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DYNAMICS OF A VECTOR-BORNE MODEL WITH DIRECT TRANSMISSION AND AGE OF INFECTION*

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Abstract. In this paper we the study of dynamics of time since infection structured vector born model with the direct transmission. We use standard incidence term to model the new infections. We analyze the corresponding system of partial differential equation and obtain an explicit formula for the basic reproduction number $\mathcal{R}_0$. The diseases-free equilibrium is locally and globally asymptotically stable whenever the basic reproduction number is less than one, $\mathcal{R}_0 < 1$. Endemic equilibrium exists and is locally asymptotically stable when $\mathcal{R}_0 > 1$. The disease will persist at the endemic equilibrium whenever the basic reproduction number is greater than one.

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1. Introduction

Zika virus (ZIKV) is a flavivirus, transmitted by the *Aedes aegypti* mosquitoes as for the other vector borne diseases such as malaria, dengue fever, and West Nile virus. ZIKV was first isolated in the Zika forest in Uganda from the rhesus monkey in 1947 [9, 42]. Outside Africa and Asia, first outbreak of Zika virus was reported in Yap State, part of the Federated States of Micronesia in 2007. After this outbreak, between 2007 and 2016, the spread of Zika virus infections have been reported around the world, including in southeast Asia; French Polynesia and other islands in the Pacific Ocean; and parts of South, Central, and North America [6, 12, 22].

The Zika virus infection causes mild or no symptoms [19]. However, Zika infection during pregnancy can cause serious birth defects and ZIKV infections were found to be connected with Guillain-Barre syndrome and Microcephaly [5, 29]. When a person develops Guillain-Barre, their body's immune system mistakenly attacks part of its peripheral nervous system whereas in Microcephaly, Zika affects the brain, causing swelling of the brain or spinal cord or a blood disorder which can result in bleeding, bruising or slow blood clotting [14, 24, 45]. To the date, there is no specific medicine or vaccine to prevent or to treat Zika disease [36]. So preventive measures is the most effective way to prevent the infection, especially to pregnant women [25, 26].

The main route of transmission for Zika virus is through the bite of an infected mosquito, but Zika has also direct transmission: through sexual contact, vertical transmission or blood infusion [1, 10, 11, 23, 30]. Direct

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transmission of Zika has been documented in nine countries- Canada, Argentina, Chile, France, Italy, New Zealand, Peru, Portugal and the United States of America [4].

Several mathematical models have been used to understand the transmission dynamics of vector borne diseases [3, 7, 27, 37, 39–41]. Ordinary differential equation (ODE) models dealing with the ZIKV disease have been proposed and extensively analyzed in past years [1, 2, 7, 15, 28, 38]. In this study, we present a mathematical model of ZIKV incorporating both vector and direct transmission where infected individuals are structured by time-since infection. Transmission and recovery varies during the infectious period. Hence infection age affects the number of secondary infected individuals [21]. In Section 2, we introduce a model of ZIKV with age of infection where vector and direct transmission are both modeled as standard incidences. Then, in Section 3, we study the local stability of the disease free and endemic equilibrium and determine the reproduction number $\mathcal{R}_0$. It is followed by Section 4, where we discuss global stability of the disease free equilibirum. In Section 5, we present the persistence of the endemic equilibrium.

2. Vector-borne model with direct transmission

To model the spread of Zika infection in a population, we divide total host population into three non intersecting classes: susceptible, infected and recovered individuals. Since infectivity for infectious individuals varies with time since infection, we structure the infected class by time since infection parameter τ. The time since infection begins when an individual becomes infected and progresses with the chronological time t. Let $i(\tau,t)$ be the density of infected individuals at time since infection τ and at time t, $S(t)$ be the number susceptible individuals and $R(t)$ be the number of recovered individuals at time t. Then $N(t)$ denotes the total human population. Here, we use an endemic model since the Zika epidemic has continued for nearly three years. Researchers estimate the reproduction number of Zika virus infection to be greater than one, and suggest that the virus is endemic [35]. Susceptible individuals are recruited at a rate Λ and μ denotes the natural death rate for humans. The vector transmission and direct transmission are both modeled as standard incidence, given by

$$\frac{\beta_v S I_v}{N} \quad \text{and} \quad \frac{S}{N}\int_0^\infty \beta_d(\tau) i(\tau,t)\mathrm{d}\tau$$

where β_v is the vector transmission rate and $\beta_d(\tau)$ is the direct transmission rate which depends on the time-since infection variable τ of the infected host. Zika infection with a given strain (lineage) is believed to offer long-term protection [8], so the model does not include the possibility of subsequent infections of recovered individuals. The recovery rate $\gamma(\tau)$ is also assumed to depend on time-since-infection The vector population is divided into 2 non-intersecting classes, susceptible vectors, $S_v(t)$, and infected vectors, $I_v(t)$. Vectors do not clear infection, so we use an SI-type model for their dynamics. Since mosquitoes feed on many other species, we assume that the abundance of vectors does not depend on the abundance of humans. In the model, μ_v is the natural death rate of vectors and Λ_v is their recruitment rate per unit time. Human to vector transmission is also modeled as standard incidence, given by

$$\frac{S_v}{N}\int_0^\infty \beta(\tau) i(\tau,t)\mathrm{d}\tau$$

where $\beta(\tau)$ is the transmission rate from infectious hosts $(i(\tau,t))$ to vectors which depends on host's age of infection.

Models of vector-borne diseases with direct transmission is not new and have been considered before as ODE models for homogeneous population [38]. Including direct transmission with the vector transmission, we obtain the following model structured with time-since-infection,

TABLE 1. Definition of the variables in the vector-borne model (2.1).

Variable	Meaning
$S_v(t)$	The number of susceptible vectors at time t
$I_v(t)$	The number of infected vectors at time t
$S(t)$	The number of susceptible individuals at time t
$i(\tau,t)$	Density of the infected host with infection age τ at time t
$R(t)$	The number of recovered individuals at time t

$$\begin{cases} S_v' = \Lambda_v - \dfrac{S_v}{N}\displaystyle\int_0^\infty \beta(\tau)i(\tau,t)\mathrm{d}\tau - \mu_v S_v, \\ I_v' = \dfrac{S_v}{N}\displaystyle\int_0^\infty \beta(\tau)i(\tau,t)\mathrm{d}\tau - \mu_v I_v, \\ S' = \Lambda - \dfrac{\beta_v S I_v}{N} - \dfrac{S}{N}\displaystyle\int_0^\infty \beta_d(\tau)i(\tau,t)\mathrm{d}\tau - \mu S, \\ \dfrac{\partial i}{\partial t} + \dfrac{\partial i}{\partial \tau} = -(\gamma(\tau)+\mu)i(\tau,t), \\ i(0,t) = \dfrac{\beta_v S I_v}{N} + \dfrac{S}{N}\displaystyle\int_0^\infty \beta_d(\tau)i(\tau,t)\mathrm{d}\tau, \\ R' = \displaystyle\int_0^\infty \gamma(\tau)i(\tau,t)\mathrm{d}\tau - \mu R, \end{cases} \tag{2.1}$$

with initial conditions $S_v(0) = S_{v_0} > 0, I_v(0) = I_{v_0} \geq 0, S(0) = S_0 > 0, i(\tau,0) = i_0(\tau) \geq 0$ and $R(0) = R_0 \geq 0$. All parameters are positive; $\Lambda_v > 0$, $\Lambda > 0$, $\mu_v > 0$, $\mu > 0$ and $\beta_v > 0$. Age of infection dependent transmission parameter-functions $\beta(\tau)$ and $\beta_d(\tau)$ are bounded and uniformly continuous defined on a compact support with a non-zero Lebesgue measure. Recovery rate $\gamma(\tau)$ belongs to $L^\infty(0,\infty)$. The dependent variables and their meaning are listed in the Table 1. The parameters and their meaning in the models are listed in the Table 2.

The total host population is $N(t) = S(t) + \int_0^\infty i(\tau,t)\mathrm{d}\tau + R(t)$ for $t \geq 0$, which satisfies the following differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(S(t) + \int_0^\infty i(\tau,t)\mathrm{d}\tau + R(t)\right) = \Lambda - \mu\left(S(t) + \int_0^\infty i(\tau,t)\mathrm{d}\tau + R(t)\right),$$

with $N(0) = S(0) + \int_0^\infty i(\tau,0)\mathrm{d}\tau + R(0)$. Thus,

$$\lim_{t\to\infty}\left(S(t) + \int_0^\infty i(\tau,t)\mathrm{d}\tau + R(t)\right) = \frac{\Lambda}{\mu}$$

TABLE 2. Definition of the parameters in the between host model.

Parameter	Meaning
Λ_v	Susceptible vector recruitment rate
μ_v	Vector natural death rate
Λ	Host recruitment rate
μ	Host natural death rate
β_v	Transmission rate of infection from an infected mosquito to susceptible host
$\beta(\tau)$	Transmission rate of infection from an infected host to susceptible vector class
$\beta_d(\tau)$	Direct transmission rate of infection from an infected host to a susceptible host
$\gamma(\tau)$	Recovery rate of an infected host

Similarly, total mosquito population $N_v(t) = S_v(t) + I_v(t)$ satisfy differential equation,

$$N_v'(t) = \Lambda_v - \mu_v N_v, \qquad N_v(0) = S_v(0) + I_v(0)$$

Hence,

$$\lim_{t\to\infty} (S_v(t) + I_v(t)) = \frac{\Lambda_v}{\mu_v}$$

Therefore the following set Ω is positively invariant for the system (2.1).

$$\begin{aligned}\Omega = \Big\{ \big(S_{v0}, I_{v0}, S_0, i_0(\cdot), R_0\big) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times L^1_+(0,\infty) \times \mathbb{R}_+ : \\ S_v(t) + I_v(t) \leq \frac{\Lambda_v}{\mu_v}, \quad S(t) + \int_0^\infty i(\tau,t)\mathrm{d}\tau + R(t) \leq \frac{\Lambda}{\mu} \Big\}.\end{aligned} \tag{2.2}$$

where $\mathbb{R}_+ = [0,\infty)$ and

$$L^1_+(0,\infty) = \{i_0(\tau) : \mathbb{R}_+ \to [0,\infty) : i_0(\tau) \geq 0 \text{ for } \tau \in \mathbb{R}_+ \text{ and } \int_0^\infty i_0(\tau) < \infty\}.$$

Since the exit rate of infectious individuals is given by the term $\gamma(\tau) + \mu$, the probability of still being infectious τ units of time after being infected is given by

$$\pi(\tau) = e^{-\int_0^\tau (\gamma(s)+\mu)\mathrm{d}s}.$$

Then the reproduction number of the disease in system (2.1) is given by

$$\mathcal{R}_0 = \beta_v \frac{\Lambda_v \mu}{\Lambda \mu_v^2} \int_0^\infty \beta(\tau)\pi(\tau)\mathrm{d}\tau + \int_0^\infty \beta_d(\tau)\pi(\tau)\mathrm{d}\tau\,.$$

Epidemiologically, reproduction number gives the number of secondary infections produced by one infected individual in a totally susceptible population during its lifetime as infection. System (2.1) has three modes of infection, vector-to-human $\mathcal{R}_v$, human-to-vector $\mathcal{R}_h$ and human-to-human $\mathcal{R}_d$ transmission, and each given by

$$\mathcal{R}_v = \frac{\beta_v}{\mu_v}, \quad \mathcal{R}_h = \frac{\Lambda_v \mu}{\Lambda \mu_v} \int_0^\infty \beta(\tau)\pi(\tau)\mathrm{d}\tau, \quad \text{and} \quad \mathcal{R}_d = \int_0^\infty \beta_d(\tau)\pi(\tau)\mathrm{d}\tau$$

Thus the reproduction number is $\mathcal{R}_0 = \mathcal{R}_v\mathcal{R}_h + \mathcal{R}_d$.

