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HOMOGENIZATION OF A MICROSCOPIC PEDESTRIANS MODEL ON A CONVERGENT JUNCTION

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Abstract. In this paper, we establish a rigorous connection between a microscopic and a macroscopic pedestrians model on a convergent junction. At the microscopic level, we consider a "follow the leader" model far from the junction point and we assume that a rule to enter the junction point is imposed. At the macroscopic level, we obtain the Hamilton-Jacobi equation with a flux limiter condition at $x = 0$ introduced in Imbert and Monneau [*Ann. Sci. l'École Normale Supér.* **50** (2017) 357–414]. To obtain our result, we inject using the "cumulative distribution functions" the microscopic model into a non-local PDE. Then, we show that the viscosity solution of the non-local PDE converges locally uniformly towards the solution of the macroscopic one.

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1. Introduction

In the literature, several papers proposed mathematical models of pedestrian flows based on partial differential equations, see [7–9, 26, 31, 34]. Macroscopic pedestrian models are effective for describing pedestrian flows at large scales but from the modeling point of view, since the dynamics of each pedestrian can't be described individually, they are hardly justifiable and incapable to predict all possible situations (for example panic during an evacuation). At the opposite, microscopic models are strong from the modeling point of view but complicated to implement at large scales. Among the works modeling pedestrians at the micro scale, we refer to [21, 23–25, 35].

A rigorous way to justify these macroscopic models is to establish a connection between them and microscopic ones, we refer for example to [3, 12, 13, 22, 37].

The "follow-the-leader" (FTL) can be used in modeling at the microscopic scale, the traffic and the pedestrian flows, see [35]. In this paper, we consider the FTL model on a convergent junction and we establish a micro-macro connection. More precisely, we consider a microscopic pedestrians model, and prove that the cumulative distribution function on each branch will converge to the solution of a Hamilton-Jacobi equation studied by Imbert and Monneau in [28] and then in [6, 33]. We obtain our convergence result by using the theory of viscosity

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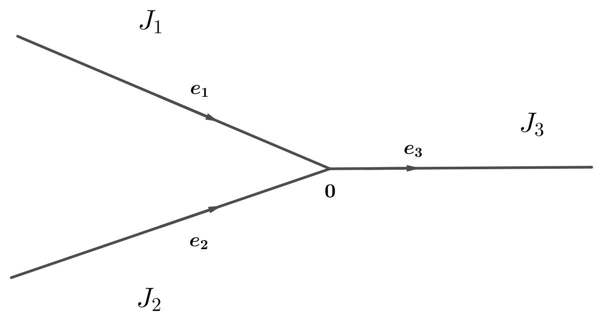

FIGURE 1. The junction J.

solution (see [10, 11]). The first homogenization result in this framework was obtained in [32] and since then, several works considered this problem. For more details, the reader can refer to [1, 15, 16, 20].

1.1. Presentation of our results

Since the global strategy to get the convergence result is similar to that in [17–19], we focus on the presentation of the new ideas and we refer the reader to [17] for some technical proofs. The control of oscillations of the solution is an important result that allows us to determine bounds on the gradient of the limit solution. In this paper, we provide a shorter proof (compared to [17]) of this result using the idea introduced in [4] and the definition of the security distance. In addition, we obtain better gradient estimates far from zero. A similar (but global) result is also needed when constructing the correctors. In this case, we should compare the components of the solution of an approximated problem on two different branches. We use here the definition of a new distance d_0 which can be seen as a security distance while crossing the junction point. We present our results in the case of a convergent junction with two incoming roads and one outgoing one to simplify the presentation of our work and lessen the article. In fact, we will see that in the case of a convergent junction with N branches (see Appendix A), the mathematical results can be easily extended but need additional "cumulative distribution functions" and additional partial differential equations.

2. MAIN RESULT

2.1. The microscopic model

We consider a microscopic pedestrians model on a convergent junction with two incoming roads and one outgoing road. The junction is defined by $J = J_1 \cup J_2 \cup J_3$ where J_1 and J_2 are the incoming roads and isometric to $\mathbb{R}^-$ and J_3 is the outgoing road and isometric to $\mathbb{R}^+$. The branches of the junction are glued at the point 0. The point 0 is called the junction point and for all $k = 1, 2, 3$, we define $J_k^* = J_k \setminus \{0\}$. To be more precise, the definitions of the branches are given by

$$J_k = (-\infty, 0] \cdot e_k \quad \text{for } k = 1, 2, \quad J_3 = [0, +\infty) \cdot e_3,$$

with e_1, e_2 and e_3 being three different unit vectors in $\mathbb{R}^2$ (see Fig. 1).

Let $x, y \in J$ with $x = x_k \cdot e_k, y = y_l \cdot e_l$: the distance $d(x, y)$ in J is defined by

$$d(x,y) = \begin{cases} |x_k - y_k| & \text{if } k = l, \\ |x_k| + |y_l| & \text{if } k \neq l. \end{cases}$$

Before writing the mathematical model, we describe its purpose.

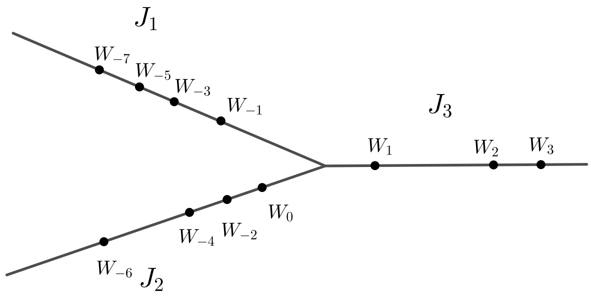

FIGURE 2. Initial distribution of pedestrians.

Purpose of the model. The microscopic model on the junction J describes the following behavior: pedestrians on J_1 and J_2 enter the branch J_3 in a periodic manner. Each one of them knows its turn to enter and will not violate the imposed rule. Such behavior is observed in many real life situations. For example, during the COVID 19 pandemic, some supermarkets decided to reduce the number of customers shopping inside the supermarket and a security agent was organizing the customers flow into the supermarket. Usually, in such situations, we observe the formation of many people lines that converge to the entrance. In order to simplify the presentation of our model, we assume that we have two lines of people and the security agent lets them enter in an alternating way, that is one from J_1 and then one from J_2 then one from J_1, etc.

Pedestrian's initial distribution. We denote by $W_i(t)$ the position of pedestrian i at time $t \geq 0$. For $i \in \mathbb{Z}$, we define the following sets:

$$\begin{cases} \mathcal{L}_1 = \{i \in \mathbb{Z}, i = -(2k+1), k \in \mathbb{N}\}, \\ \mathcal{L}_2 = \{i \in \mathbb{Z}, i = -2k, k \in \mathbb{N}\}. \end{cases} \tag{2.1}$$

We assume that at time $t = 0$, the position $W_i(0) \in J_1^*$ (resp. J_2^*, J_3) if $i \in \mathcal{L}_1$ (resp. $i \in \mathcal{L}_2, i \in \mathbb{N}^*$). Moreover, we have for all $i, j \in \mathbb{Z}$,

$$d(W_i(0), W_j(0)) > 0.$$

We refer to Figure 2 which describes pedestrian's initial distribution on the J.

The leaders. Let $W_i(t)$ be the position of the pedestrian i at time $t \geq 0$. For $i \in \mathbb{Z}$, we have

$$W_i(t) \in J \Longleftrightarrow W_i(t) = U_i(t) \cdot e_k, \tag{2.2}$$

with

$$\begin{cases} U_i(t) \geq 0 \text{ and } k = 3 & \text{if } i \in \mathbb{N}^*, \\ (U_i(t) < 0 \text{ if } k = 1) \text{ or } (U_i(t) \geq 0 \text{ if } k = 3) & \text{if } i \in \mathcal{L}_1, \\ (U_i(t) < 0 \text{ if } k = 2) \text{ or } (U_i(t) \geq 0 \text{ if } k = 3) & \text{if } i \in \mathcal{L}_2. \end{cases}$$

We denote by $d_i(t)$ the distance to its leader. For $t \geq 0$, the distance d_i is defined by

$$d_i(t) = U_{i+1}(t) - U_i(t) \quad \text{if } i \in \mathbb{N}$$

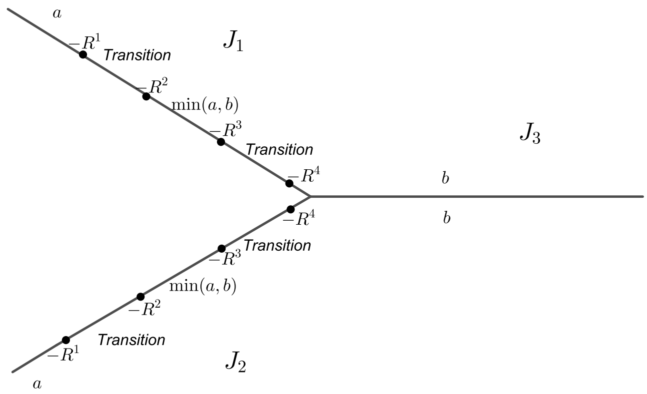

FIGURE 3. Schematic representation of $\bar{\phi}$.

and for $i \in \mathbb{Z} - \mathbb{N}$,

$$d_i(t) = \begin{cases} U_{i+2}(t) - U_i(t) & \text{if } W_i(t) \in J_1^* \cup J_2^*, d(0, W_i(t)) \geq R^1 \\ U_{i+1}(t) - U_i(t) & \text{if } (W_i(t) \in J_3) \text{ or if } (W_i(t) \in J_1^* \cup J_2^*, d(0, W_i(t)) < R^4) \end{cases}$$

where R^1 and R^4 are two positive constants defined in (2.3). This means that far before the junction point (resp. after the junction point), the leader is W_{i+2} (resp. W_{i+1}). As in the "follow-the-leader" model, the velocity of pedestrian i is a distance of $d_i(t)$. To ensure a continuous passage from $(U_{i+2}(t) - U_i(t))$ to $(U_{i+1}(t) - U_i(t))$, we introduce a function $\bar{\phi}$. The function $\bar{\phi} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as follow, for $R^1 > R^2 > R^3 > R^4$,

$$\bar{\phi}(x, a, b) = \begin{cases} a & \text{if } x \leq -R^1, \\ \left(\dfrac{a - \min(a, b)}{R^2 - R^1} \right) (x - R^1) + a & \text{if } -R^1 < x \leq -R^2, \\ \min(a, b) & \text{if } -R^2 < x \leq -R^3, \\ \left(\dfrac{b - \min(a, b)}{R^3 - R^4} \right) (x + R^4) + b & \text{if } -R^3 < x \leq -R^4, \\ b & \text{if } x > -R^4 \end{cases} \tag{2.3}$$

where $a, b \in \mathbb{R}$ (see Fig. 3).

