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ON THE INITIAL VALUE PROBLEM FOR FRACTIONAL VOLTERRA INTEGRODIFFERENTIAL EQUATIONS WITH A CAPUTO–FABRIZIO DERIVATIVE

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Abstract. In this paper, a time-fractional integrodifferential equation with the Caputo–Fabrizio type derivative will be considered. The Banach fixed point theorem is the main tool used to extend the results of a recent paper of Tuan and Zhou [*J. Comput. Appl. Math.* **375** (2020) 112811]. In the case of a globally Lipschitz source terms, thanks to the $L^p - L^q$ estimate method, we establish global in time well-posed results for mild solution. For the case of locally Lipschitz terms, we present existence and uniqueness results. Also, we show that our solution will blow up at a finite time. Finally, we present some numerical examples to illustrate the regularity and continuation of the solution based on the time variable.

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1. Introduction

Fractional calculus has many applications in mechanic, physics and engineering science... For example, a fractional diffusion equation is a generalization of a classical diffusion equation which models anomalous diffusive phenomena. For results on fractional derivatives like Riemann–Liouville type or Caputo type we refer the reader to [1, 13, 23, 29, 31, 35] and the references therein. Recently Caputo and Fabrizio [16] introduced a new fractional derivative, which called Caputo–Fabrizio fractional derivative.

Our paper focuses on studying the existence and uniqueness of a mild solution of time-fractional Volterra integrodifferential equations with the new type of fractional derivative. Let T be a positive number and $\alpha \in (0,1)$, we consider the following intial value problem

$$\begin{cases} {}_{CF}\mathscr{D}_t^\alpha u - \Delta u = G(t,x,u) + \int_0^t \varphi(t-s, u(s))\mathrm{d}s & \text{in } (0,T)\times D, \\ u(t,x) = u_0(x) & \text{in } \{0\}\times D, \end{cases} \tag{1.1}$$

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here, u_0 is the initial data, the functions G, φ and the domain D will be defined later. The symbol ${}_{CF}\mathscr{D}_t^\alpha$ stands for the Caputo–Fabrizio type time fractional derivative operator of order α (see [15, 39]). We recall the definition of the Caputo–Fabrizio type time fractional derivative as follows. Let $a > 0, 0 \le \alpha \le 1$, and for a function w belongs to $H^1(0,a)$, its Caputo-Fabrizio fractional derivative is defined as (see [15])

$$ {}_{CF}\mathscr{D}_t^\alpha w(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha s}{1-\alpha}\right)\frac{\partial w}{\partial s}\mathrm{d}s, \quad t \ge 0, \tag{1.2} $$

where M is a normalization function such that $M(0) = M(1) = 1$. When $\varphi = 0$, the authors in [39] investigated the well-posedness of an initial value problem for a fractional diffusion equation. In [2], the existence of the solution to an initial value problem for a linear differential equation was investigated and the authors applied their results to the mass-spring-damper motion in the general case. We refer the reader to [25] for results on the existence of the Korteweg-de Vries-Burgers equation with a fractional Caputo–Fabrizio derivative. For more results on time fractional partial derivative equations topic, we refer the reader to [5–7, 9–12, 26, 27, 32, 36, 38].

When ($\varphi \ne 0$), problem (1.1) can be used to model some natural phenomena with memory effects. In [3] the authors considered memory effects on the dynamics of non-Newtonian fluids and viscoelastic models for the dynamics of turbulence statistics in Newtonian fluids on the modified 3D Navier–Stokes equation

$$ \begin{array}{ll} u_t = \Delta u - \nabla p - (u\cdot\nabla)u + f(t) + \int_0^t a(t-s)(-\Delta)^\beta u(s)\mathrm{d}s, & x \in \Omega \\ \operatorname{div}(u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{array} \tag{1.3} $$

In [34] the authors compared the energy dissipation produced by the internal motion and the memory effect *i.e.* they studied the behavior between the equation

$$ u_{tt} + \Delta^2 u + \alpha\Delta u_t = 0 \quad \text{in } \Omega \tag{1.4} $$

and the equation

$$ u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\cdot,\tau)\mathrm{d}\tau = 0 \quad \text{in } \Omega. \tag{1.5} $$

We also refer the reader to [20, 28, 30, 40] and the references therein. In [20] the authors established the existence of global and exponential attractors of optimal regularity and finite fractal dimension for the related semigroup of solutions of the equation

$$ \partial_t u - \Delta u - \int_0^\infty \kappa(s)\Delta u(t-s)\mathrm{d}s + \varphi(u) = f, \tag{1.6} $$

which arises in the Coleman-Gurtin theory of heat conduction with hereditary memory.

However, to the best of our knowledge, there are few results on Volterra diffusion equations with the Caputo–Fabrizio derivative (problem (1.1)). Our first main goal is to present global results on the whole space $\mathbb{R}^N$ under a global Lipschitz condition on the source term. The second goal is to prove the existence and uniqueness of mild solutions and present a blow-up alternative for mild solutions of problem (1.1) under a locally Lipschitz condition on G and φ.

The paper is organized as follows. In Section 2, we introduce some important vector spaces, Lemma's, preliminaries, and the definition of mild solutions of (1.1). In Section 3, we prove global existence for the mild solution of problem (1.1) when $D = \mathbb{R}^N$, with globally Lipschitz source terms. Finally, in the last section, a local well-posed result will be given for problem (1.1) with locally Lipschitz condition when D is an open bounded domain with smooth boundary in $\mathbb{R}^N$.

2. Preliminary material

Assume that D is a bounded domain whose boundary is sufficiently smooth. We consider the following spectral problem for the Laplace operator Δ

$$\begin{cases} -\Delta\psi_j(x) = \lambda_j \psi_j(x) & \text{in } D, \\ \psi_j(0) = 0 & \text{on } \partial D, \end{cases}$$

which admits a family of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \cdots \nearrow \infty$ and a corresponding family of eigenvectors $\{\psi_j\}_{j\in\mathbb{N}}$.

Let $(B, \|\cdot\|_B)$ be a Banach space, then, we denote by $L^p(0,T;B)$ the Banach space of measurable functions $f:(0,T)\to B$ eqquiped with the norm

$$\|f\|_{L^p(0,T;B)} = \left(\int_0^T \|f(t)\|_B^p \mathrm{d}t\right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty \tag{2.1}$$

and

$$\|f\|_{L^\infty(0,T;B)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|f(t)\|_B, \quad \text{for } p = \infty. \tag{2.2}$$

The space of all k time's derivative continuous functions $C^k([0,T];B), k \in \mathbb{N}$ is a Banach space with the norm

$$\|f\|_{C^k([0,T];B)} = \sum_{i=1}^{k} \sup_{t\in[0,T]} \|f^i(t)\|_B. \tag{2.3}$$

We denote by $\mathbb{X}^{h,\mu}((0,T];B)$, the subspace of all functions f in $C([0,T];B)$ such that

$$\|f\|_{\mathbb{X}^{h,\mu}((0,T];B)} := \sup_{0<t\leq T} t^h e^{-\mu t} \|f(t)\|_B < \infty. \tag{2.4}$$

Assume that D is a bounded domain whose boundary is sufficiently smooth. For any $\sigma > 0$, we define the fractional Hilbert scale space by

$$\mathcal{H}^\sigma(D) = \left\{ f \in L^2(D) : \|f\|^2_{\mathcal{H}^\sigma(D)} = \sum_{j=1}^{\infty} \lambda_j^{2\sigma} \langle f, \psi_j\rangle^2_{L^2(D)} < \infty \right\}$$

and the Hilbert space $\mathcal{H}^{-\sigma}(D)$ with the norm

$$\|f\|_{\mathcal{H}^{-\sigma}(D)} = \sqrt{\sum_{j=1}^{\infty} \lambda_j^{-2\sigma} \langle f, \psi_j\rangle^2_*}, \tag{2.5}$$

which is the duality space of $\mathcal{H}^\sigma(D)$. The notation $\langle\cdot,\cdot\rangle_*$ denotes the dual inner product between $\mathcal{H}^{-\sigma}(D)$ and $\mathcal{H}^\sigma(D)$.

Lemma 2.1. *(see [15]) The Laplace transform of the Caputo–Fabrizio derivative is given by*

$$\mathscr{L}\left\{{}_{CF}\mathscr{D}_t^{\alpha} w(t)\right\}(m) = \frac{m\mathscr{L}\left\{f\right\}(m) - f(0)}{m + \alpha(1-m)}, \quad m > 0. \tag{2.6}$$

Definition 2.2. Let $\alpha\Delta\left[(1-\alpha)\Delta - I\right]^{-1}$ be the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$. Then, the formula of the mild solution for problem (1.1) is given by

$$\begin{aligned} u(t) = \mathcal{A}T(\alpha t)u_0 &+ \int_0^t \alpha\mathcal{A}^2 T(\alpha(t-z))G(u(z))\mathrm{d}z \\ &+ \int_0^t \alpha\mathcal{A}^2 T(\alpha(t-z))\int_0^z \varphi(z-s, u(s))\mathrm{d}s\mathrm{d}z, \end{aligned} \tag{2.7}$$

where we set $\mathcal{A} = \left[I - \Delta(1-\alpha)\right]^{-1}$.

