,m, \end{cases} \tag{3.1}$$

where

$$\tilde{c}_i(u_i(t))=c_i(x_i(t))-c_i(\bar{x}_i(t)),\quad \tilde{r}_j(v_j(t))=r_j(y_j(t))-r_j(\bar{y}_j(t)), \tag{3.2}$$

$$\tilde{f}_{lj}(v_j(t))=f_{lj}(y_j(t))-f_{lj}(\bar{y}_j(t)),\quad \tilde{g}_{li}(u_i(t))=g_{li}(x_i(t))-g_{li}(\bar{x}_i(t)). \tag{3.3}$$

Theorem 3.1. *If **(A1)**–**(A6)** hold, then the solutions of (2.1) are exponentially locally stable. That is the difference of any two solutions converges to zero exponentially provided that their initial data are close enough and we do not start from the equilibrium.*

Proof. From **(A1), (A4)** and **(A5)**, the upper right Dini derivatives $D^+|u_i(t)|$ and $D^+|v_j(t)|$ satisfy

$$\begin{aligned} D^+|u_i(t)| \le{}& -\beta_i|u_i(t)|+\sum_{j\neq j_0}L_{1j}|a_{ji}||v_j(t)|+\sum_{j\neq j_0}|d_{ji}|L_{2j}\int_{-\infty}^t k_{ji}(t-s)|v_j(s)|\mathrm{d}s\\ &+|a_{j_0i}||v_{j_0}(t)|\tilde{f}_{1j_0}(|v_{j_0}(t)|)+\int_{-\infty}^t|d_{j_0i}|k_{j_0i}(t-s)|v_{j_0}(s)|\tilde{f}_{2j_0}(|v_{j_0}(s)|)\mathrm{d}s, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} D^+|v_j(t)| \le{}& -\overline{\beta}_j|v_j(t)|+\sum_{i\neq i_0}M_{1i}|\bar{a}_{ij}||u_i(t)|+\sum_{i\neq i_0}|\bar{d}_{ij}|M_{2i}\int_{-\infty}^t h_{ij}(t-s)|u_i(s)|\mathrm{d}s\\ &+|\bar{a}_{i_0j}||u_{i_0}(t)|\tilde{g}_{1i_0}(|u_{i_0}(t)|)+\int_{-\infty}^t|\bar{d}_{i_0j}|h_{i_0j}(t-s)|u_{i_0}(s)|\tilde{g}_{2i_0}(|u_{i_0}(s)|)\mathrm{d}s, \end{aligned} \tag{3.5}$$

where the summations $\sum\limits_{i=1,i\neq i_0}^{n}$ and $\sum\limits_{j=1,j\neq j_0}^{m}$ are abbreviated by $\sum\limits_{i\neq i_0}$ and $\sum\limits_{j\neq j_0}$, resp. Then

$$\begin{aligned} D^+u(t) \le{}& -\beta u(t)+\sum_{i=1}^{n}\sum_{j\neq j_0}L_{1j}|a_{ji}||v_j(t)|+\sum_{i=1}^{n}\sum_{j\neq j_0}|d_{ji}|L_{2j}\int_{-\infty}^t k_{ji}(t-s)|v_j(s)|\mathrm{d}s\\ &+\sum_{i=1}^{n}|a_{j_0i}||v_{j_0}(t)|\tilde{f}_{1j_0}(|v_{j_0}(t)|)+\sum_{i=1}^{n}|d_{j_0i}|\int_{-\infty}^t k_{j_0i}(t-s)|v_{j_0}(s)|\tilde{f}_{2j_0}(|v_{j_0}(s)|)\mathrm{d}s \end{aligned}$$

and

$$D^+v(t) \leq -\overline{\beta}v(t) + \sum_{j=1}^{m}\sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}||u_i(t)| + \sum_{j=1}^{m}\sum_{i\neq i_0}|\bar{d}_{ij}|M_{2i}\int_{-\infty}^{t} h_{ij}(t-s)|u_i(s)|\mathrm{d}s$$
$$+\sum_{j=1}^{m}|\bar{a}_{i_0j}||u_{i_0}(t)|\tilde{g}_{1i_0}(|u_{i_0}(t)|) + \sum_{j=1}^{m}|\bar{d}_{i_0j}|\int_{-\infty}^{t} h_{i_0j}(t-s)|u_{i_0}(s)|\tilde{g}_{2i_0}(|u_{i_0}(s)|)\mathrm{d}s.$$

Setting $P(t) := u(t)+v(t)$, the upper right Dini derivative of $P(t)$ fulfills

$$D^+P(t) \leq -\Big[\beta - \sum_{j=1}^{m}\sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}|\Big]u(t) - \Big[\overline{\beta} - \sum_{i=1}^{n}\sum_{j\neq j_0} L_{1j}|a_{ji}|\Big]v(t) + \sum_{i=1}^{n}|a_{j_0i}|v(t)\tilde{f}_{1j_0}(v(t))$$
$$+\sum_{j=1}^{m}|\bar{a}_{i_0j}|u(t)\tilde{g}_{1i_0}(u(t)) + \int_{-\infty}^{t}\sum_{i=1}^{n}\sum_{j=1}^{m}|d_{ji}|k_{ji}(t-s)v(s)\Big[\max_{j\neq j_0} L_{2j} + \tilde{f}_{2j_0}(v(s))\Big]\mathrm{d}s$$
$$+\int_{-\infty}^{t}\sum_{i=1}^{n}\sum_{j=1}^{m}|\bar{d}_{ij}|h_{ij}(t-s)u(s)\Big[\max_{i\neq i_0} M_{2i} + \tilde{g}_{2i_0}(u(s))\Big]\mathrm{d}s,$$

or

$$D^+P(t) \leq -\Big[\beta - \sum_{j=1}^{m}\sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}|\Big]u(t) - \Big[\overline{\beta} - \sum_{i=1}^{n}\sum_{j\neq j_0} L_{1j}|a_{ji}|\Big]v(t) + \sum_{i=1}^{n}|a_{j_0i}|P(t)\tilde{f}_{1j_0}(P(t))$$
$$+\sum_{j=1}^{m}|\bar{a}_{i_0j}|P(t)\tilde{g}_{1i_0}(P(t)) + \int_{-\infty}^{t}\sum_{i=1}^{n}\sum_{j=1}^{m}|d_{ji}|k_{ji}(t-s)|P(s)\Big[\max_{j\neq j_0} L_{2j} + \tilde{f}_{2j_0}(P(s))\Big]\mathrm{d}s$$
$$+\int_{-\infty}^{t}\sum_{i=1}^{n}\sum_{j=1}^{m}|\bar{d}_{ij}|h_{ij}(t-s)P(s)\Big[\max_{i\neq i_0} M_{2i} + \tilde{g}_{2i_0}(P(s))\Big]\mathrm{d}s.$$