Optimal velocity functions. We assume different optimal velocity functions on each branch. The velocity on branch J_1 (resp. J_2, J_3) is V_1 (resp. V_2, V_3). We make the following assumptions.

Assumptions (A)

(A1) For $i = 1, 2, 3$, the function $V_i : \mathbb{R} \to \mathbb{R}^+$ is Lipschitz continuous, non-negative and non-decreasing on $\mathbb{R}$.
(A2) There exists $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V_i(h) = 0$.
(A3) There exists $h_{\max} > h_0$ such that for all $h \geq h_{\max}$, $V_i(h) = V_{\max}$.
(A4) There exists a strictly negative real number p_3 (resp. p_1, p_2) such that the function $p \mapsto pV_3(-1/p)$ (resp. $p \mapsto pV_1(-3/2p)$, $p \mapsto pV_2(-3/p)$) is decreasing on $[-1/h_0, p_3)$ (resp. $[-3/2h_0, p_1)$,$[-3/h_0, p_2)$) and increasing on $[p_3, 0)$ (resp. $[p_1, 0)$,$[p_2, 0)$).

The behavior near the junction point. Let $i, j \in \mathbb{Z} - \mathbb{N}^*$: if $i < j$, then W_j will enter the junction point before W_i. As we mentioned above, we assume that pedestrians respect the imposed rule: one passes the junction point coming from J_1, and then another one coming from J_2. This is the most restrictive assumption of the paper. This means that W_0 enters first, then W_1, then W_2, etc. To model this phenomena, we introduce the function $\omega : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\omega(x, y, p) = \alpha(x) + (1 - \alpha(x))\beta(y, p) \tag{2.4}$$

with α being a lipschitz continuous function defining the zone in which the pedestrian $W_i \in J_1^*$ (resp. $W_i \in J_2^*$) starts spotting the position of W_{i+1} on the other branch J_2 (resp. J_1) . It's defined in the following way:

$$\alpha(x) = \begin{cases} 1 & \text{if } x < -R^3 - r \text{ or } x > r, \\ 0 & \text{if } -R^3 < x < 0 \end{cases} \tag{2.5}$$

where $0 < r < R^1 - R^3$.

The function β is a lipschitz continuous function describing the fact that if pedestrian W_{i+1} is after the junction point, its influence on W_i disappears since collision is not possible in this case. It's defined by

$$\beta(y, p) = \begin{cases} 1 & \text{if } y > r \ , \\ \zeta(p) & \text{if } y \leq 0 \ , \end{cases} \tag{2.6}$$

where ζ is a lipschitz function modeling the fact that W_{i+1} will enter the junction point before W_i,

$$\zeta(p) = \begin{cases} 1 & \text{if } p > d_0 + r, \\ 0 & \text{if } p \leq d_0, \end{cases} \tag{2.7}$$

with

$$d_0 > 0. \tag{2.8}$$

Remark 2.1 (Comments on ω)**.** The role of the function ω is to manage the order of entrance through the junction point. Let $i \in \mathcal{L}_1$. If $W_i(t)$ is too close to the junction, that is $-R^3 < W_i(t) < 0$, then it will stop ($\omega = 0$) if $W_{i+1} \in J_2^*$ and if

$$|W_i(t)| - |W_{i+1}(t)| < d_0.$$

The distance d_0 can be interpreted as a security distance when crossing the junction point. Remark also that when W_{i+1} enters the branch J_3 *i.e.* $W_{i+1}(t) > r$, the influence of ω disappears ($\omega = 1$). Finally, let us remark that the definition of the function α on $[-R^3, 0]$ means that the influence of ω begins directly when the driver's velocity tends to depend on the distance to W_{i+1} (see Fig. 4).

We are now ready to write our model. We assume

$$U_0(0) > -R^4 \text{ and } U_1(0) > r. \tag{2.9}$$

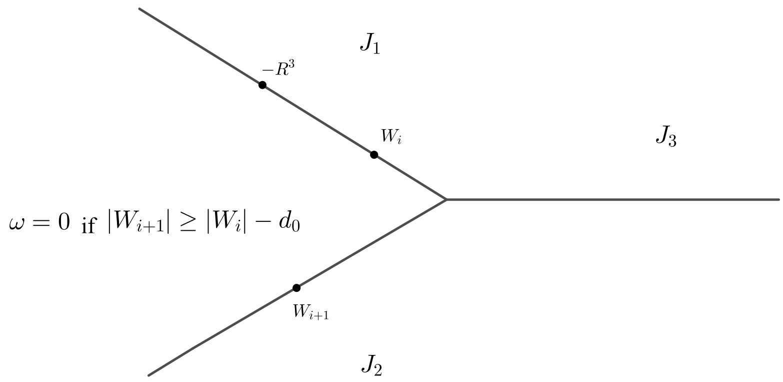

FIGURE 4. The role of ω.

Our model is described by the following differential equations,

$$\begin{cases} U_i'(t) = V_3(U_{i+1}(t) - U_i(t)) & \text{if } i \in \mathbb{N}, \\ U_i'(t) = F\left(U_i(t), U_{i+1}(t), V_1(U_{i+2}(t) - U_i(t)), V_3(|U_{i+1}(t)| - U_i(t))\right) & \text{if } i \in \mathcal{L}_1, \\ U_i'(t) = F\left(U_i(t), U_{i+1}(t), V_2(U_{i+2}(t) - U_i(t)), V_3(|U_{i+1}(t)| - U_i(t))\right) & \text{if } i \in \mathcal{L}_2 \setminus \{0\}, \end{cases}$$

with

$$F(x, y, a, b) = \bar{\phi}\left(x, a, b\right) \cdot \omega\left(x, y, |x| - |y|\right).$$

Using the definition of $\bar{\phi}$ and ω (see (2.3) and (2.4)), we can reformulate our model in the following way,

$$\begin{cases} U_i'(t) = F\left(U_i(t), U_{i+1}(t), V_1(U_{i+2}(t) - U_i(t)), V_3(|U_{i+1}(t)| - U_i(t))\right) & \text{if } i \in \mathcal{L}_1 \cup \mathbb{N}^*, \\ U_i'(t) = F\left(U_i(t), U_{i+1}(t), V_2(U_{i+2}(t) - U_i(t)), V_3(|U_{i+1}(t)| - U_i(t))\right) & \text{if } i \in \mathcal{L}_2. \end{cases} \tag{2.10}$$

Remark 2.2. Inequalities in (2.9) allow us to simplify the presentation. In fact if one of the two conditions in (2.9) is violated, we are not able to rewrite the model using only two differential equations as in (2.10). Note that the following equality does not necessarily hold

$$V_3(U_1(t) - U_0(t)) = F\left(U_0(t), U_1(t), V_2(U_2(t) - U_0(t)), V_3(U_1(t) - U_0(t))\right).$$

2.2. The macroscopic model

Let $k_0 = 1/h_0$. For $k = 1, 2, 3$, we define $\overline{H}_k : \mathbb{R} \to \mathbb{R}$ by

$$\overline{H}_1(p) = \begin{cases} -p - 2k_0 & \text{for } p < -2k_0, \\ -V_1\left(\dfrac{-2}{p}\right)|p| & \text{for } -2k_0 \leq p \leq 0, \\ p & \text{for } p > 0, \end{cases} \tag{2.11}$$

$$\overline{H}_2(p) = \begin{cases} -p - 2k_0 & \text{for } p < -2k_0, \\ -V_2\left(\dfrac{-2}{p}\right)|p| & \text{for } -2k_0 \leq p \leq 0, \\ p & \text{for } p > 0, \end{cases} \tag{2.12}$$

and

$$\overline{H}_3(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V_3\left(\dfrac{-1}{p}\right)|p| & \text{for } -k_0 \le p \le 0, \\ p & \text{for } p > 0. \end{cases} \tag{2.13}$$

Note that for all $k = 1, 2, 3$, the function $\overline{H}_k$ is continuous, coercive and because of (A4), there exists a unique point $p_k \le 0$ such that

$$\begin{cases} \overline{H}_k \text{ is decreasing on } (-\infty, p_k), \\ \overline{H}_k \text{ is increasing on } (p_k, +\infty). \end{cases} \tag{2.14}$$