Lemma 2.3. *(see [14])Assume that $D \equiv \mathbb{R}^N$. Let $1 \leq p \leq q$ and $w \in L^p(D) \cap L^q(D)$. Then, there exists a constant C such that*

$$\|T(t)w\|_{L^q(D)} \leq C(1+t)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|w\|_{L^p(D)} + Ce^{-t}\|w\|_{L^q(D)}, \quad t \geq 0. \tag{2.8}$$

$$\|\mathcal{A}w\|_{L^q(D)} \leq C\|w\|_{L^q(D)}. \tag{2.9}$$

3. Result on the whole space $\mathbb{R}^N$ with globally Lipschitz source term

In this section, we consider the existence and uniqueness of mild solution for problem (1.1) when $D \equiv \mathbb{R}^N$ under the following assumptions for the source terms

$$\begin{cases} G(0) = 0, \\ |G(a) - G(b)| \leq L|a-b|, \quad \forall a, b \in \mathbb{R} \end{cases} \tag{3.1}$$

and

$$\begin{cases} \varphi(0) = 0, \\ |\varphi(a) - \varphi(b)| \leq L|a-b|, \quad \forall a, b \in \mathbb{R}, \end{cases} \tag{3.2}$$

where L is not dependent on a, b.

Lemma 3.1. *(see [19]) Let $h > 0$ and $m, n > -1$ such that $m + n > -1$. Then, we have the following limit result*

$$\sup_{t\in[0,T]} t^h \int_0^1 s^m(1-s)^n e^{-\mu t(1-s)}\mathrm{d}s \xrightarrow{\mu\longrightarrow\infty} 0. \tag{3.3}$$

Theorem 3.2. *Let $1 \leq p \leq q$ such that*

$$\frac{1}{p} - \frac{1}{q} < \frac{1}{N}$$

and

$$h \in \left(\frac{N(q-p)}{2pq}, 1 + \frac{N(p-q)}{2pq} \right).$$

Assume that $u_0 \in Z = L^p(D) \cap L^q(D)$ *and* G, φ *satisfy* (3.1), (3.2), *respectively. Then, for any* $T > 0$, *problem* (1.1) *has a unique mild solution in* $\mathbb{X}^{h,\mu}((0,T];Z)$.

Proof. The proof begins by defining a mapping $J : \mathbb{X}^{h,\mu}((0,T];Z) \longrightarrow \mathbb{X}^{h,\mu}((0,T];Z)$ as follows

$$\begin{aligned} Jw(t) = \mathcal{A}T(\alpha t)u_0 &+ \int_0^t \alpha \mathcal{A}^2 T(\alpha(t-z)) G(w(z)) \mathrm{d}z \\ &+ \int_0^t \alpha \mathcal{A}^2 T(\alpha(t-z)) \int_0^z \varphi(z-s, w(s)) \mathrm{d}s \mathrm{d}z. \end{aligned} \tag{3.4}$$

We first use Lemma 2.8 and the inequality $e^{-r} \le c_\theta r^{-\theta} (r > 0,\ 0 < \theta < 1)$ to obtain

$$\begin{aligned} \|\mathcal{A}T(\alpha t)u_0\|_{L^q(D)} &\le C^2(1+\alpha t)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_0\|_{L^p(D)} + C^2 e^{-\alpha t}\|u_0\|_{L^q(D)} \\ &\le C^2(1+\alpha t)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_0\|_{L^p(D)} + C^2 c_\theta \alpha^{-\theta} t^{-\theta} \|u_0\|_{L^q(D)}. \end{aligned} \tag{3.5}$$

For $h > \frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$, choosing $\theta = \frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right) < 1$, we deduce that

$$\begin{aligned} t^h e^{-\mu t} \|\mathcal{A}T(\alpha t)u_0\|_{L^q(D)} &\le C^2 \alpha^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} T^{h+\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\|u_0\|_{L^p(D)} + c_\theta \|u_0\|_{L^q(D)}\right) \\ &\le \max\{1, c_\theta\} C^2 T^{h+\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_0\|_Z. \end{aligned} \tag{3.6}$$

Next, we will deal with the remain term $Jw(t) - \mathcal{A}T(\alpha t)u_0$. To this end, for any $w, v \in \mathbb{X}^{h,\mu}((0,T];Z)$, we will consider the following equation

$$Jw(t) - Jv(t) = \mathcal{I}_1 + \mathcal{I}_2, \tag{3.7}$$

where

$$\mathcal{I}_1 = \alpha \int_0^t \mathcal{A}^2 T(\alpha(t-z)) \left[G(w(z)) - G(v(z))\right] \mathrm{d}z, \tag{3.8}$$

$$\mathcal{I}_2 = \alpha \int_0^t \mathcal{A}^2 T(\alpha(t-z)) \int_0^z \left[\varphi(z-s, w(s)) - \varphi(z-s, v(s))\right] \mathrm{d}s\mathrm{d}z. \tag{3.9}$$

Using the same calculation ideas as in (3.5), we see that

$$\begin{aligned} &\left\| \mathcal{A}^2 T(\alpha(t-z)) \left[G(w(z)) - G(v(z))\right] \right\|_{L^q(D)} \\ &\le \frac{C^3}{(1+\alpha(t-z))^{\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}} \|G(w(z)) - G(v(z))\|_{L^p(D)} + \frac{C^3}{e^{\alpha(t-z)}} \|G(w(z)) - G(v(z))\|_{L^q(D)} \\ &\le \frac{C^3}{(\alpha(t-z))^{\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}} \|G(w(z)) - G(v(z))\|_{L^p(D)} + \frac{C^3 c_\theta}{\alpha^\theta (t-z)^\theta} \|G(w(z)) - G(v(z))\|_{L^q(D)}. \end{aligned} \tag{3.10}$$

Then, it follows that

$$\|\mathcal{I}_1\|_{L^q(D)} \leq M_1 \int_0^t (t-z)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|w(z)-v(z)\|_Z \mathrm{d}z, \tag{3.11}$$

where $M_1 = C^3 L\alpha^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \max\{1, c_\theta\}$. The above estimate gives us the one below

$$\begin{aligned} t^h e^{-\mu t}\|\mathcal{I}_1\|_{L^q(D)} &\leq M_1 t^h e^{-\mu t} \int_0^t (t-z)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|w(z)-v(z)\|_Z \mathrm{d}z \\ &\leq M_1 t^h \int_0^t z^{-h}(t-z)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} e^{-\mu(t-z)} \mathrm{d}z \|w-v\|_{\mathbb{X}^{h,\mu}((0,T];Z)}. \end{aligned} \tag{3.12}$$

In Lemma 2.8, taking $p=q$ and setting $M_2 = C^3 L\alpha^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}(1+c_\theta)$, we have

$$\begin{aligned} t^h e^{-\mu t}\|\mathcal{I}_1\|_{L^p(D)} &\leq t^h e^{-\mu t} M_2 \int_0^t (t-z)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|w(z)-v(z)\|_Z \mathrm{d}z \\ &\leq M_2 t^h \int_0^t z^{-h}(t-z)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} e^{-\mu(t-z)} \mathrm{d}z \|w-v\|_{\mathbb{X}^{h,\mu}((0,T];Z)}. \end{aligned} \tag{3.13}$$

These two estimates above lead us to the following result

$$t^h e^{-\mu t}\|\mathcal{I}_1\|_Z \leq (M_1+M_2) t^h \int_0^t z^{-h}(t-z)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} e^{-\mu(t-z)} \mathrm{d}z \|w-v\|_{\mathbb{X}^{h,\mu}((0,T];Z)}. \tag{3.14}$$

By similar arguments, we also obtain

$$\begin{aligned} t^h e^{-\mu t}\|\mathcal{I}_2\|_Z &\leq (M_1+M_2) t^h e^{-\mu t} \int_0^t (t-z)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \int_0^z \|w(s)-v(s)\|_Z \mathrm{d}s \mathrm{d}z \\ &\leq (M_1+M_2) t^h \int_0^t (t-z)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} e^{-\mu(t-z)} \int_0^z s^{-h} \mathrm{d}s \mathrm{d}z \|w-v\|_{\mathbb{X}^{h,\mu}((0,T];Z)} \\ &\leq \frac{(M_1+M_2)T}{1-h} t^h \int_0^t z^{-h}(t-z)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} e^{-\mu(t-z)} \mathrm{d}z \|w-v\|_{\mathbb{X}^{h,\mu}((0,T];Z)}. \end{aligned} \tag{3.15}$$

On account of the results stated above, by choosing $v=0$, it's easily check that if we take w in $\mathbb{X}^{h,\mu}((0,T];Z)$, Jw will be in $\mathbb{X}^{h,\mu}((0,T];Z)$. In addition, the standard smooth effect of the semigroup $T(t)$ will ensure the continuity of J on $(0,T]$. Therefore, we conclude that, J is invariant on $\mathbb{X}^{h,\mu}((0,T];Z)$.