Next, we use the approach first introduced by Thieme [34] and also adopted by the authors [16–18, 20, 43]. We define the semiflow $\mathcal{U}$ of the solutions of the system (2.1) as

$$\mathcal{U}\big(t; S_{v0}, I_{v0}, S_0, i_0(\cdot), R_0\big) = \big(S_v(t), I_v(t), S(t), i(\tau,t), R(t)\big).$$

Let the state space U be defined as $U = \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times L^1(0,\infty)\times\mathbb{R}$, then for any $u = (u_1, u_2, u_3, u_4(\tau), u_5) \in U$ its norm is given as

$$\|u\| = |u_1| + |u_2| + |u_3| + \int_0^\infty |u_4(\tau)|\mathrm{d}\tau + |u_5|.$$

We would like to express (2.1) as an ODE on a Banach space. First, we move the non-linearity in the boundary condition in (2.1) to a nonlinear operator by enlarging the state space U. Let $X = U \times \mathbb{R}$ and $X^0 = \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times L^1(0,\infty)\times\{0\}\times\mathbb{R}$. That is, for any vector $u = (u_1, u_2, u_3, u_4(\tau), u_5) \in U$, $u = (u_1, u_2, u_3, u_4(\tau), 0, u_5) \in X^0$. We denote the positive cone in the corresponding space by $X_+ = \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\times L^1_+(0,\infty)\times\mathbb{R}_+\times\mathbb{R}_+$ and $X^0_+ = X^0 \cap X_+$.

We define a linear operator $A : D(A) \subset X^0 \to X$ as

$$Au = \begin{pmatrix} -\mu_v u_1 \\ -\mu_v u_2 \\ -\mu u_3 \\ -u_4' - (\gamma(\tau)+\mu)u_4 \\ -u_4(0) \\ -\mu u_5 \end{pmatrix}$$

with $D(A) = \{v \in X^0_+ : u_4 \in W^1_1(0,\infty)\}$. Note that $\overline{D(A)} = X^0$, hence the linear operator A is not dense in X. Thus A can not be a generator of a C_0-semigroup. Next, we define a nonlinear map $F : X^0 \to X$ as

$$F(u)(\tau) = \begin{pmatrix} \Lambda_v - \dfrac{u_1}{N(u)} \displaystyle\int_0^\infty \beta(\tau)u_4(\tau)\mathrm{d}\tau \\ \dfrac{u_1}{N(u)} \displaystyle\int_0^\infty \beta(\tau)u_4(\tau)\mathrm{d}\tau \\ \Lambda - \beta_v \dfrac{u_3 u_2}{N(u)} - \dfrac{u_3}{N(u)} \displaystyle\int_0^\infty \beta_d(\tau)u_4(\tau)\mathrm{d}\tau \\ 0 \\ \beta_v \dfrac{u_3 u_2}{N(u)} + \dfrac{u_3}{N(u)} \displaystyle\int_0^\infty \beta_d(\tau)u_4(\tau)\mathrm{d}\tau \\ \displaystyle\int_0^\infty \gamma(\tau)u_4(\tau)\mathrm{d}\tau \end{pmatrix}$$

We now define the following semi-linear Cauchy problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au + F(u) \qquad \forall t > 0, \qquad u(0) = u_0 \in X^0 \tag{2.3}$$

Since A is not dense in X and not bounded, the abstract Cauchy problem (2.3) may not have strong solutions, since $u(t)$ is neither differentiable nor an element in $D(A)$. In other words, if $u_0 \in X^0 \setminus D(A)$, we may not obtain strong solutions of (2.3). An approach to fix this problem is obtained by integrating (2.3) in time, then we get

$$u(t) = u_0 + A\int_0^t u(s)\mathrm{d}s + \int_0^t F(u(s))\mathrm{d}s \tag{2.4}$$

A continuous solution to (2.4) is called an integral solution to (2.3).

The linear operator A is not densely defined but it satisfies the estimates of Hille-Yosida Theorem as shown in Proposition 2.1. Let $\hat{\mu} = \min\{\mu_v, \mu\}$ and $\rho(A)$ denote the resolvent of A.

Proposition 2.1. *The operator A is closed operator such that $\lambda I - A$ has a bounded inverse for $Re\lambda > -\hat{\mu}$ where $\lambda \in \rho(A)$. Furthermore,*

$$\|(\lambda I - A)^{-n}\|_A \le \frac{M_1}{(\lambda + \hat{\mu})^n}$$

for appropriate constant M_1.

Proof. Let $y = (y_1, y_2, y_3, y_4(\cdot), y_5, y_6) \in X$ and $u \in X^0$ such that $(\lambda I - A)u = y$. Setting $u = (\lambda I - A)^{-1}y$ we obtain,

$$u = \begin{pmatrix} \dfrac{y_1}{(\lambda + \mu_v)} \\ \dfrac{y_2}{(\lambda + \mu_v)} \\ \dfrac{y_3}{(\lambda + \mu)} \\ y_6 e^{-\lambda\tau}\pi(\tau) + \displaystyle\int_0^\tau e^{-\lambda(\tau - s)}\frac{\pi(\tau)}{\pi(s)}y_4(s)\mathrm{d}s \\ 0 \\ \dfrac{y_5}{(\lambda + \mu_v)} \end{pmatrix}$$

Then

$$\begin{aligned}
\|(\lambda I - A)^{-1}y\| &\le \left|\frac{y_1}{\lambda + \mu_v}\right| + \left|\frac{y_2}{\lambda + \mu_v}\right| + \left|\frac{y_3}{\lambda + \mu}\right| + \int_0^\infty \left|y_6 e^{-\lambda\tau}\pi(\tau)\right| \mathrm{d}\tau \\
&\quad + \int_0^\infty \left|\int_0^\tau e^{-\lambda(\tau - s)}y_4(s)\frac{\pi(\tau)}{\pi(s)}\right| \mathrm{d}s + \left|\frac{y_5}{\lambda + \mu_v}\right| \\
&\le \left|\frac{y_1}{\lambda + \mu_v}\right| + \left|\frac{y_2}{\lambda + \mu_v}\right| + \left|\frac{y_3}{\lambda + \mu}\right| + \left|\frac{y_6}{\lambda + \mu}\right| + \left|\frac{1}{\lambda + \mu}\right| \int_0^\infty |y_4(s)\mathrm{d}s| + \left|\frac{y_5}{\lambda + \mu_v}\right| \\
&\le \frac{1}{|\lambda + \hat{\mu}|}\left(|y_1| + |y_2| + |y_3| + \int_0^\infty |y_4(s)|\mathrm{d}s + |y_5| + |y_6|\right)
\end{aligned}$$

$$\leq \frac{1}{|\lambda+\hat{\mu}|}\|y\|$$

Since, if $\lambda = a + ib$, then $|\lambda+\mu| = \sqrt{(a+\mu)^2+b^2} \geq \sqrt{(a+\hat{\mu})^2+b^2} = |\lambda+\hat{\mu}|$. This implies $\|(\lambda I - A)^{-n}\|_A \leq \dfrac{M_1}{(\lambda+\hat{\mu})^n}$ for $Re\lambda > -\hat{\mu}$, $n \geq 1$ and appropriate constant M_1. □

Furthermore, the non linear operator $F(u)$ satisfies Lipschitz condition.

Proposition 2.2. *There exist a constant M_2 such that*

$$\|F(u^1) - F(u^2)\| \leq M_2\|u^1 - u^2\| \quad \textit{for all} \quad u^1, u^2 \in X^0.$$

Proof. Let's denote F(u) as;

$$F(u)(\tau) = \begin{pmatrix} \Lambda_v - \dfrac{u_1}{N(u)}\displaystyle\int_0^\infty \beta(\tau)u_4(\tau)\mathrm{d}\tau \\ \dfrac{u_1}{N(u)}\displaystyle\int_0^\infty \beta(\tau)u_4(\tau)\mathrm{d}\tau \\ \Lambda - \beta_v\dfrac{u_3u_2}{N(u)} - \dfrac{u_3}{N(u)}\displaystyle\int_0^\infty \beta_d(\tau)u_4(\tau)\mathrm{d}\tau \\ 0 \\ \beta_v\dfrac{u_3u_2}{N(u)} + \dfrac{u_3}{N(u)}\displaystyle\int_0^\infty \beta_d(\tau)u_4(\tau)\mathrm{d}\tau \\ \displaystyle\int_0^\infty \gamma(\tau)u_4(\tau)\mathrm{d}\tau \end{pmatrix} = \begin{pmatrix} F_1(u)(\tau) \\ F_2(u)(\tau) \\ F_3(u)(\tau) \\ 0 \\ F_4(u)(\tau) \\ F_5(u)(\tau) \end{pmatrix}$$

