CONTROLLING OF LONGWAVE OSCILLATORY MARANGONI PATTERNS ON A RHOMBIC LATTICE^{*,**}

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Abstract. We apply nonlinear feedback control to govern the stability of long-wave oscillatory Marangoni patterns. We focus on the patterns caused by instability in thin liquid film heated from below with a deformable free surface. This instability emerges in the case of substrate of low thermal conductivity, when two monotonic long-wave instabilities, Pearson's and deformational ones, are coupled. We provide weakly nonlinear analysis within the amplitude equations, which govern the evolution of the layer thickness and the temperature deviation. The action of the nonlinear feedback control on the nonlinear interaction of two standing waves is investigated. It is shown that quadratic feedback control can produce additional stable structures (standing rolls, standing squares and standing rectangles), which are subject to instability leading to traveling wave in the uncontrolled case.

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1. INTRODUCTION

The onset and development of oscillatory Marangoni convection in a thin film heated from below without control was recently investigated by Shklyaev *et al.* [5]. Among all the variety of possible patterns, only a few were stable: one-dimensional traveling waves, traveling rectangles and alternating rolls. In this paper we aim at stabilizing other kinds of patterns, *e.g.*, standing rolls and standing squares.

We have recently considered the influence of the feedback control on the oscillatory Marangoni instability in a thin film heated from below. We have shown that a linear control gain can delay the onset of instability [2] and a quadratic control gain can eliminate the subcritical excitation of instability [3]. The analysis of pattern formation was done for an infinite region, nonlinear interaction of the traveling waves was considered. In the case of traveling waves we showed that quadratic feedback control can produce additional stable structures, besides conventional traveling rolls. However, in a realistic system the reflection of waves on the lateral boundaries results in emergence of standing waves, which can interact to each other. Extending our previous investigation, we examine here the effect of nonlinear feedback control on development of Marangoni instability in a system of standing waves propagating with a definite angle between the wave vectors.

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The paper is organized as follows. We start with the mathematical formulation of the long-wave Marangoni convection problem in Section 2. There we present a set of coupled amplitude equations which governs the evolution of the layer thickness and the temperature deviation under nonlinear feedback control [3]. In Section 3 we perform the weakly nonlinear stability analysis of wave patterns within these amplitude equations. Nonlinear interaction of standing waves is investigated by means of the analysis of a system of four complex Landau equations. The paper concludes with summary in Section 4.

2. AMPLITUDE EQUATIONS

We consider a horizontal liquid layer confined between a deformable free upper surface and a solid bottom wall. The layer is heated from below; the thermal conductivity of the liquid λ is assumed to be large in comparison with that of the substrate, so that the vertical component of the heat flux λA is fixed. The unperturbed layer thickness H is assumed sufficiently small, so that the influence of buoyancy is negligible and the free surface deformation is important. The surface tension decreases linearly with the temperature: $\sigma = \sigma_0 - \sigma_T T$, where T is the deviation of the temperature from a reference one, which is the temperature of the gas above the liquid layer. The heat flux from the free surface is governed by Newton's law of cooling, which describes the rate of heat transfer from the liquid to the ambient gas phase with the heat transfer coefficient q . The Cartesian reference frame is chosen in such a way that the x - and y -axes are in the substrate plane and the z -axis is normal to the substrate.

The problem of convective instability in the given system is characterized by the following dimensionless parameters,

$$Ca = \frac{\sigma_0 H}{\rho \nu \chi}, \quad Bi = \frac{q H}{\lambda}, \quad Ga = \frac{g H^3}{\nu \chi}, \quad Ma = \frac{\sigma_T A H^2}{\rho \nu \chi},$$

which are the capillary, Biot, Galileo and Marangoni numbers, respectively. Here g is the gravitational acceleration, χ is the thermal diffusivity, ρ is the density, and ν is the kinematic viscosity.

In the uncontrolled case, the oscillatory long-wave Marangoni instability was revealed in [5]. To govern this instability we apply the feedback control based on the measurement of the temperature deviation on the free surface from its value in the conductive state. This feedback control strategy was recently demonstrated as the most effective one to delay the onset of instability under consideration [2]. The heat flux applied on the solid substrate is changed as

$$\left. \frac{\partial T}{\partial z} \right|_{z=0} = -1 - K(f)f, \quad f = T|_{z=h} - T^{(0)}|_{z=1}, \quad (2.1)$$

where $T^{(0)}$ is the temperature of no-motion state, h is the local layer thickness, K is the non-dimensional scalar control gain.

Within the lubrication approximation we employ a standard long-wave scaling

$$x = \varepsilon^{-1} X, \quad y = \varepsilon^{-1} Y, \quad t = \varepsilon^{-2} \tau \quad (2.2)$$

and restrict ourselves to following assumptions

$$Ca = \varepsilon^{-2} C, \quad Bi = \varepsilon^2 \beta, \quad K = \varepsilon^2 \kappa, \quad (2.3)$$

where $\varepsilon \ll 1$ can be thought of as the ratio of H to a typical horizontal lengthscale.