In a simpler form, we may write

$$D^+P(t) \leq -AP(t) + F(P(t)) + \int_{-\infty}^{t} H(t-s)G(P(s))\mathrm{d}s,$$

where

$$A = \min\Big\{\beta - \sum_{j=1}^{m}\sum_{i\neq i_0} L_{1j}|a_{ji}|,\ \overline{\beta} - \sum_{i=1}^{n}\sum_{j\neq j_0} M_{1i}|\bar{a}_{ij}|\Big\},$$
$$F(P(t)) = P(t)\tilde{F}(P(t)),\ \tilde{F}(P(t)) = \max\Big\{\sum_{i=1}^{n}|a_{j_0i}|, \sum_{j=1}^{m}|\bar{a}_{i_0j}|\Big\}\Big(\tilde{f}_{1j_0}(P(t)) + \tilde{g}_{1i_0}(P(t))\Big),$$
$$G(P(t)) = P(t)\tilde{G}(P(t)),\ \tilde{G}(P(t)) = \max_{j\neq j_0} L_{2j} + \max_{i\neq i_0} M_{2i} + \tilde{f}_{2j_0}(P(t)) + \tilde{g}_{2i_0}(P(t)),$$
$$H_{ij}(t) = \sup_{t\geq 0}\Big\{|d_{ji}|k_{ji}(t), |\bar{d}_{ij}|h_{ij}(t)\Big\},\ H(t) = \sum_{i=1}^{n}\sum_{j=1}^{m} H_{ij}(t).$$

It is obvious that

$$\tilde{G}(0) = \max_{j\neq j_0} L_{2j} + \max_{i\neq i_0} M_{2i} + \tilde{f}_{2j_0}(0) + \tilde{g}_{2i_0}(0).$$

By virtue of the assumption **(A5)**, we find

$$\tilde{G}(0) = \max_{j\neq j_0} L_{2j} + \max_{i\neq i_0} M_{2i}.$$

Next, from the assumption **(A6)**, it holds that

$$\tilde{G}(0)\int_0^{\infty} H(s)\mathrm{d}s < A. \tag{3.6}$$

In view of the definition of $\tilde{F}$ and the assumption **(A5)**, the inequality (3.6) yields

$$\tilde{G}(0)\int_0^{\infty} H(s)\mathrm{d}s < A - \tilde{F}(0), \tag{3.7}$$

with $\tilde{F}(0) = \max\Big\{\sum\limits_{i=1}^{n}|a_{j_0 i}|, \sum\limits_{j=1}^{m}|\bar{a}_{i_0 j}|\Big\}(\tilde{f}_{1j_0}(0)+\tilde{g}_{1i_0}(0))$.

Thus, Lemma 2.6 may be applied: Therefore, there exist $\nu > 0$ and α such that $0<\alpha<\alpha_0$ and

$$P(t) \le \sup_{-\infty<t\le 0}\Big(\sum_{i=1}^{n}|\phi_i(t)-\overline{\phi}_i(t)| + \sum_{j=1}^{m}|\varphi_j(t)-\overline{\varphi}_j(t)|\Big)e^{-\alpha t},\ t\ge 0,$$

when $\sup\limits_{-\infty<t\le 0}\Big(\sum\limits_{i=1}^{n}|\phi_i(t)-\overline{\phi}_i(t)| + \sum\limits_{j=1}^{m}|\varphi_j(t)-\overline{\varphi}_j(t)|\Big) \le \nu$.

This completes the proof. □

For the second result, we need the assumption

(A7) We have

$$\Big(\sum_{i=1}^{n}\sum_{j\neq j_0} L_{1j}|a_{ji}| + \sum_{j=1}^{m}\sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}|\Big) + \sum_{i=1}^{n}\sum_{j\neq j_0} L_{2j}|d_{ji}|\Big(\int_0^{\infty} e^{\alpha_0 s}k_{ji}(s)\mathrm{d}s\Big)$$
$$+\sum_{j=1}^{m}\sum_{i\neq i_0} M_{2i}|\bar{d}_{ij}|\Big(\int_0^{\infty} e^{\alpha_0 s}h_{ij}(s)\mathrm{d}s\Big) < A.$$

Theorem 3.2. *The conclusion of Theorem 3.1 holds with the same hypotheses except* ***(A6)*** *replaced by* ***(A7)****.*

Proof. According to estimations (3.4) and (3.5), one has

$$\begin{aligned} D^+|u_i(t)| \le & -\beta_i|u_i(t)| + \sum_{j\neq j_0} L_{1j}|a_{ji}||v_j(t)| + \sum_{j\neq j_0}|d_{ji}|L_{2j}\int_{-\infty}^{t} k_{ji}(t-s)|v_j(s)|\mathrm{d}s \\ & +|a_{j_0 i}||v_{j_0}(t)|\tilde{f}_{1j_0}(|v_{j_0}(t)|) + \int_{-\infty}^{t}|d_{j_0 i}|k_{j_0 i}(t-s)|v_{j_0}(s)|\tilde{f}_{2j_0}(|v_{j_0}(s)|)\mathrm{d}s \end{aligned} \tag{3.8}$$

and

$$D^+|v_j(t)| \le -\overline{\beta}_j|v_j(t)| + \sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}||u_i(t)| + \sum_{i\neq i_0}|\bar{d}_{ij}|M_{2i}\int_{-\infty}^{t} h_{ij}(t-s)|u_i(s)|\mathrm{d}s$$

$$+|\bar{a}_{i_0j}||u_{i_0}(t)|\tilde{g}_{1i_0}(|u_{i_0}(t)|)+\int_{-\infty}^{t}|\bar{d}_{i_0j}|h_{i_0j}(t-s)|u_{i_0}(s)|\tilde{g}_{2i_0}(|u_{i_0}(s)|)\mathrm{d}s. \tag{3.9}$$

We introduce the functionals, for $i=1,2,\ldots,n,\ j=1,2,\ldots,m$

$$V_{1i}(t)=e^{-\beta_1 t}\int_0^\infty\int_{t-s}^{t}e^{\beta_1(s+\theta)}\Big[\sum_{j\neq j_0}|d_{ji}|L_{2j}k_{ji}(s)|v_j(\theta)|+|d_{j_0i}|k_{j_0i}(s)|v_{j_0}(\theta)|\tilde{f}_{2j_0}(|v_{j_0}(\theta)|)\Big]\mathrm{d}\theta\mathrm{d}s, \tag{3.10}$$

$$V_{2j}(t)=e^{-\beta_2 t}\int_0^\infty\int_{t-s}^{t}e^{\beta_2(s+\tau)}\Big[\sum_{i\neq i_0}|\bar{d}_{ij}|M_{2i}h_{ij}(s)|u_i(\tau)|+|\bar{d}_{i_0j}|h_{i_0j}(s)|u_{i_0}(\tau)|\tilde{g}_{2i_0}(|u_{i_0}(\tau)|)\Big]\mathrm{d}\tau\mathrm{d}s, \tag{3.11}$$