And let $\bar{\beta} = \sup_{\tau\in[0,\infty)} \beta(\tau), \bar{\gamma} = \sup_{\tau\in[0,\infty)} \gamma(\tau)$ and $\bar{\beta}_d = \sup_{\tau\in[0,\infty)} \beta_d(\tau)$. Note that the total population is bounded below away from zero, and we set $N(u) > N_{u0}$. Then

$$\begin{aligned} |F_1(u^1)(\tau) - F_1(u^2)(\tau)| &= \left|\frac{u_1^1}{N(u^1)}\int_0^\infty \beta(\tau)u_4^1(\tau)\mathrm{d}\tau - \frac{u_1^2}{N(u^2)}\int_0^\infty \beta(\tau)u_4^2(\tau)\mathrm{d}\tau\right| \\ &= \left|\frac{u_1^1}{N(u^1)}\int_0^\infty \beta(\tau)u_4^1(\tau)\mathrm{d}\tau - \frac{u_1^2}{N(u^1)}\int_0^\infty \beta(\tau)u_4^1(\tau)\mathrm{d}\tau\right. \\ &\quad + \frac{u_1^2}{N(u^1)}\int_0^\infty \beta(\tau)u_4^1(\tau)\mathrm{d}\tau - \frac{u_1^2}{N(u^2)}\int_0^\infty \beta(\tau)u_4^1(\tau)\mathrm{d}\tau \\ &\quad \left. + \frac{u_1^2}{N(u^2)}\int_0^\infty \beta(\tau)u_4^1(\tau)\mathrm{d}\tau - \frac{u_1^2}{N(u^2)}\int_0^\infty \beta(\tau)u_4^2(\tau)\mathrm{d}\tau\right| \end{aligned}$$

$$
\begin{aligned}
&\le \left|\frac{\int_0^\infty \beta(\tau)u_4^1(\tau)\mathrm{d}\tau}{N(u^1)}\right||u_1^1-u_1^2| + \left|\frac{u_1^2\int_0^\infty \beta(\tau)u_4^1(\tau)\mathrm{d}\tau}{N(u^1)N(u^2)}\right||N(u^1)-N(u^2)| \\
&\quad + \left|\frac{u_1^2}{N(u^2)}\right|\int_0^\infty \beta(\tau)|u_4^1(\tau)-u_4^2(\tau)|\mathrm{d}\tau \\
&\le \bar{\beta}|u_1^1-u_1^2| + \bar{\beta}\frac{\Lambda_v}{\mu_v N_{u0}}|N(u^1)-N(u^2)| + \bar{\beta}\frac{\Lambda_v}{\mu_v N_{u0}}\int_0^\infty |u_4^1(\tau)-u_4^2(\tau)|\mathrm{d}\tau \\
&\le \bar{\kappa}_1\left(|u_1^1-u_1^2| + |u_3^1-u_3^2| + \int_0^\infty |u_4^1(\tau)-u_4^2(\tau)|\mathrm{d}\tau + |u_5^1-u_5^2|\right)
\end{aligned}
$$

Where $\bar{\kappa}_1 = \max\left\{\bar{\beta}, 2\bar{\beta}\frac{\Lambda_v}{\mu_v N_{u0}}\right\}$.

$$
\begin{aligned}
|F_3(u^1)(\tau)-F_3(u^2)(\tau)| &= \left|\beta_v\frac{u_3^2u_2^2}{N(u^2)} - \beta_v\frac{u_3^1u_2^1}{N(u^1)} + \frac{u_3^2}{N(u^2)}\int_0^\infty \beta_d(\tau)u_4^2(\tau)\mathrm{d}\tau - \frac{u_3^1}{N(u^1)}\int_0^\infty \beta_d(\tau)u_4^1(\tau)\mathrm{d}\tau\right| \\
&\le \left|\beta_v\frac{u_3^2u_2^2}{N(u^2)} - \beta_v\frac{u_3^2u_2^1}{N(u^2)} + \beta_v\frac{u_3^2u_2^1}{N(u^2)} - \beta_v\frac{u_3^2u_2^1}{N(u^1)} + \beta_v\frac{u_3^2u_2^1}{N(u^1)} - \beta_v\frac{u_3^1u_2^1}{N(u^1)}\right| \\
&\quad + \left|\frac{u_3^2}{N(u^2)}\int_0^\infty \beta_d(\tau)u_4^2(\tau)\mathrm{d}\tau - \frac{u_3^2}{N(u^2)}\int_0^\infty \beta_d(\tau)u_4^1(\tau)\mathrm{d}\tau + \frac{u_3^2}{N(u^2)}\int_0^\infty \beta_d(\tau)u_4^1(\tau)\mathrm{d}\tau\right. \\
&\quad \left. - \frac{u_3^2}{N(u^1)}\int_0^\infty \beta_d(\tau)u_4^1(\tau)\mathrm{d}\tau + \frac{u_3^2}{N(u^1)}\int_0^\infty \beta_d(\tau)u_4^1(\tau)\mathrm{d}\tau - \frac{u_3^1}{N(u^1)}\int_0^\infty \beta_d(\tau)u_4^1(\tau)\mathrm{d}\tau\right| \\
&\le \beta_v|u_2^1-u_2^2| + \frac{\beta_v\Lambda_v}{\mu_v N_{u0}}|N(u^1)-N(u^2)| + \frac{\beta_v\Lambda_v}{\mu_v N_{u0}}|u_3^2-u_3^1| \\
&\quad + \bar{\beta}_d\int_0^\infty |u_4^1(\tau)-u_4^2(\tau)|\mathrm{d}\tau + \bar{\beta}_d|N(u^1)-N(u^2)| + \bar{\beta}_d|u_3^1-u_3^2| \\
&= \beta_v|u_2^1-u_2^2| + (2\frac{\beta_v\Lambda_v}{\mu_v N_{u0}} + 2\bar{\beta}_d)|u_3^1-u_3^2| + (\frac{\beta_v\Lambda_v}{\mu_v N_{u0}} + 2\bar{\beta}_d)\int_0^\infty |u_4^1(\tau)-u_4^2(\tau)|\mathrm{d}\tau \\
&\quad + (\frac{\beta_v\Lambda_v}{\mu_v N_{u0}} + \bar{\beta}_d)|u_5^1-u_5^2| \\
&\le \bar{\kappa}_3(|u_2^1-u_2^2| + |u_3^1-u_3^2| + \int_0^\infty |u_4^1(\tau)-u_4^2(\tau)|\mathrm{d}\tau + |u_5^1-u_5^2|)
\end{aligned}
$$

Where $\bar{\kappa}_3 = \max\{\beta_v, 2\frac{\beta_v\Lambda_v}{\mu_v N_{u0}} + 2\bar{\beta}_d\}$. With similar analysis, one can show that

$$
\begin{aligned}
\|F(u^1)(\tau)-F(u^2)(\tau)\| &\le M_2(|u_1^1-u_1^2| + |u_2^1-u_2^2| + |u_3^1-u_3^2| + \int_0^\infty |u_4^2(\tau)-u_4^1(\tau)|\mathrm{d}\tau + |u_5^1-u_5^2|) \\
&= M_2\|u^1-u^2\|
\end{aligned}
$$

□

By applying the results in [34], we obtain the following result.

Theorem 2.3. *The system (2.1) represented by the integral equation (2.4) has a unique continuous solution with values in X_+^0. Thus, there exists uniquely defined semiflow $\{\mathcal{U}(t)\}_{t\geq 0}$ on X_+^0 such that for each $u_0 \in X_+^0$, there exist a continuous mapping $\mathcal{U} : [0,\infty) \times X_+^0 \to X_+^0$ which is an integral solution to Cauchy problem (2.3) i.e. for all $t \geq 0$ s.t.*

$$\int_0^t \mathcal{U}(s)u_0 \mathrm{d}s \in D(A)$$

$$\mathcal{U}(t)u_0 = u_0 + A\int_0^t \mathcal{U}(s)u_0 \mathrm{d}s + \int_0^t F(\mathcal{U}(s)u_0)\mathrm{d}s \tag{2.5}$$

Furthermore, there exist constant $M_3 \geq 1$ and $w \in \mathbb{R}$ such that

$$\|\mathcal{U}(t)u_0 - \mathcal{U}(t)v_0\| \leq M_3 e^{wt}\|u_0 - v_0\| \quad \textit{for all} \quad t \geq 0 \quad \textit{and} \quad u_0, v_0 \in X_+^0$$

Proof. Proof is a direct consequence of Theorems 2.3 and 3.2 in [34] together with Propositions 2.1 and 2.2. For the sake of completeness, we prove the details here. Thus, we first show that $\lambda(\lambda I - A)^{-1}$ maps X_+ into itself for sufficiently large λ. Let us consider $\lambda(\lambda I - A)^{-1}(z) = y$ for any $z \in X_+$. Then, $(\lambda I - A)^{-1}z = \frac{1}{\lambda}y$. Solving for z we get,

$$z = (I - \frac{A}{\lambda})y = \begin{cases} y_1 + \dfrac{\mu_v}{\lambda}y_1 \\ y_2 + \dfrac{\mu_v}{\lambda}y_2 \\ y_3 + \dfrac{\mu}{\lambda}y_3 \\ y_4(\tau) + \dfrac{y_4' + (\gamma(\tau) + \mu)}{\lambda}y_4(\tau) \\ y_4(0) + \dfrac{1}{\lambda}y_4(0) \\ y_5 + \dfrac{\mu}{\lambda}y_5 \end{cases}$$

This implies for sufficiently large λ, $\lambda(\lambda I - A)^{-1}$ maps X_+ into itself. Next, we show that $\frac{1}{h}dist(u_0 + hF(u_0), X_+) \to 0$ as $h \to 0$ for all $t \geq 0$ where $u_0 \in X_+^0$. Set

$$\hat{F}(u_0) = F(u_0) + \alpha u_0$$

where $\alpha > \max\{\beta_v, \int_0^\infty \beta(\tau)u_4(\tau)\mathrm{d}\tau, \int_0^\infty \beta_d(\tau)u_4(\tau)\mathrm{d}\tau\}$, then $\hat{F}$ maps X_+^0 into X_+. Thus for sufficiently small h and $u_0 \in X_+^0$, we have

$$\frac{1}{h}dist(u_0 + hF(u_0), X_+) = \frac{1}{h}dist(u_0 - \alpha h u_0 + h\hat{F}(u_0), X_+) \to 0$$

as $h \to 0$. □

3. Steady states and their local stability

Next, we study equilibria of (2.3). First, we recall the following Theorem from [34].

Theorem 3.1. *The following statements are equivalent for* $u^* \in X^0$
i. $u(t) = u^*$ *is time indepentent solution to* (4)*;*
ii. $\mathcal{U}(t)u^* = u^*$ *for all* $t \geq 0$*;*
iii. $u^* \in D(A)$ *and* $Au^* + F(u^*) = 0$.

Let $B = \int_0^\infty \beta(\tau)\pi(\tau)\mathrm{d}\tau$ and $B_d = \int_0^\infty \beta_d(\tau)\pi(\tau)\mathrm{d}\tau$.