The long-wave Marangoni convection in this layer is governed by the following system of dimensionless

amplitude equations [3]

$$\frac{\partial h}{\partial \tau} = \nabla \cdot \left(\frac{h^3}{3} \nabla P + Ma \frac{h^2}{2} \nabla f \right) \equiv \nabla \cdot \mathbf{j} , \quad (2.4)$$

$$h \frac{\partial \Theta}{\partial \tau} = \nabla \cdot (h \nabla \Theta) - \frac{1}{2} (\nabla h)^2 - [\beta - \varkappa(f)] f + \mathbf{j} \cdot \nabla f + \nabla \cdot \left(\frac{h^4}{8} \nabla P + \frac{h^3}{6} Ma \nabla f \right) , \quad (2.5)$$

where $\Theta(X, Y, \tau)$ is the temperature deviation from its conductive value

$$T = -z + \frac{1}{Bi} + \Theta . \quad (2.6)$$

Here $P = Gah - C\nabla^2 h$, $f = \Theta - h$ has a meaning of perturbation of the free surface temperature; $\nabla = (\partial/\partial X, \partial/\partial Y, 0)$. The vector $-\mathbf{j}$ has a meaning of the longitudinal flux of a liquid integrated across the layer.

Hereinafter we assume that the term corresponding to the feedback control in (2.5) is a quadratic polynomial of the free surface temperature perturbation:

$$\varkappa(f)f = \varkappa_l f + \varkappa_q f^2 , \quad (2.7)$$

where $\varkappa_l$ and $\varkappa_q$ are constant.

The influence of the linear part of control gain $\varkappa_l$ can be expressed as replacement $\beta \rightarrow \beta - \varkappa_l$ in formulas describing the instability threshold [2]. The quadratic part of control gain $\varkappa_q$ affects the nonlinear development of instability. In the following sections we investigate the influence of a nonlinear feedback control on the pattern formation (the linear part $\varkappa_l$ will be omitted). Specifically, we are interested in the elimination of subcritical instability.

3. WEAKLY NONLINEAR ANALYSIS

Below we study the nonlinear dynamics of small perturbations close to the threshold of the oscillatory instability Ma_0

$$Ma - Ma_0 = \delta^2 Ma_2, \delta \ll 1 , \quad (3.1)$$

where $Ma_0 = 3 + Ga + Ck^2 + 3\beta/k^2$ is obtained from the linear analysis [2].

3.1. Basic expansions

We present h , Θ and the time derivative as a series in power of the small parameter δ :

$$h = 1 + \delta \xi_1 + \delta^2 \xi_2 + \dots , \Theta = 1 + \delta \theta_1 + \delta^2 \theta_2 + \dots , \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau_0} + \delta^2 \frac{\partial}{\partial \tau_2} + \dots , \quad (3.2)$$

where two time variables, τ_0 and τ_2 , are introduced according to the multiscale approach [1] as the dynamics of wave patterns is characterized by two different time scales. The frequency of oscillations is of order of 1, while the growth rate of disturbances is of the order of $Ma - Ma_0$, *i.e.* $O(\delta^2)$

Substituting the ansatz (3.2) into equations (2.4)–(2.5), and collecting the terms of equal powers in δ , we

obtain at the first order the linear stability problem. Its solution can be presented as

$$\xi_1 = \sum_{j=1}^n A_j(\tau_2) \exp(i\mathbf{k}_j \cdot \mathbf{r} - i\omega\tau_0) + c.c., \quad \theta_1 = (\alpha + 1) \sum_{j=1}^n A_j(\tau_2) \exp(i\mathbf{k}_j \cdot \mathbf{r} - i\omega\tau_0) + c.c., \quad (3.3)$$

where c.c. denotes complex conjugate terms, $|\mathbf{k}_j| = k$ is the wavenumber, $\alpha = -2(Ga + Ck^2)/3Ma_0 + 2i\omega/Ma_0k^2$. Frequency of neutral perturbations is determined by formula

$$\omega = \frac{k^2}{12} \sqrt{(72 + Ga + Ck^2)(Ma_{mon} - Ma_0)},$$

where

$$Ma_{mon} = \frac{48(\beta + k^2)(Ga + Ck^2)}{k^2(72 + Ga + Ck^2)}$$

is the threshold of a monotonic instability [2].

The analysis can be done for any k , but the case of the critical wavenumber k_c , corresponding to the minimum of the neutral curve, is especially important, because one can expect that patterns with the wavenumber k_c will appear in the natural way by the growth of Ma . Below we consider the nonlinear interaction of disturbances and the wave patterns supported by that interaction. The computations will be done for $k = k_c$, $k_c^2 = \sqrt{3\beta}$.