for some $0<\beta_1,\beta_2<\alpha_0$, and

$$L(t):=u(t)+v(t)+V_1(t)+V_2(t),\ t\geq 0, \tag{3.12}$$

where

$$V_1(t)=\sum_{i=1}^{n}V_{1i}(t),\ V_2(t)=\sum_{j=1}^{m}V_{2j}(t).$$

The differentiation of (3.10) and (3.11) for $t>0$ yields

$$\begin{aligned}V_{1i}'(t)=&-\beta_1V_{1i}(t)+\sum_{j\neq j_0}|d_{ji}|L_{2j}\Big(\int_0^\infty e^{\beta_1 s}k_{ji}(s)\mathrm{d}s\Big)|v_j(t)|\\&+|d_{j_0i}|\Big(\int_0^\infty e^{\beta_1 s}k_{j_0i}(s)|\mathrm{d}s\Big)|v_{j_0}(t)|\tilde{f}_{2j_0}(|v_{j_0}(t)|)\\&-\int_0^\infty\Big[\sum_{j\neq j_0}|d_{ji}|L_{2j}k_{ji}(s)|v_j(t-s)|+|d_{j_0i}|k_{j_0i}(s)|v_{j_0}(t-s)|\tilde{f}_{2j_0}(|v_{j_0}(t-s)|)\Big]\mathrm{d}s\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}V_{2j}'(t)=&-\beta_2V_{2j}(t)+\sum_{i\neq i_0}|\bar{d}_{ij}|M_{2i}\Big(\int_0^\infty e^{\beta_2 s}h_{ij}(s)|\mathrm{d}s\Big)|u_i(t)|\\&+|\bar{d}_{i_0j}|\Big(\int_0^\infty e^{\beta_2 s}h_{i_0j}(s)|\mathrm{d}s\Big)|u_{i_0}(t)|\tilde{g}_{2i_0}(|u_{i_0}(t)|)\\&-\int_0^\infty\Big[\sum_{i\neq i_0}|\bar{d}_{ij}|M_{2i}h_{ij}(s)|u_i(t-s)|+|\bar{d}_{i_0j}|h_{i_0j}(s)|u_{i_0}(t-s)|\tilde{g}_{2i_0}(|u_{i_0}(t-s)|)\Big]\mathrm{d}s.\end{aligned} \tag{3.14}$$

Having in mind that, when the derivative exists, it is equal to all the Dini derivatives, from (3.8)–(3.14), the upper right Dini derivative of $L(t)$ satisfies

$$\begin{aligned}
D^+L(t) &\le -\beta u(t) - \beta_1 V_1(t) - \beta_2 V_2(t) - \overline{\beta} v(t) + \sum_{i=1}^{n}\sum_{j\neq j_0} L_{1j}|a_{ji}||v_j(t)| + \sum_{j=1}^{m}\sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}||u_i(t)| \\
&+ \sum_{i=1}^{n}\sum_{j\neq j_0} L_{2j}|d_{ji}|\int_{-\infty}^{t} k_{ji}(t-s)|v_j(s)|\mathrm{d}s + \sum_{i=1}^{n}|a_{j_0 i}||v_{j_0}(t)|\tilde{f}_{1j_0}(|v_{j_0}(t)|) \\
&+ \sum_{i=1}^{n}\int_{-\infty}^{t}|d_{j_0 i}|k_{j_0 i}(t-s)|v_{j_0}(s)|\tilde{f}_{2j_0}(|v_{j_0}(s)|)\mathrm{d}s + \sum_{j=1}^{m}\sum_{i\neq i_0} M_{2i}|\bar{d}_{ij}|\int_{-\infty}^{t} h_{ij}(t-s)|u_i(s)|\mathrm{d}s \\
&+ \sum_{j=1}^{m}|\bar{a}_{i_0 j}||u_{i_0}(t)|\tilde{g}_{1i_0}(|u_{i_0}(t)|) + \sum_{j=1}^{m}\int_{-\infty}^{t}|\bar{d}_{i_0 j}|h_{i_0 j}(t-s)|u_{i_0}(s)|\tilde{g}_{2i_0}(|u_{i_0}(s)|)\mathrm{d}s \\
&+ \sum_{i=1}^{n}\sum_{j\neq j_0} L_{2j}|d_{ji}|\Big(\int_0^\infty e^{\beta_1 s}k_{ji}(s)\mathrm{d}s\Big)|v_j(t)| - \sum_{i=1}^{n}\sum_{j\neq j_0}\int_0^\infty L_{2j}|d_{ji}|k_{ji}(s)|v_j(t-s)|\mathrm{d}s \\
&+ \sum_{i=1}^{n}|d_{j_0 i}|\Big(\int_0^\infty e^{\beta_1 s}k_{j_0 i}(s)\mathrm{d}s\Big)|v_{j_0}(t)|\tilde{f}_{2j_0}(|v_{j_0}(t)|) - \sum_{i=1}^{n}|d_{j_0 i}|\int_0^\infty k_{j_0 i}(s)|v_{j_0}(t-s)|\tilde{f}_{2j_0}(|v_{j_0}(t-s)|)\mathrm{d}s \\
&+ \sum_{j=1}^{m}\sum_{i\neq i_0} M_{2i}|\bar{d}_{ij}|\Big(\int_0^\infty e^{\beta_2 s}h_{ij}(s)\mathrm{d}s\Big)|u_i(t)| + \sum_{j=1}^{m}|\bar{d}_{i_0 j}|\Big(\int_0^\infty e^{\beta_2 s}h_{i_0 j}(s)\mathrm{d}s\Big)|u_{i_0}(t)|\tilde{g}_{2i_0}(|u_{i_0}(t)|) \\
&- \sum_{j=1}^{m}\sum_{i\neq i_0}\int_0^\infty M_{2i}|\bar{d}_{ij}|h_{ij}(s)|u_i(t-s)|\mathrm{d}s - \sum_{j=1}^{m}\sum_{i\neq i_0}\int_0^\infty |\bar{d}_{i_0 j}|h_{i_0 j}(s)|u_{i_0}(t-s)|\tilde{g}_{2i_0}(|u_{i_0}(t-s)|)\mathrm{d}s.
\end{aligned}$$

This is simplified as

$$\begin{aligned}
D^+L(t) &\le -\beta u(t) - \beta_1 V_1(t) - \overline{\beta} v(t) - \beta_2 V_2(t) + \sum_{i=1}^{n}\sum_{j\neq j_0} L_{1j}|a_{ji}||v_j(t)| \\
&+ \sum_{i=1}^{n}|a_{j_0 i}||v_{j_0}(t)|\tilde{f}_{1j_0}(|v_{j_0}(t)|) + \sum_{j=1}^{m}\sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}||u_i(t)| + \sum_{j=1}^{m}|\bar{a}_{i_0 j}||u_{i_0}(t)|\tilde{g}_{1i_0}(|u_{i_0}(t)|) \\
&+ \sum_{i=1}^{n}\sum_{j\neq j_0} L_{2j}|d_{ji}|\Big(\int_0^\infty e^{\beta_1 s}k_{ji}(s)\mathrm{d}s\Big)|v_j(t)| + \sum_{i=1}^{n}|d_{j_0 i}|\Big(\int_0^\infty e^{\beta_1 s}k_{j_0 i}(s)\mathrm{d}s\Big)|v_{j_0}(t)|\tilde{f}_{2j_0}(|v_{j_0}(t)|) \\
&+ \sum_{j=1}^{m}\sum_{i\neq i_0} M_{2i}|\bar{d}_{ij}|\Big(\int_0^\infty e^{\beta_2 s}h_{ij}(s)\mathrm{d}s\Big)|u_i(t)| + \sum_{j=1}^{m}|\bar{d}_{i_0 j}|\Big(\int_0^\infty e^{\beta_2 s}h_{i_0 j}(s)\mathrm{d}s\Big)|u_{i_0}(t)|\tilde{g}_{2i_0}(|u_{i_0}(t)|),
\end{aligned}$$