In the other hand, from the assumptions on p, q, h, we can easily deduce that

$$\begin{cases} -h > -1, \\ \frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right) > -1, \\ -h + \frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right) > -1, \\ h + \frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right) > 0. \end{cases}$$

Lemma 3.1 is now applied to yield that

$$\lim_{\mu\to\infty} \mathcal{Q}(\mu) := \lim_{\mu\to\infty} \left[\sup_{0<t\le T} t^{h+\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \int_0^1 z^{-h}(1-z)^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} e^{-\mu t(1-z)} \mathrm{d}z \right] = 0. \tag{3.16}$$

Combining (3.7), (3.14), (3.15), we have

$$t^h e^{-\mu t} \|Jw(t) - Jv(t)\|_Z \le M \left(1 + \frac{T}{1-h}\right) \mathcal{Q}(\mu) \|w - v\|_{\mathbb{X}^{h,\mu}((0,T];Z)}. \tag{3.17}$$

From (3.16), there exists a sufficiently large μ_0 such that

$$M \left(1 + \frac{T}{1-h}\right) \mathcal{Q}(\mu_0) < 1.$$

Consequently, J is a contraction on $\mathbb{X}^{h,\mu}((0,T];Z)$. Now we conclude that problem (1.1) has a unique solution in $\mathbb{X}^{h,\mu}((0,T];Z)$ and the proof is completed. □

4. Problem (1.1) under locally Lipschitz source term

Throughout this section, we suppose that $D \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary and our solution will vanish on the boundary of D. Furthermore, we consider the following assumptions for the source terms.

Let $\nu < \sigma < \nu + 2$ and $m, n > 1$. Suppose that for any $a, b \in \mathbb{R}$, functions G, φ satisfy

$$\begin{cases} \big\|G(a) - G(b)\big\|_{\mathcal{H}^\nu(D)} \le L_G \left(1 + \|a\|^{m-1}_{\mathcal{H}^\sigma(D)} + \|b\|^{m-1}_{\mathcal{H}^\sigma(D)}\right) \|a - b\|_{\mathcal{H}^\sigma(D)} \\ \big\|G(a)\big\|_{\mathcal{H}^\nu(D)} \le L_G \left(\|a\|^m_{\mathcal{H}^\sigma(D)} + 1\right) \end{cases} \tag{4.1}$$

and

$$\begin{cases} \big\|\varphi(a) - \varphi(b)\big\|_{\mathcal{H}^\nu(D)} \le L_\varphi \left(1 + \|a\|^{n-1}_{\mathcal{H}^\sigma(D)} + \|b\|^{n-1}_{\mathcal{H}^\sigma(D)}\right) \|a - b\|_{\mathcal{H}^\sigma(D)} \\ \big\|\varphi(a)\big\|_{\mathcal{H}^\nu(D)} \le L_\varphi \left(\|a\|^n_{\mathcal{H}^\sigma(D)} + 1\right). \end{cases} \tag{4.2}$$

To deal with the initial value problem with bounded domain, we will rewrite (2.7) in the form of Fourier series as follows (see [39])

$$u(t) = \mathcal{S}_0(t)u_0 + \int_0^t \mathcal{S}(t-z) G(u(z)) \mathrm{d}z + \int_0^t \mathcal{S}(t-z) \int_0^z \varphi(z-s, u(s)) \mathrm{d}s \mathrm{d}z$$

where

$$\mathcal{S}_0(t) f(x) := \sum_{j=1}^\infty \frac{1}{1+\lambda_j(1-\alpha)} \exp\left(\frac{-\lambda_j \alpha t}{1+\lambda_j(1-\alpha)}\right) \langle f, \psi_j \rangle_{L^2(D)} \psi_j(x), \tag{4.3}$$

and

$$\mathcal{S}(t) f(x) := \sum_{j=1}^\infty \frac{\alpha}{[1+\lambda_j(1-\alpha)]^2} \exp\left(\frac{-\lambda_j \alpha t}{1+\lambda_j(1-\alpha)}\right) \langle f, \psi_j \rangle_{L^2(D)} \psi_j(x). \tag{4.4}$$

4.1. Local existence and uniqueness of the mild solution

Lemma 4.1. *Let $\mathcal{K}, T_0$ be positive constants. Assume that $u \in C([0,T_0];\mathcal{H}^\sigma(D))$ and $f \in \mathcal{H}^\sigma(D)$ such that $\sup_{t\in[0,T_0]} \|u(t)-f\|_{\mathcal{H}^\sigma(D)} \leq \mathcal{K}$, then for $0 \leq t_1 < t_2 \leq T_0$, we have the following estimates*

$$\left\| \int_{t_1}^{t_2} \mathcal{S}(t_2-z)G(u(z))\mathrm{d}z \right\|_{\mathcal{H}^\sigma(D)} \leq \frac{L_G(t_2-t_1)}{(1-\alpha)^2} \left[(\|f\|_{\mathcal{H}^\sigma(D)} + \mathcal{K})^m + 1 \right] \tag{4.5}$$

and

$$\left\| \int_{t_1}^{t_2} \mathcal{S}(t-z) \int_0^z \varphi(z-s,u(s))\mathrm{d}s\mathrm{d}z \right\|_{\mathcal{H}^\sigma(D)} \leq \frac{L_\varphi(t_2-t_1)}{T_0^{-1}(1-\alpha)^2} \left[(\|f\|_{\mathcal{H}^\sigma(D)} + \mathcal{K})^n + 1 \right]. \tag{4.6}$$

Proof. Using the triangle inequality, we have

$$\|u(t)\|_{\mathcal{H}^\sigma(D)} \leq \|f\|_{\mathcal{H}^\sigma(D)} + \sup_{0\leq t\leq T_0} \left\|u(t)-f\right\|_{\mathcal{H}^\sigma(D)} \leq \|f\|_{\mathcal{H}^\sigma(D)} + \mathcal{K}. \tag{4.7}$$

Thanks to the Parseval identity, the following estimate holds

$$\begin{aligned}
&\left\|\mathcal{S}(t_2-z)G(u(z))\right\|^2_{\mathcal{H}^\sigma(D)} \\
&\leq \sum_{j=1}^\infty \frac{\lambda_j^{2p}}{(1+\lambda_j(1-\alpha))^4} \exp\left(-\frac{2\alpha\lambda_j(t_2-z)}{1+\lambda_j(1-\alpha)}\right) \langle G(u(z)),\psi_j\rangle^2_{L^2(\Omega)} \\
&\leq \frac{1}{(1-\alpha)^4} \sum_{j=1}^\infty \lambda_j^{2p-4} \langle G(u(z)),\psi_j\rangle^2_{L^2(\Omega)} \leq \frac{1}{(1-\alpha)^4} \|G(u(z))\|^2_{\mathcal{H}^\nu(D)}.
\end{aligned} \tag{4.8}$$

The above result gives

$$\begin{aligned}
\left\| \int_{t_1}^{t_2} \mathcal{S}(t_2-z)G(u(z))\mathrm{d}z \right\|_{\mathcal{H}^\sigma(D)} &\leq \tfrac{L_G}{(1-\alpha)^2} \int_{t_1}^{t_2} \left(\|u(z)\|^m_{\mathcal{H}^\sigma(D)} + 1 \right) \mathrm{d}z \\
&\leq \tfrac{L_G}{(1-\alpha)^2} \left[(\|f\|_{\mathcal{H}^\sigma(D)} + \mathcal{K})^m + 1 \right] (t_2-t_1).
\end{aligned}$$

Similarly, the second conclusion of this lemma can also be drawn

$$\begin{aligned}
&\left\| \int_{t_1}^{t_2} \mathcal{S}(t-z) \int_0^z \varphi(z-s,u(s))\mathrm{d}s\mathrm{d}z \right\|_{\mathcal{H}^\sigma(D)} \\
&\leq \frac{L_\varphi}{(1-\alpha)^2} \int_{t_1}^{t_2} \int_0^z \left(\|u(s)\|^n_{\mathcal{H}^\sigma(D)} + 1 \right) \mathrm{d}s\mathrm{d}z \\
&\leq \frac{t_2 L_\varphi}{(1-\alpha)^2} \left[(\|f\|_{\mathcal{H}^\sigma(D)} + \mathcal{K})^n + 1 \right] (t_2-t_1).
\end{aligned} \tag{4.9}$$

□

Theorem 4.2. *Let G and φ sastify the locally Lipschitz condition (4.1), (4.2), respectively. Assume that $u_0 \in \mathcal{H}^\sigma(D)$. Then, there exists $T_0 > 0$ such that problem (1.1) has a unique mild solution in $C([0,T_0];\mathcal{H}^\sigma(D))$.*

Proof. Set $\widetilde{u_0} = \frac{u_0}{1+\lambda_j(1-\alpha)}$ and we can check immediately that $\widetilde{u_0} \in \mathcal{H}^\sigma(D)$. Next, for a fixed positive constant$\mathcal{K}$, we consider the following space

$$\mathbb{A} := \left\{ u \in C([0,T_0];\mathcal{H}^\sigma(D)) : \|u(\cdot;t) - \widetilde{u_0}\|_{\mathcal{H}^\sigma(D)} \le \mathcal{K}, \forall t \in [0,T_0] \right\}$$

and define the operator $J : \mathbb{A} \to \mathbb{A}$ by

$$Jw(t,x) = \mathcal{S}_0(t)u_0 + \alpha \int_0^t \mathcal{S}(t-z)G(w(z))\mathrm{d}z + \alpha \int_0^t \mathcal{S}(t-z) \int_0^z \varphi(z-s,w(s))\mathrm{d}s\mathrm{d}z. \tag{4.10}$$