We denote by

$$H_0 = \max_{k\in\{1,2,3\}} H_0^k \tag{2.15}$$

with

$$H_0^k = \min_{p\in\mathbb{R}} \overline{H}_k(p). \tag{2.16}$$

We introduce now the definition of the gradient of a function defined on J. If $x \in J$, then we define

$$u_x(x) = \begin{cases} \partial_i u(x) & \text{if } x \in J_i^*, \\ (\partial_1 u(0), \partial_2 u(0), \partial_3 u(0)) & \text{if } x = 0, \end{cases}$$

with $\partial_i u(x)$ the derivative of the function u with respect to $x \in J_i$. We denote by $J^* = J\backslash\{0\}$. The macroscopic model is the Hamilton-Jacobi equation with flux limiting condition at the junction point as considered by Imbert and Monneau in [28] and given by

$$\begin{cases} u_t^0 + \overline{H}_k(u_x^0) = 0 \ \text{ for } (t,x) \in (0,+\infty) \times J_k^*, \\ u_t^0 + F_{\overline{A}}\left(u_x^0\right) = 0 \text{ for } (t,x) \in (0,+\infty) \times \{0\}, \\ u^0(0,x) = \overline{u}_0(x), \end{cases} \tag{2.17}$$

where $\overline{A}$ is the flux limiter and $F_{\overline{A}}$ is defined by

$$F_{\overline{A}}(p_1, p_2, p_3) = \max\left(\overline{A}, \overline{H}_1^+(p_1), \overline{H}_2^+(p_2), \overline{H}_3^-(p_3)\right), \tag{2.18}$$

with

$$\overline{H}_k^-(p) = \begin{cases} \overline{H}_k(p) & \text{if } p \le p_k, \\ \overline{H}_k(p_k) & \text{if } p \ge p_k, \end{cases} \quad \text{and} \quad \overline{H}_k^+(p) = \begin{cases} \overline{H}_k(p_k) & \text{if } p \le p_k, \\ \overline{H}_k(p) & \text{if } p \ge p_k. \end{cases} \tag{2.19}$$

Remark 2.3. In [29], authors showed that equation (2.17) is equivalent (deriving in space) to the model introduced in [30].

2.3. The micro-macro connection

To establish the micro-macro connection, we need to introduce two cumulative functions ρ and σ. Unlike the case of simple road (like in [19]), the pedestrians are not well ordered. In fact, if $i \in \mathcal{L}_1$ and $j \in \mathcal{L}_2$, and $i < j$, we don't necessarily have $U_i(t) < U_j(t)$. However, we know that if $i, j \in \mathcal{L}_1 \cup \mathbb{N}^*$, and $i < j$, then

$$U_i(t) < U_j(t). \tag{2.20}$$

Similarly, (2.20) holds if $i, j \in \mathcal{L}_2 \cup \mathbb{N}^*$. We have $\rho(t, U_i(t)) = -i$ (resp. $\sigma(t, U_i(t)) = -i$) for $i \in \mathcal{L}_1 \cup \mathbb{N}^*$ (resp. $i \in \mathcal{L}_2 \cup \mathbb{N}^*$). The functions ρ and σ are defined as follows,

$$\rho\,(t, y) = -\left(\sum_{i\geq 1} H(y - U_i(t)) + G(y - U_{-1}(t)) + 2 \sum_{i=-2k-1, k\in\mathbb{N}^*} G(y - U_i(t))\right) \tag{2.21}$$

and

$$\sigma\,(t, y) = -\left(\sum_{i\geq 1} H\,(y - U_i(t)) + 2 \sum_{i=-2k, k\in\mathbb{N}^*} G(y - U_i(t)\right), \tag{2.22}$$

with

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

and

$$G(x) = \begin{cases} 0 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0. \end{cases}$$

The main result of this paper is given by the following theorem.

Theorem 2.4 (Junction condition by homogenization: application to pedestrians flow)**.** *Assume (A) and that at initial time, we have*

$$\begin{cases} U_i(0) \leq U_{i+1}(0) - h_0 & \text{if } i \in \mathbb{N}\ , \\ U_i(0) \leq U_{i+2}(0) - h_0 & \text{if } i \in \mathbb{Z} - \mathbb{N}\ . \end{cases} \tag{2.23}$$

We also assume that there exists $R > 0$ such that

$$\begin{cases} U_{i+2}(0) - U_i(0) = h_1 & \text{if } i \in \mathcal{L}_1 \text{ and } U_i(0) < -R, \\ U_{i+2}(0) - U_i(0) = h_2 & \text{if } i \in \mathcal{L}_2 \text{ and } U_i(0) < -R, \\ U_{i+1}(0) - U_i(0) = h_3 & \text{if } i \in \mathbb{N}^* \text{ and } U_i(0) \geq R. \end{cases} \tag{2.24}$$

with $h_1, h_2, h_3 \geq h_0$. We define two functions u_0 and v_0 as follows

$$u_0(x) = -\frac{2}{h_1} x 1_{\{x<0\}} - \frac{1}{h_3} x 1_{\{x\geq 0\}},$$
$$v_0(x) = -\frac{2}{h_2} x 1_{\{x<0\}} - \frac{1}{h_3} x 1_{\{x\geq 0\}}.$$

Let $\varepsilon > 0$ and $\chi^\varepsilon : \mathbb{R}^+ \times J \to \mathbb{R}$ be the function defined by

$$\chi^\varepsilon(t,x) = \begin{cases} \rho^\varepsilon(t, -d(0,x)) & \text{if } x \in J_1^*, \\ \sigma^\varepsilon(t, -d(0,x)) & \text{if } x \in J_2^*, \\ \rho^\varepsilon(t, d(0,x)) & \text{if } x \in J_3. \end{cases} \tag{2.25}$$

with

$$\rho^\varepsilon(t,x) = \varepsilon\rho\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \quad \sigma^\varepsilon(t,x) = \varepsilon\sigma\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

We define

$$\overline{u}_0(x) = \begin{cases} u_0(-d(0,x)) & \text{if } x \in J_1^*, \\ v_0(-d(0,x)) & \text{if } x \in J_2^*, \\ u_0(d(0,x)) & \text{if } x \in J_3. \end{cases}$$

Then there exists a unique $\overline{A} \in [H_0, 0]$ such that the function χ^ε converges towards the unique solution u^0 of (2.17).

Remark 2.5. The proof of this result will be done through three steps.

1) Inject the cumulative distributions function into a non-local PDE.
2) Couple the PDE with suitable initial conditions and prove the convergence to the solution of (2.17).
3) Deduce Theorem 2.4.

3. The non-local equation

In this section, we first define the non-local operators and then show that (ρ, σ, τ) is a viscosity solution of a non-local PDE. To do this, we use the function $\phi(x,a,b) = -\bar{\phi}(x,-a,-b)$ where $\bar{\phi}$ is defined in (2.3). We remark that $\phi(x,a,\cdot)$ and $\phi(x,\cdot,b)$ are non-decreasing functions which is a crucial property to obtain the comparison principle (see Prop. 4.2).

3.1. Definition of the non-local operators

In this subsection, we give the definition of the non-local operators. To do this, we first introduce the following functions. Let $a \in \mathbb{R}$. We define

$$E_a(z) = \begin{cases} 0 & \text{if } z \geq a \\ 1 & \text{if } z < a, \end{cases} \quad \tilde{E}_a(z) = \begin{cases} 0 & \text{if } z > a \\ 1 & \text{if } z \leq a, \end{cases} \tag{3.1}$$

and

$$F(z) = \begin{cases} 0 & \text{if } z > -1 \\ -1 & \text{if } z \leq -1, \end{cases} \quad \tilde{F}(z) = \begin{cases} 0 & \text{if } z \geq -1 \\ -1 & \text{if } z < -1. \end{cases} \tag{3.2}$$

Let $U, V : \mathbb{R} \to \mathbb{R}$. We define the following non-local operators: for $i = 1, 2$,

$$M_i(U)(x) = \int_0^{+\infty} V_i'(z) E_{-2}\left(U(x+z) - U(x)\right) \mathrm{d}z - V_{\max}, \tag{3.3}$$

$$N(U, [V])(x) = \int_{z \geq x} V_3'(|z| - x) E_{-1}\left(V(z) - U(x)\right) \mathrm{d}z - V_{\max}, \tag{3.4}$$

and

$$K(U, [V])(x) = \int_{\mathbb{R}} \omega_z(x, z, |x| - |z|) F\left(V(z) - U(x)\right) \mathrm{d}z + 1, \tag{3.5}$$

with ω defined in (2.4). Using the non-local operators defined above, we introduce

$$R_1\left(x, U, [V]\right)(x) = \phi\left(x, M_1(U)(x), N(U, [V])(x)\right) K(U, [V])(x), \tag{3.6}$$
$$R_2\left(x, V, [U]\right)(x) = \phi\left(x, M_2(V)(x), N(V, [U])(x)\right) \cdot K(V, [U])(x). \tag{3.7}$$

Remark 3.1 (Comments on the non-local operators definition)**.** Using the non-local operators defined above, we can inject the ODE (2.10) into a non-local PDE (see Lem. 3.4). In particular, we have that

$$\begin{cases} M_1\left(\rho(t,\cdot)\right)\left(U_i(t)\right) = -V_1(U_{i+2}(t) - U_i(t)) & \text{if } i \in \mathcal{L}_1, \\ M_2\left(\sigma(t,\cdot)\right)\left(U_i(t)\right) = -V_2(U_{i+2}(t) - U_i(t)) & \text{if } i \in \mathcal{L}_2 \backslash \{0\}, \\ N(\rho(t, U_i(t)), [\sigma(t,\cdot)])(U_i(t)) = -V_3(|U_{i+1}(t)| - U_i(t)) & \text{if } i \in \mathcal{L}_1 \cup \mathbb{N}^*, \\ N\left(\sigma(t, U_i(t)), [\rho(t,\cdot)]\right)\left(U_i(t)\right) = -V_3(|U_{i+1}(t)| - U_i(t)) & \text{if } i \in \mathcal{L}_2, \\ K(\rho(t, U_i(t)), [\sigma(t,\cdot)])(U_i(t)) = \omega\left(U_i(t), U_{i+1}(t), |U_i(t)| - |U_{i+1}(t)|\right) & \text{if } i \in \mathcal{L}_1 \cup \mathbb{N}^*, \\ K\left(\sigma(t, U_i(t)), [\rho(t,\cdot)]\right)\left(U_i(t)\right) = \omega\left(U_i(t), U_{i+1}(t), |U_i(t)| - |U_{i+1}(t)|\right) & \text{if } i \in \mathcal{L}_2. \end{cases}$$

Remark that in the definition of M_i, the variable z is positive since it models the distance between U_i and its leader whose position is on the same branch or on J_3 (looking ahead). In the definition of N, the variable z models the leader on the other branch, U_{i+1}. We take $z > x$ since for $z \leq x$, the velocity of U_i is zero due to the function ω. In the definition of K the variable $z \in \mathbb{R}$ since it's modeling the pedestrian U_{i+1}.