or

$$\begin{aligned}
D^+L(t) \le &-\left[\beta - \Big(\sum_{j=1}^{m}\sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}|\Big) - \sum_{j=1}^{m}\sum_{i\neq i_0} M_{2i}|\bar{d}_{ij}|\Big(\int_0^\infty e^{\beta_2 s}h_{ij}(s)\mathrm{d}s\Big)\right]u(t) \\
&-\left[\overline{\beta} - \Big(\sum_{i=1}^{n}\sum_{j\neq j_0} L_{1j}|a_{ji}|\Big) - \sum_{i=1}^{n}\sum_{j\neq j_0} L_{2j}|d_{ji}|\Big(\int_0^\infty e^{\beta_1 s}k_{ji}(s)\mathrm{d}s\Big)\right]v(t) \\
&-\beta_1 V_1(t) - \beta_2 V_2(t) + \sum_{i=1}^{n}|a_{j_0 i}||v_{j_0}(t)|\tilde{f}_{1j_0}(|v_{j_0}(t)|) + \sum_{j=1}^{m}|\bar{a}_{i_0 j}||u_{i_0}(t)|\tilde{g}_{1i_0}(|u_{i_0}(t)|) \\
&+\sum_{i=1}^{n}|d_{j_0 i}|\Big(\int_0^\infty e^{\beta_1 s}k_{j_0 i}(s)\mathrm{d}s\Big)|v_{j_0}(t)|\tilde{f}_{2j_0}(|v_{j_0}(t)|) \\
&+\sum_{j=1}^{m}|\bar{d}_{i_0 j}|\Big(\int_0^\infty e^{\beta_2 s}h_{i_0 j}(s)\mathrm{d}s\Big)|u_{i_0}(t)|\tilde{g}_{2i_0}(|u_{i_0}(t)|).
\end{aligned}$$

We may further estimate $D^+L(t)$ as follows

$$
\begin{aligned}
D^+L(t) \le & -\Big[\beta - \Big(\sum_{j=1}^{m}\sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}|\Big) - \sum_{j=1}^{m}\sum_{i\neq i_0} M_{2i}|\bar{d}_{ij}|\Big(\int_0^\infty e^{\beta_2 s} h_{ij}(s)\mathrm{d}s\Big)\Big]u(t) \\
& -\Big[\overline{\beta} - \Big(\sum_{i=1}^{n}\sum_{j\neq j_0} L_{1j}|a_{ji}|\Big) - \sum_{i=1}^{n}\sum_{j\neq j_0} L_{2j}|d_{ji}|\Big(\int_0^\infty e^{\beta_1 s} k_{ji}(s)\mathrm{d}s\Big)\Big]v(t) \\
& -\beta_1 V_1(t) - \beta_2 V_2(t) + \Big(\sum_{i=1}^{n}|a_{j_0 i}|\Big)L(t)\tilde{f}_{1j_0}(L(t)) + \Big(\sum_{j=1}^{m}|\bar{a}_{i_0 j}|\Big)L(t)\tilde{g}_{1i_0}(L(t)) \\
& +\tilde{f}_{2j_0}(L(t))\Big(\sum_{i=1}^{n}|d_{j_0 i}|\Big(\int_0^\infty e^{\beta_1 s} k_{j_0 i}(s)\mathrm{d}s\Big)\Big)L(t) \\
& +\tilde{g}_{2i_0}(L(t))\Big(\sum_{j=1}^{m}|\bar{d}_{i_0 j}|\Big(\int_0^\infty e^{\beta_2 s} h_{i_0 j}(s)\mathrm{d}s\Big)\Big)L(t),
\end{aligned}
$$

which we can write in short in the form

$$
D^+L(t) \le -\Big[A - F(L(t))\Big]L(t), \tag{3.15}
$$

with

$$
\begin{aligned}
A = \min\Big\{ & \beta_1, \beta_2, \beta - \Big(\sum_{j=1}^{m}\sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}|\Big) - \sum_{j=1}^{m}\sum_{i\neq i_0} M_{2i}|\bar{d}_{ij}|\Big(\int_0^\infty e^{\beta_2 s} h_{ij}(s)\mathrm{d}s\Big), \\
& \overline{\beta} - \Big(\sum_{i=1}^{n}\sum_{j\neq j_0} L_{1j}|a_{ji}|\Big) - \sum_{i=1}^{n}\sum_{j\neq j_0} L_{2j}|d_{ji}|\Big(\int_0^\infty e^{\beta_1 s} k_{ji}(s)\mathrm{d}s\Big)\Big\}, \\
F(L(t)) = & \Big(\sum_{i=1}^{n}|a_{j_0 i}|\Big)\tilde{f}_{1j_0}(L(t)) + \tilde{f}_{2j_0}(L(t))\Big(\sum_{i=1}^{n}|d_{j_0 i}|\Big(\int_0^\infty e^{\beta_1 s} k_{j_0 i}(s)\mathrm{d}s\Big)\Big) \\
& +\Big(\sum_{j=1}^{m}|\bar{a}_{i_0 j}|\Big)\tilde{g}_{1i_0}(L(t)) + \tilde{g}_{2i_0}(L(t))\Big(\sum_{j=1}^{m}|\bar{d}_{i_0 j}|\Big(\int_0^\infty e^{\beta_2 s} h_{i_0 j}(s)\mathrm{d}s\Big)\Big).
\end{aligned}
$$

We shall compare solutions of (3.15) to solutions of the differential problem

$$
D^+z(t) = -[A - F(z(t))]z(t), \quad t > 0, \tag{3.16}
$$

with

$$
\begin{aligned}
z(0) = z_0 := L(0) = u(0) + v(0) + & \sum_{i=1}^{n}\int_0^\infty\int_{-s}^{0} e^{\beta_1(s+\theta)}\Big[\sum_{j\neq j_0}|d_{ji}|L_{2j}k_{ji}(s)|\tilde{\varphi}_{j_0}(\theta)| \\
& +|d_{j_0 i}|k_{j_0 i}(s)|\tilde{\varphi}_{j_0}(\theta)|\tilde{f}_{2j_0}(|\tilde{\varphi}_{j_0}(\theta)|)\Big]\mathrm{d}\theta\mathrm{d}s + \sum_{j=1}^{m}\int_0^\infty\int_{-s}^{0} e^{\beta_2(s+\tau)}\Big[\sum_{i\neq i_0}|\bar{d}_{ij}|M_{2i}h_{ij}(s) \\
& +|\bar{d}_{i_0 j}h_{i_0 j}(s)|\tilde{\phi}_{i_0}(\tau)|\tilde{g}_{2i_0}(|\tilde{\phi}_{i_0}(\tau)|)\Big]\mathrm{d}\tau\mathrm{d}s.
\end{aligned}
$$