Note that $\mathbb{A}$ is a complete space with respect to the usual sup norm. For the purpose of using the Banach fixed point theorem, we first need to show that $Jw \in C([0,T_0];\mathcal{H}^\sigma(D))$ for any $w \in C([0,T_0];\mathcal{H}^\sigma(D))$. Let $0 < t \le t+\varepsilon \le T_0$. Using the inequality $|e^{-a} - e^{-b}| \le |a-b|$, for $a,b > 0$, we have

$$\begin{aligned}\left\| \left(\mathcal{S}_0(t+\varepsilon) - \mathcal{S}_0(t)\right) u_0 \right\|_{\mathcal{H}^\sigma(D)} &\le \sqrt{\frac{\alpha^2\varepsilon^2}{\lambda_1^2(1-\alpha)^4} \sum_{j=1}^\infty \lambda_j^{2p} \left\langle u_0, e_j \right\rangle_{L^2(D)}^2} \\ &= \frac{\alpha\varepsilon}{\lambda_1(1-\alpha)^2} \|u_0\|_{\mathcal{H}^\sigma(D)}. \end{aligned} \tag{4.11}$$

and

$$\begin{aligned}&\left\| \int_0^t \left(\mathcal{S}(t+\varepsilon-z) - \mathcal{S}(t-z)\right) G(w(z))\mathrm{d}z \right\|_{\mathcal{H}^\sigma(D)} \\ &\le \frac{\alpha\varepsilon}{(1-\alpha)^3} \int_0^t \left\|G(w(z))\right\|_{\mathcal{H}^\nu(D)}\mathrm{d}z \le \left[\left(\|\widetilde{u_0}\|_{\mathcal{H}^\sigma(D)} + \mathcal{K}\right)^m + 1\right] \frac{\alpha\varepsilon T_0 L_G}{(1-\alpha)^3}. \end{aligned} \tag{4.12}$$

In the other hand, it follows from Lemma 4.1 that

$$\left\| \int_t^{t+\varepsilon} \mathcal{S}(t+\varepsilon-z)G(w(z))\mathrm{d}z \right\|_{\mathcal{H}^\sigma(D)} \le \frac{L_G\varepsilon}{(1-\alpha)^2} \left[(\|\widetilde{u_0}\|_{\mathcal{H}^\sigma(D)} + \mathcal{K})^m + 1 \right]. \tag{4.13}$$

By similar arguments, we also obtain

$$\begin{aligned}&\left\| \int_t^{t+\varepsilon} \mathcal{S}(t+\varepsilon-z) \int_0^z \varphi(z-s,w(s))\mathrm{d}s\mathrm{d}z \right\|_{\mathcal{H}^\sigma(D)} \\ &\le \frac{L_\varphi T_0\varepsilon}{(1-\alpha)^2} \left[(\|\widetilde{u_0}\|_{\mathcal{H}^\sigma(D)} + \mathcal{K})^n + 1 \right] \end{aligned} \tag{4.14}$$

and

$$\begin{aligned}&\left\| \int_0^t \left(\mathcal{S}(t+\varepsilon-z) - \mathcal{S}(t-z)\right) \int_0^z \varphi(z-s,w(s))\mathrm{d}s\mathrm{d}z \right\|_{\mathcal{H}^\sigma(D)} \\ &\le \left[\left(\|\widetilde{u_0}\|_{\mathcal{H}^\sigma(D)} + \mathcal{K}\right)^n + 1\right] \frac{\alpha\varepsilon T_0^2 L_\varphi}{(1-\alpha)^3}. \end{aligned} \tag{4.15}$$

Hence, if w belongs to $C([0,T_0];\mathcal{H}^\sigma(D))$, then, Jw is in $C([0,T_0];\mathcal{H}^\sigma(D))$. Next, we set

$$\mathcal{M} = \max\left\{ \left(\frac{\|u_0\|_{\mathcal{H}^\sigma(D)}}{\lambda_1(1-\alpha)} + \mathcal{K} \right)^m, \left(\frac{\|u_0\|_{\mathcal{H}^\sigma(D)}}{\lambda_1(1-\alpha)} + \mathcal{K} \right)^n \right\}$$

and choose a sufficiently small T_0 such that

$$\begin{cases} \dfrac{\alpha T_0 \lambda_1^{-1} \|u_0\|_{\mathcal{H}^\sigma(D)}}{(1-\alpha)^2} + \dfrac{(L_G T_0 + L_\varphi T_0^2)(\mathcal{M}+1)}{(1-\alpha)^2} \le \mathcal{K}, \\ (L_G T_0 + L_\varphi T_0^2) < \dfrac{(1-\alpha)^2}{1+2\mathcal{M}}. \end{cases}$$

Using the inequality $1 - e^r \le r$, we have

$$\begin{aligned} \|\mathcal{S}_0(t)u_0 - \widetilde{u_0}\|^2_{\mathcal{H}^\sigma(D)} &\le \sum_{j=1}^\infty \frac{\lambda_j^{2\sigma}}{(1+\lambda_j(1-\alpha))^2} \left[\exp\left(\frac{-\alpha\lambda_j t}{1+\lambda_j(1-\alpha)} \right) - 1 \right]^2 \langle u_0, \psi_j \rangle^2_{L^2(D)} \\ &\le \frac{(\alpha t)^2}{\lambda_1^2(1-\alpha)^4} \sum_{j=1}^\infty \lambda_j^{2\sigma} \langle u_0, \psi_j \rangle^2_{L^2(D)}. \end{aligned} \tag{4.16}$$

Let $w \in \mathbb{A}$. From Lemma 4.1 and the above estimate, we get

$$\begin{aligned} \|Jw(t) - \widetilde{u_0}\|_{\mathcal{H}^\sigma(D)} &\le \|\mathcal{S}_0(t)u_0 - \widetilde{u_0}\|_{\mathcal{H}^\sigma(D)} \\ &\quad + \left\| \int_0^t \mathcal{S}(t-z) G(w(z)) \mathrm{d}z \right\|_{\mathcal{H}^\sigma(D)} \\ &\quad + \left\| \int_0^t \mathcal{S}(t-z) \int_0^z \varphi(z-s, w(s)) \mathrm{d}s \mathrm{d}z \right\|_{\mathcal{H}^\sigma(D)} \\ &\le \frac{\alpha T_0}{\lambda_1(1-\alpha)^2} \|u_0\|_{\mathcal{H}^\sigma(D)} + \frac{L_G T_0 + L_\varphi T_0^2}{(1-\alpha)^2} (\mathcal{M}+1). \end{aligned} \tag{4.17}$$

Thus, $Jw \in \mathbb{A}$. Now, we show that J is a contraction. Let $v, w \in \mathbb{A}$, for any $t \in [0,T_0]$, we have

$$\begin{aligned} \big\|Jw(t) - Jv(t)\big\|_{\mathcal{H}^\sigma(D)} &\le \int_0^t \Big\| \mathcal{S}(t-z) \left(G(w(z)) - G(v(z)) \right) \Big\|_{\mathcal{H}^\sigma(D)} \mathrm{d}z \\ &\quad + \int_0^t \Big\| \mathcal{S}(t-z) \int_0^z \left(\varphi(z-s, w(s)) - \varphi(z-s, v(s)) \right) \mathrm{d}s \Big\|_{\mathcal{H}^\sigma(D)} \mathrm{d}z. \end{aligned} \tag{4.18}$$

Note for the first term (one can apply a similar argument for the second term)

$$\begin{aligned} &\int_0^t \Big\| \mathcal{S}(t-z) \left(G(w(z)) - G(v(z)) \right) \Big\|_{\mathcal{H}^\sigma(D)} \mathrm{d}z \\ &\le \frac{1}{(1-\alpha)^2} \int_0^t \big\| G(w(\cdot,z)) - G(v(\cdot,z)) \big\|_{\mathcal{H}^\nu(D)} \mathrm{d}z \\ &\le \frac{L_G}{(1-\alpha)^2} \int_0^t \left(1 + \|v\|^{m-1}_{\mathcal{H}^\sigma(D)} + \|w\|^{m-1}_{\mathcal{H}^\sigma(D)} \right) \|v - w\|_{\mathcal{H}^\sigma(D)} \mathrm{d}z \end{aligned} \tag{4.19}$$

$$\leq \frac{L_G}{(1-\alpha)^2} \int_0^t \|v-w\|_{\mathcal{H}^\sigma(D)} \left[1 + 2\left(\mathcal{K} + \|\widetilde{u_0}\|_{\mathcal{H}^\sigma(D)}\right)^{m-1}\right] \mathrm{d}z$$
$$\leq \frac{L_G T_0}{(1-\alpha)^2}(1+2\mathcal{M})\|v-w\|_{C([0,T_0];\mathcal{H}^\sigma(D))}.$$

Thus

$$\big\|Jw(t) - Jv(t)\big\|_{C([0,T_0];\mathcal{H}^\sigma(D))} \leq \frac{(L_G T_0 + L_\varphi T_0^2)}{(1-\alpha)^2}(1+2\mathcal{M})\|v-w\|_{C([0,T_0];\mathcal{H}^\sigma(D))}. \tag{4.20}$$

From this point of view, we can conclude that J is a contraction on $\mathbb{A}$. Therefore, there exists a unique solution u of (4.10) and the theorem is proved. □

4.2. Continuation and blow-up alternative

In this subsection, we present a continuation result and a blow-up alternative for the mild solution of problem (1.1).