Theorem 3.2. *Let* $\mathcal{R}_0 = \beta_v \dfrac{\Lambda_v \mu}{\Lambda \mu_v^2} B + B_d$*, then the following statement are true.*

i. $E_0 = \big(\dfrac{\Lambda_v}{\mu_v}, 0, \dfrac{\Lambda}{\mu}, 0, 0, 0\big) \in X^0$ *is a disease free equilibrium of* (2.3)*;*

ii. When $\mathcal{R}_0 > 1$*, then there exist a unique endemic equilibrium given by* $E^* = \big(S_v^*, I_v^*, S^*, i^*(\tau), R^*\big)$ *where*

$$S_v^* = \frac{\Lambda_v}{\dfrac{i^*(0)B}{N^*} + \mu_v} \qquad I_v^* = \frac{N_v^* i^*(0)B}{i^*(0)B + N^*\mu_v}$$

$$S^* = \frac{N^*}{\beta_v \dfrac{N_v^* B}{i^*(0)B + N^*\mu_v} + B_d} \qquad R^* = \frac{i^*(0)\int_0^\infty \gamma(\tau)\pi(\tau)\mathrm{d}\tau}{\mu}$$

and $i^*(0)$ *is unique positive solution of*

$$\frac{B_d B}{\Lambda} {i^*}^2(0) + \frac{\mu_v}{\mu}\big(\mathcal{R}_0 + B\frac{\mu}{\mu_v} - B_d B\frac{\mu}{\mu_v}\big) i^*(0) + N^*\mu_v(1 - \mathcal{R}_0) = 0.$$

Proof. Suppose $u^* \in X^0$ is an equilibrium of (2.3), then $Au^* + F(u^*) = 0$. The system for the equilibria takes the following form

$$\begin{aligned}
&\Lambda_v - \frac{S_v^*}{N^*}\int_0^\infty \beta(\tau) i^*(\tau)\mathrm{d}\tau - \mu_v S_v^* = 0, \\
&\frac{S_v^*}{N^*}\int_0^\infty \beta(\tau) i^*(\tau)\mathrm{d}\tau - \mu_v I_v^* = 0, \\
&\Lambda - \frac{\beta_v S^* I_v^*}{N^*} - \frac{S^*}{N^*}\int_0^\infty \beta_d(\tau) i^*(\tau)\mathrm{d}\tau - \mu S^* = 0, \\
&\frac{di^*}{\mathrm{d}\tau} = -(\gamma(\tau) + \mu) i^*(\tau), \\
&i^*(0) = \frac{\beta_v S^* I_v^*}{N^*} + \frac{S^*}{N^*}\int_0^\infty \beta_d(\tau) i^*(\tau)\mathrm{d}\tau, \\
&\int_0^\infty \gamma(\tau) i^*(\tau)\mathrm{d}\tau - \mu R^* = 0.
\end{aligned} \tag{3.1}$$

Clearly, $i^*(\tau) = i^*(0)\pi(\tau)$. If $i^*(0) = 0$, then we obtain disease free equilibrium $u^0 = \big(\frac{\Lambda_v}{\mu_v}, 0, \frac{\Lambda}{\mu}, 0, 0, 0\big)$. Let $i^*(0) \neq 0$ and let $N_v^* = \frac{\Lambda_v}{\mu_v}$, then from second equation in (3.1) we obtain

$$\frac{N_v^* - I_v^*}{N^*} i^*(0) \int_0^\infty \beta(\tau)\pi(\tau)\mathrm{d}\tau - \mu_v I_v^* = 0$$

Solving for I_v^*, we get,

$$I_v^* = \frac{N_v^* i^*(0) B}{i^*(0)B + N^*\mu_v},$$

Solving first equation of (3.1) for S_v^* gives,

$$S_v^* = \frac{\Lambda_v}{\dfrac{i^*(0)B}{N^*} + \mu_v}$$

and solving for S^*, we get

$$S^* = \frac{N^*}{\beta_v \dfrac{N_v^* B}{i^*(0)B + N^*\mu_v} + B_d}.$$

Since $\Lambda - i^*(0) - \mu S^* = 0$, substituting S^*, we obtain

$$i^*(0) = \Lambda - \frac{\Lambda}{\beta_v \dfrac{mBN^*}{i^*(0)B + \mu_v N^*} + B_d}$$

which can be written as quadratic equation

$$a_2 {i^*}^2(0) + a_1 i^*(0) + a_0 = 0$$

where

$$\begin{aligned}
a_2 &= \frac{B_d B}{\Lambda},\\
a_1 &= \frac{\mu_v}{\mu}\big(\mathcal{R}_0 + B\frac{\mu}{\mu_v} - B_d B\frac{\mu}{\mu_v}\big),\\
a_0 &= N^*\mu_v(1 - \mathcal{R}_0).
\end{aligned}$$

and $i^*(0) = \frac{-a_1 \mp \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$. Clearly, $a_2 > 0$. When $\mathcal{R}_0 \geq 1$, $a_0 = N^*\mu_v(1 - \mathcal{R}_0) \leq 0$. Thus, there exists a unique positive $i^*(0)$ when $\mathcal{R}_0 \geq 1$. When $\mathcal{R}_0 < 1$ then $a_0 > 0$ and

$$a_1 = \frac{\mu_v}{\mu}\big(\mathcal{R}_0 + B\frac{\mu}{\mu_v} - B_d B\frac{\mu}{\mu_v}\big) = \frac{\mu_v}{\mu}\mathcal{R}_0 + B(1 - B_d) > 0,$$

since $B_d < 1$. Thus the endemic equilibrium only exists when $\mathcal{R}_0 \geq 1$. □

3.1. Linearized system

Let $u^* \in D(A)$ be an equilibrium of the abstract Cauchy Problem (2.3). Let $u(t) = u^* + v(t)$ then the linearized equation of (2.3) around the equilibrium u^* is

$$\begin{aligned} &\frac{\mathrm{d}v(t)}{\mathrm{d}t} = Av(t) + F'(u^*)v(t) \quad \text{for all} \quad t \geq 0 \\ &v(0) = v_0 \in \overline{D(A)}. \end{aligned}$$

First we will be interested in A_0, the part of A in $D(A)$ which is defined as

$$A_0 v = Av \quad \text{for all} \quad v \in D(A_0)$$

where

$$D(A_0) = \{v \in D(A) : Av \in \overline{D(A)}\}.$$

Thus,

$$D(A_0) = \left\{ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ 0 \\ u_5 \end{pmatrix} \in D(A) : \begin{pmatrix} -\mu_v u_1 \\ -\mu_v u_2 \\ -\mu u_3 \\ -u_4' - (\gamma(\tau) + \mu)u_4 \\ -u_4(0) \\ -\mu u_5 \end{pmatrix} \in \overline{D(A)} \right\}$$

That is

$$D(A_0) = \big\{v \in D(A) : u_4(0) = 0\big\}.$$

Clearly, A_0 is an infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ of a bounded linear operators in $\overline{D(A)}$. Furthermore, the semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ generated by A_0 is a contraction. That is there exists $M_4 \geq 0$ such that

$$\|T_{A_0}(t)\| \leq M_4 e^{-\hat{\mu}t} \quad \text{for all} \quad t \geq 0. \tag{3.2}$$

To continue with the analysis, we recall the following definitions from [17, 34].

Definition 3.3. [17, 34] Let $A : D(A) \subset X \to X$ be a infinitesimal generator of a linear C_0 semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X. We define the *growth bound* $\omega_0(A) \in [-\infty, \infty]$ of A as,

$$\omega_0(A) = \lim_{t\to\infty} \frac{\ln \|T_A(t)\|_X}{t}.$$

We define the *essential growth bound* $\omega_{0,ess}(A) \in [-\infty, \infty]$ of A as,

$$\omega_{0,ess}(A) = \lim_{t\to\infty} \frac{\ln \|T_A(t)\|_{ess}}{t}.$$

where $\|\cdot\|_{ess}$denotes an appropriate measure of noncompactness of an operator.

It follows from (3.2) that essential growth rate $\omega_{ess}(A_0)$ of $\{T_{A_0}(t)\}_{t\geq 0}$ is no more than $-\hat{\mu}$.

Theorem 3.4. *The DFE* $E_0 = \big(\frac{\Lambda_v}{\mu_v}, 0, \frac{\Lambda}{\mu}, 0, 0, 0\big)$ *is locally asymptotically stable when* $\mathcal{R}_0 < 1$ *and unstable when* $\mathcal{R}_0 > 1$.

Proof. The linearized equation of (2.3) around the DFE E_0 is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = Av(t) + DF(E_0)\big(v(t)\big)$$

where

$$DF(E_0)v(t) = \begin{pmatrix} -m\int_0^\infty \beta(\tau)v_4(\tau)\mathrm{d}\tau \\ m\int_0^\infty \beta(\tau)v_4(\tau)\mathrm{d}\tau \\ -\beta_v v_2 - \int_0^\infty \beta_d(\tau)v_4(\tau)\mathrm{d}\tau \\ 0 \\ \beta_v v_2 + \int_0^\infty \beta_d(\tau)v_4(\tau)\mathrm{d}\tau \\ \int_0^\infty \gamma(\tau)v_4(\tau)\mathrm{d}\tau \end{pmatrix}$$

with $m = \frac{\Lambda_v \mu}{\Lambda \mu_v}$ and $v \in X^0$. Now let $(A + DF(E_0))_0$ denote the part of $A + DF(E_0)$ in $\overline{D(A)}$. Since $DF(E_0)$ is a compact bounded linear operator, it follows from [17, 34] that

$$\omega_{0,ess}\big((A + DF(E_0))_0\big) \leq -\hat{\mu}.$$

By Corollary 4.3 in [34], local stability is determined by point spectrum of $(A + DF(E_0))_0$.

Note that the resolvent set of the operator $(A + DF(E_0))_0$ consists of $\lambda \in \mathbb{C}$ such that

$$\lambda v - (A + DF(E_0))_0 v = u$$

has a unique solution $v \in \overline{D(A)}$ for any $u \in \overline{D(A)}$. That is,

$$\begin{aligned} \lambda v_1 + \mu_v v_1 + m\int_0^\infty \beta(\tau)v_4(\tau)\mathrm{d}\tau &= u_1 \\ \lambda v_2 + \mu_v v_2 - m\int_0^\infty \beta(\tau)v_4(\tau)\mathrm{d}\tau &= u_2 \\ \lambda v_3 + \mu v_3 + \beta_v v_2 + \int_0^\infty \beta_d(\tau)v_4(\tau)\mathrm{d}\tau &= u_3 \\ \lambda v_4 + v_4'(\tau) + (\gamma(\tau) + \mu)v_4 &= u_4(\tau) \\ \lambda v_5 + \mu v_5 + \int_0^\infty \gamma(\tau)v_4(\tau)\mathrm{d}\tau &= u_5 \\ v_4(0) - \beta_v v_2 - \int_0^\infty \beta_d(\tau)v_4(\tau)\mathrm{d}\tau &= 0. \end{aligned} \tag{3.3}$$

