Math. Model. Nat. Phenom. 17 (2022) 24
https://doi.org/10.1051/mmnp/2022028

Mathematical Modelling of Natural Phenomena
www.mmnp-journal.org

NONLOCAL LONGITUDINAL VIBRATION IN A NANOROD, A SYSTEM THEORETIC ANALYSIS

Hanif Heidari[1,**] and Hans Zwart[2,3]

Abstract. Analysis of longitudinal vibration in a nanorod is an important subject in science and engineering due to its vast application in nanotechnology. This paper introduces a port-Hamiltonian formulation for the longitudinal vibrations in a nanorod, which shows that this model is essentially hyperbolic. Furthermore, it investigates the spectral properties of the associated system operator. Standard distributed control and feedback are shown not to be controllable nor stabilizing.

Mathematics Subject Classification. 74H45, 47D03, 74-10, 93C20.

Received December 17, 2021. Accepted June 27, 2022.

1. Introduction

The physics of micro- and nano-scale are fundamentally different from macro-scale. For example, carbon nanotubes have hollow structure [31], low density defect [25], high electrical and thermal conductivity [20], ZnO nanowires have wide band gap of 3.37 eV and an exciton binding energy of 60 meV [22] and boron nitride (BN) nanotubes are light in weight, stable at high temperatures, resistant to oxidation, and have outstanding thermal and electrical conductivity [16]. Nitrogen-doped carbon nanotube sponge can load capacity 100 times larger than its weight [24]. These distinguished properties of nanomaterials have led to its practical usages in NEMS.

According to the number of dimensions less than 100 nm, nanostructures can be classified into two-dimensional (2-D, nanofilm), 1-D (nanorod, nanotube or nanowire), and 0-D (nanoparticle) structures. In particular, 1-D nanostructures have attracted much attention due to the confinement of the other two dimensions perpendicular to the longitudinal direction. Due to the combination of quantum confinement in the nanoscaled dimensions and the bulk properties in another dimension, a host of interesting properties and applications can be expected based on a wide variety of 1-D nanostructures.

Therefore, dynamical analysis and control of nanorods are essential tools for industrial applications. There are many interdisciplinary research works related to vibrations of nanoroads. Hadi *et al.* investigate axial vibration analysis of nonlocal Rayleigh nanorod [4]. Nonlocal longitudinal vibration of viscoelastic coupled double-nanorod

Keywords and phrases: Port-Hamiltonian, Riesz basis, contraction semigroup, nanotubes, controllability, stabilisability.

[1] Department of Applied Mathematics, School of Mathematics and Computer Science, Damghan University, Damghan, P.O. Box 36715-364, Iran.

[2] Department of Applied Mathematics, Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, 7500 AE Enschede, The Netherlands.

[3] Faculty of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands.

** Corresponding author: heidari@du.ac.ir

systems is studied by Karlicic *et al.* [13]. Eren and Aydogdu introduced a PDE model for describing nonlinear free vibration of nanotube with finite strain [7]. Akbas studied the vibration dynamic of cracked nanotube under an axial force [3]. Numanoglu *et al.* studied vibration of nanorod under different boundary conditions using Eringen's nonlocal theory [19]. Wang and Wand analyzed the effect of surface energy on dynamics of vibrating cracked nanorod [30].

The dynamics of many physical systems can be suitably modeled using the port-Hamiltonian (pH) formulation. This formulation gives valuable information on the energy function, the interconnection structure and the passivity and dissipation of the system [12, 28]. The ports model the connection of the system to its environment, and this makes these models very suitable for control design. We refer to the work of *e.g.* [8, 27, 29] for applications on macro-scale models. For more details we refer to [28].

Investigating the dynamics of interconnected nanorods is essential due to nanotechnology rapid development and its wide applications in different areas such as cable-payload system [14] (space elevator [21], drug delivery [24]), nanomedicine [23], NEMS circuits [26], energy harvesting [19] optical [11] and chemical sensors [5]. Since the interconnection of pH systems *via* the ports is again a pH system, pH-models will form a powerful tool for analyzing the dynamics and control of interconnected nanorods.

To the best of our knowledge, in spite of the large amount of research on vibration of nanorod and pH system, there is little research on pH modelling of vibration of nanorods. As for Hamiltonian systems, there can be one than one pH-models. In [10], we introduced two, but they are less suitable for system analysis and control design, therefor we introduce a new pH model. We show how its ports are related to the boundary conditions of the original PDE, and show that its eigenfunctions form a Riesz basis. This latter property implies that simple tests can be used to check system properties like controllability and stabilizability, but is also very desirable when designing and analysing controllers, see *e.g.* [6]. Our controllability and stability analysis complements that of [9].

The paper is organized as follows. In the next section, A short review on nonlocal theory and governing equation are given. The port-Hamiltonian formulation, its semigroup, spectral properties of the port-Hamiltonian model, controllability and feedback stabilization are investigated in next sections. Conclusions and some future works are given in the last section.

2. Model formulation

In this section, we recall the reference [18] to obtain the mathematical modelling of vibration of nanorods.

We consider a nanorod with length ℓ and cross-sectional area A. In our case, the cross-sectional area is constant along the x-coordinate, but in general it could have arbitrary shapes along this x coordinate. We assume the material of the nanorod is elastic and homogeneous. Also, we consider the free longitudinal vibration of nanorod in x-direction. An infinitesimal element of length $\mathrm{d}x$ is taken at a typical coordinate location x. Further, we take that a force N is the resultant of an axial stress σ_{xx} acting internally on A, where σ_{xx} is assumed to be uniform over the cross-section. The stress resultant N may vary along the length, and is also a function of time $N = N(x,t)$. Using our assumptions, we find that

$$N(x,t) = \int_A \sigma_{xx}(x,t)\mathrm{d}A = \sigma_{xx}(x,t)A. \tag{2.1}$$

In addition, an axially distributed force $\widetilde{F}$ is assumed, having dimensions of force per unit length, which results from external sources, either internally or externally applied.

The equilibrium of forces in the x-direction is

$$-N + \left(N + \frac{\partial N}{\partial x}\mathrm{d}x\right) - \tilde{F}\mathrm{d}x = \frac{\partial^2 w}{\partial t^2}\mathrm{d}m, \tag{2.2}$$

where $\mathrm{d}m = \rho A dx$ is the mass of an infinitesimal element and w is the displacement in the x-direction. Substituting $\mathrm{d}m = \rho A dx$ and simplifying (2.2) gives

$$\frac{\partial N}{\partial x} = \tilde{F} + \rho A \frac{\partial^2 w}{\partial t^2}. \tag{2.3}$$

Next we must model the stress-strain relation. Based on nonlocal Eringen's theory [1, 2], it is assumed that the stress at a point is related to the strain at all other points in the domain. The nonlocal constitutive equations for an elastic medium are as follows

$$\sigma_{xx} - \mu \frac{d^2 \sigma_{xx}}{\mathrm{d}x^2} = E\epsilon_{xx}, \tag{2.4}$$

where E is the elastic modulus and $\mu = e_0 a^2$ is the nonlocal parameter (length scales).

By substituting equation (2.1) into equation (2.4), the stress resultant for the nonlocal theory is obtained as

$$N - \mu \frac{d^2 N}{\mathrm{d}x^2} = EA \frac{\partial w}{\partial x}. \tag{2.5}$$

The equation of motion can be expressed in terms of the displacement w for nonlocal constitutive relation. By differentiating both sides of equation (2.5) with respect to x and using equation (2.3), we get the following equation of motion

$$\tilde{F} + \rho A \frac{\partial^2 w}{\partial t^2} - \mu \frac{\partial^2}{\partial x^2}\left(\tilde{F} + \rho A \frac{\partial^2 w}{\partial t^2}\right) = EA \frac{\partial^2 w}{\partial x^2}. \tag{2.6}$$

We consider $\tilde{F} = a^2 w$ as the distributional axial force. For simplicity, without loss of generality, we assume that $A = 1$. Therefore,

$$a^2 w + \rho \frac{\partial^2 w}{\partial t^2} - \mu \frac{\partial^2}{\partial x^2}\left(a^2 w + \rho \frac{\partial^2 w}{\partial t^2}\right) = E \frac{\partial^2 w}{\partial x^2}. \tag{2.7}$$

Next, we analyze equation (2.7) for the development of a port-Hamiltonian model.