Theorem 4.3. *Assume that G sastifies (4.1) and φ satisfies (4.2). Then, the mild solution u of problem (1.1) on $[0,T_0]$ can be extended to $[0,T_0+h]$, for some $h>0$.*

Proof. Consider the following space

$$\mathbb{B} = \{v \in C([0,T_0+h], \mathcal{H}^\sigma(D)) : v(t) = u(t), \forall t \in [0,T_0]$$
$$\text{and } \|v(t) - u(T_0)\|_{\mathcal{H}^\sigma(D)} \leq \mathcal{M}, \forall t \in [T_0, T_0+h]\big\}.$$

It's obvious that $\mathbb{B}$ is a complete space. Define a mapping $\mathscr{J} : \mathbb{B} \to \mathbb{B}$ by

$$\mathscr{J} w(t,x) = \mathcal{S}_0(t)u_0 + \alpha \int_0^t \mathcal{S}(t-z)G(w(z))\mathrm{d}z + \alpha \int_0^t \mathcal{S}(t-z)\int_0^z \varphi(z-s,w(s))\mathrm{d}s\mathrm{d}z. \tag{4.21}$$

Let $v \in \mathbb{B}$. If $t \in [0,T_0]$ we have $\mathscr{J} w(t) = \mathscr{J} u(t) = u(t)$. If $t \in [T_0, T_0+h]$, then

$$\begin{aligned}
\|\mathscr{J} v(t) - u(T_0)\|_{\mathcal{H}^\sigma(D)} \leq{}& \|\left(\mathcal{S}_0(t) - \mathcal{S}_0(T_0)\right)u_0\|_{\mathcal{H}^\sigma(D)} \\
&+ \int_{T_0}^t \Big\|\mathcal{S}(t-z)G(v(z))\Big\|_{\mathcal{H}^\sigma(D)} \mathrm{d}z \\
&+ \int_{T_0}^t \left\|\mathcal{S}(t-z)\int_0^z \varphi(z-s,v(s))\mathrm{d}s\right\|_{\mathcal{H}^\sigma(D)} \mathrm{d}z \\
&+ \int_0^{T_0} \Big\|\left(\mathcal{S}(t-z) - \mathcal{S}(T_0-z)\right)G(u(z))\Big\|_{\mathcal{H}^\sigma(D)} \mathrm{d}z \\
&+ \int_0^{T_0} \left\|\left(\mathcal{S}(t-z) - \mathcal{S}(T_0-z)\right)\int_0^z \varphi(z-s,u(s))\mathrm{d}s\right\|_{\mathcal{H}^\sigma(D)} \mathrm{d}z.
\end{aligned} \tag{4.22}$$

Using some arguments as in the proof of the previous theorem, we have
Claim 1.

$$\|\left(\mathcal{S}_0(t) - \mathcal{S}_0(T_0)\right)u_0\|_{\mathcal{H}^\sigma(D)} \leq \frac{\alpha(t-T_0)}{\lambda_1(1-\alpha)^2}\|u_0\|_{\mathcal{H}^\sigma(D)}$$

$$\leq \frac{h\alpha}{\lambda_1(1-\alpha)^2}\|u_0\|_{\mathcal{H}^\sigma(D)}. \tag{4.23}$$

Claim 2.

$$\int_{T_0}^t \left\|\mathcal{S}(t-z)G(v(z))\right\|_{\mathcal{H}^\sigma(D)}\mathrm{d}z \leq \frac{hL_G}{(1-\alpha)^2}\left[(\|u(T_0)\|_{\mathcal{H}^\sigma(D)}+\mathcal{M})^m+1\right]. \tag{4.24}$$

Claim 3.

$$\int_0^{T_0} \left\|\left(\mathcal{S}(t-z)-\mathcal{S}(T_0-z)\right)G(u(z))\right\|_{\mathcal{H}^\sigma(D)}\mathrm{d}z \leq \frac{hL_GT_0}{(1-\alpha)^2}\left(\|u\|^m_{(C[0,T_0],\mathcal{H}^\sigma(D))}+1\right). \tag{4.25}$$

Claim 4.

$$\int_{T_0}^t \left\|\mathcal{S}(t-z)\int_0^z \varphi(z-s,v(s))\mathrm{d}s\right\|_{\mathcal{H}^\sigma(D)}\mathrm{d}z \leq \frac{hT_0L_\varphi}{(1-\alpha)^2}\left[(\|u(T_0)\|_{\mathcal{H}^\sigma(D)}+\mathcal{M})^n+1\right]. \tag{4.26}$$

Claim 5.

$$\begin{aligned}\int_0^{T_0} &\left\|\left(\mathcal{S}(t-z)-\mathcal{S}(T_0-z)\right)\int_0^z \varphi(z-s,u(s))\mathrm{d}s\right\|_{\mathcal{H}^\sigma(D)}\mathrm{d}z\\ &\leq \frac{hT_0L_\varphi T_0}{(1-\alpha)^2}\left(\|u\|^n_{(C[0,T_0],\mathcal{H}^\sigma(D))}+1\right).\end{aligned} \tag{4.27}$$

Combining the above claims, we can choose some $h>0$ and $\mathcal{M}>0$ such that the right hand side of the above estimate is less than $\mathcal{M}$, so $\mathscr{J}v$ is in $\mathbb{B}$.

Next, take $v,w\in\mathbb{B}$. Then

$$\begin{aligned}&\|\mathscr{J}v(t)-\mathscr{J}w(t)\|_{\mathcal{H}^\sigma(D)}\\ &\quad\leq \int_{T_0}^t \left\|\mathcal{S}(t-z)\left(G(v(z))-G(w(z))\right)\right\|_{\mathcal{H}^\sigma(D)}\mathrm{d}z\\ &\qquad+\int_{T_0}^t \left\|\mathcal{S}(t-z)\int_0^z\left(\varphi(z-s,v(s))-\varphi(z-s,w(s))\right)\mathrm{d}s\right.\\ &\quad\leq \frac{L_G}{(1-\alpha)^2}\int_{T_0}^t\left(1+\|v\|^{p-1}_{\mathcal{H}^\sigma(D)}+\|w\|^{p-1}_{\mathcal{H}^\sigma(D)}\right)\|v-w\|_{\mathcal{H}^\sigma(D)}\mathrm{d}z.\\ &\qquad+\frac{(L_\varphi T_0}{(1-\alpha)^2}\int_{T_0}^t\left(1+\|v\|^{p-1}_{\mathcal{H}^\sigma(D)}+\|w\|^{p-1}_{\mathcal{H}^\sigma(D)}\right)\|v-w\|_{\mathcal{H}^\sigma(D)}\mathrm{d}z.\end{aligned} \tag{4.28}$$

Since $v,w\in\mathbb{B}$, for $t\in[T_0,T_0+h]$ we have

$$\|v\|^{p-1}_{\mathcal{H}^\sigma(D)}+\|w\|^{p-1}_{\mathcal{H}^\sigma(D)}\leq 2\left(\|u(T_0)\|_{\mathcal{H}^\sigma(D)}+\mathcal{M}\right)^{p-1}. \tag{4.29}$$

Thus

$$\|\mathscr{J}v(t)-\mathscr{J}w(t)\|_{\mathcal{H}^\sigma(D)}$$

$$
\begin{aligned}
&\leq \frac{hL_G}{(1-\alpha)^2}\left[1+2\left(\|u(T_0)\|_{\mathcal{H}^\sigma(D)}+\mathcal{M}\right)^{p-1}\right]\|v-w\|_{C([0,T_0+h],\mathcal{H}^\sigma(D))} \\
&\leq \frac{hL_\varphi T_0}{(1-\alpha)^2}\left[1+2\left(\|u(T_0)\|_{\mathcal{H}^\sigma(D)}+\mathcal{M}\right)^{p-1}\right]\|v-w\|_{C([0,T_0+h],\mathcal{H}^\sigma(D))}. \qquad (4.30)
\end{aligned}
$$

We can find a $C_h < 1$ (depending on h) such that

$$
\|\mathscr{J}v(t)-\mathscr{J}w(t)\|_{C([0,T_0+h],\mathcal{H}^\sigma(D))} \leq C_h\|v-w\|_{C([0,T_0+h],\mathcal{H}^\sigma(D))}. \qquad (4.31)
$$

Then $\mathscr{J}$ is a contraction on $\mathbb{B}$. Now apply the Banach Fixed Point Theorem to get the desired result. □

Theorem 4.4. *Assume that G,φ satisfy (4.1) and (4.2). Let u be the mild solution of problem (1.1) defined on $[0,T_{\max})$, where $T_{\max}$ is the maximal time of existence of u. Then we have*

$$
T_{\max}=\infty \quad \text{or} \quad \lim_{t\to T_{\max}^-}\|u(\cdot,t)\|_{\mathcal{H}^\sigma(D)}=\infty.
$$

Proof. Suppose that $T_{\max}<\infty$ and there exists a constant $\Im>0$ such that

$$
\max\left\{\|u_0\|_{\mathcal{H}^{\sigma-2}(D)},\sup_{0\leq t<T_{\max}}\|u(t)\|_{\mathcal{H}^\sigma(D)}\right\}\leq \Im.
$$