3.2. Interaction of waves

In order to investigate the nonlinear interaction of waves, consider the class of solutions corresponding to two pairs of waves with the wave vectors $\pm\mathbf{k}_1$, $\pm\mathbf{k}_2$, where $\mathbf{k}_1 = (k, 0)$ and $\mathbf{k}_2 = (k \cos \phi, k \sin \phi)$, that propagate with a phase velocity ω/k and complex amplitudes $A_{1,2}$ and $B_{1,2}$

$$\xi_1 = [A_1(\tau_2)e^{ikX} + A_2(\tau_2)e^{-ikX} + B_1(\tau_2)e^{i\mathbf{k}_2 \cdot \mathbf{r}} + B_2(\tau_2)e^{-i\mathbf{k}_2 \cdot \mathbf{r}}] e^{i\omega\tau_0} + c.c. \quad (3.4)$$

Here ϕ is an arbitrary angle different from 0 and π . That class of solutions includes travelling and standing waves as particular cases.

At the second order we obtain

$$\frac{\partial \xi_2}{\partial \tau_0} - \Delta \left(\frac{1}{3} P_2 + \frac{Ma_0}{2} f_2 \right) = \nabla \cdot (\xi_1 \nabla P_1 + Ma_0 \xi_1 \nabla f_1), \quad (3.5)$$

$$\begin{aligned} \frac{\partial \theta_2}{\partial \tau_0} - \Delta \left(\theta_2 + \frac{1}{8} P_2 + \frac{Ma_0}{6} f_2 \right) + \beta f_2 = & -\xi_1 \frac{\partial \theta_1}{\partial \tau_0} + \nabla \cdot (\xi_1 \nabla \theta_1) - \frac{1}{2} (\nabla \xi_1)^2 + \varkappa_q f_1^2 \\ & + \left(\frac{1}{3} P_1 + \frac{Ma_0}{2} f_1 \right) \cdot \nabla f_1 + \nabla \cdot \left(\frac{\xi_1}{2} \nabla P_1 + \frac{Ma_0}{2} \xi_1 \nabla f_1 \right), \end{aligned} \quad (3.6)$$

where $P_{1,2} = Ga\xi_{1,2} - C\Delta\xi_{1,2}$, $f_{1,2} = \theta_{1,2} - \xi_{1,2}$. The solution can be chosen in the form

$$\begin{aligned} \xi_2 = & a_{10} (A_1 B_2^* + A_2 B_1^*) e^{i\psi_+} + a_{1-0} (A_1 B_1^* + A_2 B_1^*) e^{i\psi_-} \\ & + [a_{11} (A_1 B_1 e^{i\psi_+} + A_2 B_2 e^{-i\psi_+}) + a_{1-1} (A_1 B_2 e^{i\psi_-} + A_2 B_1 e^{-i\psi_-})] \\ & + a_{22} (A_1^2 e^{2ikX} + A_2^2 e^{-2ikX} + B_1^2 e^{2i\mathbf{k}_2 \cdot \mathbf{r}} + B_2^2 e^{-2i\mathbf{k}_2 \cdot \mathbf{r}}) e^{2i\omega\tau_0} \\ & + a_{20} (A_1 A_2^* e^{2ikX} + B_1 B_2^* e^{2i\mathbf{k}_2 \cdot \mathbf{r}}) + c.c. \end{aligned} \quad (3.7)$$

$$\begin{aligned}
\theta_2 = & b_{20} (A_1 A_2^* e^{2ikX} + B_1 B_2^* e^{2i\mathbf{k}_2 \cdot \mathbf{r}}) + b_{02} (A_1 A_2 + B_1 B_2) e^{2i\omega\tau_0} \\
& + b_{10} (A_1 B_2^* + A_2 B_1^*) e^{i\psi_+} + b_{1-0} (A_1 B_1^* + A_2 B_1^*) e^{i\psi_-} \\
& + [b_{11} (A_1 B_1 e^{i\psi_+} + A_2 B_2 e^{-i\psi_+}) + b_{1-1} (A_1 B_2 e^{i\psi_-} + A_2 B_1 e^{-i\psi_-}) \\
& + b_{22} (A_1^2 e^{2ikX} + A_2^2 e^{-2ikX} + B_1^2 e^{2i\mathbf{k}_2 \cdot \mathbf{r}} + B_2^2 e^{-2i\mathbf{k}_2 \cdot \mathbf{r}})] e^{2i\omega\tau_0} \\
& + b_{00} (|A_1|^2 + |A_2|^2 + |B_1|^2 + |B_2|^2) + c.c.,
\end{aligned} \tag{3.8}$$

where $\psi_+ = kX + \mathbf{k}_2 \cdot \mathbf{r}$, $\psi_- = kX - \mathbf{k}_2 \cdot \mathbf{r}$. Hereafter the asterisk denotes the complex-conjugate term; b_{00} , b_{02} , a_{10} , b_{10} , ..., b_{1-1} are constants, which are very cumbersome and therefore they are not given here.