3. Port-Hamiltonian formulation

To find a port-Hamiltonian formulation of equation (2.7), we proceed formally first. To simplify notation we write w_{tt} for $\frac{\partial^2 w}{\partial t^2}$ and w_{xx} for $\frac{\partial^2 w}{\partial x^2}$. With this notation and the formal operator $J = 1 - \mu \frac{\mathrm{d}^2}{\mathrm{d}x^2}$, we can write the PDE (2.7) as

$$J(\rho w_{tt}) = -a^2 J(w) + E w_{xx}. \tag{3.1}$$

This equation can equivalently be written as

$$\rho w_{tt} = -a^2 w + E J^{-1}(w_{xx}). \tag{3.2}$$

To design a port-Hamiltonian system, we begin by finding an operator Q such that

$$-Q' Q(w) = J^{-1}(w_{xx}), \tag{3.3}$$

where Q' is the (formal) adjoint of Q. Since the (formal) adjoint of $\frac{\mathrm{d}}{\mathrm{d}x}$ equal $-\frac{\mathrm{d}}{\mathrm{d}x}$, we have that if we choose Q as

$$Q = \frac{\frac{\mathrm{d}}{\mathrm{d}x}}{1-\sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}}, \tag{3.4}$$

then

$$Q' = \frac{-\frac{\mathrm{d}}{\mathrm{d}x}}{1+\sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}}$$

and so

$$Q'Q = \frac{-\frac{\mathrm{d}}{\mathrm{d}x}}{1+\sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}} \frac{\frac{\mathrm{d}}{\mathrm{d}x}}{1-\sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}} = \frac{-\frac{\mathrm{d}^2}{\mathrm{d}x^2}}{1-\mu\frac{\mathrm{d}^2}{\mathrm{d}x^2}} = -J^{-1}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right). \tag{3.5}$$

Using this, our PDE (3.2) becomes

$$\rho w_{tt} = -a^2 w - EQ'Q(w). \tag{3.6}$$

Let us introduce the state

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} w \\ \rho w_t \\ Q(w) \end{pmatrix}, \tag{3.7}$$

then

$$\begin{aligned} \dot{z}(t) = \begin{pmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{pmatrix} &= \begin{pmatrix} w_t \\ \rho w_{tt} \\ (Qw)_t \end{pmatrix} = \begin{pmatrix} w_t \\ -a^2 w - EQ'Q(w) \\ Q(w_t) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -Q' \\ 0 & Q & 0 \end{pmatrix} \begin{pmatrix} a^2 w \\ w_t \\ EQ(w) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -Q' \\ 0 & Q & 0 \end{pmatrix} \begin{pmatrix} a^2 & 0 & 0 \\ 0 & \rho^{-1} & 0 \\ 0 & 0 & E \end{pmatrix} \begin{pmatrix} w \\ \rho w_t \\ Q(w) \end{pmatrix}. \end{aligned} \tag{3.8}$$

The last equation can be rewritten as

$$\dot{z}(t) = P_{1,Q}\mathcal{H}z(t) + P_0\mathcal{H}z(t), \tag{3.9}$$

with

$$P_{1,Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -Q' \\ 0 & Q & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.10}$$

and

$$\mathcal{H} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & \rho^{-1} & 0 \\ 0 & 0 & E \end{pmatrix}. \tag{3.11}$$

We see that this format is very similar to the general port-Hamiltonian format as given in [15]. The difference is that the first derivative in that paper is replaced by Q and Q'. It is easy to see that $P_{1,Q}$ and P_0 are (formally) skew-adjoint, and hence $\int z^T \mathcal{H} z \mathrm{d}x$ is a conserved quantity.

To prove that this is a conserved quantity, we must, like in [15], define the state space and the proper domain of the operators.

4. Contraction semigroup

In this section we show that by imposing proper boundary conditions, the abstract differential equation given in (3.9)–(3.11) generates a (contraction) semigroup on a state space. As the state space X, we choose the Hilbert space $X = L^2((0,\ell);\mathbb{R}^3)$ with inner product

$$\langle f, g\rangle_{\mathcal{H}} = \int_0^\ell g(x)^T \mathcal{H}(x) f(x) \mathrm{d}x, \tag{4.1}$$

with $\mathcal{H}$ ginven by (3.11). By Lemma 7.2.3 of [12] we know that $A_{\mathcal{H}} := P_{1,Q}\mathcal{H} + P_0\mathcal{H}$ generates a contraction semigroup on X if and only if $A := P_{1,Q} + P_0$ generates a contraction semigroup on $L^2((0,\ell);\mathbb{R}^3)$ with the standard inner product. Hence without loss of generality, we may assume that $\mathcal{H}$ is the identity.

Before we formulate and prove the contraction property, we need some more notation. By R and R' we denote the (formal) operators

$$R = \frac{1}{1 - \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}} \text{ and } R' = \frac{1}{1 + \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}}. \tag{4.2}$$

Thus the operator R' is the formal adjoint of R, and

$$Q = \frac{-1}{\sqrt{\mu}} I + \frac{1}{\sqrt{\mu}} R, \qquad Q' = \frac{-1}{\sqrt{\mu}} I + \frac{1}{\sqrt{\mu}} R'. \tag{4.3}$$

We can write

$$P_{1,Q} + P_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \frac{1}{\sqrt{\mu}} I - \frac{1}{\sqrt{\mu}} R' \\ 0 & \frac{-1}{\sqrt{\mu}} I + \frac{1}{\sqrt{\mu}} R & 0 \end{pmatrix}. \tag{4.4}$$

Again formally we see that $R(f) = g$ if and only if $f = R^{-1}(g)$. The last expression is nothing else than the ordinary differential equation

$$g - \sqrt{\mu}\frac{\mathrm{d}g}{\mathrm{d}x} = f. \tag{4.5}$$

The solution of this equation is given by

$$g(x) = c e^{\frac{x}{\sqrt{\mu}}} - \int_0^x e^{\frac{x-\tau}{\sqrt{\mu}}} \frac{1}{\sqrt{\mu}} f(\tau) \mathrm{d}\tau. \tag{4.6}$$

The above expression defines a mapping, when we fix c. This can be done by imposing boundary conditions on g. By $R_c(f)$, we denote the right-hand side of the above equation, *i.e.*,

$$\left(R_c(f)\right)(x) = ce^{\frac{x}{\sqrt{\mu}}} - \int_0^x e^{\frac{x-\tau}{\sqrt{\mu}}} \frac{1}{\sqrt{\mu}} f(\tau)\mathrm{d}\tau. \tag{4.7}$$

Similarly, we define

$$R'_{c_d}(f_d) = c_d e^{\frac{-x}{\sqrt{\mu}}} + \int_0^x e^{\frac{-x+\tau}{\sqrt{\mu}}} \frac{1}{\sqrt{\mu}} f_d(\tau)\mathrm{d}\tau. \tag{4.8}$$

It is not hard to see that for f, f_d smooth

$$\left(I - \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(-I + R_c\right) f = \sqrt{\mu}\frac{\mathrm{d}f}{\mathrm{d}x}, \tag{4.9}$$

$$\left(I + \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(I - R'_{c_d}\right) f_d = \sqrt{\mu}\frac{df_d}{\mathrm{d}x}. \tag{4.10}$$

With the mappings R_c and R'_{c_d} we define the A as, see (4.4),

$$A\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} := \begin{pmatrix} 0 & I & 0 \\ -I & 0 & \frac{1}{\sqrt{\mu}}I - \frac{1}{\sqrt{\mu}}R'_{c_d} \\ 0 & \frac{-1}{\sqrt{\mu}}I + \frac{1}{\sqrt{\mu}}R_c & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{4.11}$$

The following result is almost immediate.

Lemma 4.1. *If c and c_d are bounded linear mappings from $L^2((0,\ell);\mathbb{R}^3)$ to $\mathbb{R}$, then the operator A of (4.11) with $c = c(f_1, f_2, f_3)$, $c_d = c(f_1, f_2, f_3)$ is a bounded linear operator. Hence it is the infinitesimal generator of a group on $L^2((0,\ell);\mathbb{R}^3)$ and thus on X.*

Now we consider the question when this semigroup on X will be a contraction. Therefor we assume that c and c_d are such that

$$W\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = 0, \tag{4.12}$$

where W is a full-rank 2×4 matrix and $f_\partial, e_\partial \in \mathbb{R}^2$ are defined as

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} g_2(\ell) + g_2(0) \\ g_3^d(\ell) + g_3^d(0) \\ g_3^d(\ell) - g_3^d(0) \\ g_2(\ell) - g_2(0) \end{pmatrix}, \tag{4.13}$$

with $g_2 = \frac{1}{\sqrt{\mu}}R_c(f_2)$ and $g_3^d = \frac{1}{\sqrt{\mu}}R'_{c_d}(f_3)$, see equations (4.7) and (4.8), respectively.