Consider a sequence of positive numbers $\{t_k\}_{k\in\mathbb{N}}\subset[0,T_{\max})$ such that $t_k \xrightarrow{k\to\infty} T_{\max}^-$ and the sequence $\{u(t_k)\}$ is a subset of $\mathcal{H}^\sigma(D)$. We show that $\{u(t_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{H}^\sigma(D)$. Let $t_m,t_n\in\mathcal{H}^\sigma(D)$, and without loss of generality suppose that $t_m<t_n$. Then, we have

$$
\begin{aligned}
u(t_n)-u(t_m) &= [\mathcal{S}_0(t_n)-\mathcal{S}_0(t_m)]u_0+\int_0^{t_m}\left[\mathcal{S}(t_n-z)-\mathcal{S}(t_m-z)\right]G(u(z))\mathrm{d}z \\
&\quad+\int_{t_m}^{t_n}\mathcal{S}(t_n-z)G(u(z))\mathrm{d}z+\int_{t_m}^{t_n}\mathcal{S}(t_n-z)\int_0^z\varphi(z-s,u(s))\mathrm{d}s\mathrm{d}z \\
&\quad+\int_0^{t_m}\left[\mathcal{S}(t_n-z)-\mathcal{S}(t_m-z)\right]\int_0^z\varphi(z-s,u(s))\mathrm{d}s\mathrm{d}z \\
&=: \mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3+\mathcal{P}_4+\mathcal{P}_5. \qquad (4.32)
\end{aligned}
$$

Using the inequality $|e^{-a}-e^{-b}|\leq|a-b|$, we have

$$
\begin{aligned}
\|\mathcal{P}_1\|_{\mathcal{H}^\sigma(D)} &\leq \sqrt{\frac{\alpha^2(t_n-t_m)^2}{(1-\alpha)^4}\sum_{j=1}^\infty\lambda_j^{2\sigma-4}\left\langle u_0,\psi_j\right\rangle_{L^2(D)}^2} \\
&\leq \frac{\alpha|t_n-t_m|}{(1-\alpha)^2}\|u_0\|_{\mathcal{H}^{\sigma-2}(D)}\leq\frac{\Im\alpha}{(1-\alpha)^2}|t_n-t_m|. \qquad (4.33)
\end{aligned}
$$

Also

$$
\begin{aligned}
\|\mathcal{P}_2\|^2_{\mathcal{H}^\sigma(D)} &\leq \int_0^{t_m}\sum_{j=1}^\infty\frac{\lambda_j^{2\sigma}\left\langle G(u(z)),\psi_j\right\rangle_{L^2(D)}^2}{(1+\lambda_j(1-\alpha))^4}\left[\frac{\alpha\lambda_j(t_n-z)}{1+\lambda_j(1-\alpha)}-\frac{\alpha\lambda_j(t_m-z)}{1+\lambda_j(1-\alpha)}\right]^2dz \\
&\leq \frac{\alpha^2}{(1-\alpha)^4}(t_n-t_m)^2\int_0^{t_m}\left\|G(u(z))\right\|^2_{\mathcal{H}^\nu(D)}dz
\end{aligned}
$$

$$\leq \frac{L_G^2 \alpha^2 |t_n - t_m|^2}{(1-\alpha)^6 \left(\Im^p + 1\right)^{-2}} T_{\max}. \tag{4.34}$$

From the same method as in the proof of Lemma 4.1 we have

$$\|\mathcal{P}_3\|_{\mathcal{H}^\sigma(D)} \leq \frac{L_G}{(1-\alpha)^2} \int_{t_m}^{t_n} \left(\|u(z)\|^p_{\mathcal{H}^\sigma(D)} + 1 \right) \mathrm{d}z \leq \frac{L_G \left(\Im^p + 1\right)}{(1-\alpha)^2} |t_n - t_m|. \tag{4.35}$$

Next, we have

$$\begin{aligned} \|\mathcal{P}_4\|_{\mathcal{H}^\sigma(D)} &\leq \frac{L_\varphi}{(1-\alpha)^2} \int_{t_m}^{t_n} \int_0^z \left(\|u(s)\|^p_{\mathcal{H}^\sigma(D)} + 1 \right) \mathrm{d}s\mathrm{d}z \\ &\leq \frac{L_\varphi \left(\Im^p + 1\right)}{(1-\alpha)^2} T_{\max} |t_n - t_m|. \end{aligned} \tag{4.36}$$

Finally note

$$\begin{aligned} \|\mathcal{P}_5\|_{\mathcal{H}^\sigma(D)} &\leq \frac{\alpha |t_n - t_m|}{(1-\alpha)^2} \sqrt{\int_0^{t_m} \int_0^z \left(\|u(z)\|^p_{\mathcal{H}^\sigma(D)} + 1 \right)^2 \mathrm{d}s\mathrm{d}z} \\ &\leq \frac{L_\varphi \alpha |t_n - t_m|}{(1-\alpha)^2} \left(\Im^p + 1\right) T_{\max}. \end{aligned} \tag{4.37}$$

From the above estimates we have

$$\|u(t_n) - u(t_m)\|_{\mathcal{H}^\sigma(D)} \xrightarrow{m,n \to \infty} 0. \tag{4.38}$$

Hence, $\{u(t_k)\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}^\sigma(D)$. We deduce that $\{u(t_k)\}_{k \in \mathbb{N}}$ has a limit $u_{T_{\max}}$ in $\mathcal{H}^\sigma(D)$. Because $\{t_k\}_{k \in \mathbb{N}}$ is arbitrary, we have the following limit

$$\lim_{t \to T_{\max}^-} \|u\|_{\mathcal{H}^\sigma(D)} = \|u_{T_{\max}}\|_{\mathcal{H}^\sigma(D)}. \tag{4.39}$$

Therefore, we can define u over $[0, T_{\max}]$. By applying Theorem 4.3, the mild solution of problem (1.1) on $[0, T_{\max}]$ can be extended to some larger interval and this leads to a contradiction with the definition of $T_{\max}$. The result follows. □

5. Numerical example

In this section, we consider some numerical examples to illustrate two properties of the mild solution in the Fourier form (4), namely the regularity and continuation of the solution u based on the time variable.

Choose $T = 1$ and the domain $D = [0, \pi]$. We focus on the equation

$${}_{CF}\mathscr{D}_t^\alpha u - \Delta u = G(t, x, u) + \int_0^t \varphi(t - s, u(s)) \mathrm{d}s, \quad (t, x) \in (0, 1) \times (0, \pi), \tag{5.1}$$

with the intial condition

$$u(0, x) = u_0(x), \ x \in (0, \pi). \tag{5.2}$$

Then we have the orthogonal basis in $L^2(0,\pi)$ is $\psi_j(x) = \sqrt{\frac{2}{\pi}}\sin(jx)$ and the corresponding eigenvalues $\{\lambda_j\} = \{j^2 \mid j = 1, 2, ...\}$. The spatial variable are discretized as follows

$$x_l = \frac{l\pi}{X_l + 1}, \quad \text{for } l = 0, 1, ..., X_l + 1,$$

where $X_l > 0$ is a given integer number which is the number of partitions.

In addition, the scalar product of g and h in $L^2(0,\pi)$ is given by

$$\langle g, h\rangle_{L^2(0,\pi)} = \int_0^\pi gh \,\mathrm{d}x.$$

According to (4), we have the solution as follows

$$\begin{aligned} &u(x,t)\\ =&\sum_{j=1}^{J}\left[\frac{1}{1+j^2(1-\alpha)}\exp\left(\frac{-j^2\alpha t}{1+j^2(1-\alpha)}\right)\int_0^\pi u_0(\xi)\psi_j(\xi)\mathrm{d}\xi\right.\\ &+\int_0^t \frac{\alpha}{\left[1+j^2(1-\alpha)\right]^2}\exp\left(\frac{-j^2\alpha(t-z)}{1+j^2(1-\alpha)}\right)\int_0^\pi G(z,\xi)\psi_j(\xi)\mathrm{d}\xi\mathrm{d}z\\ &\left.+\int_0^t \frac{\alpha}{\left[1+j^2(1-\alpha)\right]^2}\exp\left(\frac{-j^2\alpha(t-z)}{1+j^2(1-\alpha)}\right)\int_0^\pi\int_0^z \varphi(z-s,u(s,\xi))\mathrm{d}s\psi_j(\xi)\mathrm{d}\xi\mathrm{d}z\right]\psi_j(\xi), \end{aligned} \tag{5.3}$$

where

$$u_0 = \sqrt{2/\pi}\sin(x), \quad x \in [0,\pi], \tag{5.4}$$

$$G(t,x) = \sqrt{2/\pi}\sin(2x), \quad (t,x) \in [0,1]\times[0,\pi], \tag{5.5}$$

$$\varphi(w,u) = \sqrt{2/\pi}w\sin(3x), \quad (t,x) \in [0,1]\times[0,\pi], \tag{5.6}$$

Next, to consider the regularity of the solution based on the time variable, we focus on the absolute error estimation between the solution at t and the solution at t^* for some values of α as follows

$$AEE_t^{t^*}(\alpha) = \left(\frac{1}{X_k+1}\sum_{k=1}^{X_k+1}\left|u(t^*,x_k) - u(t,x_k)\right|^2\right)^{1/2}, \text{ for } t \text{ and } t^* \text{ in } [0,1]. \tag{5.7}$$

Numerical results are presented in Tables 1–4 and Figures 1–3, *i.e.*, we show the solutions in the 2D and 3D graph for $t \in \{0.2, 0.4, 0.6, 0.8\}$, $\alpha \in \{0.1, 0.5, 0.9\}$, respectively. From the above results, it can be concluded that the solution u at t^* approaches the solution u at t when t^* tends to t. In addition, we also show a 3D graph of the solution u for different values of α in $\{0.1, 0.5, 0.9\}$, see Figure 4.