Let

$$p := \sup\Big\{\sigma \geq 0 : F(\sigma) < A\Big\}.$$

In light of our assumptions $\tilde{f}_{1j_0}(0) = \tilde{f}_{2j_0}(0) = \tilde{g}_{1i_0}(0) = \tilde{g}_{2i_0}(0) = 0$, we have $p > 0$. For $q,\ 0 < q < p$, we define

$$Q(\xi) := -A + \xi + F(q).$$

Due to the non-decreasingness of the function F and $q < p$, then

$$Q(0) = -A + F(q) < 0. \tag{3.17}$$

Besides,

$$\lim_{\xi\to\infty} Q(\xi) = \infty. \tag{3.18}$$

In view of (3.17) and (3.18), then for any positive real number λ that satisfies $-\lambda > -A + F(q)$, one can find a $\xi_\lambda > 0$ such that

$$Q(\xi_\lambda) := -A + \xi_\lambda + F(q) = -\lambda < 0. \tag{3.19}$$

We claim that, if $z_0 < q$, then

$$z(t) \leq q e^{-\xi_\lambda t},\ t \geq 0. \tag{3.20}$$

We argue by contradiction. Assume that $t^* > 0$ is the first time

$$z(t) < q e^{-\xi_\lambda t},\ 0 \leq t < t^*,$$

$z(t^*) = q e^{-\xi_\lambda t^*}$ and $\psi'(t^*) \geq 0$, where $\psi(t) := z(t)e^{\xi_\lambda t}$. It is easy to see that

$$\begin{aligned}
0 \leq \psi'(t^*) &= \Big[z(t)e^{\xi_\lambda t}\Big]'\Big|_{t=t^*} = z'(t^*)e^{\xi_\lambda t^*} + \xi_\lambda z(t^*)e^{\xi_\lambda t^*}\\
&\leq e^{\xi_\lambda t^*}\Big[-Az(t^*) + F(z(t^*))z(t^*)\Big] + \xi_\lambda z(t^*)e^{\xi_\lambda t^*}\\
&= \Big(-A + \xi_\lambda + F(z(t^*))\Big)z(t^*)e^{\xi_\lambda t^*} = \Big(-A + \xi_\lambda + F(z(t^*))\Big)\psi(t^*)\\
&\leq \Big(-A + \xi_\lambda + F(q)\Big)\psi(t^*)
\end{aligned}$$

and therefore, from (3.16) and (3.19), we infer

$$0 \leq \psi'(t^*) \leq \Big(-A + \xi_\lambda + F(q)\Big)\psi(t^*) < 0.$$

This contradiction confirms the relation (3.20). By comparison, we obtain

$$L(t) \leq Be^{-\xi^* t}, \quad t \geq 0$$

and thus

$$u(t) \le Be^{-\xi^* t}, \quad v(t) \le Be^{-\xi^* t},\ t \ge 0,$$

for some positive constants ξ^* and B. This completes the proof. □

Remark 3.3. **1.** We note that it is not necessary to verify the assumption

$$\tilde{f}_{1j_0}(0) = \tilde{f}_{2j_0}(0) = \tilde{g}_{1i_0}(0) = \tilde{g}_{2i_0}(0) = 0,$$

we need in the nonlinear version of Halanay inequality (Lemma 2.6) the following estimation

$$\tilde{G}(0) \int\limits_0^\infty H(s)\mathrm{d}s < A - \tilde{F}(0),$$

which implies

$$\begin{aligned} &\Big[\max_{j\neq j_0} L_{2j} + \tilde{f}_{2j_0}(0)\Big] \sum_{i=1}^n \sum_{j=1}^m |d_{ji}| \int\limits_0^\infty k_{ji}(s)\mathrm{d}s + \Big[\max_{i\neq i_0} M_{2i} + \tilde{g}_{2i_0}(0)\Big] \sum_{i=1}^n \sum_{j=1}^m |\bar{d}_{ij}| \int\limits_0^\infty h_{ij}(s)\mathrm{d}s \\ &< \Big[\beta - \sum_{j=1}^m \sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}|\Big] + \Big[\bar{\beta} - \sum_{i=1}^n \sum_{j\neq j_0} L_{1j}|a_{ji}|\Big] - \Big(\sum_{i=1}^n |a_{j_0 i}|\Big)\tilde{f}_{1j_0}(0) - \Big(\sum_{j=1}^m |\bar{a}_{i_0 j}|\Big)\tilde{g}_{1i_0}(0) \end{aligned}$$

for Theorem 3.1.

In the proof of Theorem 3.2, we need $F(0) < A$, which leads to

$$\begin{aligned} &\Big(\sum_{i=1}^n |a_{j_0 i}|\Big)\tilde{f}_{1j_0}(0) + \tilde{f}_{2j_0}(0)\Big(\sum_{i=1}^n |d_{j_0 i}|\Big(\int_0^\infty e^{\beta_1 s} k_{j_0 i}(s)\mathrm{d}s\Big)\Big) \\ &\quad + \Big(\sum_{j=1}^m |\bar{a}_{i_0 j}|\Big)\tilde{g}_{1i_0}(0) + \tilde{g}_{2i_0}(0)\Big(\sum_{j=1}^m |\bar{d}_{i_0 j}|\Big(\int_0^\infty e^{\beta_2 s} h_{i_0 j}(s)\mathrm{d}s\Big)\Big) \\ &\quad < \min\Big\{\beta_1, \beta_2, \beta - \Big(\sum_{j=1}^m \sum_{i\neq i_0} M_{1i}|\bar{a}_{ij}|\Big) - \sum_{j=1}^m \sum_{i\neq i_0} M_{2i}|\bar{d}_{ij}|\Big(\int_0^\infty e^{\beta_2 s} h_{ij}(s)\mathrm{d}s\Big), \\ &\quad \overline{\beta} - \Big(\sum_{i=1}^n \sum_{j\neq j_0} L_{1j}|a_{ji}|\Big) - \sum_{i=1}^n \sum_{j\neq j_0} L_{2j}|d_{ji}|\Big(\int_0^\infty e^{\beta_1 s} k_{ji}(s)\mathrm{d}s\Big)\Big\}. \end{aligned}$$

2. The obtained results improve the ones in [42, 43], where the conditions

$$yf_{1j_0}(y) > 0,\ yf_{2j_0}(y) > 0,\ yg_{1i_0}(y) > 0,\ yg_{2i_0}(y) > 0,\ \text{for } y \neq 0$$

and

$$\sup_{y\neq 0} \frac{f_{1j_0}(y)}{y},\ \sup_{y\neq 0} \frac{f_{2j_0}(y)}{y},\ \sup_{y\neq 0} \frac{g_{1i_0}(y)}{y},\ \sup_{y\neq 0} \frac{g_{2i_0}(y)}{y}$$

corresponding to our functions $f_{1j_0}(y), f_{2j_0}(y), g_{1i_0}(y), g_{2i_0}(y)$ exist.