Solving the system (3.3), we obtain,

$$v_1 = \frac{u_1}{\lambda+\mu_v} - \frac{mv_4(0)}{\lambda+\mu_v}\int_0^\infty \beta(\tau)e^{-\lambda\tau}\pi(\tau)\mathrm{d}\tau - \frac{m}{\lambda+\mu_v}\int_0^\infty \beta(\tau)\int_0^\tau u_4(s)e^{-\lambda(\tau-s)}e^{-\int_s^\tau(\gamma(\sigma)+\mu)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}\tau$$

$$v_2 = \frac{u_2}{\lambda+\mu_v} + \frac{mv_4(0)}{\lambda+\mu_v}\int_0^\infty \beta(\tau)e^{-\lambda\tau}\pi(\tau)\mathrm{d}\tau + \frac{m}{\lambda+\mu_v}\int_0^\infty \beta(\tau)\int_0^\tau u_4(s)e^{-\lambda(\tau-s)}e^{-\int_s^\tau(\gamma(\sigma)+\mu)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}\tau$$

$$v_3 = \frac{u_3}{\lambda+\mu} - \frac{v_4(0)}{\lambda+\mu}$$

$$v_4(\tau) = v_4(0)e^{-\lambda\tau}\pi(\tau) + \int_0^\tau u_4(s)e^{-\lambda(\tau-s)}e^{-\int_s^\tau(\gamma(\sigma)+\mu)\mathrm{d}\sigma}\mathrm{d}s$$

$$v_5 = \frac{u_5}{\lambda+\mu} + \frac{v_4(0)}{\lambda+\mu}\int_0^\infty \gamma(\tau)e^{-\lambda\tau}\pi(\tau)\mathrm{d}\tau - \frac{1}{\lambda+\mu}\int_0^\infty \gamma(\tau)\int_0^\tau u_4(s)e^{-\lambda(\tau-s)}e^{-\int_s^\tau(\gamma(\sigma)+\mu)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}\tau$$

and

$$v_4(0) = \Bigg(\beta_v\frac{u_2}{\lambda+\mu_v} + \beta_v m\int_0^\infty \beta(\tau)\int_0^\tau u_4(s)e^{-\lambda(\tau-s)}e^{-\int_s^\tau(\gamma(\sigma)+\mu)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}\tau + \int_0^\infty \beta_d(\tau)\int_0^\tau u_4(s)e^{-\lambda(\tau-s)}e^{-\int_s^\tau(\gamma(\sigma)+\mu)\mathrm{d}\sigma}\mathrm{d}s\mathrm{d}\tau\Bigg)/(1-F(\lambda))$$

Thus, the system (3.3) has a unique solution if and only if $1 \neq F(\lambda)$, where

$$F(\lambda) = \frac{\beta_v m}{\lambda+\mu_v}\int_0^\infty \beta(\tau)e^{-\lambda\tau}\pi(\tau)\mathrm{d}\tau + \int_0^\infty \beta_d(\tau)e^{-\lambda\tau}\pi(\tau)\mathrm{d}\tau.$$

Next, we solve the characteristic equation $F(\lambda) = 1$ to determine the stability of the disease-free equilibrium E_0. It is clear that for any real λ, $\lim\limits_{\lambda\to\infty} F(\lambda) = \infty$ and $F'(\lambda) < 0$. Since $F(0) = \mathcal{R}_0$ and $\mathcal{R}_0 < 1$, the equation $F(\lambda) = 1$ has no positive real root. Now, suppose that all complex roots have non-negative real parts, that is $\lambda = a + ib$ with $a \geq 0$, then $|F(\lambda)| \leq \mathcal{R}_0 < 1$. Thus all roots of $F(\lambda) = 1$ have negative real parts when $\mathcal{R}_0 < 1$. When $\mathcal{R}_0 > 1$, since $F(0) > 1$, the equation $F(\lambda) = 1$ has at least one positive solution. That is the disease-free equilibrium E_0 is locally asymptotically stable when $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$. □

Theorem 3.5. *The endemic equilibrium $E^* = \big(S_v^*, I_v^*, S^*, i^*(\tau), R^*\big)$ is locally stable asymptotically stable when it exists, that is when $\mathcal{R}_0 > 1$.*

Proof. The linearized equation of (2.3) around the endemic equilibrium $E^* = \big(S_v^*, I_v^*, S^*, i^*(\tau), R^*\big)$ is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = Av(t) + DF(E^*)\big(v(t)\big)$$

where

$$DF(E^*)v(t) = \begin{pmatrix} -\frac{S_v^*}{N^*}\int_0^\infty \beta(\tau)v_4(\tau)\mathrm{d}\tau - \frac{v_1}{N^*}\int_0^\infty \beta(\tau)i^*(\tau)\mathrm{d}\tau + \frac{S_v^*}{(N^*)^2}N(v)\int_0^\infty \beta(\tau)i^*(\tau)\mathrm{d}\tau \\ \frac{S_v^*}{N^*}\int_0^\infty \beta(\tau)v_4(\tau)\mathrm{d}\tau + \frac{v_1}{N^*}\int_0^\infty \beta(\tau)i^*(\tau)\mathrm{d}\tau - \frac{S_v^*}{(N^*)^2}N(v)\int_0^\infty \beta(\tau)i^*(\tau)\mathrm{d}\tau \\ -\frac{\beta_v S^* v_2}{N^*} - \frac{\beta_v I_v^* v_3}{N^*} + \frac{\beta_v S^* I_v^* N(v)}{(N^*)^2} - \frac{S^*}{N^*}\int_0^\infty \beta_d(\tau)v_4(\tau)\mathrm{d}\tau - \frac{v_3}{N^*}\int_0^\infty \beta_d(\tau)i^*(\tau)\mathrm{d}\tau + \frac{S^*N(v)}{(N^*)^2}\int_0^\infty \beta_d(\tau)i^*(\tau)\mathrm{d}\tau \\ 0 \\ \int_0^\infty \gamma(\tau)v_4(\tau)\mathrm{d}\tau \\ \frac{\beta_v S^* v_2}{N^*} + \frac{\beta_v I_v^* v_3}{N^*} - \frac{\beta_v S^* I_v^* N(v)}{(N^*)^2} + \frac{S^*}{N^*}\int_0^\infty \beta_d(\tau)v_4(\tau)\mathrm{d}\tau + \frac{v_3}{N^*}\int_0^\infty \beta_d(\tau)i^*(\tau)\mathrm{d}\tau - \frac{S^*N(v)}{(N^*)^2}\int_0^\infty \beta_d(\tau)i^*(\tau)\mathrm{d}\tau \end{pmatrix}$$

Following the steps in Theorem 3.4, by Corollary 4.3 in [34], local stability is determined by point spectrum of $(A + DF(E^*))_0$. Thus, setting

$$(A + DF(E^*))_0 v = \lambda v$$

we obtain,

$$\begin{aligned} -\frac{S_v^*}{N^*}\int_0^\infty \beta(\tau)v_4(\tau)\mathrm{d}\tau - \frac{v_1}{N^*}\int_0^\infty \beta(\tau)i^*(\tau)\mathrm{d}\tau + \frac{S_v^*}{(N^*)^2}N(v)\int_0^\infty \beta(\tau)i^*(\tau)\mathrm{d}\tau - \mu_v v_1 &= \lambda v_1 \\ \frac{S_v^*}{N^*}\int_0^\infty \beta(\tau)v_4(\tau)\mathrm{d}\tau + \frac{v_1}{N^*}\int_0^\infty \beta(\tau)i^*(\tau)\mathrm{d}\tau - \frac{S_v^*}{(N^*)^2}N(v)\int_0^\infty \beta(\tau)i^*(\tau)\mathrm{d}\tau - \mu_v v_2 &= \lambda v_2 \\ -v_4(0) - \mu v_3 &= \lambda v_3 \\ -v_4'(\tau) - (\gamma(\tau) + \mu)v_4 &= \lambda v_4(\tau) \\ \int_0^\infty \gamma(\tau)v_4(\tau)\mathrm{d}\tau - \mu v_5 &= \lambda v_5 \end{aligned}$$

and

$$\begin{aligned} v_4(0) = \frac{\beta_v S^* v_2}{N^*} &+ \frac{\beta_v I_v^* v_3}{N^*} - \frac{\beta_v S^* I_v^* N(v)}{(N^*)^2} + \frac{S^*}{N^*}\int_0^\infty \beta_d(\tau)v_4(\tau)\mathrm{d}\tau \\ &+ \frac{v_3}{N^*}\int_0^\infty \beta_d(\tau)i^*(\tau)\mathrm{d}\tau - \frac{S^*N(v)}{(N^*)^2}\int_0^\infty \beta_d(\tau)i^*(\tau)\mathrm{d}\tau \end{aligned}$$

Clearly, $v_4(\tau) = v_4(0)e^{-\lambda\tau}\pi(\tau)$ and $N(v) = v_3 + \int_0^\infty v_4(\tau)\mathrm{d}\tau + v_5$. Since

$$v_3 = -\frac{v_4(0)}{\lambda + \mu} \quad \text{and} \quad v_5 = \frac{v_4(0)}{\lambda + \mu}\int_0^\infty \gamma(\tau)e^{\lambda\tau}\pi(\tau)\mathrm{d}\tau,$$

we have $N(v) = 0$. Let $B(\lambda) = \int_0^\infty \beta(\tau)e^{-\lambda\tau}\pi(\tau)\mathrm{d}\tau$ and $B_d(\lambda) = \int_0^\infty \beta_d(\tau)e^{-\lambda\tau}\pi(\tau)\mathrm{d}\tau$ and $B^* = \frac{1}{N^*}\int_0^\infty \beta(\tau)i^*(\tau)\mathrm{d}\tau$ and $B_d^* = \frac{1}{N^*}\int_0^\infty \beta_d(\tau)i^*(\tau)\mathrm{d}\tau$. Since $v_1 = -v_2$, we have;

$$\begin{aligned} (\lambda + \mu_v + B^*)\, v_2 - \frac{S_v^*}{N^*}B(\lambda)v_4(0) &= 0 \\ -\frac{\beta_v S^*}{N^*}v_2 + \left(1 + \frac{\beta_v I_v^*}{N^*(\lambda + \mu)} - \frac{S^* B_d(\lambda)}{N^*} + \frac{B_d^*}{\lambda + \mu}\right) v_4(0) &= 0. \end{aligned}$$

Requiring $v_2 \neq 0$ and $v_4(0) \neq 0$ gives the characteristic equation as $F(\lambda) = G(\lambda)$ where

$$\begin{aligned} F(\lambda) =& 1 + \frac{\beta_v I_v^*}{N^*(\lambda+\mu)} + \frac{B_d^*}{\lambda+\mu} \\ G(\lambda) =& \frac{S^*}{N^*}\left(\frac{\beta_v S_v^* B(\lambda)}{N^*(\lambda+\mu_v+B^*)} + B_d(\lambda)\right). \end{aligned}$$

Let $\lambda = a + ib$ with $a > 0$. Clearly $|F(\lambda)| > 1$ and

$$|G(\lambda)| \leq \frac{S^*}{N^*}\left(\frac{\beta_v S_v^* B(a)}{N^*\sqrt{(a+\mu_v+B^*)^2+b^2}} + B_d(a)\right) \leq \frac{S^*}{N^*}\left(\frac{\beta_v S_v^* B}{N^*\mu_v} + B_d\right) = 1.$$

Since $\frac{S_v^*}{N^*}\int_0^\infty \beta(\tau) i^*(\tau) \mathrm{d}\tau - \mu_v I_v^* = 0$ gives $I_v^* = \frac{S_v^* i^*(0) B}{N^*\mu_v}$, and substituting it into

$$i^*(0) = \frac{\beta_v S^* I_v^*}{N^*} + \frac{S^*}{N^*}\int_0^\infty \beta_d(\tau) i^*(\tau)\mathrm{d}\tau \quad \text{yields} \quad \frac{S^*}{N^*}\left(\frac{\beta_v S_v^* B}{N^*\mu_v} + B_d\right) = 1.$$

Thus, any λ with positive real parts cannot satisfy the equation $F(\lambda) = G(\lambda)$. Therefore, the endemic equilibrium is locally asymptotically stable in this case. □

4. Global stability of disease free equilibrium

We next establish the global stability of the disease-free equilibrium E_0. In showing the global stability of disease free equilibrium, we need following Fluctiation Lemma (Lem. 4.1) and Lem. 4.2.