In order to link the constants c and c_d to the boundary conditions, we introduce the matrices

$$E_1 = \begin{pmatrix} 1 + e^{\frac{\ell}{\sqrt{\mu}}} & 0 \\ 0 & 1 + e^{\frac{-\ell}{\sqrt{\mu}}} \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & e^{\frac{-\ell}{\sqrt{\mu}}} - 1 \\ e^{\frac{\ell}{\sqrt{\mu}}} - 1 & 0 \end{pmatrix}.$$

Theorem 4.2. *Assume that $W\left(\begin{smallmatrix}E_1\\ E_2\end{smallmatrix}\right)$ is invertible. Then the operator A as defined in (4.11)–(4.13) is a well-defined bounded operator. It generates a contraction semigroup if and only if*

$$W\Sigma W^T \geq 0,$$

where $\Sigma = \left(\begin{smallmatrix}0_2 & I_2\\ I_2 & 0_2\end{smallmatrix}\right)$, with I_2 the 2×2 identity matrix and 0_2 the 2×2 zero matrix. In addition, It generates a unitary group if and only if $W\Sigma W^T = 0$.

Proof. We begin by showing that A is well-defined. Since R_c and R'_{c_d} are defined using the constants c and c_d, respectively, and A is defined *via* boundary values of g_2 and g_3^d, respectively, this is not immediately clear.

Using the definitions of g_2 and g_3^d, we see that

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{\mu}}\begin{pmatrix} ce^{\frac{\ell}{\sqrt{\mu}}}+c \\ c_d e^{\frac{-\ell}{\sqrt{\mu}}}+c_d \\ c_d e^{\frac{-\ell}{\sqrt{\mu}}}-c_d \\ ce^{\frac{\ell}{\sqrt{\mu}}}-c \end{pmatrix} + \begin{pmatrix} -\int_0^\ell e^{\frac{\ell-\tau}{\sqrt{\mu}}}\frac{1}{\mu}f_2(\tau)\mathrm{d}\tau \\ \int_0^\ell e^{\frac{-\ell+\tau}{\sqrt{\mu}}}\frac{1}{\mu}f_3(\tau)\mathrm{d}\tau \\ \int_0^\ell e^{\frac{-\ell+\tau}{\sqrt{\mu}}}\frac{1}{\mu}f_3(\tau)\mathrm{d}\tau \\ -\int_0^\ell e^{\frac{\ell-\tau}{\sqrt{\mu}}}\frac{1}{\mu}f_2(\tau)\mathrm{d}\tau \end{pmatrix}.$$

Now $\left(\begin{smallmatrix}f_\partial\\ e_\partial\end{smallmatrix}\right) \in \ker W$ if and only if

$$W\begin{pmatrix} ce^{\frac{\ell}{\sqrt{\mu}}}+c \\ c_d e^{\frac{-\ell}{\sqrt{\mu}}}+c_d \\ c_d e^{\frac{-\ell}{\sqrt{\mu}}}-c_d \\ ce^{\frac{\ell}{\sqrt{\mu}}}-c \end{pmatrix} = -\sqrt{\mu}W\begin{pmatrix} -\int_0^\ell e^{\frac{\ell-\tau}{\sqrt{\mu}}}\frac{1}{\mu}f_2(\tau)\mathrm{d}\tau \\ \int_0^\ell e^{\frac{-\ell+\tau}{\sqrt{\mu}}}\frac{1}{\mu}f_3(\tau)\mathrm{d}\tau \\ \int_0^\ell e^{\frac{-\ell+\tau}{\sqrt{\mu}}}\frac{1}{\mu}f_3(\tau)\mathrm{d}\tau \\ -\int_0^\ell e^{\frac{\ell-\tau}{\sqrt{\mu}}}\frac{1}{\mu}f_2(\tau)\mathrm{d}\tau \end{pmatrix}. \tag{4.14}$$

We see that this is uniquely solvable for all f_2 and f_3 if and only if

$$W\begin{pmatrix} E_1 \\ E_2 \end{pmatrix}\begin{pmatrix} c \\ c_d \end{pmatrix} = 0$$

implies that $\left(\begin{smallmatrix}c\\ c_d\end{smallmatrix}\right) = 0$. By our assumption, this holds, and so (4.14) uniquely determines c and c_d. Furthermore, we see that c and c_d are linear bounded mappings of f_2 and f_3 and so by Lemma 4.1, it follows that A is a bounded linear operator from X to X. Hence A generates a contraction semigroup if and only if for all $f \in X$ the following inequality holds:

$$\langle f, Af\rangle + \langle Af, f\rangle \leq 0. \tag{4.15}$$

Since A equals

$$A = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & \frac{1}{\sqrt{\mu}}I \\ 0 & \frac{-1}{\sqrt{\mu}}I & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{\mu}}R'_{c_d} \\ 0 & \frac{1}{\sqrt{\mu}}R_c & 0 \end{pmatrix},$$

and since the first operator is skew-adjoint, we find that

$$\langle f,Af\rangle + \langle Af, f\rangle =$$

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{\mu}} R'_{c_d}(f_3) \\ \frac{1}{\sqrt{\mu}} R_c(f_2) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{\mu}} R'_{c_d}(f_3) \\ \frac{1}{\sqrt{\mu}} R_c(f_2) \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \right\rangle.$$

To study the sign of this expression, we use that, see (4.13),

$$g_2 = \frac{1}{\sqrt{\mu}} R_c(f_2), \qquad g_3^d = \frac{1}{\sqrt{\mu}} R'_{c_d}(f_3).$$

With this notation we have that

$$\langle f, Af \rangle + \langle Af, f \rangle = -\langle f_2, g_3^d \rangle + \langle f_3, g_2 \rangle - \langle g_3^d, f_2 \rangle + \langle g_2, f_3 \rangle. \tag{4.16}$$

By construction, we have that

$$\sqrt{\mu} \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{-\frac{x}{\sqrt{\mu}}} g_2(x) \right) = -e^{-\frac{x}{\sqrt{\mu}}} f_2(x) \qquad \sqrt{\mu} \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{\frac{x}{\sqrt{\mu}}} g_3^d(x) \right) = e^{\frac{x}{\sqrt{\mu}}} f_3(x). \tag{4.17}$$

Using this we find

$$\begin{aligned}
\langle g_3^d, -f_2 \rangle =& \langle e^{\frac{x}{\sqrt{\mu}}} g_3^d, -e^{-\frac{x}{\sqrt{\mu}}} f_2 \rangle \\
=& \langle e^{\frac{x}{\sqrt{\mu}}} g_3^d, \sqrt{\mu} \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{-\frac{x}{\sqrt{\mu}}} g_2(x) \right) \rangle \\
=& \sqrt{\mu} \int_0^\ell e^{\frac{x}{\sqrt{\mu}}} g_3^d(x) \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{-\frac{x}{\sqrt{\mu}}} g_2(x) \right) \mathrm{d}x \\
=& \sqrt{\mu} \left[e^{\frac{x}{\sqrt{\mu}}} g_3^d(x) e^{-\frac{x}{\sqrt{\mu}}} g_2(x) \right]_0^\ell - \sqrt{\mu} \int_0^\ell \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{\frac{x}{\sqrt{\mu}}} g_3^d(x) \right) e^{-\frac{x}{\sqrt{\mu}}} g_2(x) \mathrm{d}x \\
=& \sqrt{\mu} \left[g_3^d(x) g_2(x) \right]_0^\ell - \langle e^{\frac{x}{\sqrt{\mu}}} f_3, e^{-\frac{x}{\sqrt{\mu}}} g_2 \rangle \\
=& \sqrt{\mu} \left[g_3^d(x) g_2(x) \right]_0^\ell - \langle f_3, g_2 \rangle.
\end{aligned}$$

Using this in (4.16) have the following

$$\begin{aligned}
\langle f, Af \rangle + \langle Af, f \rangle =& \sqrt{\mu} \left[g_2(x) g_3^d(x) \right]_0^\ell - \langle g_2, f_3 \rangle + \langle f_3, g_2 \rangle \\
&+ \sqrt{\mu} \left[g_3^d(x) g_2(x) \right]_0^\ell - \langle f_3, g_2 \rangle + \langle g_2, f_3 \rangle \\
=& 2\sqrt{\mu} g_2(\ell) g_3^d(\ell) - 2\sqrt{\mu} g_2(0) g_3^d(0).
\end{aligned} \tag{4.18}$$

Applying the definition of f_∂ and e_∂ the last expression equals $\sqrt{\mu} f_\partial^T e_\partial$. By Theorem 7.2.4 of [12] or Theorem 4.1 of [15] we have that $f_\partial^T e_\partial \leq 0$ for all $\left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix} \right) \in \ker W$ if and only if $W \Sigma W^T \geq 0$. This proves the assertion for the contraction semigroup.