In the same way, we define $\tilde{M}_i, \tilde{N}$ and $\tilde{K}$ replacing E_a and F respectively by $\tilde{E}_a$ and $\tilde{F}$ and then we get the definition of $\tilde{R}_1$ and $\tilde{R}_2$. Finally, we can easily remark that

$$-V_{\max} \leq R_1\left(x, U, [V]\right)(x), R_2\left(x, V, [U]\right)(x) \leq 0. \tag{3.8}$$

For $\varepsilon > 0$, we define the following non-local operators: for $i = 1, 2$,

$$M_i^{\varepsilon}(U)(x) = \int_0^{+\infty} V_i'(z) E_{-2}\left(U(x + \varepsilon z) - U(x)\right) \mathrm{d}z - V_{\max} \tag{3.9}$$

$$N^{\varepsilon}(U, [V])(x) = \int_{z \geq \frac{x}{\varepsilon}} V_3'\left(|z| - \frac{x}{\varepsilon}\right) E_{-1}\left(V(\varepsilon z) - U(x)\right) \mathrm{d}z - V_{\max}, \tag{3.10}$$

and

$$K^{\varepsilon}(U, [V])(x) = \int_{\mathbb{R}} \omega_z\left(\frac{x}{\varepsilon}, z, \left|\frac{x}{\varepsilon}\right| - |z|\right) F\left(V(\varepsilon z) - U(x)\right) \mathrm{d}z + 1. \tag{3.11}$$

We then define

$$R_1^\varepsilon\left(\frac{x}{\varepsilon}, U, [V]\right)(x) = \phi\left(\frac{x}{\varepsilon}, M_1^\varepsilon(U)(x), N^\varepsilon(U,[V])(x)\right) \cdot K^\varepsilon(U,[V])(x), \tag{3.12}$$

$$R_2^\varepsilon\left(\frac{x}{\varepsilon}, V, [U]\right)(x) = \phi\left(\frac{x}{\varepsilon}, M_2^\varepsilon(V)(x), N^\varepsilon(V,[U])(x)\right) \cdot K^\varepsilon(V,[U])(x). \tag{3.13}$$

In the same way, we define $\tilde{M}_i^\varepsilon, \tilde{N}^\varepsilon$ and $\tilde{K}^\varepsilon$ replacing E_a and F respectively by $\tilde{E}_a$ and $\tilde{F}$ and then we get the definition of $\tilde{R}_1^\varepsilon$ and $\tilde{R}_2^\varepsilon$.

3.2. The non-local PDE

In this subsection, we prove that (ρ,σ) is a discontinuous viscosity solutions of the following non-local PDE,

$$\begin{cases} \rho_t(t,x) + R_1\left(x, \rho(t,x), [\sigma(t,.)]\right)(x)|\rho_x(t,x)| = 0 & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R} \ , \\ \sigma_t(t,x) + R_2\left(x, \sigma(t,x), [\rho(t,.)]\right)(x)|\sigma_x(t,x)| = 0 & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R} \ , \end{cases} \tag{3.14}$$

with R_1 and R_2 defined in (3.6) and (3.7). We use the definition of viscosity solutions introduced in [36]. This definition allows to have a stability result for the non-local term. We refer to Proposition 4.2 of [15] for the corresponding stability result. We give now the definition of viscosity solutions of (3.14).

Definition 3.2 (Viscosity solutions for (3.14))**.** Let $u, v : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ be upper semi-continuous (resp. lower semi-continuous) functions. We say that (u,v) is a viscosity sub-solution (resp. super-solution) of (3.14) on $(0,+\infty)\times\mathbb{R}$, if we have the following:

1) if $\varphi \in C^1([0,+\infty)\times\mathbb{R})$ such that $u-\varphi$ reaches a maximum (resp. a minimum) at the point (t,x), we have

$$\varphi_t(t,x) + R_1\left(x, u(t,x), [v(t,.)]\right)|\varphi_x(t,x)| \leq 0,$$
$$(\text{resp. } \varphi_t(t,x) + \tilde{R}_1\left(x, u(t,x), [v(t,.)]\right)|\varphi_x(t,x)| \geq 0) \ .$$

2) If $\varphi \in C^1([0,+\infty)\times\mathbb{R})$ such that $v-\varphi$ reaches a maximum (resp. a minimum) at the point (t,x), we have

$$\varphi_t(t,x) + R_2\left(x, v(t,x), [u(t,.)]\right)|\varphi_x(t,x)| \leq 0,$$
$$(\text{resp. } \varphi_t(t,x) + \tilde{R}_2\left(x, v(t,x), [u(t,.)]\right)|\varphi_x(t,x)| \geq 0) \ .$$

We say that (u,v) is a viscosity solution of (3.14) if (u^*,v^*) and (u_*,v_*) are respectively a sub-solution and a super-solution of (3.14).

We have the following theorem.

Theorem 3.3. *The function (ρ,σ) is a discontinuous viscosity solution of (3.14). Conversely, if (u,v) are bounded and continuous viscosity solution of (3.14) satisfying for some time $T>0$, and for all $t\in(0,T)$*

$$u(t,\cdot), v(t,\cdot) \textit{ are decreasing},$$

then the points $U_i(t)$, defined by

$$\begin{cases} u(t,U_i(t)) = -i & \textit{for } i \in \mathcal{L}_1 \cup \mathbb{N}^*, \\ v(t,U_i(t)) = -i & \textit{for } i \in \mathcal{L}_2 \cup \mathbb{N}^*, \end{cases}$$

are viscosity solutions of (2.10) *on* $(0,T)$.

The proof of Theorem 3.3 is an easy adaptation of the proof of Theorem 7.1 in [17] using the following lemma.

Lemma 3.4. *Let* $(U_i)_{i\in\mathbb{Z}}$ *be the solution of* (2.10). *Then we have*

$$\begin{cases} U_i'(t) = -R_1\left(U_i(t), \rho(t,U_i(t)), [\sigma(t,\cdot)]\right)(U_i(t)) & \text{if } i \in \mathcal{L}_1 \cup \mathbb{N}^*, \\ U_i'(t) = -R_2\left(U_i(t), \sigma(t,U_i(t)), [\rho(t,\cdot)]\right)(U_i(t)) & \text{if } i \in \mathcal{L}_2. \end{cases} \tag{3.15}$$

Proof. We skip the detailed proof of this technical lemma since it can be obtained by simple calculations using the definition of the non-local operators. We refer to the Proof of Lemma 7.2 in [17]. □

4. Results for the non-local PDE with initial conditions

In this section, we consider the following PDE:

$$\begin{cases} u_t(t,x) + R_1\left(x, u(t,x), [v(t,.)]\right)(x)|u_x(t,x)| = 0 & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R} , \\ v_t(t,x) + R_2\left(x, v(t,x), [u(t,.)]\right)(x)|v_x(t,x)| = 0 & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R} , \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}, \\ v(0,x) = v_0(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{4.1}$$

We will give first some classical results like the comparison principle and existence *via* Perron's method, then we will prove a gradient estimate result far from the junction point. The initial conditions u_0 and v_0 are lipschitz continuous functions. In addition, to control the gradient of the limit solution of (2.17), we need the following assumption. Let $k_0 = 1/h_0$.

(A0) The functions u_0 and v_0 are non-increasing. Let R^1 the positive parameter appearing in the definition (2.3) of $\bar{\phi}$. We assume that for $y < x < -R^1$,

$$\begin{cases} -2k_0(x-y) - 2 \le u_0(x) - u_0(y), \\ -2k_0(x-y) - 2 \le v_0(x) - v_0(y), \end{cases}$$

and for $x > y > 0$,

$$-k_0(x-y) - 1 \le u_0(x) - u_0(y).$$

Remark 4.1. The constante k_0 is the maximal pedestrian's density. The initial conditions u_0 and v_0 in Theorem 2.4 satisfy condition (A0). The definition of u_0 and v_0 is crucial to get the homogenization result in Theorem 2.4.

4.1. Recall some existence and uniqueness results for (4.1)

The definition of viscosity solutions for (4.1) is the same as in Definition 3.2 for $t > 0$ and for $t = 0$, we will add the following inequality for the sub-solution (resp. super-solution),

$$u(0,x) \le u_0(x) \text{ and } v(0,x) \le v_0(x),$$
$$(\text{resp. } u(0,x) \ge u_0(x) \text{ and } v(0,x) \ge v_0(x)).$$

We begin first with the comparison principle.