Lemma 4.1. *(Fluctuation Lemma, Proposition A.22, p431, [32]) Let $f(t) : \mathbb{R}_+ \to \mathbb{R}$ be a bounded and continuously differentiable function. Then there exist sequences s_n and t_n such that*

$$\lim_{s_n\to\infty} f(s_n) = \limsup_{t\to\infty} f(t) \text{ and } \lim_{s_n\to\infty} f'(s_n) = 0$$

$$\lim_{t_n\to\infty} f(t_n) = \liminf_{t\to\infty} f(t) \text{ and } \lim_{t_n\to\infty} f'(t_n) = 0$$

Lemma 4.2. *Let $f(t) : \mathbb{R}_+ \to \mathbb{R}$ be a bounded function, then*

$$\limsup_{t\to\infty} \int_0^t \beta(\tau) f(t-\tau)\mathrm{d}\tau < \limsup_{t\to\infty} f(t) \int_0^\infty \beta(\tau)\mathrm{d}\tau$$

Theorem 4.3. *When $\mathcal{R}_0 < 1$, the disease-free equilibrium E_0 is globally asymptotically stable.*

Proof. Integrating the infected class $i(\tau, t)$ along the characteristic lines, we obtain

$$i(\tau,t) = \begin{cases} B(t-\tau)\pi(\tau), & t > \tau, \\ i_0(\tau-t)\dfrac{\pi(\tau)}{\pi(\tau-t)}, & t < \tau, \end{cases} \tag{4.1}$$

where $B(t) = \dfrac{\beta_v S I_v}{N} + \dfrac{S}{N}\int_0^\infty \beta_d(\tau) i(\tau,t)\mathrm{d}\tau$. Since $\dfrac{S}{N} < 1$, substituting (4.1) into $B(t)$ we obtain,

$$\begin{aligned}B(t) &\leq \beta_v I_v + \int_0^t \beta_d(\tau)B(t-\tau)\pi(\tau)\mathrm{d}\tau + \int_t^\infty \beta_d(\tau) i_0(\tau - t)\frac{\pi(\tau)}{\pi(\tau-t)}\mathrm{d}\tau \\ &\leq \beta_v I_v + \int_0^t \beta_d(\tau)B(t-\tau)\pi(\tau)\mathrm{d}\tau + e^{-\mu t}\int_t^\infty \beta_d(\tau) i_0(\tau - t)\mathrm{d}\tau\end{aligned}$$

Using Lemma 4.2, and $\int_0^\infty \beta_d(\tau)\pi(\tau)\mathrm{d}\tau < 1$, when $\mathcal{R}_0 < 1$ we obtain,

$$\limsup_{t\to\infty} B(t) \leq \frac{\beta_v}{1 - \int_0^\infty \beta_d(\tau)\pi(\tau)\mathrm{d}\tau}\limsup_{t\to\infty} I_v(t) \tag{4.2}$$

Integrating the infected vector class, and $S_v \leq \dfrac{\Lambda_v}{\mu_v}$ yields

$$\begin{aligned}I_v(t) &\leq e^{-\mu_v t}I_v(0) + \frac{\Lambda_v}{\mu_v}\int_0^t e^{-\mu_v(t-s)}\frac{1}{N(s)}\int_0^\infty \beta(\tau) i(\tau,s)\mathrm{d}\tau\mathrm{d}s \\ &\leq e^{-\mu_v t}I_v(0) + \frac{\Lambda_v}{\mu_v}\int_0^t e^{-\mu_v(t-s)}\limsup_{s\to\infty}\left(\frac{1}{N(s)}\int_0^\infty \beta(\tau) i(\tau,s)\mathrm{d}\tau\right)\mathrm{d}s\end{aligned}$$

Substituting (4.1) and applying Lemma 4.2 gives

$$\begin{aligned}\limsup_{s\to\infty}\left(\frac{1}{N(s)}\int_0^\infty \beta(\tau) i(\tau,s)\mathrm{d}\tau\right) &\leq \frac{\mu}{\Lambda}\limsup_{s\to\infty}\left(\int_0^s \beta(\tau)B(s-\tau)\pi(\tau)\mathrm{d}\tau + e^{-\mu s}\int_s^\infty \beta(\tau) i_0(\tau - s)\mathrm{d}\tau\right) \\ &\leq \frac{\mu}{\Lambda}\limsup_{s\to\infty} B(s)\int_0^\infty \beta(\tau)\pi(\tau)\mathrm{d}\tau.\end{aligned}$$

Thus

$$I_v(t) \leq e^{-\mu_v t}I_v(0) + \frac{\Lambda_v \mu}{\mu_v \Lambda}\limsup_{s\to\infty} B(s)\int_0^\infty \beta(\tau)\pi(\tau)\mathrm{d}\tau\int_0^t e^{-\mu_v(t-s)}\mathrm{d}s$$

Substituting (4.2) and taking lim sup yields

$$\begin{aligned}\limsup_{t\to\infty} I_v(t) &\leq \frac{\Lambda_v\mu}{\mu_v^2\Lambda}\frac{\beta_v\int_0^\infty \beta(\tau)\pi(\tau)\mathrm{d}\tau}{1 - \int_0^\infty \beta_d(\tau)\pi(\tau)\mathrm{d}\tau}\limsup_{t\to\infty} I_v(t) \\ &\leq \left(\frac{\beta_v\Lambda_v\mu}{\mu_v^2\Lambda}\int_0^\infty \beta(\tau)\pi(\tau)\mathrm{d}\tau + \int_0^\infty \beta_d(\tau)\pi(\tau)\mathrm{d}\tau\right)\limsup_{t\to\infty} I_v(t) \\ &= \mathcal{R}_0 \limsup_{t\to\infty} I_v(t),\end{aligned}$$

which gives $\limsup_{t\to\infty} I_v(t) = 0$, since $\mathcal{R}_0 < 1$. Hence $\lim_{t\to\infty} I_v(t) = 0$. From (4.2), this concludes that $\limsup_{t\to\infty} B(t) = 0$.

Next, we show that $\lim_{t\to\infty}\int_0^\infty i(\tau,t)\mathrm{d}\tau = 0$. Substituting (4.1), yields

$$\begin{aligned}\int_0^\infty i(\tau,t)\mathrm{d}\tau &= \int_0^t B(t-\tau)\pi(\tau)\mathrm{d}\tau + \int_t^\infty i_0(\tau-t)\frac{\pi(\tau)}{\pi(\tau-t)}\mathrm{d}\tau \\ &\leq \int_0^t B(t-\tau)\pi(\tau)\mathrm{d}\tau + e^{-\mu t}\int_t^\infty i_0(\tau-t)\mathrm{d}\tau\end{aligned}$$

By Lemma 4.2 we obtain,

$$\begin{aligned}\limsup_{t\to\infty}\int_0^\infty i(\tau,t)\mathrm{d}\tau &\leq \limsup_{t\to\infty} B(t)\int_0^\infty \pi(\tau)\mathrm{d}\tau \\ &\leq \limsup_{t\to\infty} B(t)\frac{1}{\mu} = 0,\end{aligned}$$

which implies that $\lim_{t\to\infty}\int_0^\infty i(\tau,t)\mathrm{d}\tau = 0$. Furthermore, since $\beta(\tau)$ is bounded, $\lim_{t\to\infty}\int_0^\infty \beta(\tau)i(\tau,t)\mathrm{d}\tau = 0$. Similar results are true for bounded functions $\beta_d(\tau)$ and $\gamma(\tau)$. Next, we show that $\lim_{t\to\infty} S_v(t) = \frac{\Lambda_v}{\mu_v}$. By Lemma 4.1 there exists a sequence t_n such that

$$\lim_{t_n\to\infty} S_v(t_n) = \liminf_{t\to\infty} S_v(t) \text{ and } \lim_{t_n\to\infty} S_v'(t_n) = 0.$$

It follows from

$$S_v'(t_n) = \Lambda_v - \frac{S_v(t_n)}{N(t_n)}\int_0^\infty \beta(\tau)i(\tau,t_n)\mathrm{d}\tau - \mu_v S_v(t_n)$$

that $\liminf_{t\to\infty} S_v(t) = \frac{\Lambda_v}{\mu_v} = \limsup_{t\to\infty} S_v(t)$. Similar argument shows $\lim_{t\to\infty} R(t) = 0$. □

5. Persistence

When $\mathcal{R}_0 > 1$, the disease free equilibrium is unstable. We will show that, when $\mathcal{R}_0 > 1$ the system (2.1) is persistent and hence the disease will establish. That is we need to first make sure that the initial conditions are non-trivial and they lead to new infections. To be precise, suppose that

$$\hat{M} = \{i_0(\tau) \in L^1_+(0,\infty) : \int_0^\infty \beta(\tau)i_0(\tau) > 0 \text{ and } \int_0^\infty \beta_d(\tau)i_0(\tau) > 0\}.$$

Notice that the space X^0_+ can be identified with the space $\mathcal{M}^0 = \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\times L^1_+(0,\infty)\times\mathbb{R}_+$. As stated in (2.2), $\mathcal{M}^0$ is positively invariant under the semiflow $\mathcal{U}(t)$. Furthermore, let $\hat{\mathcal{M}} = \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\times\hat{M}\times\mathbb{R}_+$, then set

$$\mathcal{M} = \mathcal{M}^0 \cap \hat{\mathcal{M}}$$

Let $\rho_i : \mathcal{M} \to \mathbb{R}_+$ for $i = 1, 2, 3$ be defined as follows,

$$\rho_1(\mathcal{U}(t)u_0) = \frac{\beta_v I_v}{N}$$
$$\rho_2(\mathcal{U}(t)u_0) = \frac{1}{N}\int_0^\infty \beta(\tau)i(\tau,t)\mathrm{d}\tau$$
$$\rho_3(\mathcal{U}(t)u_0) = \frac{1}{N}\int_0^\infty \beta_d(\tau)i(\tau,t)\mathrm{d}\tau$$

Definition 5.1. ([31]) We say that the vector-borne disease in system (2.1) is uniformly weakly ρ-persistent, if there exists some $\epsilon > 0$ such that

$$\limsup_{t\to\infty} \frac{\beta_v I_v}{N} > \epsilon \quad \text{whenever} \quad I_{v0} > 0 \quad \text{and}$$
$$\limsup_{t\to\infty} \frac{1}{N}\int_0^\infty \beta(\tau)i(\tau,t)\mathrm{d}\tau > \epsilon \quad \text{whenever} \quad \int_0^\infty \beta(\tau)i_0(\tau)\mathrm{d}\tau > 0$$
$$\limsup_{t\to\infty} \frac{1}{N}\int_0^\infty \beta_d(\tau)i(\tau,t)\mathrm{d}\tau > \epsilon \quad \text{whenever} \quad \int_0^\infty \beta_d(\tau)i_0(\tau)\mathrm{d}\tau > 0$$