3. The existence and uniqueness may be deduced from the following argument

$$\begin{aligned}
&\Big|\int_{-\infty}^{t_1} k_{ji}(t_1-s)f_{2j}(y_j(s))\mathrm{d}s-\int_{-\infty}^{t_2} k_{ji}(t_2-s)f_{2j}(y_j(s))\mathrm{d}s\Big|\\
&\le\Big|\int_{-\infty}^{t_1} k_{ji}(t_1-s)f_{2j}(y_j(s))\mathrm{d}s-\int_{-\infty}^{t_1} k_{ji}(t_2-s)f_{2j}(y_j(s))\mathrm{d}s\Big|\\
&\quad+\Big|\int_{-\infty}^{t_1} k_{ji}(t_2-s)f_{2j}(y_j(s))\mathrm{d}s-\int_{-\infty}^{t_2} k_{ji}(t_2-s)f_{2j}(y_j(s))\mathrm{d}s\Big|\\
&\le\int_{-\infty}^{t_1}|k_{ji}(t_1-s)-k_{ji}(t_2-s)||f_{2j}(y_j(s))|\mathrm{d}s+\Big|\int_{t_2}^{t_1} k_{ji}(t_2-s)f_{2j}(y_j(s))\mathrm{d}s\Big|,\\
&\Big|\int_{-\infty}^{t_1} h_{ij}(t_1-s)g_{2i}(x_i(s))\mathrm{d}s-\int_{-\infty}^{t_2} h_{ij}(t_2-s)g_{2i}(x_i(s))\mathrm{d}s\Big|\\
&\le\Big|\int_{-\infty}^{t_1} h_{ij}(t_1-s)g_{2i}(x_i(s))\mathrm{d}s-\int_{-\infty}^{t_1} h_{ij}(t_2-s)g_{2i}(x_i(s))\mathrm{d}s\Big|\\
&\quad+\Big|\int_{-\infty}^{t_1} h_{ij}(t_2-s)g_{2i}(x_i(s))\mathrm{d}s-\int_{-\infty}^{t_2} h_{ij}(t_2-s)g_{2i}(x_i(s))\mathrm{d}s\Big|\\
&\le\int_{-\infty}^{t_1}|h_{ij}(t_1-s)-h_{ij}(t_2-s)||g_{2i}(x_i(s))|\mathrm{d}s+\Big|\int_{t_2}^{t_1} h_{ij}(t_2-s)g_{2i}(x_i(s))\mathrm{d}s\Big|.
\end{aligned}$$

If k_{ji} and h_{ij} are Lipschitz and bounded, then

$$\begin{aligned}
&\Big|\int_{-\infty}^{t_1} k_{ji}(t_1-s)f_{2j}(y_j(s))\mathrm{d}s-\int_{-\infty}^{t_2} k_{ji}(t_2-s)f_{2j}(y_j(s))\mathrm{d}s\Big|\\
&\le K_{ji}|t_1-t_2|\int_{-\infty}^{t_1} f_{2j}(y_j(s))\mathrm{d}s+\Lambda|t_1-t_2|,\\
&\Big|\int_{-\infty}^{t_1} h_{ij}(t_1-s)g_{2i}(x_i(s))\mathrm{d}s-\int_{-\infty}^{t_2} h_{ij}(t_2-s)g_{2i}(x_i(s))\mathrm{d}s\Big|\\
&\le H_{ij}|t_1-t_2|\int_{-\infty}^{t_1} g_{2i}(x_i(s))\mathrm{d}s+\bar{\Lambda}|t_1-t_2|.
\end{aligned}$$

This, added to our a priori estimates, implies the Lipschitz continuity with respect to t and Cid's Theorem 2.4 [4] implies the well-posedness.

4. The stability of the equilibrium needs additional conditions on the i_0 and j_0 components that ensure the uniqueness of the equilibrium (such as the Osgood condition). However, our argument will then depend on the existence of $\tilde{f}_{1j_0}, \tilde{f}_{2j_0}, \tilde{g}_{1i_0}$ and $\tilde{g}_{2i_0}$ (see (1.2)) fulfilling the conditions (see 2. in Rem. 3.3).

4. Numerical illustration

In this section, we present a numerical example to validate the above results. Observe that, the Hölder continuous functions (having an exponent between 0 and 1) cannot be covered by our results. These were studied in [7, 27–33]. It is known that when the exponents exceed one, the functions are constant. Neither are the Log-Lipschitz continuous functions, for example, functions with $x|ln(x)|$ as a continuity modulus (because $ln(x)$ is unbounded near 0). These functions do not satisfy the Lipschitz condition, whereas Osgood's condition is verified (which guarantees the uniqueness of the equilibrium). Nevertheless, these results can be applied to locally Lipschitz continuous functions.

Example 4.1. Consider the following BAM neural network system with distributed delays that contains four neurons

$$\begin{cases} x_1'(t) = -c_1(x_1(t)) + \sum\limits_{j=1}^{2} a_{j1} f_{1j}(y_j(t)) + \sum\limits_{j=1}^{2} d_{j1} \int_0^t k_{j1}(s) f_{2j}(y_j(t-s))\mathrm{d}s + 1.5, \quad t \in [0,5], \\ x_2'(t) = -c_2(x_2(t)) + \sum\limits_{j=1}^{2} a_{j2} f_{1j}(y_j(t)) + \sum\limits_{j=1}^{2} d_{j2} \int_0^t k_{j2}(s) f_{2j}(y_j(t-s))\mathrm{d}s + 1, \quad t \in [0,5], \\ y_1'(t) = -r_1(y_1(t)) + \sum\limits_{i=1}^{2} \bar{a}_{i1} g_{1i}(x_i(t)) + \sum\limits_{i=1}^{2} \bar{d}_{i1} \int_0^t h_{i1}(s) g_{2i}(x_i(t-s))\mathrm{d}s + 2, \quad t \in [0,5], \\ y_2'(t) = -r_2(y_2(t)) + \sum\limits_{i=1}^{2} \bar{a}_{i2} g_{1i}(x_i(t)) + \sum\limits_{i=1}^{2} \bar{d}_{i2} \int_0^t h_{i2}(s) g_{2i}(x_i(t-s))\mathrm{d}s + 0.5, \quad t \in [0,5], \end{cases} \tag{4.1}$$

where $a_{11} = 0.12,\ a_{12} = 0.23,\ a_{21} = 0.5,\ a_{22} = 1,\ d_{11} = 0.5,\ d_{12} = 0.25,\ d_{21} = 0.5,\ d_{22} = 2.5,\ \bar{a}_{11} = 0.25,\ \bar{a}_{12} = 0.2,\ \bar{a}_{21} = 0.75,\ \bar{a}_{22} = 1.5,\ \bar{d}_{11} = \frac{1}{3},\ \bar{d}_{12} = 1,\ \bar{d}_{21} = \frac{1}{3},\ \bar{d}_{22} = 2,\ c_1(x) = 4x,\ c_2(x) = 7x,\ r_1(x) = 5x,\ r_2(x) = 8x,\ f_{11}(x) = f_{21}(x) = \frac{1}{2}(|x+1| - |x-1|),\ g_{11}(x) = g_{21}(x) = tanh(x),\ f_{12}(x) = x^2,\ f_{22}(x) = x^3,\ g_{12}(x) = x^2+1,\ g_{22}(x) = x^3+2,\ k_{ji}(t) = e^{-5t},\ h_{ij}(t) = e^{-6t},\ x_1(t) = 1,\ x_2(t) = 1.5,\ y_1(t) = 2,\ y_2(t) = 3,\ t \in [-5,0]$.

Taking $\alpha_0 = 4,\ \beta_1 = \beta_2 = 3,\ i_0 = j_0 = 2$. Through simple calculations, one can obtain $\beta = 4,\ \bar{\beta} = 5,\ L_{1j} = L_{2j} = M_{1i} = M_{2i} = 1,\ \int_0^\infty e^{\alpha_0 t} k_{ji}(t)\mathrm{d}t = 1,\ \int_0^\infty e^{\alpha_0 t} h_{ij}(t)\mathrm{d}t = 0.5$.