Proposition 4.2 (Comparison principle for (4.1))**.** *Assume (A). Let (u,v) be a visocsity sub-solution of (4.1) and $(\hat{u},\hat{v})$ be a viscosity super-solution of (4.1) in the sense introduced of Definition 3.2. Let us also assume that there exists a constant $K>0$ such that for all $t>0$,*

$$\begin{aligned} u(t,x) &\le u_0(x)+Kt \quad \text{and} \quad -\hat{u}(t,x)\le -u_0(x)+Kt \quad \text{for } x\in\mathbb{R},\\ v(t,x) &\le v_0(x)+Kt \quad \text{and} \quad -\hat{v}(t,x)\le -v_0(x)+Kt \quad \text{for } x\in\mathbb{R}. \end{aligned}$$

Then we have for all $t>0$,

$$\begin{aligned} u(t,x) &\le \hat{u}(t,x) \quad \text{for } x\in\mathbb{R},\\ v(t,x) &\le \hat{v}(t,x) \quad \text{for } x\in\mathbb{R}. \end{aligned}$$

Proof. The proof is very similar to the one in [18] and uses the monotony of E_{-2}, F, ϕ and w, so we skip it. □

In the proof of convergence (see Sect. 6), we need a comparison principle on bounded sets. For a given point $(t_0,x_0)\in(0,T)\times\mathbb{R}$ and for $\overline{r},\overline{R}>0$, we define the set

$$\mathcal{P}_{\overline{r},\overline{R}}(t_0,x_0)=(t_0-\overline{r},t_0+\overline{r})\times(x_0-\overline{R},x_0+\overline{R}). \tag{4.2}$$

Theorem 4.3 (Comparison principle on bounded sets for (4.1))**.** *Assume (A). If (u,v) and $(\hat{u},\hat{v})$ are respectively sub-solution and super-solution of (4.1) on the open set $\mathcal{P}_{\overline{r},\overline{R}}$ such that*

$$u\le\hat{u},\quad v\le\hat{v}\quad \text{outside } \mathcal{P}_{\overline{r},\overline{R}},$$

then

$$u\le\hat{u},\quad v\le\hat{v}\quad \text{in } \mathcal{P}_{\overline{r},\overline{R}}.$$

Proof. The proof of this theorem is similar to the one of Proposition 4.2, so we skip it. □

Lemma 4.4 (Existence of barriers for (4.1))**.** *Assume (A0) and (A). Let $K=2V_{\max}k_0$. We define*

$$\begin{aligned} (u^+(t,x),v^+(t,x)) &= (Kt+u_0(x),Kt+v_0(x)),\\ (u^-(t,x),v^-(t,x)) &= (u_0(x),v_0(x)). \end{aligned}$$

Then (u^+,v^+) and (u^-,v^-) are respectively super and sub solution of (4.1).

Proof. The proof is very simple. We just use the bounds of R_1 and R_2 (see (3.8)) and the gradient bounds (A0). □

Applying Perron's method (see [2] or [27] to see how to apply Perron's method for problems with non-local terms), joint to the comparison principle, we obtain the following result.

Theorem 4.5 (Existence and uniqueness of viscosity solutions for (4.1))**.** *Assume (A0) and (A). Then, there exists a unique continuous solution of* (4.1) *which satisfies*

$$|u(t,x) - u_0(x)| \leq Kt, \quad \text{and} \quad |v(t,x) - v_0(x)| \leq Kt. \tag{4.3}$$

4.2. Gradient estimates

In the following theorem, we obtain gradient estimates for the solution of (4.1). We use a localization argument in order to get precise gradient bounds on each branch.

Theorem 4.6 (Control of the oscillations)**.** *Assume (A0)-(A). Let $T > 0$ and (u, v) be the solution of* (4.1) *on $(0, T)$ given in Theorem 4.5. For all $t > 0$, the functions $u(t, \cdot)$ and $v(t, \cdot)$ are non-increasing. Let $x < -R^1$ and $h > 0$ small enough such that $x + h < -R^1$ and $h < h_0$. For all $t \in (0, T)$, we have*

$$-2k_0h - 2 \leq u(t, x + h) - u(t, x), \tag{4.4}$$
$$-2k_0h - 2 \leq v(t, x + h) - v(t, x). \tag{4.5}$$

Let $x > 0$ and $0 < h < h_0$. For all $t \in (0, T)$, we have

$$-k_0h - 1 \leq u(t, x + h) - u(t, x). \tag{4.6}$$

Proof. The proof of monotony is similar to the one in Theorem 4.10 of [19]. We will only do the proof of (4.4) since the other inequalities can be proved in the same way. We use the technique introduced in [4] for the proof of local gradient estimates. Let $x_0 < -R^1$ and $\delta > 0$ small enough such that

$$\begin{cases} h + 2\delta < h_0, \\ x_0 + h + \delta < -R^1. \end{cases} \tag{4.7}$$

We will prove for all $t \in (0, T)$, $y \in B_\delta(x_0)$ and $x \in B_\delta(x_0 + h)$, that

$$-2k_0(x - y) - 2 - L_\delta(y - x_0)^2 - L_\delta(x - x_0 - h)^2 \leq u(t, x) - u(t, y) \tag{4.8}$$

with $L_\delta = \dfrac{KT}{\delta^2}$ and K defined in (4.3). In particular, taking $y = x_0$ and $x = x_0 + h$, we obtain (4.4). To prove (4.8), we introduce

$$\Delta = [0, T) \times B_\delta(x_0 + h) \times B_\delta(x_0)$$

and consider the following supremum

$$M = \sup_{(t,x,y) \in \Delta} \left\{ u(t, y) - u(t, x) - 2k_0(x - y) - 2 - L_\delta(x - x_0 - h)^2 - L_\delta(y - x_0)^2 \right\}.$$

We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 1: the test function. For $\eta > 0$ small, we define

$$\varphi(t, x, y) = u(t, y) - u(t, x) - 2k_0(x - y) - 2 - L_\delta(x - x_0 - h)^2 - L_\delta(y - x_0)^2 - \frac{\eta}{T - t}.$$

Since φ is continuous, it reaches a maximum on Δ at a point that we denote by $(\bar{t}, \bar{x}, \bar{y})$. Classically we have for η small enough,

$$0 < \frac{M}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}).$$

Step 2: $\bar{t} > 0$. By contradiction, assume first that $\bar{t} = 0$. Then we have

$$\frac{\eta}{T} < u_0(\bar{y}) - u_0(\bar{x}) - 2k_0(\bar{x} - \bar{y}) - 2 \leq 0,$$

where we used (A0).

Step 3: $|\bar{x} - x_0 - h| \neq \delta$ and $|\bar{y} - x_0| \neq \delta$. By contradiction, assume that $|\bar{x} - x_0 - h| = \delta$. Using the barriers and (A0), we get that

$$\begin{aligned} 0 < \frac{\eta}{T} &\leq u_0(\bar{y}) - u_0(\bar{x}) - 2k_0(\bar{x} - \bar{y}) - 2 - KT - L_\delta \delta^2 \\ &\leq K_1 T - L_\delta \delta^2 = 0 \end{aligned}$$

where we used the definition of L_δ. In the same way, we have $|\bar{y} - x_0| \neq \delta$.

Step 4: using of the equation. By doing a duplication of the time variable and passing to the limit in this duplication parameter, we get that

$$\begin{aligned} \frac{\eta}{(T - \bar{t})^2} &\leq \tilde{M}_1\left(u(\bar{t}, \cdot)\right)(\bar{x}) \cdot |-2k_0 - 2L_\delta(\bar{x} - x_0 - h)| \\ &\quad - M_1\left(u(\bar{t}, \cdot)\right)(\bar{y}) \cdot |-2k_0 + 2L_\delta(\bar{y} - x_0)| \end{aligned} \tag{4.9}$$

where we used the fact that $\bar{\phi}(x, a, b) = a$ and $\omega(x, y, p) = 1$ for $x < -R^1$. We claim that

$$M_1\left(u(\bar{t}, \cdot)\right)(\bar{y}) = 0. \tag{4.10}$$

In fact if (4.10) is true we will obtain a contradiction in (4.9) since $\tilde{M}_1 \leq 0$. The definition of M_1 can be written for $z \geq h_0$ since for $z < h_0$, we have $V_1' = 0$. We have

$$M_1\left(u(\bar{t}, \cdot)\right)(\bar{y}) = \int_{h_0}^{+\infty} V_1'(z) E_{-2}(u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}))\mathrm{d}z - V_{\max}.$$

Using that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, we have

$$u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) < -2.$$

If $z > h_0$, using (4.7), we have

$$\bar{y} + z > x_0 - \delta + h_0 > x_0 + \delta + h > \bar{x}.$$

Since $u(t, \cdot)$ is non-increasing, we get for $z > h_0$,

$$u(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) < -2$$

which implies $E_{-2}(u(\bar{t},\bar{y}+z)-u(\bar{t},\bar{y}))=1$ and then (4.10). This ends the proof. □

5. Construction of correctors

In this section, we construct the corrector far and near the junction point, which allow us to use the perturbed test method introduced by Evans in [14].

5.1. Corrector far from the junction point

Proposition 5.1 (Homogenization far from the junction point)**.** *Assume (A). For $p\le 0$, we define the following non-local operator:*

$$M_i^p(U)(x)=\int_0^{+\infty} V_i'(z)E_{-2}\left(U(x+z)-U(x)+pz\right)\mathrm{d}z-V_{\max}, \tag{5.1}$$

$$M^p(U,[V])(x)=\int_0^{+\infty} V_3'(z)E_{-1}\left(V(x+z)-U(x)+pz\right)\mathrm{d}z-V_{\max} \tag{5.2}$$

with E_a defined in (3.1).