Proposition 5.2. *Let $\mathcal{R}_0 > 1$, then the system (2.1) is uniformly weakly persistent. That is there exists $\epsilon > 0$ such that,*

$$\limsup_{t\to\infty} \frac{\beta_v I_v}{N} \geq \epsilon$$
$$\limsup_{t\to\infty} \frac{1}{N}\int_0^\infty \beta(\tau)i(\tau,t)\mathrm{d}\tau \geq \epsilon$$
$$\limsup_{t\to\infty} \frac{1}{N}\int_0^\infty \beta_d(\tau)i(\tau,t)\mathrm{d}\tau \geq \epsilon$$

Proof. We will argue by contradiction. Assume that for each $\epsilon > 0$, then either one of the inequalities will have

$$\limsup_{t\to\infty} \frac{\beta_v I_v}{N} \leq \epsilon$$
$$\limsup_{t\to\infty} \frac{1}{N}\int_0^\infty \beta(\tau)i(\tau,t)\mathrm{d}\tau \leq \epsilon$$
$$\limsup_{t\to\infty} \frac{1}{N}\int_0^\infty \beta_d(\tau)i(\tau,t)\mathrm{d}\tau \leq \epsilon$$

Hence there exists $T > 0$ such that for each $t > T$, we have

$$\frac{\beta_v I_v}{N} \leq \epsilon, \quad \frac{1}{N}\int_0^\infty \beta(\tau)i(\tau,t)\mathrm{d}\tau \leq \epsilon, \quad \text{and} \quad \frac{1}{N}\int_0^\infty \beta_d(\tau)i(\tau,t)\mathrm{d}\tau \leq \epsilon$$

Withous loss of generality, we can assume that above inequality hods true for all $t \geq 0$, (that is $T = 0$) by shifting the dynamical system. From the equations involving susceptible vectors and susceptible hosts in (2.1), we obtain

$$S_v' > \Lambda_v - \epsilon S_v - \mu_v S, \quad \text{and} \quad S' > \Lambda - \epsilon S - \epsilon S - \mu S$$

which gives,

$$\liminf_{t\to\infty} S_v \geq \frac{\Lambda_v}{\epsilon+\mu_v} \quad \text{and} \quad \liminf_{t\to\infty} S \geq \frac{\Lambda}{2\epsilon+\mu}$$

Using (4.1), we have

$$\begin{aligned}
B(t) &= \frac{\beta_v S I_v}{N} + \frac{S}{N}\int_0^\infty \beta_d(\tau) i(\tau,t)\mathrm{d}\tau \\
&> \frac{\beta_v I_v}{N}\frac{\Lambda}{2\epsilon+\mu} + \frac{1}{N}\frac{\Lambda}{2\epsilon+\mu}\int_0^\infty \beta_d(\tau) i(\tau,t)\mathrm{d}\tau \\
&> \frac{\beta_v\mu}{2\epsilon+\mu} I_v + \frac{\mu}{2\epsilon+\mu}\int_0^\infty \beta_d(\tau) i(\tau,t)\mathrm{d}\tau \\
&= \frac{\beta_v\mu}{2\epsilon+\mu} I_v + \frac{\mu}{2\epsilon+\mu}\left(\int_0^t \beta_d(\tau)B(t-\tau)\pi(\tau)\mathrm{d}\tau + \int_t^\infty \beta_d(\tau)\frac{\pi(\tau)}{\pi(\tau-t)}\mathrm{d}\tau\right) \\
&> \frac{\beta_v\mu}{2\epsilon+\mu} I_v + \frac{\mu}{2\epsilon+\mu}\int_0^t \beta_d(\tau)B(t-\tau)\pi(\tau)\mathrm{d}\tau
\end{aligned}$$

And similarly for the infected vector class we have,

$$I_v' = \frac{S_v}{N}\int_0^\infty \beta(\tau) i(\tau,t)\mathrm{d}\tau - \mu_v I_v > \frac{\Lambda_v\mu}{(\epsilon+\mu_v)\Lambda}\int_0^t \beta(\tau)B(t-\tau)\pi(\tau)\mathrm{d}\tau - \mu_v I_v$$

We next take the Laplace transform of both $B(t)$ and $I_v(t)$. We denote the Laplace transforms of $B(t)$ and $I_v(t)$ as $\hat{B}(\lambda)$ and $\hat{I}_v(\lambda)$ respectively. We obtain,

$$\begin{aligned}
\hat{B}(\lambda) &> \frac{\beta_v\mu}{2\epsilon+\mu}\hat{I}_v(\lambda) + \frac{\mu}{2\epsilon+\mu}\hat{K}_d(\lambda)\hat{B}(\lambda) \\
\hat{I}_v(\lambda) - I_v(0) &> \frac{\Lambda_v\mu}{(\epsilon+\mu_v)\Lambda}\hat{K}(\lambda)\hat{B}(\lambda) - \mu_v\hat{I}_v(\lambda)
\end{aligned}$$

where $\hat{K}(\lambda) = \int_0^t \beta(\tau)\pi(\tau)e^{-\lambda\tau}\mathrm{d}\tau$ and $\hat{K}_d(\lambda) = \int_0^t \beta_d(\tau)\pi(\tau)e^{-\lambda\tau}\mathrm{d}\tau$. Eliminating $\hat{I}_v(\lambda)$ from the sytem above, we obtain

$$\hat{B}(\lambda) > \underbrace{\frac{\beta_v\mu^2\Lambda_v}{(2\epsilon+\mu)(\epsilon+\mu_v)\Lambda(\lambda+\mu_v)}\hat{K}(\lambda)\hat{B}(\lambda) + \frac{\mu}{2\epsilon+\mu}\hat{K}_d(\lambda)\hat{B}(\lambda)}_{\approx\mathcal{R}_0>1} + \frac{\beta_v\mu}{(2\epsilon+\mu)(\lambda+\mu_v)}I_v(0)$$

which gives a contradiction. □

A consequence of the system (2.1) being uniformly weakly persistent is that the disease-free equilibrium is unstable. We proceed by showing that the semiflow $\mathcal{U}(t) : [0,\infty)\times\mathcal{M}\to\mathcal{M}$ of the system (2.1) has a global compact attractor $\mathcal{A}$. An invariant set $\mathcal{A}$ in $\mathcal{M}$ is called a *global compact attractor* for $\mathcal{U}(t)$, if $\mathcal{A}$ is a maximal compact invariant set which attracts each bounded set $\mathcal{B}\subset\mathcal{M}$ [13]. To show that there exists a global compact attractor, we will use Lemma 3.2.3 and Theorem 3.4.6 in [13]. For sake of completeness, we present Lemma 3.2.3 and Theorem 3.4.6 in [13] here.

Lemma 5.3. *(Lemma 3.2.3 in [13]) For each $t > 0$ suppose $\mathcal{U}(t) = \tilde{\mathcal{U}}(t) + \hat{\mathcal{U}}(t)$, where $\tilde{\mathcal{U}}$ is completely continuous. Suppose there exists a function $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $k(t, r) \to 0$ as $t \to 0$ and $\|\hat{\mathcal{U}}(t)u_0\| \leq k(t, r)$. Then $\mathcal{U}(t)$ is asymptotically smooth.*

Lemma 5.4. *(Theorem 3.4.6 in [13]) If $\mathcal{U}(t)$ is asymptotically smooth, point dissipative and orbits of bounded sets are bounded, then there exists a global attractor.*

First, we show that the semiflow $\mathcal{U}(t)$ is asymptotically smooth.

Proposition 5.5. *The semiflow $\mathcal{U}(t)$ is asymptotically smooth.*

Proof. We will apply Lemma 5.3 to show that $\mathcal{U}(t)$ is asymptotically smooth. We begin by splitting the semiflow into two compartments, $\mathcal{U}(t) = \tilde{\mathcal{U}}(t) + \hat{\mathcal{U}}(t)$ as follows:

$$\begin{aligned}\tilde{\mathcal{U}}(t; S_{v0}, I_{v0}, S_0, i_0, R_0) &= (S_v(t), I_v(t), S(t), \tilde{i}(\cdot, t), R(t))\\ \hat{\mathcal{U}}(t; S_{v0}, I_{v0}, S_0, i_0, R_0) &= (0, 0, 0, \hat{i}(\cdot, t), 0)\end{aligned}$$

where

$$\tilde{i}(\tau, t) = \begin{cases} i(\tau, t), & t > \tau, \\ 0, & t < \tau, \end{cases} = \begin{cases} B(t-\tau)\pi(\tau), & t > \tau, \\ 0, & t < \tau, \end{cases} \tag{5.1}$$

and $\hat{i}(\tau, t) = i(\tau, t) - \tilde{i}(\tau, t)$, that is

$$\hat{i}(\tau, t) = \begin{cases} 0, & t > \tau, \\ i_0(\tau - t)\dfrac{\pi(\tau)}{\pi(\tau - t)}, & t < \tau, \end{cases} \tag{5.2}$$

Note that $S_v(t), I_v(t), S(t)$ and $R(t)$ satisfy the sytem (2.1) with $i(\tau, t)$. Since, both $\tilde{i}$ and $\hat{i}$ are non-negative, $\tilde{\mathcal{U}} \leq \mathcal{U}$ and $\hat{\mathcal{U}} \leq \mathcal{U}$. Note that,

$$\begin{aligned}\|\hat{\mathcal{U}}(t)u_0\| &= \|\hat{i}(\cdot, t)\|_{L^1}\\ &= \int_t^\infty i_0(\tau - t)\frac{\pi(\tau)}{\pi(\tau - t)}\mathrm{d}\tau\\ &= \int_0^\infty i_0(\tau)\frac{\pi(\tau + t)}{\pi(\tau)}\mathrm{d}\tau\\ &\leq e^{-\mu t}\|i_0\|_{L^1}\\ &\leq e^{-\mu t}\|u_0\| = k(t, \|u_0\|) \text{ (as in Lem. 5.3)}\end{aligned}$$