At the third order in δ , we obtain

$$\frac{\partial \xi_3}{\partial \tau_0} - \Delta \left(\frac{1}{3} P_3 + \frac{Ma_0}{2} f_3 \right) = F^{(1)}, \tag{3.9}$$

$$\frac{\partial \theta_3}{\partial \tau_0} - \Delta \left(\theta_3 + \frac{1}{8} P_3 + \frac{Ma_0}{6} f_3 \right) + \beta f_3 = F^{(2)}, \tag{3.10}$$

where $P_3 = Ga\xi_3 - C\Delta\xi_3$, $f_3 = \theta_3 - \xi_3$; inhomogeneities $F^{(1,2)}$ are defined as

$$\begin{aligned}
F^{(1)} = & -\frac{\partial \xi_1}{\partial \tau_2} + \frac{1}{2} Ma_2 \Delta f_1 + \nabla \cdot (Ma_0 \xi_1 \nabla f_2 + \xi_1 \nabla P_2) \\
& + \nabla \cdot \left[\xi_1^2 \left(\nabla P_1 + \frac{Ma_0}{2} \nabla f_1 \right) + \xi_2 (\nabla P_2 + Ma_0 \nabla f_1) \right],
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
F^{(2)} = & -\frac{\partial \theta_1}{\partial \tau_2} - \xi_2 \frac{\partial \theta_1}{\partial \tau_0} - \xi_1 \frac{\partial \theta_2}{\partial \tau_0} + 2\kappa_q f_1 f_2 + \frac{1}{6} Ma_2 \Delta f_1 - \nabla \xi_1 \cdot \nabla \xi_2 + \nabla \cdot (\xi_1 \nabla \theta_2 + \xi_2 \nabla \theta_1) \\
& + \frac{1}{3} \nabla P_2 \cdot \nabla f_1 + \nabla P_1 \cdot \left(\xi_1 \nabla f_1 + \frac{1}{3} \nabla f_2 \right) + \frac{3}{4} \nabla \cdot (\xi_1^2 \nabla P_1) + \frac{1}{2} \nabla \cdot (\xi_1 \nabla P_2 + \xi_2 \nabla P_1) \\
& + Ma_0 \left[\xi_1 \nabla f_1^2 + \nabla f_1 \cdot \nabla f_2 + \frac{1}{2} \nabla \cdot [(\xi_1^2 + \xi_2) \nabla f_1] + \frac{1}{2} \nabla \cdot (\xi_1 \nabla f_2) \right].
\end{aligned} \tag{3.12}$$

The solvability condition at the third order can be formulated as

$$\left(i\omega + \frac{Ma_0 k^2}{6} + k^2 + \beta \right) F_{sec}^{(1)} = \frac{Ma_0 k^2}{2} F_{sec}^{(2)}, \tag{3.13}$$

where $F_{sec}^{(1,2)}$ are secular parts of inhomogeneities. It yields a set of four complex differential equations that govern the evolution of wave amplitudes $A_{1,2}$ and $B_{1,2}$

$$\begin{cases} \frac{dA_1}{d\tau_2} = \left(\gamma - K_0|A_1|^2 - K_1|A_2|^2 - K_2(\phi)|B_1|^2 - K_2(\pi - \phi)|B_2|^2 \right) A_1 - K_4(\phi) A_2^* B_1 B_2 \\ \frac{dA_2}{d\tau_2} = \left(\gamma - K_0|A_2|^2 - K_1|A_1|^2 - K_2(\phi)|B_2|^2 - K_2(\pi - \phi)|B_1|^2 \right) A_2 - K_4(\phi) A_1^* B_1 B_2 \\ \frac{dB_1}{d\tau_2} = \left(\gamma - K_0|B_1|^2 - K_1|B_2|^2 - K_2(\phi)|A_1|^2 - K_2(\pi - \phi)|A_2|^2 \right) B_1 - K_4(\phi) B_2^* A_1 A_2 \\ \frac{dB_2}{d\tau_2} = \left(\gamma - K_0|B_2|^2 - K_1|B_1|^2 - K_2(\phi)|A_2|^2 - K_2(\pi - \phi)|A_1|^2 \right) B_2 - K_4(\phi) B_1^* A_1 A_2 \end{cases} \quad (3.14)$$

Here

$$\gamma = \frac{k^2 M a_2}{2} \left(1 - i \frac{3k^2 (Ga + Ck^2 + 72)}{2\omega_0} \right),$$

expressions for Landau coefficients K_0 , K_1 , $K_2(\phi)$ and $K_4(\phi)$ are presented in the Appendix. Below we use notation $K_3(\phi) = K_2(\pi - \phi)$.

Equations (3.14) have six types of stable solutions in general case $\phi \neq \pi/2$ (which corresponds to the rhombic lattice in the Fourier space):

$$(i) \quad \text{Traveling rolls (TR)} \quad |A_1|^2 = \gamma_r / K_{0r}, \quad A_2 = B_1 = B_2 = 0. \quad (3.15)$$

$$(ii) \quad \text{Standing rolls (SR)} \quad |A_1|^2 = \gamma_r / (K_{0r} + K_{1r}), \quad A_1 = A_2, \quad B_1 = B_2 = 0. \quad (3.16)$$

$$(iiia) \quad \text{Traveling rectangles (TRa1)} \quad |A_1|^2 = \gamma_r / (K_{0r} + K_{2r}), \quad A_1 = B_1, \quad A_2 = B_2 = 0. \quad (3.17)$$