Since A is bounded, we have that it generates a unitary group if and only if we have equality in (4.15). Or equivalently, when $f_\partial^T e_\partial = 0$ for all $\left(\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix} \right) \in \ker W$. Again by using Theorem 4.1 of [15] we see that this holds if and only if $W \Sigma W^T = 0$. □

In the above theorem we see that W has to satisfy two conditions. However, in a special case we have that whenever the second one is satisfied, so is the first.

Lemma 4.3. *Let W be a 2×4 matrix satisfying $W\Sigma W^T > 0$, then $W\left(\begin{smallmatrix} E_1 \\ E_2 \end{smallmatrix}\right)$ is invertible. If $W\Sigma W^T$ is not strictly positive, then $W\left(\begin{smallmatrix} E_1 \\ E_2 \end{smallmatrix}\right)$ is invertible if and only if*

$$\det(I - U_0(W_1 - W_2)^{-1}(W_1 + W_2)) \neq 0, \tag{4.19}$$

where $W = (W_1 \; W_2)$ and

$$U_0 = \begin{pmatrix} \frac{-1}{\cosh(\frac{\ell}{\sqrt{\mu}})} & -\tanh(\frac{\ell}{\sqrt{\mu}}) \\ \tanh(\frac{\ell}{\sqrt{\mu}}) & \frac{-1}{\cosh(\frac{\ell}{\sqrt{\mu}})} \end{pmatrix}.$$

Proof. By Lemma 7.3.1 of [12] we see that any W satisfying $W\Sigma W^T \geq 0$ can be written as

$$W = S\begin{pmatrix} I + V & I - V \end{pmatrix},$$

where S is invertible, and $VV^* \leq I$. Furthermore, $W\Sigma W^T > 0$ if and only if $VV^* < I$.

Assume now that $W\left(\begin{smallmatrix} E_1 \\ E_2 \end{smallmatrix}\right)$ is not invertible. Then there exists a non-zero vector $v \in \mathbb{R}^2$ such that

$$(I + V)E_1 v + (I - V)E_2 v = 0,$$

or equivalently

$$(E_1 + E_2)v = V(E_2 - E_1)v. \tag{4.20}$$

A direct calculation gives that

$$\begin{aligned} (E_1 + E_2)(E_2 - E_1)^{-1} &= \frac{-1}{e^{2\frac{\ell}{\sqrt{\mu}}} + 1} \begin{pmatrix} 2e^{\frac{\ell}{\sqrt{\mu}}} & -e^{2\frac{\ell}{\sqrt{\mu}}} + 1 \\ e^{2\frac{\ell}{\sqrt{\mu}}} - 1 & 2e^{\frac{\ell}{\sqrt{\mu}}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-1}{\cosh(\frac{\ell}{\sqrt{\mu}})} & \tanh(\frac{\ell}{\sqrt{\mu}}) \\ -\tanh(\frac{\ell}{\sqrt{\mu}}) & \frac{-1}{\cosh(\frac{\ell}{\sqrt{\mu}})} \end{pmatrix}, \end{aligned}$$

which is an unitary matrix. We see that this unitary matrix equals U_0^*. Defining $\tilde{v} := (E_2 - E_1)v$ we write equation (4.20) as

$$U_0^* \tilde{v} = V\tilde{v}. \tag{4.21}$$

Hence if $VV^* < I$, then there does not exists a non-zero v such that (4.21) holds, which proves the assertion for $W\Sigma W^T > 0$. Since $V = (W_1 - W_2)^{-1}(W_1 + W_2)$ and since U_0 is unitary, we see that there exists a non-zero $\tilde{v}$ such that (4.21) holds if and only if (4.19) holds. □

5. More on the boundary conditions

In (4.11), Lemma 4.1, and Theorem 4.2, we see that we have defined A *via* c and c_d. However, it is not a-priori clear how c and c_d are related to "normal" boundary conditions like

$$\alpha_1 w(0,t) + \alpha_2 w_x(0,t) = 0 \qquad \beta_1 w(\ell,t) + \beta_2 w_x(\ell,t) = 0, \tag{5.1}$$

with $|\alpha_1|+|\alpha_2|\neq 0$ and $|\beta_1|+|\beta_2|\neq 0$. Note that when we are considering A, this is equivalent to the PDE (2.7) with $a=1, E=1, \rho=1$. So we investigate the relation between c, c_d and the boundary conditions (5.1) under these conditions. Looking at (3.8) and (4.11) we see that

$$(f_2)_{tt}=w_{ttt}=-w_t+(I+R'_{c_d})(-I+R'_c)w_t=-f_2+(I-R'_{c_d})(-I+R_c)f_2. \tag{5.2}$$

Using (4.9) and (4.10) we find

$$(I-\mu\frac{d^2}{\mathrm{d}x^2})f_{2tt}=-(I-\mu\frac{d^2}{\mathrm{d}x^2})f_2+\mu f_{2xx}. \tag{5.3}$$

or equivalently

$$(I-\mu\frac{d^2}{\mathrm{d}x^2})(f_{2,tt}+2f_2)=f_2. \tag{5.4}$$

This can equivalently be written as

$$(f_{2,tt}+2f_2)(x)=c_1\sinh(\gamma x)+c_2\cosh(\gamma x)-\gamma\int_0^x\sinh(\gamma(x-\xi))f_2(\xi)\mathrm{d}\xi,$$

with $\gamma=\frac{1}{\sqrt{\mu}}$ and

$$(f_{2,tt}+2f_2)(x)(0)=c_2,\quad \gamma c_1=(f_{2,tt}+2f_2)_x(0). \tag{5.5}$$

Thus

$$f_{2,tt}(x)=-2f_2(x)+c_1\sinh(\gamma x)+c_2\cosh(\gamma x)-\gamma\int_0^x\sinh(\gamma(x-\xi))f_2(\xi)\mathrm{d}\xi. \tag{5.6}$$

So we have an expression for the left-hand side of (5.2). Letting $(I+\sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x})$ operate on (5.2) and using (4.10) we find

$$(I+\sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x})f_{2,tt}=-(I+\sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x})f_2+\sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\left((-I+R_c)f_2\right).$$

Using (5.6) we find

$$\begin{aligned}-2f_2(x)+&\ c_1\sinh(\gamma x)+c_2\cosh(\gamma x)-\gamma\int_0^x\sinh(\gamma(x-\xi))f_2(\xi)\mathrm{d}\xi\\ +&\ -2\sqrt{\mu}f_{2,x}(x)+c_1\cosh(\gamma x)+c_2\sinh(\gamma x)-\gamma\int_0^x\cosh(\gamma(x-\xi))f_2(\xi)\mathrm{d}\xi\\ =&\ -f_2(x)-\sqrt{\mu}f_{2,x}(x)-\sqrt{\mu}f_{2,x}(x)+ce^{\frac{x}{\sqrt{\mu}}}-f_2(x)-\frac{1}{\sqrt{\mu}}\int_0^x e^{\frac{x-\tau}{\sqrt{\mu}}}f_2(\tau)\mathrm{d}\tau.\end{aligned}$$

Since $\sinh(y)+\cosh(y)=e^y$ and $\gamma=\frac{1}{\sqrt{\mu}}$, we can simplify this as

$$(c_1+c_2)e^{\gamma x}=ce^{\frac{x}{\sqrt{\mu}}}.$$

Thus

$$c_1 + c_2 = c. \tag{5.7}$$

Now we repeat the procedure and let $(I - \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x})$ operate on (5.2) and using (4.9) we find

$$(I - \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x})f_{2,tt} = -(I - \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x})f_2 + \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\left((I - R'_{c_d})f_2\right).$$

Using (5.6) we find

$$\begin{aligned}
-2f_2(x)+ & \, c_1\sinh(\gamma x) + \tilde{c}_2\cosh(\gamma x) - \gamma\int_0^x \sinh(\gamma(x-\xi))f_2(\xi)\mathrm{d}\xi \\
& + 2\sqrt{\mu}f_{2,x}(x) - c_1\cosh(\gamma x) - c_2\sinh(\gamma x) + \gamma\int_0^x \cosh(\gamma(x-\xi))f_2(\xi)\mathrm{d}\xi \\
= & \, -f_2(x) + \sqrt{\mu}f_{2,x}(x) + \sqrt{\mu}f_{2,x}(x) + c_d e^{\frac{-x}{\sqrt{\mu}}} - f_2(x) + \frac{1}{\sqrt{\mu}}\int_0^x e^{\frac{-x+\tau}{\sqrt{\mu}}}f_2(\tau)\mathrm{d}\tau
\end{aligned}$$

Since $\cosh(y) - \sinh(y) = e^{-y}$, we can simplify this as

$$(c_2 - c_1)e^{-\gamma x} = c_d e^{\frac{-x}{\sqrt{\mu}}}.$$

Thus

$$c_2 - c_1 = c_d. \tag{5.8}$$

From (5.7) and (5.8) we see that the pairs (c, c_d) and $(\tilde{c}_1, \tilde{c}_2)$ are one-to-one correspondence with each other.