Hence, $\|\hat{\mathcal{U}}(t)u_0\| \to 0$ as $t \to \infty$. Next, we show that $\tilde{\mathcal{U}}(t)$ is completely continuous. A semiflow $\tilde{\mathcal{U}}(t)$ is called completely continuous if for each fixed t, the family of functions $\mathcal{K}_t = \{\tilde{\mathcal{U}}(t)u_0 : u_0 \in \mathcal{B}\}$ is precompact for a bounded set $\mathcal{B}$ contained in $\mathcal{M}$ (p. 36 [13]). So, assume that the initial conditions are in a abounded set $\mathcal{B}$, that is $\|u_0\| \in \mathcal{B}$, such that $\|u_0\| = |S_{v0}| + |I_{v0}| + |S_0| + \|i_0\|_{L^1} + |R_0| \leq r$ for some constant r. By (2.2), we have $\mathcal{K}_t = \{\tilde{\mathcal{U}}(t)u_0 = (S_v(t), I_v(t), S(t), \tilde{i}(\cdot, t), R(t))\} \subset \Omega$ and hence bounded. We will show that the family of functions, $\mathcal{K} = \{\tilde{i}(\cdot, t) : \tilde{\mathcal{U}}(t)u_0 \in \mathcal{K}_t\}$ for any fixed t is precompact by Frechet-Kolmogorov Theorem [44]. Frechet-Kolmogorov Theorem states that $\mathcal{K} \subset L^1_+(0, \infty)$ is precompact, iff

(i) $\sup_{\tilde{i} \in \mathcal{K}} \int_0^\infty \tilde{i}(\tau, t)\mathrm{d}\tau < \infty$

(ii) $\lim_{t\to\infty}\int_t^\infty \tilde{i}(\tau,t)\mathrm{d}\tau = 0$ uniformly in $\tilde{i}\in\mathcal{K}$

(iii) $\lim_{h\to 0}\int_0^\infty |\tilde{i}(\tau+h,t)-\tilde{i}(\tau,t)|\mathrm{d}\tau = 0$ uniformly in $\tilde{i}\in\mathcal{K}$

By (2.2) $\int_0^\infty \tilde{i}(\tau,t)\mathrm{d}\tau < \frac{\Lambda}{\mu}$, hence (i) is satisfied. By definition (5.1), $\int_t^\infty \tilde{i}(\tau,t)\mathrm{d}\tau = 0$, hence (ii) is satisfied. We only need to show (iii). To show (iii), we need to bound the L^1 norm of $\frac{\partial\tilde{i}}{\partial\tau}$, since

$$\int_0^\infty |\tilde{i}(\tau+h,t)-\tilde{i}(\tau,t)|\mathrm{d}\tau \le \left\|\frac{\partial\tilde{i}}{\partial\tau}\right\|_{L^1}|h|$$

We differentiate (5.1):

$$\left|\frac{\partial\tilde{i}(\tau,t)}{\partial\tau}\right| \le \begin{cases} |B'(t-\tau)|\pi(\tau)+B(t-\tau)|\pi'(\tau)|, & t>\tau,\\ 0, & t<\tau,\end{cases} \tag{5.3}$$

Thus, we first need to show that both $B(t)$ and $B'(t)$ are bounded.

$$\begin{aligned} B(t) &= \frac{\beta_v S I_v}{N} + \int_0^\infty \beta_d(\tau)\tilde{i}(\tau,t)\mathrm{d}\tau \\ &= \frac{\beta_v S I_v}{N} + \frac{S}{N}\int_0^t \beta_d(\tau)B(t-\tau)\pi(\tau)\mathrm{d}\tau \\ &\le \beta_v I_v + \int_0^t \beta_d(\tau)B(t-\tau)\pi(\tau)\mathrm{d}\tau \end{aligned}$$

Thus for some positive constants, $|B(t)|\le k_1+k_2t$. Since t is fixed, $|B(t)|\le\tilde{k}_1$. On the other hand,

$$\begin{aligned} B'(t) =& \beta_v\frac{S'I_v}{N} + \frac{\beta_v S I_v'}{N} + \frac{S}{N}\beta_d(t)\pi(t)B(0) + \frac{S'}{N}\int_0^t \beta_d(\tau)B'(t-\tau)\pi(\tau)\mathrm{d}\tau \\ &- \frac{N'}{N}\left(\frac{\beta_v S I_v}{N} + \frac{S}{N}\int_0^t \beta_d(\tau)B(t-\tau)\pi(\tau)\mathrm{d}\tau\right) \end{aligned}$$

Hence $|B'(t)|\le k_3+k_4t$ and similarly, $|B'(t)|\le\tilde{k}_2$. Then,

$$\begin{aligned} \left\|\frac{\partial\tilde{i}}{\partial\tau}\right\|_{L^1} &\le \int_0^\infty |B'(t-\tau)|\pi(\tau)\mathrm{d}\tau + \int_0^\infty |B(t-\tau)|\pi'(\tau)\mathrm{d}\tau \\ &\le \tilde{k}_2\int_0^\infty \pi(\tau)\mathrm{d}\tau + \tilde{k}_1\int_0^\infty (\gamma(\tau)+\mu)\pi(\tau)\mathrm{d}\tau \\ &\le \tilde{k}_2\int_0^\infty \pi(\tau)\mathrm{d}\tau + \tilde{k}_1(\hat{\gamma}+\mu)\int_0^\infty \pi(\tau)\mathrm{d}\tau \\ &\le \tilde{k}_2 + \tilde{k}_1(\hat{\gamma}+\mu)\|\pi(\tau)\|_{L^1} \le k_5 \end{aligned}$$

where $\hat{\gamma} = \sup_\tau \gamma(\tau)$. □

Next, we show that the semiflow $\mathcal{U}(t)$ has a global compact attractor.

Proposition 5.6. *When $\mathcal{R}_0 > 1$, then the semiflow $\mathcal{U}(t)$ has a global compact attractor.*

Proof. We will apply Lemma 5.4. Thus, we need to show that $\mathcal{U}(t)$ is point dissipative and orbits of bounded sets are bounded. A semiflow $\mathcal{U}(t)$ is called point dissipative, if there exists a bounded set $\mathcal{B} \subset \mathcal{M}$ that attracts each point of $\mathcal{M}$ [13]. Clearly by (2.2), the semiflow $\mathcal{U}(t)$ is point dissipative. Again by (2.2), orbits of bounded sets are bounded. That is $\|\mathcal{U}(t)u_0\| \leq \frac{\Lambda}{\mu} + \frac{\Lambda_v}{\mu_v}$ for all $t \geq 0$ whenever $\|u_0\| \leq c_1$ for some constant c_1. Thus semiflow $\mathcal{U}(t)$ has a global compact attractor. □

We continue by showing that the disease is uniformly strongly persistent.

Definition 5.7. ([31]) We say that the vector-borne disease in system (2.1) is uniformly strongly ρ-persistent, if there exists some $\epsilon > 0$ such that

$$\begin{aligned}
\liminf_{t\to\infty} \frac{\beta_v I_v}{N} > \epsilon \quad &\text{whenever} \quad I_{v0} > 0 \quad \text{and} \\
\liminf_{t\to\infty} \frac{1}{N}\int_0^\infty \beta(\tau) i(\tau,t)\mathrm{d}\tau > \epsilon \quad &\text{whenever} \quad \int_0^\infty \beta(\tau) i_0(\tau)\mathrm{d}\tau > 0 \\
\liminf_{t\to\infty} \frac{1}{N}\int_0^\infty \beta_d(\tau) i(\tau,t)\mathrm{d}\tau > \epsilon \quad &\text{whenever} \quad \int_0^\infty \beta_d(\tau) i_0(\tau)\mathrm{d}\tau > 0
\end{aligned}$$

We will prove that the vector-borne disease in system (2.1) is uniformly strongly ρ-persistent, by applying the Theorem 2.6 in [33]. The Theorem 2.6 in [33] states the following.

Lemma 5.8. *(Theorem 2.6 in [33]) Let $\mathcal{U}(t)$ be a continuous semiflow on a metric space $\mathcal{M}$ which has a compact attractor $\mathcal{A}$. Furthermore, we assume for any total orbit $\phi(t) : \mathbb{R} \to \mathcal{M}$, if $s \in \mathbb{R}$ and $\rho(\phi(s)) > 0$, then $\rho(\phi(t)) > 0$ for all $t > s$. Then $\mathcal{U}(t)$ is strongly ρ-persistent whenever it is uniformly weakly persistent.*

Proposition 5.9. *Let $\mathcal{R}_0 > 1$, then the system (2.1) is uniformly strongly ρ-persistent. That is there exists $\epsilon > 0$ such that,*

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{\beta_v I_v}{N} \geq \epsilon \\
&\liminf_{t\to\infty} \frac{1}{N}\int_0^\infty \beta(\tau) i(\tau,t)\mathrm{d}\tau \geq \epsilon \\
&\liminf_{t\to\infty} \frac{1}{N}\int_0^\infty \beta_d(\tau) i(\tau,t)\mathrm{d}\tau \geq \epsilon
\end{aligned}$$

Proof. We will apply Theorem 2.6 in [33]. Total orbits are the solutions of the system (2.1) defined for all time $t \in \mathbb{R}$. By (2.2), the solutions of the semiflow is non-negative. Hence, integrating the differential equation of I_v, we obtain

$$I_v(t) \geq I_v(s)e^{-\mu_v(t-s)} \quad \text{for all} \quad t > s.$$

Therefore,

$$\frac{\beta_v I_v(t)}{N} \geq \frac{\beta_v I_v(s)}{N} e^{-\mu_v(t-s)} \quad \text{for all} \quad t > s$$

$$\frac{1}{N}\int_0^\infty \beta(\tau)i(\tau,t)\mathrm{d}\tau \geq c_1\int_0^t B(\tau)\mathrm{d}\tau > c_1\int_0^t \frac{\beta_v S I_v}{N}\mathrm{d}\tau > c_2\int_0^t I_v(\tau)\mathrm{d}\tau$$
$$> c_2\int_0^t I_v(s)e^{-\mu_v(\tau-s)}\mathrm{d}\tau = c_2\frac{I_v(s)}{\mu_v}e^{\mu_v s}(1-e^{-\mu_v t})$$

Similarly,

$$\frac{1}{N}\int_0^\infty \beta_d(\tau)i(\tau,t)\mathrm{d}\tau \geq c_3\frac{I_v(s)}{\mu_v}e^{\mu_v s}(1-e^{-\mu_v t})$$

Thus, $\rho(\phi(t)) > 0$ for all $t > s$, whenever $I_v(s) > 0$. So, by Theorem 2.6 in [33], semiflow $\mathcal{U}(t)$ is uniformly strongly ρ-persistent, whenever it is uniformly weakly-persistent, that is when $\mathcal{R}_0 > 1$. □

6. Conclusion

Infection age plays a vital role in the transmission of ZIKV. In this study, we formulate a hyperbolic PDE model of Zika virus infections, which includes both vector and direct transmissions and where the new infections are modeled as standard incidence. We obtain the explicit representation of the reproduction number, $\mathcal{R}_0$. We showed that the disease-free-equilibria is locally and globally stable when $\mathcal{R}_0 < 1$. We also showed endemic equilibrium is locally stable when $\mathcal{R}_0 > 1$. Persistence of endemic equilibrium is established when $\mathcal{R}_0 > 1$, however global stability of endemic equilibrium for $\mathcal{R}_0 > 1$ is still an open question.

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