(based on $\mathbf{k}_1$ and $\mathbf{k}_2$)

$$(iiib) \quad \text{Traveling rectagles (TRa2)} \quad |A_1|^2 = \gamma_r / (K_{0r} + K_{3r}), \quad A_1 = B_1, \quad A_2 = B_2 = 0. \quad (3.18)$$

(based on $\mathbf{k}_1$ and $-\mathbf{k}_2$)

$$(iv) \quad \text{Standing rectangles (SRa)} \quad |A_1|^2 = \gamma_r / (K_{0r} + K_{1r} + K_{2r} + K_{3r} + K_{4r}), \quad (3.19)$$

$$A_1 = A_2 = B_1 = B_2.$$

$$(v) \quad \text{Alternating rolls (AR-R)} \quad |A_1|^2 = \gamma_r / (K_{0r} + K_{1r} + K_{2r} + K_{3r} - K_{4r}), \quad (3.20)$$

$$A_1 = A_2 = iB_1 = iB_2.$$

For any parameters, we use notation $K_r = \text{Re}K$, $K_i = \text{Im}K$.

A stability analysis for these patterns shows that they are selected if they emerge through the direct Hopf bifurcation ($\gamma_r > 0$). Conditions for supercritical excitation for each pattern are as follows:

$$K_{super}^{TR} = K_{0r} > 0, \quad (3.21)$$

$$K_{super}^{SR} = K_{0r} + K_{1r} > 0, \quad (3.22)$$

$$K_{super}^{TRa1} = K_{0r} + K_{2r} > 0, \quad (3.23)$$

$$K_{super}^{TRa2} = K_{0r} + K_{3r} > 0, \quad (3.24)$$

$$K_{super}^{SRa} = K_{0r} + K_{1r} + K_{2r} + K_{3r} + K_{4r} > 0, \quad (3.25)$$

$$K_{super}^{AR-R} = K_{0r} + K_{1r} + K_{2r} + K_{3r} - K_{4r} > 0. \quad (3.26)$$

The remain stability conditions for the patterns are listed below:

$$K_{TR} : \text{ (1) } K_{0r} > 0, \text{ (2) } K_{1r} - K_{0r} > 0, \text{ (3) } K_{2r} - K_{0r} > 0, \text{ (4) } K_{3r} - K_{0r} > 0 \quad (3.27)$$

$$K_{SR} : \text{ (1) } K_{0r} + K_{1r} > 0, \text{ (2) } K_{0r} + K_{1r} > 0, \text{ (3) } K_{2r} + K_{3r} - K_{0r} - K_{1r} > 0, \quad (3.28)$$

$$\text{ (4) } |K_0 + K_1 - K_2 - K_3|^2 - |K_4|^2 > 0.$$

$$K_{TRa1} : \text{ (1) } K_{0r} - K_{2r} > 0, \text{ (2) } K_{1r} - K_{0r} - K_{2r} + K_{3r} + K_{4r} > 0, \quad (3.29)$$

$$\text{ (3) } K_{1r} - K_{0r} - K_{2r} + K_{3r} - K_{4r} > 0.$$

$$K_{TRa2} : \text{ (1) } K_{0r} - K_{3r} > 0, \text{ (2) } K_{1r} - K_{0r} + K_{2r} - K_{3r} + K_{4r} > 0, \quad (3.30)$$

$$\text{ (3) } K_{1r} - K_{0r} + K_{2r} - K_{3r} - K_{4r} > 0.$$

$$K_{SRa} : \text{ (1) } K_{0r} + K_{1r} - K_{2r} - K_{3r} - 3K_{4r} > 0, \text{ (2) } K_{0r} - K_{1r} + K_{2r} - K_{3r} - K_{4r} > 0, \quad (3.31)$$

$$\text{ (3) } K_{0r} - K_{1r} - K_{2r} + K_{3r} - K_{4r} > 0, \text{ (4) } |K_4|^2 - [K_4^* (K_0 + K_1 - K_2 - K_3)]_r > 0,$$

$$K_{AR-R} : \text{ (1) } K_{0r} + K_{1r} - K_{2r} - K_{3r} + 3K_{4r} > 0, \text{ (2) } K_{0r} - K_{1r} + K_{2r} - K_{3r} + K_{4r} > 0, \quad (3.32)$$

$$\text{ (3) } K_{0r} - K_{1r} - K_{2r} + K_{3r} + K_{4r} > 0, \text{ (4) } |K_4|^2 + [K_4^* (K_0 + K_1 - K_2 - K_3)]_r > 0$$

In particular case of square symmetry ($\phi = \pi/2$) conditions (3.21)–(3.32) simplifies due to that $K_3(\phi) = K_2(\phi)$. Classification (3.15)–(3.20) can be adopted except for the traveling rectangles 1 and 2, which become traveling squares (TS), and the standing rectangles, which become standing squares (SSq) on the square lattice.