Since $f_2 = w_t$, we find from (5.1) that

$$\alpha_1 f_2(0) + \alpha_2 f_{2,x}(0) = 0 \qquad \beta_1 f_2(\ell) + \beta_2 f_{2,x}(\ell) = 0.$$

and similarly,

$$\alpha_1 f_{2,tt}(0) + \alpha_2 f_{2,tt,x}(0) = 0 \qquad \beta_1 f_{2,tt}(\ell) + \beta_2 f_{2,tt,x}(\ell) = 0.$$

Thus with (5.5)

$$0 = \alpha_1(f_{2,tt}(0) + 2f_2(0)) + \alpha_2(f_{2,tt,x}(0) + f_{2,x}(0)) = \alpha_1 c_2 + \alpha_2\gamma c_1$$

and using (5.6) we derive from

$$0 = \beta_1(f_{2,tt}(\ell) + 2f_2(\ell)) + \beta_2(f_{2,tt,x}(\ell) + f_{2,x}(\ell))$$

that

$$0 = \beta_1\left[c_1\sinh(\gamma\ell) + c_2\cosh(\gamma\ell) - \gamma\int_0^\ell \sinh((\gamma(\ell-\xi))f_2(\xi)\mathrm{d}\xi\right]$$

$$+\beta_2\left[c_1\gamma\cosh(\gamma\ell)+c_2\gamma\sinh(\gamma\ell)-\gamma^2\int_0^\ell\cosh(\gamma(\ell-\xi))f_2(\xi)\mathrm{d}\xi\right].$$

The two equations for the pair (c_1, c_2) can be written as

$$\begin{bmatrix}\alpha_2\gamma & \alpha_1\\ \beta_1\sinh(\gamma\ell)+\beta_2\gamma\cosh(\gamma\ell) & \beta_1\cosh(\gamma\ell)+\beta_2\gamma\sinh(\gamma\ell)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}0\\ \int_0^\ell\left[\beta_1\gamma\sinh(\gamma(\ell-\xi))+\beta_2\gamma^2\cosh(\gamma(\ell-\xi))\right]f_2(\xi)\mathrm{d}\xi\end{bmatrix}. \tag{5.9}$$

If the matrix on the left is invertible, then we can uniquely derived c_1 and c_2 and thus c and c_d. This will happen in most cases, for instance, when $(\alpha_1,\alpha_2)=(\beta_1,\beta_2)=(1,0)$. We summarise the above in a theorem

Theorem 5.1. *Consider the PDE (2.7) with boundary conditions (5.1) and let $\gamma=\frac{1}{\sqrt{\mu}}$. If*

$$det := (\alpha_2\beta_2\gamma^2-\alpha_1\beta_1)\sinh(\gamma\ell)+(\alpha_2\beta_1\gamma-\alpha_1\beta_2\gamma)\cosh(\gamma\ell)\neq 0, \tag{5.10}$$

then the PDE can be represented as $\dot{z}(t)=A\mathcal{H}z(t)$ with $\mathcal{H}$ given by (3.11) and A by (4.11) with

$$\begin{bmatrix}c\\ c_d\end{bmatrix}=\frac{\int_0^\ell\left[\beta_1\gamma\sinh(\gamma(\ell-\xi))+\beta_2\gamma^2\cosh(\gamma(\ell-\xi))\right]f_2(\xi)\mathrm{d}\xi}{det}\begin{bmatrix}-\alpha_1+\alpha_2\gamma\\ \alpha_1+\alpha_2\gamma\end{bmatrix}.$$

From the above we see that the condition for the one-to-one relation between the PDE and the abstract formulation depends on many parameters. Thus if it does not hold, then for a slightly different value of μ and/or ℓ it will hold. However, when $\alpha_1=\gamma\alpha_2$ and $\beta_1=\gamma\beta_2$ then there is no value of ℓ for which (5.10) will hold.

6. Spectral properties of A

In this section we study the spectrum of the operator associated to the PDE (2.7). That is we study the (bounded) operator, see (3.8) and (4.11)

$$A_H := A\mathcal{H}. \tag{6.1}$$

with boundary conditions given by (5.1). To uniquely link these boundary conditions to the c and c_d in (4.11) we assume that $\det\neq 0$, see Theorem 5.1. We begin by determining the eigenfunctions and eigenvalues of A_H.

Using (4.11) we see that $A_Hf=\lambda f$ becomes

$$\begin{pmatrix}0 & I & 0\\ -I & 0 & \frac{1}{\sqrt{\mu}}I-\frac{1}{\sqrt{\mu}}R'_{c_d}\\ 0 & \frac{-1}{\sqrt{\mu}}I+\frac{1}{\sqrt{\mu}}R_c & 0\end{pmatrix}\begin{pmatrix}a^2f_1\\ \rho^{-1}f_2\\ Ef_3\end{pmatrix}=\lambda\begin{pmatrix}f_1\\ f_2\\ f_3\end{pmatrix}, \tag{6.2}$$

which concludes

$$\begin{cases}\rho^{-1}f_2=\lambda f_1\\ -a^2f_1+\left(\frac{1}{\sqrt{\mu}}I-\frac{1}{\sqrt{\mu}}R'_{c_d}\right)Ef_3=\lambda f_2\\ \rho^{-1}\left(\frac{-1}{\sqrt{\mu}}I+\frac{1}{\sqrt{\mu}}R_c\right)f_2=\lambda f_3\end{cases} \tag{6.3}$$

$$\Rightarrow \begin{cases} -a^2\lambda f_1 + E\lambda\left(\frac{1}{\sqrt{\mu}}I - \frac{1}{\sqrt{\mu}}R'_{c_d}\right)f_3 = \lambda^2 f_2 \\ \rho^{-1}\left(\frac{1}{\sqrt{\mu}}I - \frac{1}{\sqrt{\mu}}R'_{c_d}\right)\left(\frac{-1}{\sqrt{\mu}}I + \frac{1}{\sqrt{\mu}}R_c\right)f_2 = \lambda\left(\frac{1}{\sqrt{\mu}}I - \frac{1}{\sqrt{\mu}}R'_{c_d}\right)f_3 \end{cases} \tag{6.4}$$

Thus we find the following equation of f_2

$$\left(-\mu a^2 I + E\left(I - R'_{c_d}\right)(-I + R_c)\right)f_2 = \mu\rho\lambda^2 f_2. \tag{6.5}$$

Since $J = I - \mu\frac{\mathrm{d}^2}{\mathrm{d}x^2} = (I - \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x})(I + \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x})$ and since (6.3) implies that f_2 is smooth, we obtain by using (4.9) and (4.10) that

$$\begin{aligned} J(\mu\rho\lambda^2 f_2) &= -\mu a^2 J(f_2) + E(I - \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x})\left(\sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}(-I + R_c)f_2\right) \\ &= -\mu a^2 J(f_2) + E\sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\left((I - \sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x})(-I + R_c)f_2\right) \\ &= -\mu a^2 J(f_2) + \mu E\frac{d^2 f_2}{\mathrm{d}x^2}. \end{aligned}$$

Using the expression for J, this becomes the differential equation

$$\left(E + \mu\rho\lambda^2 + \mu a^2\right)\frac{d^2 f_2}{\mathrm{d}x^2} = (a^2 + \rho\lambda^2)f_2. \tag{6.6}$$

The boundary conditions (5.1) give that, see also Section 5,

$$\begin{cases} \alpha_1 f_2(0) + \alpha_2 f_{2,x}(0) = 0 \\ \beta_1 f_2(\ell) + \beta_2 f_{2,x}(\ell) = 0. \end{cases} \tag{6.7}$$

Equation (6.6) and (6.7) form a special case of the Sturm-Liouville eigenvalue problem, and so we have that there exists infinite infinitely many solutions, $\phi_n, n \in \mathbb{N}$ and when normalised they form an orthonormal basis of $L^2(0,\ell)$.