1) For every $p\le 0$, there exists a unique $\lambda_1\in\mathbb{R}$ such that there exists a viscosity solution u of

$$\begin{cases} M_1^p(u)(x)|p+u_x|=\lambda_1 & x\in\mathbb{R}\\ u \text{ is bounded.}\end{cases} \tag{5.3}$$

2) For every $p\le 0$, there exists a unique $\lambda_2\in\mathbb{R}$ such that there exists a viscosity solution v of

$$\begin{cases} M_2^p(v)(x)|p+v_x|=\lambda_2 & x\in\mathbb{R}\ ,\\ v \text{ is bounded.}\end{cases} \tag{5.4}$$

3) For every $p\le 0$, there exists a unique $\lambda_3\in\mathbb{R}$ such that there exists a viscosity solution (u,v) of

$$\begin{cases} M^p(u(x),[v])(x)|p+u_x|=\lambda_3 & x\in\mathbb{R}\ ,\\ M^p(v(x),[u])(x)|p+v_x|=\lambda_3 & x\in\mathbb{R}\ .\\ u,v \text{ are bounded.}\end{cases} \tag{5.5}$$

In particular, $\lambda_i=\overline{H}_i(p)$ with $\overline{H}_i$ defined in (2.11),(2.12) and (2.13).

Proof. We can easily verify that

$$\begin{cases} u=0 \text{ is a solution of (5.3) with } \lambda_1=-|p|V_1\left(\dfrac{-2}{p}\right)\\ v=0 \text{ is a solution of (5.4) with } \lambda_2=-|p|V_2\left(\dfrac{-2}{p}\right)\\ (u,v)=(0,0) \text{ is a solution of (5.5) with } \lambda_3=-|p|V_3\left(\dfrac{-1}{p}\right).\end{cases}$$

The uniqueness of λ_i is a classical result and the reader can refer to [32] for the proof. □

5.2. Correctors at the junction point

In this subsection, we will construct correctors near the junction point. To do this, we will consider the following equation: for $\lambda \in \mathbb{R}$, we consider a viscosity solution $(\mathtt{u}, \mathtt{v})$ of

$$\begin{cases} R_1\left(x, \mathtt{u}(x), [\mathtt{v}]\right)(x)|\mathtt{u}_x| = \lambda & x \in \mathbb{R}, \\ R_2\left(x, \mathtt{v}(x), [\mathtt{u}]\right)(x)|\mathtt{v}_x| = \lambda & x \in \mathbb{R} \end{cases} \tag{5.6}$$

where the non-local operators R_1 and R_2 are defined in (3.6) and (3.7). We now prove the existence result of the corrector. For $\overline{A} \geq H_0$ with H_0 defined in (2.15), we introduce the real numbers $\overline{p}^1_-, \overline{p}^2_-$ and $\overline{p}^3_+$ satisfying

$$\overline{H}_1(\overline{p}^1_-) = \overline{H}_1^-(\overline{p}^1_-) = \overline{A}, \quad \overline{H}_2(\overline{p}^2_-) = \overline{H}_2^-(\overline{p}^2_-) = \overline{A}, \quad \overline{H}_3(\overline{p}^3_+) = \overline{H}_3^+(\overline{p}^3_+) = \overline{A}.$$

Theorem 5.2 (Existence of a global corrector for the junction)**.** *Assume (A).*

i) (General properties) There exists a constant $\bar{A} \in [H_0, 0]$ such that there exists a solution $(\mathtt{u}, \mathtt{v})$ of (5.6) with $\lambda = \overline{A}$ and such that there exists a constant C and a globally Lipschitz continuous function m such that

$$|\mathtt{u}(x) - m(x)| \leq C, \quad |\mathtt{v}(x) - m(x)| \leq C. \tag{5.7}$$

ii) (Rescaling) For $\varepsilon > 0$, we set

$$\mathtt{u}^\varepsilon(x) = \varepsilon \mathtt{u}\left(\frac{x}{\varepsilon}\right), \quad \mathtt{v}^\varepsilon(x) = \varepsilon \mathtt{v}\left(\frac{x}{\varepsilon}\right).$$

Then (along a subsequence $\varepsilon_n \to 0$), we have the following convergence locally uniformly: $\mathtt{u}^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} U$ and $\mathtt{v}^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} V$ with U and V satisfying

$$\begin{cases} |U(x) - U(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \overline{H}_1(U_x) = \overline{A} & \text{for all } x < 0, \\ \overline{H}_3(U_x) = \overline{A} & \text{for all } x > 0, \end{cases} \tag{5.8}$$

and

$$\begin{cases} |V(x) - V(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \overline{H}_2(V_x) = \overline{A} & \text{for all } x < 0, \\ \overline{H}_3(V_x) = \overline{A} & \text{for all } x > 0. \end{cases} \tag{5.9}$$

In particular, we have (with $U(0) = V(0) = 0$)

$$U(x) = \overline{p}^3_+ x 1_{\{x>0\}} + \overline{p}^1_- x 1_{\{x<0\}}, \tag{5.10}$$

$$V(x) = \overline{p}^3_+ x 1_{\{x>0\}} + \overline{p}^2_- x 1_{\{x<0\}}. \tag{5.11}$$

iii)(Uniqueness of the flux limiter $\overline{A}$) We define the following set of functions

$$\mathcal{S} = \{(\mathtt{u}, \mathtt{v}) \text{ s.t. } \exists \text{Lipschitz function } m \text{ and } C \geq 0 \text{ satisfying } (5.7)\}.$$

Then we have

$$\overline{A} = \inf\{\lambda \in \mathbb{R} : \exists\ (\mathtt{u}, \mathtt{v}) \in \mathcal{S} \text{ solution of } (5.6)\}.$$

5.3. Proof of Theorem 5.2

This subsection contains the proof of Theorem 5.2. Let $R > R^1$ (R^1 the constant appearing in (2.3)), we want to find λ_R, such that there exists a solution (u^R, v^R) of

$$\begin{cases} G^1_R\left(x, u^R(x), [v^R], u^R_x\right) = \lambda_R \text{ if } x \in [-l_R, l_R], \\ G^2_R\left(x, v^R(x), [u^R], v^R_x\right) = \lambda_R \text{ if } x \in [-l_R, l_R], \end{cases} \tag{5.12}$$

with

$$\begin{aligned} G^1_R(x, U, [V], q) = {} & \psi^-_R(x)\psi^+_R(x)\phi\left(x, M_1(U)(x), N(U, [V])(x)\right) K(U, [V])(x)|q| \\ & + (1 - \psi^-_R(x)) \cdot \overline{H}^-_1(q) + (1 - \psi^+_R(x)) \cdot \overline{H}^+_3(q), \end{aligned} \tag{5.13}$$

$$\begin{aligned} G^2_R(x, V, [U], q) = {} & \psi^-_R(x)\psi^+_R(x)\phi\left(x, M_2(V)(x), N(V, [U])(x)\right) K(V, [U])(x)|q| \\ & + (1 - \psi^-_R(x)) \cdot \overline{H}^-_1(q) + (1 - \psi^+_R(x)) \cdot \overline{H}^+_3(q), \end{aligned} \tag{5.14}$$

and $\psi^+_R, \psi^-_R \in C^\infty$, $\psi^\pm_R : \mathbb{R} \to [0, 1]$, with $\psi^-_R(x) = \psi^+_R(-x)$,

$$\psi^+_R \equiv \begin{cases} 1 & \text{on } [-\infty, R] \\ 0 & \text{outside } [R+1, +\infty), \end{cases} \qquad \text{and } \psi^+_R \text{ non-increasing,}$$

and $l_R >> R + 1 \gg \max(h_{\max}, R^3 + r)$. To each operator G^i_R, we associate $\tilde{G}^i_R$ which is defined in the same way but replacing the non-local operators M_i, N and K by $\tilde{M}_i, \tilde{N}$ and $\tilde{K}$.