These conditions provide the following boundaries of selection between two stable patterns:

$$\begin{aligned} &\text{between TR and SR } K_{0r} = K_{1r} \\ &\text{between TR and TRa1 (TR and TS for } \phi = \pi/2) K_{0r} = K_{2r} \\ &\text{between TR and TRa2 } K_{0r} = K_{3r} \\ &\text{between TRa1 and SRa } K_{0r} - K_{1r} + K_{2r} - K_{3r} = K_{4r} \\ &\text{between TRa1 and AR-R } K_{0r} - K_{1r} + K_{2r} - K_{3r} = -K_{4r} \\ &\text{between TRa2 and SRa } K_{0r} - K_{1r} + K_{2r} + K_{3r} = K_{4r} \\ &\text{between TRa2 and AR-R } K_{0r} - K_{1r} - K_{2r} + K_{3r} = -K_{4r} \\ &\text{between TS and SSq } K_{0r} - K_{1r} = K_{4r} \\ &\text{between TS and AR } K_{0r} - K_{1r} = -K_{4r} \end{aligned}$$

Below we apply the general results described above to the particular problem, which is the subject of the present paper. Our goal is the computation of coefficients K_{0r} , K_{1r} , K_{2r} , K_{3r} and K_{4r} for different ϕ as functions of the problem parameters, which are β , Ga and $\varkappa_q$. Parameter C can be chosen arbitrary, which corresponds to the definition $\epsilon = \sqrt{C/Ca}$. In the computations below, C is equal to 1000.

3.3. Uncontrolled convection

For uncontrolled convection, pattern selection was investigated previously for $\phi = \pi/2$ and $\phi = \pi/3$ [5]. In [4] maps of the stable patterns in a wide range of angle ϕ was obtained within the domain on $Ga - \beta$ plane, where the oscillatory mode is critical (dashed line in Fig. 1 and in all other figures below on $Ga - \beta$ plane). A small area of stable alternating rolls was discovered on a square lattice, $\phi = \pi/2$ (see Fig. 1a). However, this area intersects with the domain of subcritical traveling rolls, so here depending on the initial condition the system either approaches AR or demonstrates the infinite growth of one of the amplitudes. Note, that the boundary of stability for alternating rolls here is defined by condition $K_{0r} - K_{1r} = -K_{4r}$ corresponding to the boundary between AR and TS. Thus, alternating rolls first become unstable against traveling squares, that in turn become unstable against traveling rolls. For $\phi = \pi/3$ stable traveling rectangles emerge within the domain

FIGURE 1. Pattern selection for uncontrolled convection. $\phi = \pi/2$ (a), $\phi = \pi/3$ (b), $\phi = \pi/8$ (c). Panel (d) shows zoomed-in region of small Ga from the panel (c). Domains of stability for traveling rolls, traveling rectangles and alternating rolls are marked by “TR”, “TR” and “AR”, respectively. The domain of subcriticality for traveling rolls is marked by “subTR”. The domain of bistability of traveling rolls and alternating rolls on a rhombic lattice is marked by “AR-R/TR”.

of small Ga and β (see Fig. 1b). For $\phi = \pi/8$ besides traveling rolls and traveling rectangles, the alternating rolls can emerge on a rhombic lattice at small Ga (see Fig. 1c,d). Note, that domain of stable alternating rolls intersect the domain of stable traveling rolls, providing the bistability.

3.4. Nonlinear feedback control

Quadratic control gain varies Landau coefficients, resulting in a changing of stability boundaries for the patterns and boundaries of domains of subcritical instability. In Figure 2 the variation of Landau coefficients with the control gain $\varkappa_q$ is presented in wide range of angles ϕ for fixed parameters $Ga = 10$ and $\beta = 0.03$. Here we focus on the combinations of Landau coefficients (3.21)–(3.26), which signs determine the type of the Hopf bifurcation for each of six solutions. Note that quadratic control influences the combinations of Landau coefficients differently for different values of angle ϕ . For example, for TRa1 the quadratic control increases combination $K_{0r} + K_{2r}$ (thus it eliminate the subcritical bifurcation) for acute angles ϕ , whereas for obtuse angles ϕ it acts contrary (see Fig. 2c and vice versa for TRa2 in Fig. 2d). For AR-R the quadratic control increases combination $K_{0r} + K_{1r} + K_{2r} + K_{3r} - K_{4r}$ for any ϕ except for the small region of angles around $\phi = \pi/2$ (see Fig. 2f). Eventually, one can eliminate the subcritical excitation for all six patterns by applying the quadratic feedback control with gain $\varkappa \approx -0.2$.

Indeed, one can see from Figure 3 that for fixed $Ga = 10$, $\beta = 0.03$ the subcritical excitation is eliminated for all patterns at any ϕ under control gain $\varkappa_q = -0.18$. Besides that, the quadratic feedback control effects on pattern selection: stable traveling rectangles with $\phi \approx \pi/4$ (see Fig. 3c) and stable standing rectangles with angle ϕ around $\pi/2$ (see Fig. 3d) emerge. However, from comparison Figures 3b and 3d it is clear, that ranges

FIGURE 2. Combinations of Landau coefficients (3.21)–(3.26) depending on the angle ϕ and quadratic control gain κ_q for fixed $Ga = 10$, $\beta = 0.03$. Horizontal shaded plane correspond to zero value of each combination K_{super} .

of stable SRa and stable TR overlap, thus there is the bistability of SRa and TR. Below we investigate the influence of quadratic feedback control on pattern selection for fixed $\phi = \pi/4$ and $\phi = \pi/2$ within entire domain on $Ga - \beta$ plane, where oscillatory longwave mode is critical.