6. Conclusion

This paper considers the initial value problem for a time-fractional equation with the Caputo - Fabrizio derivative and memory effect on the source term. In our work, the formula of the mild solution is given and based on it, we investigate the global in time results when the source terms are globally Lipschitz and when the source terms are locally Lipschitz, the local in time well-posedness and finite-time blow-up results are obtained.

TABLE 1. The error estimation between the solutions for $\alpha \in \{0.1, 0.5, 0.9\}$ at $t = 0.2$, $t^* \in \{0.21, 0.205, 0.201\}$.

$\{t, t^*\}$	$X_k = 150$			
	$\|t - t^*\|$	$AEE_t^{t^*}(0.1)$	$AEE_t^{t^*}(0.5)$	$AEE_t^{t^*}(0.9)$
{0.2, 0.21}	0.01	0.0012270674377	0.0105582080558	0.0636880980074
{0.2, 0.205}	0.005	0.0006154094608	0.0054654258649	0.0358135333112
{0.2, 0.201}	0.001	0.0001233859136	0.0011269353172	0.0079505003648

TABLE 2. The error estimation between the solutions for $\alpha \in \{0.1, 0.5, 0.9\}$ at $t = 0.4$, $t^* \in \{0.41, 0.405, 0.401\}$.

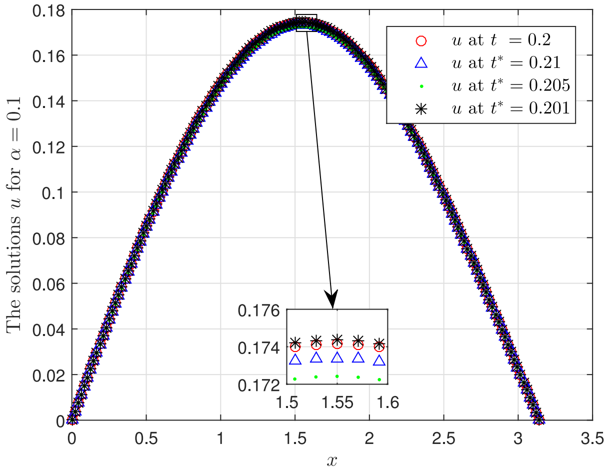
$\{t, t^*\}$	$X_k = 150$			
	$\|t - t^*\|$	$AEE_t^{t^*}(0.1)$	$AEE_t^{t^*}(0.5)$	$AEE_t^{t^*}(0.9)$
{0.4, 0.41}	0.01	0.0011983825229	0.00855479959428	0.0306090673693
{0.4, 0.405}	0.005	0.0006009162962	0.00436512479262	0.0162930949190
{0.4, 0.401}	0.001	0.0001204620857	0.00088821424025	0.0034450187237

TABLE 3. The error estimation between the solutions for $\alpha \in \{0.1, 0.5, 0.9\}$ at $t = 0.6$, $t^* \in \{0.61, 0.605, 0.601\}$.

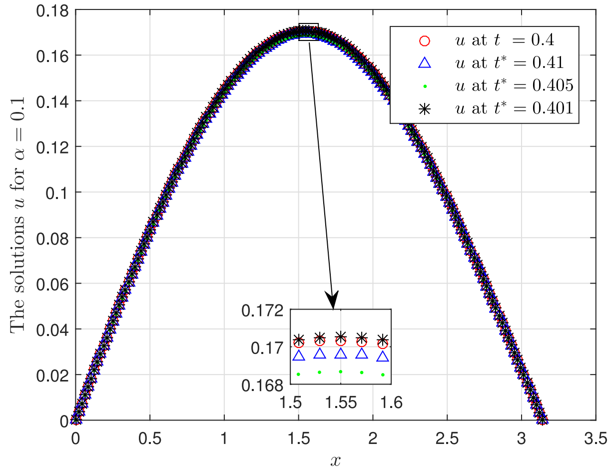
$\{t, t^*\}$	$X_k = 150$			
	$\|t - t^*\|$	$AEE_t^{t^*}(0.1)$	$AEE_t^{t^*}(0.5)$	$AEE_t^{t^*}(0.9)$
{0.6, 0.61}	0.01	0.00117173866119	0.0074400470519	0.0208244528158
{0.6, 0.605}	0.005	0.00058748494398	0.0037778671419	0.0108012709801
{0.6, 0.601}	0.001	0.00011775757958	0.0007652301935	0.0022290448785

TABLE 4. The error estimation between the solutions for $\alpha \in \{0.1, 0.5, 0.9\}$ at $t = 0.8$, $t^* \in \{0.81, 0.805, 0.801\}$.

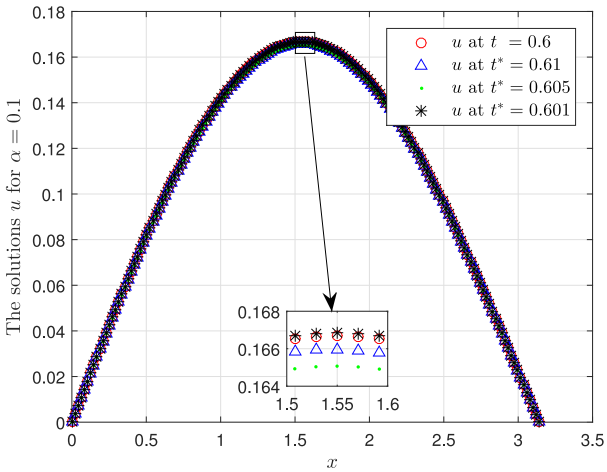
$\{t, t^*\}$	$X_k = 150$			
	$\|t - t^*\|$	$AEE_t^{t^*}(0.1)$	$AEE_t^{t^*}(0.5)$	$AEE_t^{t^*}(0.9)$
{0.8, 0.81}	0.01	0.00114659653167	0.00662422332304	0.0161788710335
{0.8, 0.805}	0.005	0.00057483220274	0.00335797555066	0.0083143933938
{0.8, 0.801}	0.001	0.00011521347457	0.00067913121516	0.0017010871479



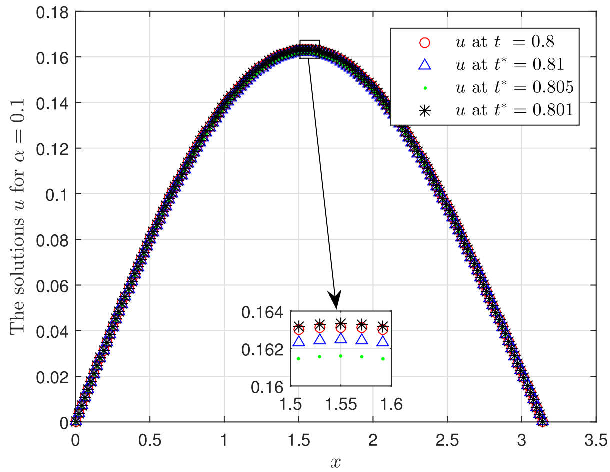
(a) The solution u at $t = 0.2$, $\alpha = 0.1$