Proposition 5.3 (Comparison principle)**.** *Let $\lambda \in \mathbb{R}$,*

1) Let us consider the following problem,

$$\begin{cases} \tilde{G}^1_R\left(x, \hat{u}(x), [\hat{v}], \hat{u}_x\right) \geq \lambda \textit{ if } x \in (0, l_R], \\ \tilde{G}^2_R\left(x, \hat{v}(x), [\hat{u}], \hat{v}_x\right) \geq \lambda \textit{ if } x \in (0, l_R], \end{cases}$$

and for $\varepsilon > 0$

$$\begin{cases} G^1_R\left(x, u(x), [v], u_x\right) \leq \lambda - \varepsilon \textit{ if } x \in (0, l_R], \\ G^2_R\left(x, v(x), [u], v_x\right) \leq \lambda - \varepsilon \textit{ if } x \in (0, l_R]. \end{cases}$$

If

$$u(0) \leq \hat{u}(0), \quad v(0) \leq \hat{v}(0),$$

then for all $x \in [0, l_R]$, we have

$$u(x) \leq \hat{u}(x), \quad v(x) \leq \hat{v}(x).$$

2) Let us consider the following problem,

$$\tilde{G}^1_R\left(x, \hat{u}(x), [\hat{u}], \hat{u}_x\right) \geq \lambda \textit{ if } x \in [-l_R, -R^1),$$

and for $\varepsilon > 0$

$$G^1_R\,(x, u(x), [u], u_x) \le \lambda - \varepsilon \text{ if } x \in [-l_R, -R^1).$$

If

$$u(x) \le \hat{u}(x) \quad \text{for } x \in [-R^1 - h_{\max}, -R^1],$$

then for all $x \in [-l_R, -R^1]$ *, we have*

$$u(x) \le \hat{u}(x).$$

3) Let us consider the following problem,

$$\tilde{G}^2_R\,(x, \hat{v}(x), [\hat{v}], \hat{w}_x) \ge \lambda \text{ if } x \in [-l_R, -R^1),$$

and for $\varepsilon > 0$

$$G^2_R\,(x, v(x), [v], v_x) \le \lambda - \varepsilon \text{ if } x \in [-l_R, -R^1).$$

If

$$v(x) \le \hat{v}(x) \quad \text{for } x \in [-R^1 - h_{\max}, -R^1],$$

then for all $x \in [-l_R, -R^1]$ *, we have*

$$v(x) \le \hat{v}(x).$$

Proof. The proof of this proposition can be done using classical arguments of viscosity solutions theory, (see [5, 10]. Let us remark that the initial comparison between sub-solution and super-solution in points 2)-3) is given on the interval $[-R^1 - h_{\max}, -R^1]$. This condition will ensure, when doing the proof, that the maximum points (x, y) satisfy $x + z, y + z < -R^1$ for $z \in [0, h_{\max}]$. □

Proposition 5.4 (Existence of correctors)**.** *There exists a unique* $\lambda_R \in \mathbb{R}$ *such that there exists a solution* (u^R, v^R) *of (5.12). Moreover, there exists a constant* C *, and a Lipschitz continuous function* m^R *such that*

$$\begin{cases} H_0 \le \lambda_R \le 0, \\ |m^R(x) - m^R(y)| \le C|x - y| & \text{for } x \in [-l_R, l_R], \\ |u^R(x) - m^R(x)| \le C & \text{for } x \in [-l_R, l_R], \\ |v^R(x) - m^R(x)| \le C & \text{for } x \in [-l_R, l_R]. \end{cases} \tag{5.15}$$

Proof. In order to construct a corrector on the truncated domain, we will consider the approximated problem

$$\begin{cases} \delta u^\delta + G^1_R\left(x, u^\delta(x), [v^\delta], u^\delta_x\right) = 0 \text{ if } x \in [-l_R, l_R], \\ \delta v^\delta + G^2_R\left(x, v^\delta(x), [u^\delta], v^\delta_x\right) = 0 \text{ if } x \in [-l_R, l_R]. \end{cases} \tag{5.16}$$

Step 1: Construction of the solution.

Lemma 5.5. *There exists a unique viscosity solution* (u^δ, v^δ) *of* (5.16) *such that*

$$\begin{cases} 0 \le u^\delta(x) \le \dfrac{|H_0^1|}{\delta}, \\ 0 \le v^\delta(x) \le \dfrac{|H_0^2|}{\delta}. \end{cases} \tag{5.17}$$

Proof. We can easily show that,

$$(u^-, v^-) = (0,0) \text{ and } (u^+, v^+) = \left(\frac{|H_0^1|}{\delta}, \frac{|H_0^2|}{\delta}\right)$$

are respectively sub-solution and super-solution of (5.16). Using the comparison principle and the Perron's method, we construct a unique viscosity solution (u^δ, v^δ) satisfying (5.17).

□

Step 2: A new super-solution.

Lemma 5.6. *Let* $b_0 = \max(g_0, k_0)$ *and* $g_0 = 1/d_0$ *with* d_0 *defined in* (2.7)*. Let*

$$(u^{++}, v^{++}) = \left(-2b_0(x - l_R) + \frac{|H_0^3|}{\delta}, -2b_0(x - l_R) + \frac{|H_0^3|}{\delta}\right).$$

Then (u^{++}, v^{++}) *is a super-solution of* (5.16)*.*

Proof. Since $x \in [-l_R, l_R]$, we have $u^{++}, v^{++} \ge \dfrac{|H_0^3|}{\delta}$. In addition, using the definition of H_k^-, we remark that for $k = 1, 2, 3$,

$$H_k(-2b_0) \ge H_k(-2k_0) = H_k^-(-2k_0) \ge 0.$$

We recall that for $i = 1, 2, 3$, we have

$$V_i(h) = 0 \text{ if } h \le h_0, \quad \text{and } \omega(x, y, p) = 0 \text{ for } x \in [-R^3, 0], y < 0 \text{ and } p \le d_0. \tag{5.18}$$

We claim that

$$\begin{array}{lll} M_1(u^{++})(x) = 0 & \text{if } x \le -R^3, & \\ M_2(v^{++})(x) = 0 & \text{if } x \le -R^3, & \\ K(u^{++}(x), [v^{++}])(x) = 0 & K(v^{++}(x), [u^{++}])(x) = 0 & \text{if } x \in (-R^3, -d_0], \\ N(u^{++}(x), [v^{++}])(x) = 0 & N(v^{++}(x), [u^{++}])(x) = 0 & \text{if } x \ge 0. \end{array}$$

In addition, on $(-d_0, 0)$, we have

$$\begin{array}{lll} K(u^{++}(x), [v^{++}])(x) = 0 & K(v^{++}(x), [u^{++}])(x) = 0 & \text{if } \frac{h_0}{2} < d_0, \\ N(u^{++}(x), [v^{++}])(x) = 0 & N(v^{++}(x), [u^{++}])(x) = 0 & \text{if } \frac{h_0}{2} \ge d_0. \end{array}$$

We prove only four equalities since others are done in the same way. First, let $x \leq -R^3$,

$$\begin{aligned} M_1(u^{++})(x) + V_{\max} &= \int_0^{+\infty} V_1'(z) E_{-2}\left(u^{++}(x+z) - u^{++}(x)\right) \mathrm{d}z \\ &= \int_{h_0}^{+\infty} V_1'(z) E_{-2}\left(-2b_0 z\right) \mathrm{d}z. \end{aligned}$$

For $z \geq h_0$, $-2b_0 z \leq -2$. Using the definition of E_{-2} (see (3.1)), we get

$$M_1(u^{++})(x) = \int_{h_0}^{h_{\max}} V_1'(z) \mathrm{d}z - V_{\max} = 0.$$

Second, let $x \in (-R^3, -d_0]$,

$$\begin{aligned} K(u^{++}(x), [v^{++}])(x) - 1 &= \int_{\mathbb{R}} \omega_z(x, z, |x| - |z|) F\left(v^{++}(z) - u^{++}(x)\right) \mathrm{d}z \\ &= \int_{\mathbb{R}} \omega_z(x, z, |x| - |z|) F\left(-2b_0(z - x)\right) \mathrm{d}z \\ &= \int_{\mathbb{R}^-} \omega_z(x, z, |x| - |z|) F\left(-2b_0(z - x)\right) \mathrm{d}z \\ &\quad + \int_{\mathbb{R}^+} \omega_z(x, z, |x| - |z|) F\left(-2b_0(z - x)\right) \mathrm{d}z. \end{aligned}$$

Let $z \leq 0$: using (5.18), we remark that we can assume that $z \geq x + d_0$. In that case, we have $-2b_0(z - x) \leq -2b_0 d_0 = -2 < -1$. The last inequality is also true if we take $z > 0$. Using the definition of F (see (3.2)), we obtain

$$\begin{aligned} K(u^{++}(x), [v^{++}])(x) &= -\int_{x+d_0}^{0} \omega_z(x, z, |x| - |z|) \mathrm{d}z - \int_{\mathbb{R}^+} \omega_z(x, z, |x| - |z|) \mathrm{d}z + 1 \\ &= -\omega(x, 0, |x|) + \omega(x, x + d_0, d_0) - 1 + \omega(x, 0, |x|) + 1 = 0 \end{aligned}$$

where we used in the last line the fact that $\omega(x, z, p) = 1$ if $z \gg 0$. Finally, let $x \in (-d_0, 0)$: if $\frac{h_0}{2} \leq d_0$, then using (5.18), we evaluate $K(u^{++}(x), [v^{++}])(x)$ only for $z \in \mathbb{R}^+$ since $x + d_0 > 0 > z$. If $z \geq 0$, then $-2b_0(z - x) \leq 2b_0 x \leq -2b_0 \frac{h_0}{2} \leq -1$, and we get

$$K(u^{++}(x), [v^{++}])(x) = -\int_{\mathbb{R}^+} \omega_z(x, z, |x| - |z|) \mathrm{d}z = -1 + \omega(x, 0, |x|) + 1 = 0$$

where we used that $|x| = -x < d_0$. Assume now $\frac{h_0}{2} > d_0$. We have

$$\begin{aligned} N(u^{++}(x), [v^{++}])(x) + V_{\max} &= \int_{z \geq x} V_3'(|z| - x) E_{-1}\left(v^{++}(z) - u^{++}(x)\right) \mathrm{d}z \\ &= \int_{z \geq x} V_3'(|z| - x) E_{-1}\left(-2b_0(z - x)\right) \mathrm{d}z \\ &= \int_{0 > z \geq x} V_3'(-z - x) E_{-1}\left(-2b_0(z - x)\right) \mathrm{d}z \\ &\quad + \int_{z \geq 0} V_3'(z - x) E_{-1}\left(-2b_0(z - x)\right) \mathrm{d}z. \end{aligned}$$