Besides, assumptions **(A6)** and **(A7)** are satisfied

$$\begin{aligned} &\Big(\max_{i \neq i_0} M_{2i} + \max_{j \neq j_0} L_{2j}\Big) \int_0^\infty H(t)\mathrm{d}t = 1.64 \\ &< \min\Big\{\beta - \sum_{i=1}^{2} \sum_{j \neq j_0} L_{1j}|a_{ji}|, \overline{\beta} - \sum_{j=1}^{2} \sum_{i \neq i_0} M_{1i}|\bar{a}_{ij}|\Big\} = 3.65 \end{aligned}$$

and

$$\begin{aligned} &\Big(\sum_{i=1}^{2} \sum_{j \neq j_0} L_{1j}|a_{ji}| + \sum_{j=1}^{2} \sum_{i \neq i_0} M_{1i}|\bar{a}_{ij}|\Big) + \sum_{i=1}^{2} \sum_{j \neq j_0} L_{2j}|d_{ji}|\Big(\int_0^\infty e^{\alpha_0 s} k_{ji}(s)\mathrm{d}s\Big) \\ &+ \sum_{j=1}^{2} \sum_{i \neq i_0} M_{2i}|\bar{d}_{ij}|\Big(\int_0^\infty e^{\alpha_0 s} h_{ij}(s)\mathrm{d}s\Big) = 2.21 < A = 3. \end{aligned}$$

The difference between solutions $(x(t), y(t))$ and $(\bar{x}(t), \bar{y}(t))$ of system (4.1) is given by

$$u_i(t) = x_i(t) - \bar{x}_i(t),\ i = 1,2, \quad v_j(t) = y_j(t) - \bar{y}_j(t),\ j = 1,2$$

and satisfies

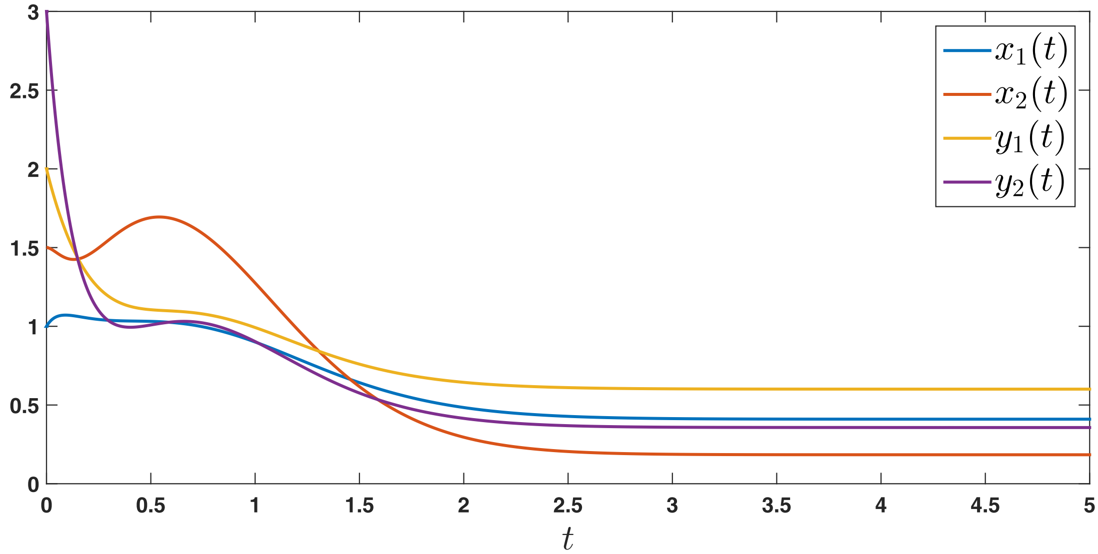

FIGURE 1. Trajectories of the states $x_1(t), x_2(t), y_1(t)$ and $y_2(t)$.

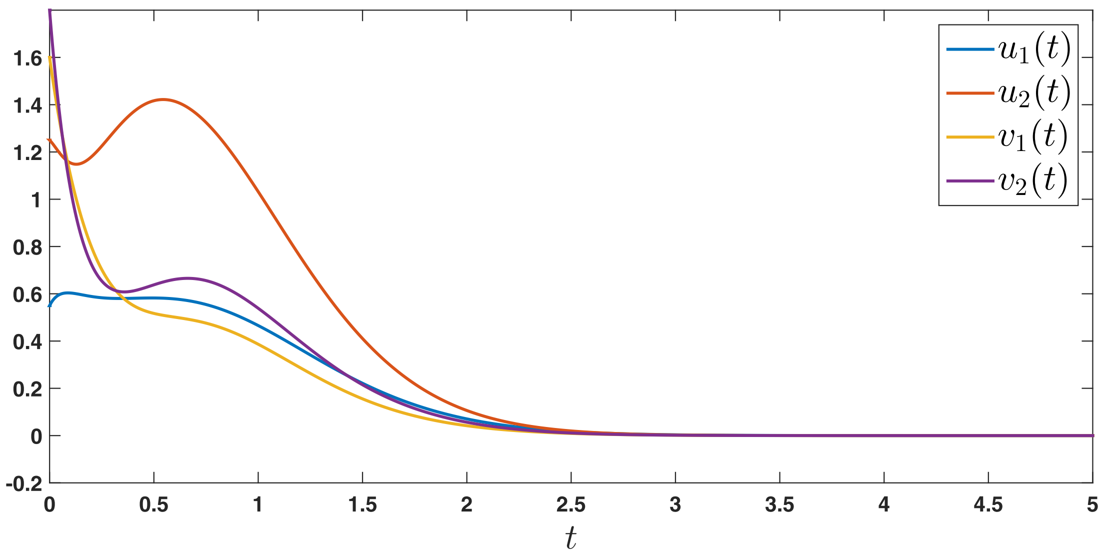

FIGURE 2. Trajectories of the states $u_1(t), u_2(t), v_1(t)$ and $v_2(t)$.