However since we are studying the eigenfunctions of A_H, we cannot directly exclude the case that $f_2 = 0$. From (6.3) we see that $f_2 = 0$ implies that $\lambda = 0$ or $f_1 = 0, f_3 = 0$. The latter we can ignore, since we are looking for eigenfunctions of A_H. When $\lambda = 0$, then (6.3) implies that

$$-a^2 f_1 + \left(\frac{1}{\sqrt{\mu}}I - \frac{1}{\sqrt{\mu}}R'_{c_d}\right)Ef_3 = 0$$

Since $f_2 = 0$, we find from Theorem 5.1 that $c_d = 0$, and so the above equation becomes

$$-a^2\sqrt{\mu}f_1(x) + Ef_3(x) - \frac{E}{\sqrt{\mu}}\int_0^x e^{\frac{-x+\tau}{\sqrt{\mu}}}f_3(\tau)\mathrm{d}\tau. \tag{6.8}$$

Hence the eigenspace corresponding to $\lambda = 0$ is infinite-dimensional.

We summarise and extend the above results in a theorem. For that we need the following notation. Let $\{\phi_n, n \in \mathbb{N}\}$ be the orthonormal basis of eigenfunctions corresponding to the eigenvalues problem

$$\frac{d^2\phi}{\mathrm{d}x^2} = \kappa\phi \text{ with } \alpha_1\phi(0) + \alpha_2\phi_x(0) = 0 \text{ and } \beta_1\phi(\ell) + \beta_2\phi_x(\ell) = 0. \tag{6.9}$$

We denote the corresponding eigenvalues by κ_n, $n \in \mathbb{N}$.

Theorem 6.1. *Consider the operator A_H of equation (6.1) with boundary conditions given by (5.1). Assume that* det *as defined in Theorem 5.1 is unequal to zero, $\kappa_n \neq \mu^{-1}$, and $\frac{a^2}{E+\mu a^2} \neq \kappa_n$ for all n, where κ_n are the eigenvalues/solutions of (6.9).*

The eigenvalues are given by $\lambda_0 = 0$, and

$$\lambda_n = \sqrt{\frac{(E+\mu a^2)\kappa_n}{\rho(1-\mu\kappa_n)}}, \quad \lambda_{-n} = -\sqrt{\frac{(E+\mu a^2)\kappa_n}{\rho(1-\mu\kappa_n)}}, \quad n \in \mathbb{N}. \tag{6.10}$$

For $n \neq 0$ the corresponding eigenvectors are

$$f_n = \begin{bmatrix} \frac{1}{\rho\lambda_n}\phi_n \\ \phi_n \\ \frac{1}{\rho\lambda_n\sqrt{\mu}}(-I+R_c)(\phi_n) \end{bmatrix}, \quad n \in \mathbb{Z}\setminus\{0\}, \tag{6.11}$$

where $\phi_{-n} := \phi_n, n \in \mathbb{N}$ and λ_n is given in (6.10).

For $n = 0$ the eigenspace is given by

$$V_0 = \left\{ \begin{bmatrix} f_1 \\ 0 \\ f_3 \end{bmatrix} \in L^2((0,\ell);\mathbb{R}^3) \mid f_1(x) = \frac{E}{a^2\sqrt{\mu}}\left(f_3(x) - \frac{1}{\sqrt{\mu}}\int_0^x e^{\frac{-x+\tau}{\sqrt{\mu}}} f_3(\tau) d\tau \right) \right\}.$$

The only accumulation points of λ_n are $\pm\sqrt{\frac{E+\mu a^2}{\mu\rho}}i$.

The eigenfunctions $f_n, n \in \mathbb{Z}\setminus\{0\}$ form a Riesz basis of the linear subspace

$$V_1 = \left\{ \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \in L^2((0,\ell);\mathbb{R}^3) \mid f_3 = \frac{1}{\sqrt{\mu}}(-I+R_c)f_1 \right\}.$$

If $\{q_n, n \in \mathbb{N}\}$ are chosen such that they form a Riesz basis of V_0, then the union of $\{f_n, n \in \mathbb{Z}\setminus\{0\}\}$ and $\{q_n, n \in \mathbb{N}\}$ form a Riesz basis of X or $L^2((0,\ell);\mathbb{R}^3)$.

Proof. a. Since (6.6) and (6.7) are the same as (6.9), we see that that λ is an eigenvalue of A_H only if

$$\frac{a^2+\rho\lambda^2}{E+\mu\rho\lambda^2+\mu a^2} = \kappa_n \tag{6.12}$$

for some $n \in \mathbb{N}$. This gives as solutions (6.10). In that case f_2 can be chosen as ϕ_n. By (6.4) we have that $f_{1,n} = \frac{1}{\rho\lambda_n} f_{2,n}$, and $\frac{1}{\rho\lambda_n\sqrt{\mu}}(-I+R_c)(\phi_n)$, $n \in \mathbb{Z}_0 := \mathbb{Z}\setminus\{0\}$. Note that by our assumption $\frac{a^2}{E+\mu a^2} \neq \kappa_n$, $\lambda_n \neq 0$ for $n \neq 0$. By the assumption $\kappa_n \neq \mu^{-1}$ equation (6.12) is solvable for λ.

b. The eigenspace expression for the eigenvalue zero follows directly from (6.8).

c. Since κ_n lie on the real axis and $|\kappa_n| \to \infty$, we have that $\pm\sqrt{\frac{E+\mu a^2}{\mu\rho}}i$ are the only accumulation points of the λ_n's.

d. We first show that V_1 is a closed subspace of $L^2((0,\ell);\mathbb{R}^3)$. It is easy to see that this holds if and only if $(-I+R_c)$ has closed range. Since R_c is a compact operator, this holds.

On V_1 we have the following equivalent norm

$$\left\| \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \right\|_e = \sqrt{\|f_1\|^2 + \|f_2\|^2}. \tag{6.13}$$

Using the fact that a set remains a Riesz basis when the norm is replaced by and equivalent one, we can check the Riesz basis property of $f_n, n \in \mathbb{Z} \setminus \{0\}$ with respect to this norm. Note that $f_n \in V_1$. For $a_n \in \mathbb{C}, n \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned} \| \sum_{n=-N, n\neq 0}^{N} a_n f_n \|_e^2 &= \| \sum_{n=-N, n\neq 0}^{N} \frac{a_n}{\rho\lambda_n} \phi_n \|^2 + \| \sum_{n=-N, n\neq 0}^{N} a_n \phi_n \|^2 \\ &= \| \sum_{n=1}^{N} (\frac{a_n}{\rho\lambda_n} + \frac{a_{-n}}{\rho\lambda_{-n}}) \phi_n \|^2 + \| \sum_{n=1}^{N} (a_n + a_{-n}) \phi_n \|^2 \\ &= \sum_{n=1}^{N} | \frac{a_n}{\rho\lambda_n} + \frac{a_{-n}}{\rho\lambda_{-n}} |^2 + |a_n + a_{-n}|^2 \\ &= \sum_{n=1}^{N} \frac{1}{\rho^2 |\lambda_n|^2} |a_n - a_{-n}|^2 + |a_n + a_{-n}|^2, \end{aligned} \tag{6.14}$$

where we used that $\phi_{-n} = \phi_n$, $\lambda_{-n} = -\lambda_n$, and that $\phi_n, n \in \mathbb{N}$ is an orthonormal basis of $L^2(0, \ell)$. Since λ_n is bounded and bounded away from zero, we have that there are positive constants m, M independent of a_n and N such that

$$\begin{aligned} m \sum_{n=1}^{N} |a_n - a_{-n}|^2 + |a_n + a_{-n}|^2 \leq & \sum_{n=1}^{N} \frac{1}{\rho^2 |\lambda_n|^2} |a_n - a_{-n}|^2 + |a_n + a_{-n}|^2 \\ & \leq M \sum_{n=1}^{N} |a_n - a_{-n}|^2 + |a_n + a_{-n}|^2. \end{aligned}$$