Pattern selection on a rhombic lattice with angle $\phi = \pi/4$ under negative quadratic feedback control $\kappa_q = -0.18$ is presented in Figure 4. In the case of a single traveling wave such control gain eliminates subcritical excitation of instability within entire domain, where oscillatory mode is critical [3]. However, nonlinear interaction of standing waves results in additional three-dimensional patterns, as for example traveling rectangles and standing rectangles. Under control gain $\kappa_q = -0.18$ both of additional patterns emerge through the subcritical excitation within most of the domain, where oscillatory node is critical. However, there are small region of Ga and β , where traveling rolls, traveling rectangles or standing rectangles are stable. Note, that there is also the domain of bistability of traveling rolls and standing rectangles (see Fig. 4b).

Influence of the quadratic feedback control on pattern selection for $\phi = \pi/2$ is presented in Figure 5. Positive control gain reduces the domain of stability for traveling wave, whereas the domain of subcriticality for traveling wave is enlarged. Additional domain of subcriticality arises due to the standing squares. Stable standing squares emerge for $\kappa_q = 0.1$ instead of alternating rolls in the uncontrolled case (see Fig. 1a). However, the domain of

FIGURE 3. Combinations of Landau coefficients depending on the angle ϕ for fixed $Ga = 10$, $\beta = 0.03$ and quadratic control gain $\varkappa_q = -0.18$. Panel (a) shows variation of combinations (3.23), (3.25) and (3.26), such that their positive values correspond to emergence through the direct Hopf bifurcation for TRa1, SRa and AR-R. Panels (b), (c) and (d) shows variation of combinations (3.27), (3.29) and (3.31), such that their positive values correspond to stability of TR, TRa1 and SRa, respectively (lines (1), (2), (3) correspond to the same number of combination in formula).

stable standing squares intersects with the domain of subcritical traveling rolls, so here depending on the initial condition the system either approaches SSq or demonstrates the infinite growth of one of the amplitudes. Note, that the boundary of stability for standing squares here is defined by condition $K_{0r} - K_{1r} = K_{4r}$ corresponding to the boundary between SSq and TS. Thus, standing squares first become unstable against traveling squares, that in turn become unstable against traveling rolls.

For negative control gain traveling rolls are stable within the whole domain, where the oscillatory mode is critical. However, there is a domain of subcriticality for standing squares. Moreover, there are two small areas of stable standing squares, which intersect the domain of stable traveling rolls, resulting in the bistability (see Fig. 5b).

Although, standing squares can be selected for larger negative control gain, see Figure 5c,d. Besides stable traveling rolls within most of the domain, where oscillatory mode is critical, there are also small areas of bistabilities SR/SSq and SSq/TR. Note, that small area of subcritical standing squares exists for a small values of β .

FIGURE 4. Pattern selection on a rhombic lattice with angle $\phi = \pi/4$ under quadratic feedback control with gain $\kappa_q = -0.18$. Panel (b) shows zoomed-in shaded domain from panel (a). Domains of stability for traveling rolls, traveling rectangles and standing rectangles are marked by “TR”, “TRa1” and “SRa”, respectively (corresponding domains of subcriticality are marked by “subTR”, “subTRa1” and “subSRa”). The domain of bistability of traveling rolls and standing rectangles is marked by “SRa/TR”.

FIGURE 5. Pattern selection for $\phi = \pi/2$ for $\kappa_q = 0.1$ (a), for $\kappa_q = -0.1$ (b) and for $\kappa_q = -0.2$ (c,d). Panel (d) shows zoomed-in shaded domain from panel (c). Domains of stability for traveling rolls, standing rolls, standing squares and alternating rolls are marked by “TR”, “SR”, “SSq” and “AR”, respectively. The domains of subcriticality for traveling rolls and standing squares are marked by “subTR” and “subSSq”, respectively. “SSq/TR” and “SR/SSq” mark the domains of bistability of traveling rolls/standing squares and standing rolls/standing squares, respectively.

4. CONCLUSIONS AND DISCUSSION

We have studied pattern formation of oscillatory Marangoni instability in a thin film under nonlinear feedback control.

We have performed a weakly nonlinear analysis within the amplitude equations, which describe coupled evolution of the thickness and temperature of thin film in the presence of the nonlinear control. Our analysis is based on the consideration of the nonlinear interaction of a pair of standing waves propagating at the angle ϕ between the wave vectors. That consideration leads to a set of four complex Landau equations that govern the evolution of wave amplitudes. The coefficients of Landau equations, which define pattern formation, have been calculated in a wide range of angles ϕ for different values of the control gain, Galileo and Biot numbers. We have demonstrated, that besides conventional traveling rolls an additional stable patterns (such as standing rolls, standing squares and standing rectangles) emerges under nonlinear feedback control. In the case of negative control gain, we have shown that a quadratic control can eliminate the subcritical excitation of instability within entire domain, where oscillatory mode is critical.