$$\begin{cases} D^+u_1(t) \leq -(c_1(x_1(t)) - c_1(\bar{x}_1(t))) + |a_{11}||v_1(t)| + |a_{21}||v_2(t)||\tilde{f}_{12}(y_2(t), \bar{y}_2(t))| \\ +|d_{11}| \int_{-\infty}^{t} k_{11}(t-s)|v_1(s)|\mathrm{d}s + |d_{21}| \int_{-\infty}^{t} k_{21}(t-s)|v_2(s)||\tilde{f}_{22}(y_2(s), \bar{y}_2(s))|\mathrm{d}s,\ t \in [0,5], \\ D^+u_2(t) \leq -(c_2(x_2(t)) - c_2(\bar{x}_2(t))) + |a_{12}||v_1(t)| + |a_{22}||v_2(t)||\tilde{f}_{12}(y_2(t), \bar{y}_2(t))| \\ +|d_{12}| \int_{-\infty}^{t} k_{12}(t-s)|v_1(s)|\mathrm{d}s + |d_{22}| \int_{-\infty}^{t} k_{22}(t-s)|v_2(s)||\tilde{f}_{22}(y_2(s), \bar{y}_2(s))|\mathrm{d}s,\ t \in [0,5], \\ D^+v_1(t) \leq -(r_1(y_1(t)) - r_1(\bar{y}_1(t))) + |\bar{a}_{11}||u_1(t)| + |\bar{a}_{21}||u_2(t)||\tilde{g}_{12}(x_2(t), \bar{x}_2(t))| \\ +|\bar{d}_{11}| \int_{-\infty}^{t} h_{11}(t-s)|u_1(s)|\mathrm{d}s + |\bar{d}_{21}| \int_{-\infty}^{t} h_{21}(t-s)|u_2(s)||\tilde{g}_{22}(x_2(s), \bar{x}_2(s))|\mathrm{d}s,\ t \in [0,5], \\ D^+v_1(t) \leq -(r_2(y_2(t)) - r_2(\bar{y}_2(t))) + |\bar{a}_{12}||u_1(t)| + |\bar{a}_{22}||u_2(t)||\tilde{g}_{12}(x_2(t), \bar{x}_2(t))| \\ +|\bar{d}_{12}| \int_{-\infty}^{t} h_{12}(t-s)|u_1(s)|\mathrm{d}s + |\bar{d}_{22}| \int_{-\infty}^{t} h_{22}(t-s)|u_2(s)||\tilde{g}_{22}(x_2(s), \bar{x}_2(s))|\mathrm{d}s,\ t \in [0,5], \\ u_1(t) = 0.5, \quad u_2(t) = 1.25,\ t \in [-5, 0], \\ v_1(t) = 1.6, \quad v_2(t) = 1.8,\ t \in [-5, 0], \end{cases}$$

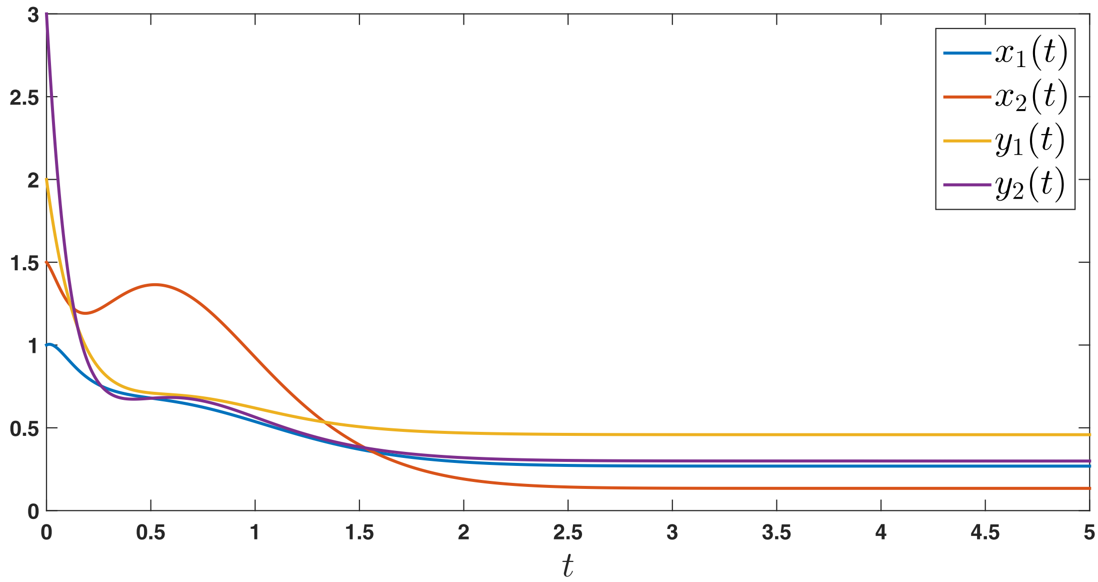

FIGURE 3. Trajectories of the states $x_1(t), x_2(t), y_1(t)$ and $y_2(t)$.

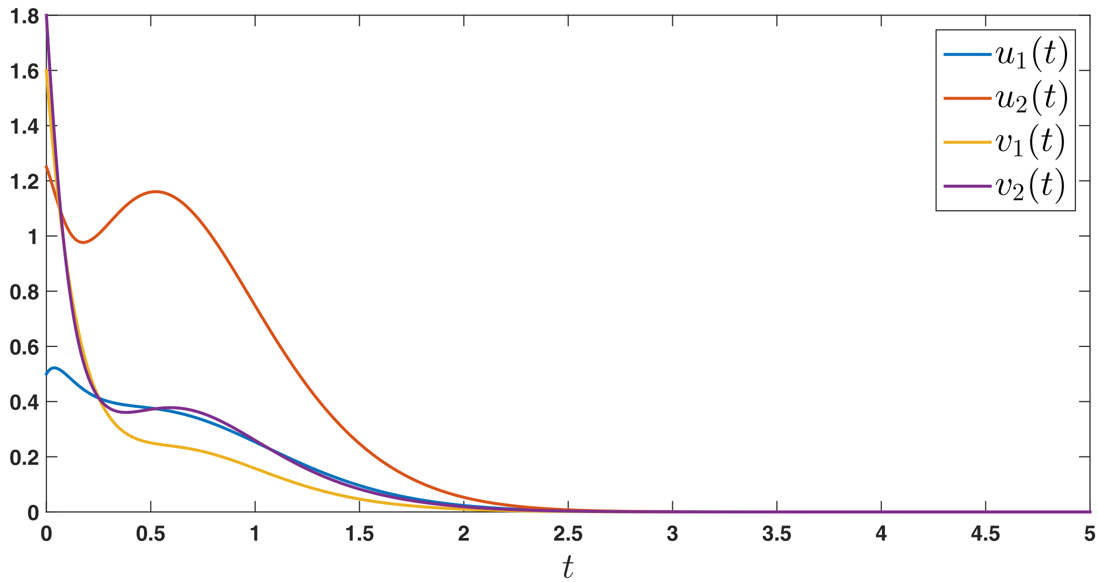

FIGURE 4. Trajectories of the states $u_1(t), u_2(t), v_1(t)$ and $v_2(t)$.

where

$$\tilde{f}_{12}(y_2(t), \bar{y}_2(t)) = y_2(t) + \bar{y}_2(t), \quad \tilde{f}_{22}(y_2(t), \bar{y}_2(t)) = y_1^2(t) + y_2(t)\bar{y}_2(t) + \bar{y}_2^2(t),$$
$$\tilde{g}_{12}(x_2(t), \bar{x}_2(t)) = x_2(t) + \bar{x}_2(t), \quad \tilde{g}_{22}(x_2(t), \bar{x}_2(t)) = \bar{x}^2(t) + x_2(t)\bar{x}_2(t).$$

As, assumptions **(A1)**–**(A7)** are met, Theorems 3.1 and 3.2 imply that solutions of system (4.1) are exponentially stable. This is depicted by Figures 1 and 2.

Example 4.2. Consider system (4.1) of Example 4.1 with the same parameters and functions. The functions $c_1(x), c_2(t), r_1(x)$ and $r_2(x)$ are selected as $c_1(x) = 5x + xcos(x)$, $c_2(x) = 8x + xcos(x)$, $r_1(x) = 6x + xsin(x)$, $r_2(x) = 9x + xsin(x)$. By choosing the same parameters as in Example 4.1, the assumptions **(A1)**–**(A7)** are satisfied, and *via* Theorems 3.1 and 3.2, solutions of system (4.1) are exponentially stable. This is shown in Figures 3 and 4.

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