Using that $\sum_{n=1}^{N} |a_n - a_{-n}|^2 + |a_n + a_{-n}|^2 = 2 \sum_{n=-N, n\neq 0}^{N} |a_n|^2$ we see that

$$2m \sum_{n=-N, n\neq 0}^{N} |a_n|^2 \leq \| \sum_{n=-N, n\neq 0}^{N} a_n f_n \|_e^2 \leq 2M \sum_{n=-N, n\neq 0}^{N} |a_n|^2.$$

So if we can show that $f_n, n \in \mathbb{Z} \setminus \{0\}$ span V_1, then we have shown that it is a Riesz basis for this subspace. Let $\begin{bmatrix} f_1 \\ f_2 \\ \frac{1}{\sqrt{\mu}}(-I + R_c) f_1 \end{bmatrix}$ be an element of V_1. We write

$$f_1 = \sum_{n=1}^{\infty} b_n \phi_n \text{ and } f_2 = \sum_{n=1}^{\infty} d_n \phi_n.$$

This is possible since $\phi_n, n \in \mathbb{N}$ is an orthonormal basis of $L^2(0,\ell)$. Next we choose $a_n = \frac{1}{2}\left(\rho\lambda_n b_{|n|} + d_{|n|}\right)$, $n \in \mathbb{Z}\setminus\{0\}$. With this choice we have

$$f_1 = \sum_{n=-\infty, n\neq 0}^{\infty} \frac{a_n}{\rho\lambda_n}\phi_n \text{ and } f_2 = \sum_{n=-\infty, n\neq 0}^{\infty} a_n\phi_n$$

Hence we obtain

$$\begin{aligned}\begin{bmatrix} f_1 \\ f_2 \\ \frac{1}{\sqrt{\mu}}(-I+R_c)f_1 \end{bmatrix} &= \begin{bmatrix} \sum_{n=-\infty,n\neq 0}^{\infty} \frac{a_n}{\rho\lambda_n}\phi_n \\ \sum_{n=-\infty,n\neq 0}^{\infty} a_n\phi_n \\ \frac{1}{\sqrt{\mu}}(-I+R_c)(\sum_{n=-\infty,n\neq 0}^{\infty} \frac{a_n}{\rho\lambda_n}\phi_n) \end{bmatrix} \\ &= \sum_{n=-\infty,n\neq 0}^{\infty} a_n \begin{bmatrix} \frac{1}{\rho\lambda_n}\phi_n \\ \phi_n \\ \frac{1}{\sqrt{\mu}}(-I+R_c)(\frac{1}{\rho\lambda_n}\phi_n) \end{bmatrix} = \sum_{n=-\infty,n\neq 0}^{\infty} a_n f_n.\end{aligned}$$

This concludes the assertion that $f_n, n \in \mathbb{Z}\setminus\{0\}$ is a Riesz basis of V_1.

e. Since V_0 is the eigenspace of A_H corresponding to eigenvalue zero, and since zero is not in the spectrum of A_H restricted to V_1, the sets must have trivial intersection.

Let $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ be an element of $L^2((0,\ell);\mathbb{R}^3)$. It lies in $V_0 + V_1$ if and only if there exists $f_{1,1}, f_{2,1}$, and $f_{3,0}$ all elements of $L^2(0,\ell)$ such that

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{E}{a^2\sqrt{\mu}}(f_{3,0} - \frac{1}{\sqrt{\mu}}\int_0^x e^{\frac{-x+\tau}{\sqrt{\mu}}} f_{3,0}(\tau)\mathrm{d}\tau) \\ 0 \\ f_{3,0} \end{bmatrix} + \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ \frac{1}{\sqrt{\mu}}(-I+R_c)f_{1,1} \end{bmatrix},$$

where $c = c(f_{2,1})$. Hence $f_2 = f_{2,1}$ and $f_{1,1}$, $f_{3,0}$ should satisfy

$$\begin{bmatrix} f_1(x) \\ f_3(x) \end{bmatrix} = \begin{bmatrix} \frac{E}{a^2\sqrt{\mu}}(I - R_0')(f_{3,0})(x) + f_{1,1}(x) \\ f_{3,0}(x) + c\frac{1}{\sqrt{\mu}}e^{\frac{x}{\sqrt{\mu}}} + \frac{1}{\sqrt{\mu}}(-I+R_0)(f_{1,1})(x) \end{bmatrix}, \tag{6.15}$$

with $R_0 := R_{c=0}$, $R_0' := R'_{c_d=0}$. Since c depends on $f_{2,1}$ we may consider that given, and we obtain the following set of equation for $f_{11}, f_{3,0} \in L^2(0,\ell)$

$$\begin{bmatrix} a^2\sqrt{\mu}f_{1,1} + E(I-R_0')f_{3,0} \\ \sqrt{\mu}f_{3,0} + (-I+R_0)f_{1,1} \end{bmatrix} = \begin{bmatrix} a^2\sqrt{\mu}f_1 \\ \sqrt{\mu}\mu f_3 - ce^{\frac{x}{\sqrt{\mu}}} \end{bmatrix} =: \begin{bmatrix} g_1 \\ g_3 \end{bmatrix}. \tag{6.16}$$

Using the second, the first equation gives

$$a^2\sqrt{\mu}f_{1,1} + E(I-R_0')\left(\frac{1}{\sqrt{\mu}}(I-R_0)f_{1,1} + \frac{1}{\sqrt{\mu}}g_3\right) = g_1,$$

which is equivalent to

$$\left(a^2\sqrt{\mu} + \frac{E}{\sqrt{\mu}}\right)f_{1,1} + \frac{E}{\sqrt{\mu}}\left(-R_0' - R_0 + R_0'R_0\right)f_{1,1} = g_1 - \frac{E}{\sqrt{\mu}}(I-R_0')g_3. \tag{6.17}$$

Since R_0 and R_0' are compact operators, we have that this equation is uniquely solvable for any right hand side if and only if

$$\left(a^2\sqrt{\mu}+\frac{E}{\sqrt{\mu}}\right)f_{1,1}+\frac{E}{\sqrt{\mu}}\left(-R_0'-R_0+R_0'R_0\right)f_{1,1}=0\Rightarrow f_{1,1}=0 \tag{6.18}$$

The equation implies that $f_{1,1}(0)=0$ and it can be rewritten as

$$a^2\mu f_{1,1}+E(I-R_0')(I-R_0)f_{1,1}=0.$$

Thus with (4.10)

$$a^2\mu\left(I+\sqrt{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\right)f_{1,1}+E\sqrt{\mu}\left(\frac{\mathrm{d}f_{1,1}}{\mathrm{d}x}+f_{11}+\frac{1}{\sqrt{\mu}}\int_0^x e^{\frac{x-\tau}{\sqrt{\mu}}}f_{1,1}(\tau)\mathrm{d}\tau\right)=0,$$

which gives $\frac{\mathrm{d}f_{1,1}}{\mathrm{d}x}(0)=0$. Differentiating this once more leads to a second order differential equation in $f_{1,1}$. From the zero initial conditions, we conclude that $f_{1,1}$ is identically zero, and thus (6.18) holds. Hence (6.17) gives a unique $f_{1,1}$, and substituting this in (6.16) gives $f_{3,0}$. Concluding we see that any $\begin{bmatrix} f_1\\ f_2\\ f_3\end{bmatrix}\in L^2((0,\ell);\mathbb{R}^3)$ lies in V_0+V_1 as well.

f. Since $V_0+V_1=L^2((0,\ell);\mathbb{R}^3)$, and since $\{f_n, n\in\mathbb{Z}\setminus\{0\}\}$ is a Riesz basis of V_1, and $\{q_n, n\in\mathbb{N}\}$ is a Riesz basis of V_0, it follows almost directly the their union is a Riesz basis of $L^2((0,\ell);\mathbb{R}^3)$. Since the norm of X and that of $L^2((0,\ell);\mathbb{R}^3)$ are equivalent, this union is also a Riesz basis of X. □

7. Controllability and stability

In this section, the control system is considered in the following form

$$a^2w+\rho\frac{\partial^2 w}{\partial t^2}-\mu\frac{\partial^2}{\partial x^2}\left(a^2w+\rho\frac{\partial^2 w}{\partial t^2}\right)-E\frac{\partial^2 w}{\partial x^2}+b(x)u(t)=0, \tag{7.1}$$

$$w(x,0)=f(x),\ \frac{\partial w}{\partial t}(x,0)=g(x), \tag{7.2}$$

where $u(t)$ is the control function. The boundary conditions are the same as in (5.1).