APPENDIX

The coefficients K_0 , K_1 , K_2 and K_4 of Landau equations (3.14) can be presented as

$$K_j = \frac{K_{j1}}{2I\omega} \left(i\omega + \frac{Ma_0}{6}k^2 + k^2 + \beta \right) - \frac{K_{j2}}{4I\omega} Ma_0 k^2,$$

where $j = 0, 1, 2, 4$. Here

$$K_{01} = Gk^2 + Ma_0 k^2 \alpha - \frac{1}{2} Ma_0 k^2 \alpha^* + Gk^2 A_{22} + 6Ck^4 A_{22} + 2Ma_0 k^2 B_{22} - Ma_0 k^2 \alpha^* A_{22},$$

$$K_{11} = 2Gk^2 + Ma_0 k^2 \alpha^* + Gk^2 A_{20} + 6Ck^4 A_{20} + 2Ma_0 k^2 B_{20} - Ma_0 k^2 \alpha A_{20},$$

$$K_{21} = Gk^2 (2 + A_{1-0} + A_{11}) + Ma_0 k^2 (\alpha A_{1-0} - \alpha^* A_{11} - \alpha^*) \cos \phi \\ + [Ma_0 k^2 (\alpha + B_{11}) + Ck^4 A_{11} (2 \cos \phi + 1)] (1 + \cos \phi) + [Ma_0 k^2 B_{1-0} - Ck^4 A_{1-0} (2 \cos \phi - 1)] (1 - \cos \phi),$$

$$K_{22} = \frac{Gk^2}{2} (3 + A_{11} + A_{1-0}) + 2k^2 (A_{11} + A_{1-0}) \\ + I\omega (A_{1-0} + A_{11} + 2B_{11} + \alpha A_{1-0} - \alpha^* A_{11}) - 2\kappa_q (\alpha B_{1-0} + \alpha^* B_{11} + \alpha B_{00}) \\ + \left[Ma_0 k^2 (2\alpha^2 - \alpha^*) + \frac{k^2}{2} (\alpha A_{1-0} - \alpha^* A_{11}) (Ma_0 + 2) + k^2 (A_{11} - A_{1-0}) \right] \cos \phi \\ + \left[\frac{Ma_0 k^2}{2} (1 - 2\alpha^*) (2\alpha + B_{11}) - Gk^2 \alpha^* - \frac{Gk^2}{3} (B_{11} + \alpha^* A_{11}) + k^2 B_{11} \right] (1 + \cos \phi) \\ + \left[\frac{Ma_0 k^2}{2} B_{1-0} (1 - 2\alpha) - Gk^2 \alpha - \frac{Gk^2}{3} (B_{1-0} + \alpha A_{1-0}) + k^2 B_{1-0} \right] (1 - \cos \phi) \\ + \frac{Ck^4}{6} (3 - 2\alpha^*) [A_{11} (2 \cos \phi + 1) (1 + \cos \phi) - A_{1-0} (2 \cos \phi - 1) (1 - \cos \phi)],$$

$$\begin{aligned}
K_{41} = & Gk^2 (2 + A_{10} + A_{1-0}) + Ma_0 k^2 \alpha^* + Ma_0 k^2 (\alpha A_{1-0} - \alpha^* A_{10}) \cos \phi \\
& + [Ma_0 k^2 B_{10} + Ck^4 A_{10} (2 \cos \phi + 1)] (1 + \cos \phi) \\
& + [Ma_0 k^2 B_{1-0} - Ck^4 A_{1-0} (2 \cos \phi - 1)] (1 - \cos \phi),
\end{aligned}$$

$$\begin{aligned}
K_{42} = & \frac{Gk^2}{2} (3 + A_{10} + A_{1-0} + 4\alpha) + 2k^2 (A_{11} + A_{1-0}) + Ma_0 k^2 (\alpha^* - 2\alpha^2) \\
& + I\omega (A_{1-0} + A_{10} + 2B_{02} + \alpha A_{1-0} + \alpha A_{10}) - 2\kappa_q (\alpha B_{1-0} + \alpha^* B_{02} + \alpha B_{10}) \\
& + \left[\frac{Ma_0 k^2}{2} \alpha + k^2 (\alpha - 1) \right] (A_{1-0} - A_{10}) \cos \phi \\
& + \left[\frac{Ma_0 k^2}{2} (1 - 2\alpha) B_{10} - \frac{Gk^2}{3} (B_{10} + \alpha A_{10}) + k^2 B_{10} \right] (1 + \cos \phi) \\
& + \left[\frac{Ma_0 k^2}{2} (1 - 2\alpha) B_{1-0} - \frac{Gk^2}{3} (B_{1-0} + \alpha A_{1-0}) + k^2 B_{1-0} \right] (1 - \cos \phi) \\
& + \frac{Ck^4}{6} (3 - 2\alpha) [A_{10} (2 \cos \phi + 1) (1 + \cos \phi) - A_{1-0} (2 \cos \phi - 1) (1 - \cos \phi)].
\end{aligned}$$

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