(b) The solution u at $t = 0.4$, $\alpha = 0.1$

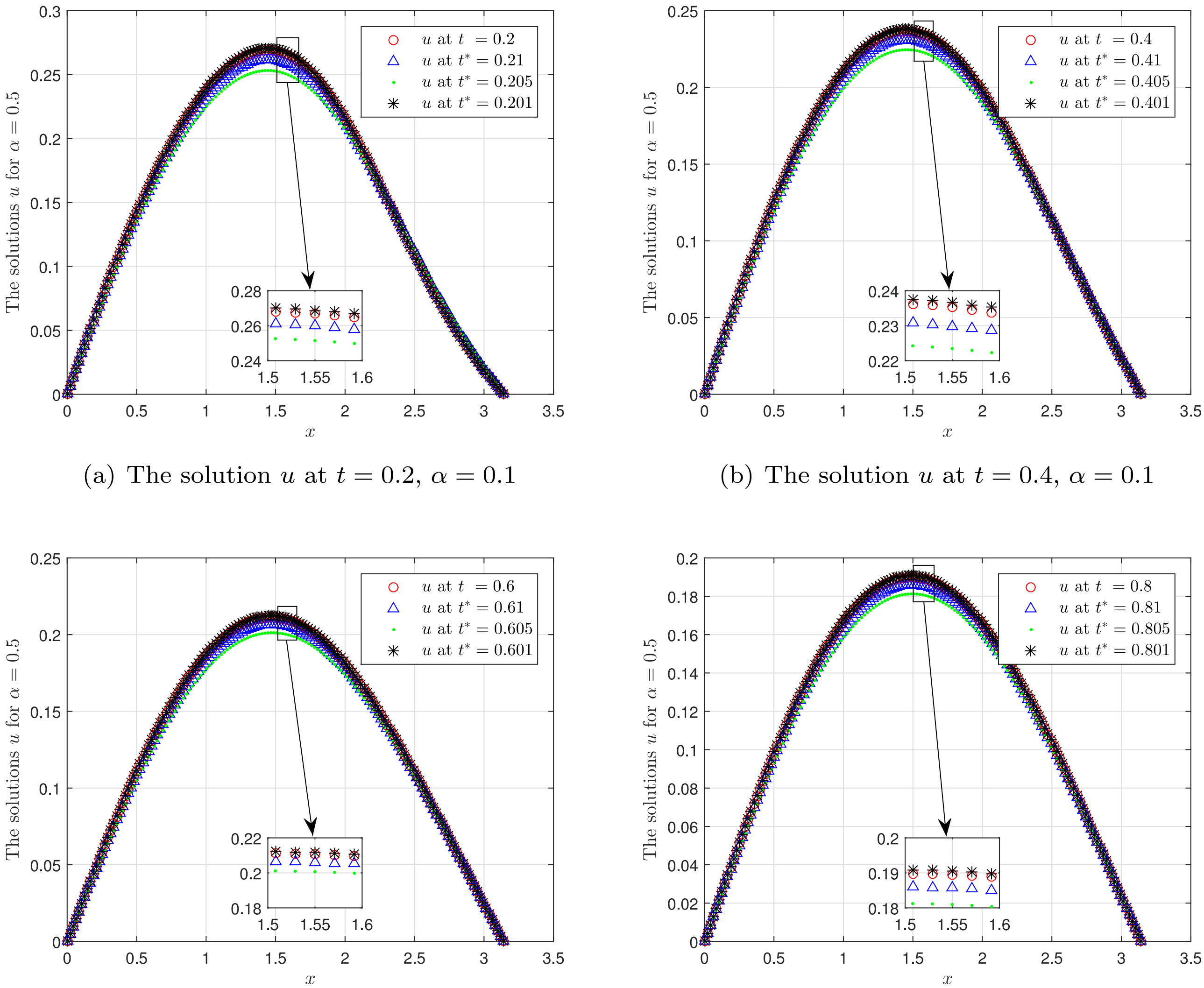


(c) The solution u at $t = 0.6$, $\alpha = 0.1$



(d) The solution u at $t = 0.8$, $\alpha = 0.1$

FIGURE 1. A comparison between the solutions u_α and $u_{\alpha'}$ for $\alpha = 0.3$, $\alpha^* \in \{0.31, 0.305, 0.301\}$.



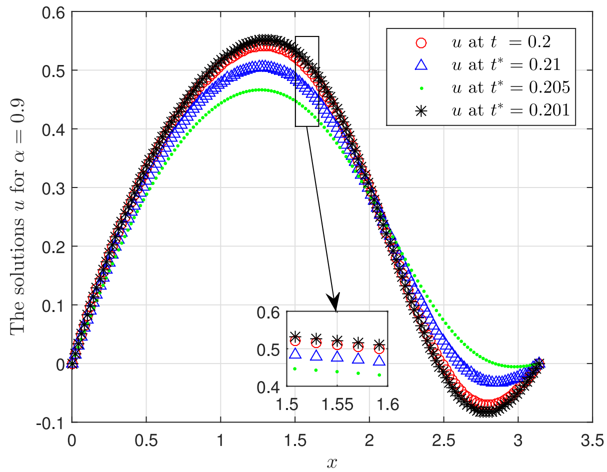
(a) The solution u at $t = 0.2$, $\alpha = 0.1$

(b) The solution u at $t = 0.4$, $\alpha = 0.1$

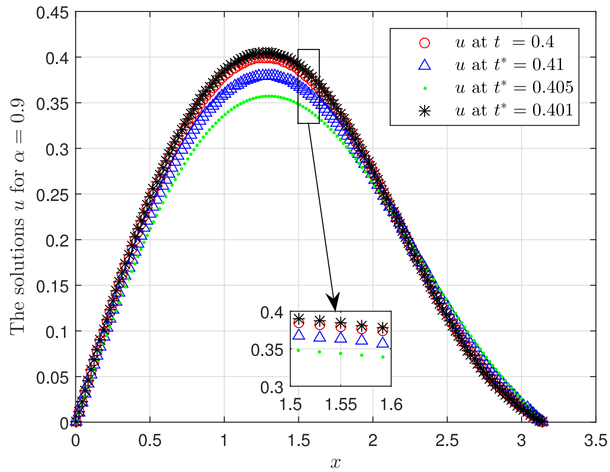
(c) The solution u at $t = 0.6$, $\alpha = 0.1$

(d) The solution u at $t = 0.8$, $\alpha = 0.1$

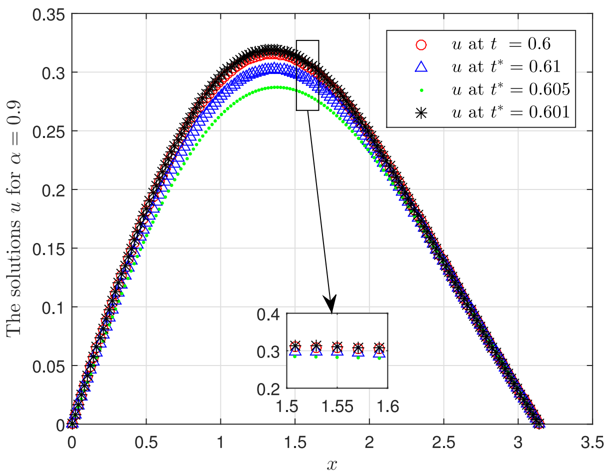
FIGURE 2. A comparison between the solutions u_α and $u_{\alpha'}$ for $\alpha = 0.3$, $\alpha^* \in \{0.31, 0.305, 0.301\}$.



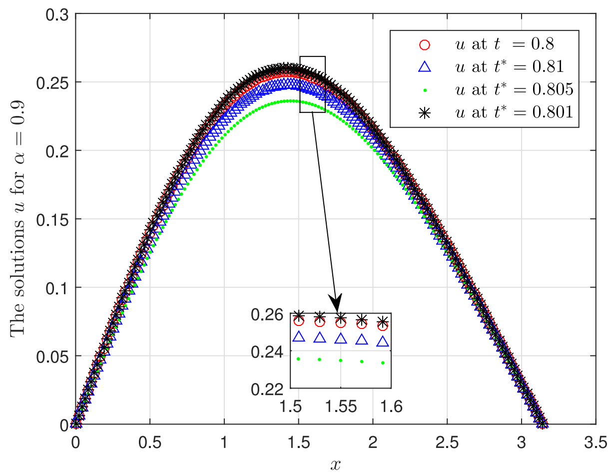
(a) The solution u at $t = 0.2$, $\alpha = 0.1$



(b) The solution u at $t = 0.4$, $\alpha = 0.1$

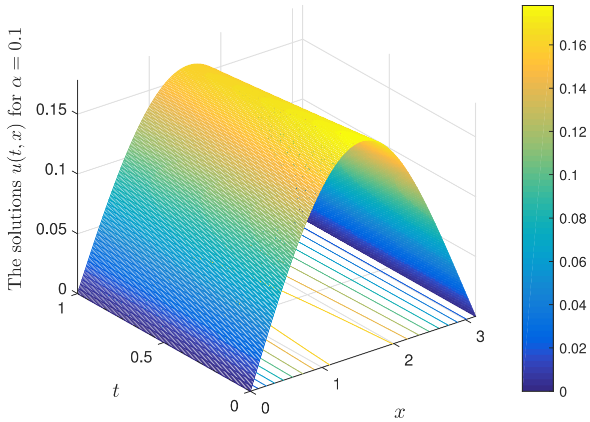


(c) The solution u at $t = 0.6$, $\alpha = 0.1$

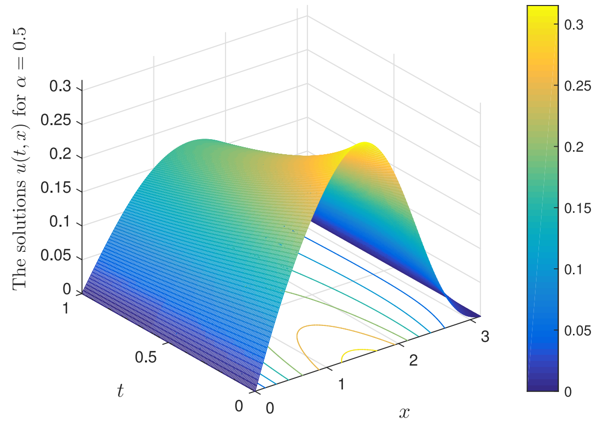


(d) The solution u at $t = 0.8$, $\alpha = 0.1$

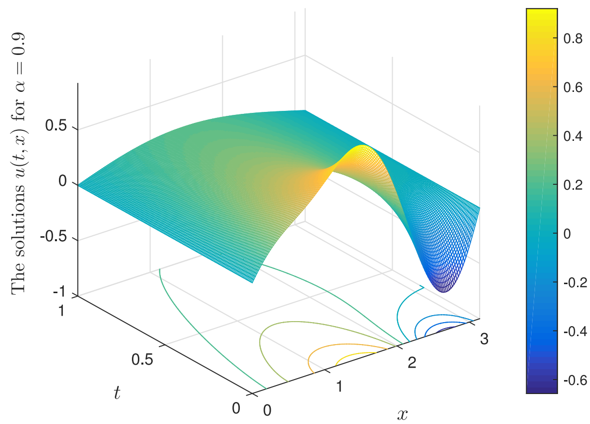
FIGURE 3. A comparison between the solutions u_α and $u_{\alpha'}$ for $\alpha = 0.3$, $\alpha^* \in \{0.31, 0.305, 0.301\}$.



(a) 3D graph of u for $\alpha = 0.1$



(b) 3D graph of u for $\alpha = 0.5$



(c) 3D graph of u for $\alpha = 0.9$

FIGURE 4. The 3D graph of the solutions u on $(t, x) \in (0, 1) \times (0, \pi)$ for $\alpha \in \{0.1, 0.5, 0.9\}$.

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