Similar as in (3.8) we can rewrite this as $\dot{z}(t)=A_Hz(t)+Bu(t)$, where A_H is given by (6.1) and $B\in\mathcal{L}(\mathbb{R},X)$ equals

$$Bu=\begin{bmatrix} 0\\ -R_{c_d}'R_cbu\\ 0\end{bmatrix},$$

see also (3.6)–(3.8), (4.2), and (4.5)–(4.7).

Using the notions and results from the previous section, we see that B maps into V_1. Since V_1 is spanned by eigenfunctions of A_H, we see that any solution of $\dot{z}(t)=A_Hz(t)+Bu(t)$, $z_0=0$ will stay in V_1 for all $t\geq 0$. Hence the system (7.1) is not controllable.

Next we take as input the velocity feedback, *i.e.*, $u(t)=\alpha\frac{\partial w}{\partial t}$. With this choice, the operator A_H becomes

$$A_{fb}=\begin{pmatrix} 0 & I & 0\\ -I & -\alpha R_{c_d}'R_cb & \frac{1}{\sqrt{\mu}}I-\frac{1}{\sqrt{\mu}}R_{c_d}'\\ 0 & \frac{-1}{\sqrt{\mu}}I+\frac{1}{\sqrt{\mu}}R_c & 0\end{pmatrix}\mathcal{H}.$$

It is easy to see that $A_{fb}V_0 = 0$, and so zero remains an eigenvalue of the closed loop system operator, and thus the system (7.1) has not become asymptotically stable.

8. Conclusion and future works

This paper investigates longitudinal vibration of a nanorod by proposing a new port-Hamiltonian model. This gives the energy (Hamiltonian) function, which is usually a good Lyapunov function, explicitly appears in the dynamics of the system [17]. This model always generates a strongly continuous group, but for special boundary conditions it can generate a contraction semigroup. Moreover, we showed that the eigenfunctions from a Riesz basis, but that that the eigenspace associated to the eigenvalue zero was infinite-dimensional, and this prevented controllability and stabilizability. For the future work, we plan to extend this work for viscoelastic vibrating nanorod [13], and to study the system theoretic properties of the port-Hamiltonian formulations given in [10].

References

[1] S. Akbas, Modal analysis of viscoelastic nanorods under an axially harmonic load. *Adv Nano Res* **8** (2020) 277–282.

[2] S.D. Akbas, Forced vibration analysis of cracked nanobeams. *J. Brazilian Soc. Mech. Sci. Eng.* **40** (2018) 1–11.

[3] S.D. Akbas, Axially forced vibration analysis of cracked a nanorod. *J. Comput. Appl. Mech.* **50** (2019) 63–68.

[4] P. Alasvand Hadi, H. Heidari and R. Nazemnezhad, Approximate controllability of nonlocal Rayleigh beam. *Comp. M. Diff. Eq.* **9** (2021) 180–186.

[5] S.B. Ali, A.B. Oshido, A. Houlton and B.R. Horrocks, Models for sensing by nanowire networks: application to organic vapour detection by multiwall carbon nanotube—DNA films. *Nanotechnology* **33** (2021) 045502.

[6] R.F. Curtain and H.J. Zwart, Vol. 71 of *An Introduction to Infinite-Dimensional Linear System Theory: A state-space approach*. Springer-Nature (2020).

[7] M. Eren and M. Aydogdu, Finite strain nonlinear longitudinal vibration of nanorods. *Adv. Nano Res.* **6** (2018) 323–337.

[8] G. Golo, A. van der Schaft and S. Stramigioli, Hamiltonian formulation of planar beams. *IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control* (2003) 169–174.

[9] H. Heidari, Dynamical analysis of an axially vibrating nanorod. *Int. J. Appl. Math.* **29** (2016) 263–270.

[10] H. Heidari and H. Zwart, Port-Hamiltonian modelling of nonlocal longitudinal vibrations in a viscoelastic nanorod. *Math. Comput. Model. Dyn. Syst.* **25** (2019) 447–462.

[11] S. Inguva, R.K. Vijayaraghavan, E. McGlynn and J.P. Mosnier, High quality interconnected core/shell ZnO nanorod architectures grown by pulsed laser deposition on ZnO-seeded Si substrates. *Superlattices Microstruct.* **101** (2017) 8–14.

[12] B. Jacob and H.J. Zwart, Linear port-Hamiltonian systems on infinite-dimensional spaces. Springer, Basel (2012).

[13] D. Karličić, M. Cajić, T. Murmu and S. Adhikari, Nonlocal longitudinal vibration of viscoelastic coupled double-nanorod systems. *Eur. J. Mech. A/Solids* **49** (2015) 183–196.

[14] A.L. Kozlovskiy, I.V. Korolkov, G. Kalkabay, M.A. Ibragimova, A.D. Ibrayeva, M.V. Zdorovets, V.S. Mikulich, D.V. Yakimchuk, A.E. Shumskaya and E.Y. Kaniukov, Comprehensive study of Ni nanotubes for bioapplications: from synthesis to payloads attaching. *J. Nanomater.* **2017** (2017) 3060972.

[15] Y. Le Gorrec, H. Zwart and B. Maschke, Dirac structures and boundary control systems associated with skew-symmetric differential operators. *SIAM J. Control Optim.* **44** (2006) 1864–1892.

[16] D.L. Liu, W. Lei, S. Qin and Y. Chen, Template-free synthesis of functional 3D BN architecture for removal of dyes from water. *Sci. Rep.* **4** (2014) 1–5.

[17] A. Macchelli, A.J. van der Schaft and C. Melchiorri, Port Hamiltonian formulation of infinite dimensional systems II. Boundary control by interconnection. *Proc. IEEE Conf. Decis. Control* **4** (2004) 3768–3773.

[18] T. Murmu and S. Adhikari, Nonlocal effects in the longitudinal vibration of double-nanorod systems. *Phys. E Low-Dimensional Syst. Nanostructures* **43** (2010) 415–422.

[19] H.M. Numanoglu, B. Akgöz and O. Civalek, On dynamic analysis of nanorods. *Int. J. Eng. Sci.* **130** (2018) 33–50.

[20] V.N. Popov, Carbon nanotubes: properties and application. *Mater. Sci. Eng. R Reports* **43** (2004) 61–102.

[21] N.M. Pugno, The role of defects in the design of space elevator cable: from nanotube to megatube. *Acta Mater.* **55** (2007) 5269–5279.

[22] S. Rafique, L. Han and H. Zhao, Growth and electrical properties of free-standing zinc oxide nanomembranes. *Cryst. Growth Des.* **16** (2016) 1654–1661.

[23] A. Raunika, S.A. Raj, K. Jayakrishna and M.T. Sultan, Carbon nanotube: A review on its mechanical properties and application in aerospace industry. *IOP Conf. Ser. Mater. Sci. Eng.* **270** (2017) 012027.

[24] A. Salazar, V. Pérez-De la Cruz, E. Muñoz-Sandoval, V. Chavarria, M.d.L. García Morales, A. Espinosa-Bonilla, J. Sotelo, A. Jiménez-Anguiano and B. Pineda, Potential use of nitrogen-doped carbon nanotube sponges as payload carriers against malignant glioma. *Nanomaterials* **11** (2021) 1–11.

[25] J.P. Salvetat-Delmotte and A. Rubio, Mechanical properties of carbon nanotubes: a fiber digest for beginners. *Carbon N. Y.* **40** (2002) 1729–1734.
[26] M. Simsek, Nonlinear free vibration of a functionally graded nanobeam using nonlocal strain gradient theory and a novel Hamiltonian approach. *Int. J. Eng. Sci.* **105** (2016) 12–27.
[27] V. Trenchant, Y. Fares, H. Ramirez and Y. Le Gorrec, A port-Hamiltonian formulation of a 2D boundary controlled acoustic system. *IFAC* **48** (2015) 235–240.
[28] A. van der Schaft and D. Jeltsema, Port-Hamiltonian systems theory: an introductory overview. *Found. Trends Syst. Control* **1** (2014) 173–378.
[29] T. Voß and J. Scherpen, Port-Hamiltonian modeling of a nonlinear Timoshenko beam with piezo actuation. *SIAM J. Control Optim.* **52** (2014) 493–519.
[30] K. Wang and B. Wang, Timoshenko beam model for the vibration analysis of a cracked nanobeam with surface energy. *J. Vib. Control* **21** (2015) 2452–2464.
[31] J. Yoon, C. Ru and A. Mioduchowski, Terahertz vibration of short carbon nanotubes modeled as Timoshenko beams. *J. Appl. Mech.* **72** (2005) 10–17.