We evaluate the above integral only if $z \geq 0$ since if $x \leq z < 0$, we get $-z - x \leq -2x < 2d_0 < h_0$ and so $V_3'(-z-x) = 0$. If $z \geq 0$ and $z - x \geq h_0$, we get that $-2b_0(z-x) \leq 2b_0 h_0 \leq -1$ and using the definition of E_{-1}, we get

$$N(u^{++}(x), [v^{++}])(x) = \int_{z \geq x + h_0} V_3'(z-x)\mathrm{d}z - V_{\max} = 0.$$

Recalling the definition of G_R^1 and G_R^2 (see (5.13) and (5.14)) and using the lower bound of u^{++} and v^{++}, we deduce that (u^{++}, v^{++}) is a super-solution of the approximated problem for $x \in (-l_R, l_R)$. Let us now treat the boundary conditions. Consider a test function φ touching u^{++} from below at $x = -l_R$ (resp. $x = l_R$). We have that

$$\delta\varphi(-l_R) + H_1^-(\varphi_x(-l_R)) \geq H_1^-(u_x^{++}(-l_R)) = H_1^-(-2b_0) \geq 0,$$
$$(\text{resp. } \delta\varphi(l_R) + H_3^+(\varphi_x(l_R)) \geq \delta u^{++}(l_R) + H_0^3 \geq 0).$$

We can argue in the same way for the function v^{++}. This implies that (u^{++}, v^{++}) is a super-solution of the approximated problem for $x \in [-l_R, l_R]$. Using the stability of viscosity super-solution, we can construct a new super-solution,

$$(\min(u^+, u^{++}), \min(v^+, v^{++})).$$

The comparison principle then implies that

$$\begin{cases} 0 \leq u^\delta(x) \leq \min(u^+, u^{++}), \\ 0 \leq v^\delta(x) \leq \min(v^+, v^{++}). \end{cases} \tag{5.19}$$

□

Remark 5.7. We need the non-constant new super-solution to ensure (after passing to the limit in δ and R) that the flux limiter $\overline{A}$ satisfies $\overline{A} \geq H_0$.

Step 3: bounds for the solution.

Lemma 5.8. *Let (u^δ, v^δ) be the solution of (5.16). The functions u^δ and v^δ are non-increasing and we have the following bounds,*

$$-1 \leq u^\delta(x) - v^\delta(x) \leq 1 \qquad \text{if } 0 \leq x \leq l_R, \tag{5.20}$$

and

$$-1 \leq u^\delta(x) - v^\delta(x) \leq 1 \qquad \text{if } -R^3 \leq x \leq 0,. \tag{5.21}$$

Proof. The proof of monotony is similar to that in Lemma 5.5 of [19] so we skip it.

We will do the proof of the lower bound in (5.21) (the proof of the upper bound is treated in the same way). Let $-R^3 < z_0 < 0$, $a > 0$ small and $L_a > \dfrac{|H_0^2|}{\delta a^2}$. We will prove that for $x \in B_a(z_0)$,

$$u^\delta(x) - v^\delta(x) \geq -1 - L_a(x - z_0)^2.$$

If the above inequality is true, then taking $x = z_0$, we obtain the first inequality in (5.21). For $\varepsilon > 0$, we define

$$M_\varepsilon = \sup_{x,y \in B_a(z_0)} \left\{ v^\delta(y) - u^\delta(x) - 1 - L_a(x - z_0)^2 - \frac{(x-y)^2}{2\varepsilon} \right\}.$$

We have that $M_\varepsilon \geq M > 0$ and since the functions v^δ and u^δ are continuous, M_ε is reached at a point $(\bar{x}, \bar{y})$. Classically, we have

$$0 < M_\varepsilon < \frac{|H_0^2|}{\delta} - \frac{(\bar{x}-\bar{y})^2}{2\varepsilon}$$

which implies $|\bar{x} - \bar{y}| \to 0$ as $\varepsilon \to 0$. If $|\bar{x} - z_0| = a$, and using the definition of L_a, we obtain

$$0 < \frac{|H_0^2|}{\delta} - L_a a^2 \leq 0$$

which gives a contradiction and implies that $|\bar{x} - z_0| \neq a$ and $|\bar{y} - z_0| \neq a$ for ε small enough. Writing the sub-solution viscosity inequality, we get

$$\delta < \delta v^\delta(\bar{y}) \leq -G_R^2 \left(\bar{y}, v^\delta(\bar{y}), [u^\delta], \frac{\bar{x} - \bar{y}}{\varepsilon} \right).$$

Using the definition of G_R^2 for $\bar{y} \in (-R^3, 0)$, we can see that we obtain a contradiction if we prove that

$$K\left(v^\delta(\bar{y}), [u^\delta]\right)(\bar{y}) = \int_{\mathbb{R}} \omega_z\left(\bar{y}, z, |\bar{y}| - |z|\right) F\left(u^\delta(z) - v^\delta(\bar{y})\right) \mathrm{d}z + 1 = 0. \tag{5.22}$$

Recalling that $\omega(x, y, p) = 0$ if $x \in (-R^3, 0), y < 0$ and $p < 0$, we can evaluate the above integral for $z \geq 0$. If we prove $u^\delta(z) - v^\delta(\bar{y}) < -1$ for all $z \geq \bar{y}$, we will get

$$\int_{\bar{y}}^{+\infty} \omega_z\left(\bar{y}, z, |\bar{y}| - |z|\right) F\left(u^\delta(z) - v^\delta(\bar{y})\right) \mathrm{d}z = -1$$

which implies (5.22). Using the supremum M_ε, we have

$$u^\delta(\bar{x}) - v^\delta(\bar{y}) < -1$$

and since $|\bar{x} - \bar{y}| \to 0$ as $\varepsilon \to 0$, we have for ε small enough

$$u^\delta(\bar{y}) - v^\delta(\bar{y}) < -1.$$

Using the fact that u^δ is non-increasing, we have for $z \geq \bar{y}$,

$$u^\delta(z) - v^\delta(\bar{y}) \leq u^\delta(\bar{y}) - v^\delta(\bar{y}) < -1$$

which gives the desired result.

The proof of (5.20) is similar. The difference that in this case (always to prove the lower bound), we prove $N\left(v^\delta(\bar{y}), [u^\delta]\right)(\bar{y}) = 0$.

□

Step 4: control of oscillations.

Lemma 5.9. *Let* (u^δ, v^δ) *be a solution of* (5.16). *Let* $g_0 = 1/d_0$ *and* $b_0 = \max(g_0, k_0)$. *We have that*

$$\begin{cases} -2b_0(x-y) - 2 \le u^\delta(x) - u^\delta(y) \le 0 & \text{for } -l_R \le y \le x \le l_R, \\ -2b_0(x-y) - 2 \le v^\delta(x) - v^\delta(y) \le 0 & \text{for } -l_R \le y \le x \le l_R. \end{cases} \tag{5.23}$$

Proof. We will only prove the first inequality in (5.23). To prove the other inequality, we proceed similarly.
We define

$$N = \sup_{-l_R \le y \le x \le l_R} \left\{ u^\delta(y) - u^\delta(x) - 2b_0(x-y) - 2 \right\}.$$

We want to prove that $N \le 0$. Assume by contradiction that $N > 0$. Classicaly, N is reached at a point $(\bar{x}, \bar{y})$ with $\bar{x} \ge \bar{y}$. Moreover $\bar{x} \ne \bar{y}$ since $N > 0$. Writing the viscosity inequalities, we get

$$\begin{aligned} \delta u^\delta(\bar{y}) - \delta u^\delta(x) &\le \tilde{G}^1_R\left(\bar{x}, u^\delta(\bar{x}), [v^\delta], -2b_0\right) \\ &\quad - G^1_R\left(\bar{y}, u^\delta(\bar{y}), [v^\delta], -2b_0\right) \\ &\le (1 - \psi^-_R(\bar{x}))\overline{H}^-_1(-2b_0) + (1 - \psi^+_R(\bar{x}))\overline{H}^+_3(-2b_0) \\ &\quad - G^1_R\left(\bar{y}, u^\delta(\bar{y}), [v^\delta], -2b_0\right). \end{aligned}$$

We claim that

$$\phi\left(\bar{y}, M_1\left(u^\delta\right)(\bar{y}), N\left(u^\delta(\bar{y}), [v^\delta]\right)(\bar{y})\right) K\left(u^\delta(\bar{y}), [v^\delta]\right)(\bar{y}) = 0. \tag{5.24}$$

If we prove (5.24), using the definition of G^1_R, we get

$$\begin{aligned} \delta u^\delta(\bar{y}) - \delta u^\delta(x) &\le (1 - \psi^-_R(\bar{x}))\overline{H}^-_1(-2b_0) + (1 - \psi^+_R(\bar{x}))\overline{H}^+_3(-2b_0) \\ &\quad - (1 - \psi^-_R(\bar{y}))\overline{H}^-_1(-2b_0) - (1 - \psi^+_R(\bar{y}))\overline{H}^+_3(-2b_0) \\ &= (\psi^-_R(\bar{y})) - \psi^-_R(\bar{x}))\overline{H}^-_1(-2b_0) + (\psi^+_R(\bar{y})) - \psi^+_R(\bar{x}))\overline{H}^+_3(-2b_0). \end{aligned}$$

Using that $H^-_1(-2b_0) \ge 0, H^+_3(-2b_0) \le 0$ and that ψ^- and ψ^+ are respectively non-decreasing and non-increasing, we get a contradiction.

Let us now prove (5.24). Using the definition of $\bar{\phi}$ (see (2.3)), we obtain (5.24) if we prove:

$$\begin{cases} M_1\left(u^\delta\right)(\bar{y}) = 0 & \text{if } \bar{y} < -R^3, \\ N\left(u^\delta(\bar{y}), [v^\delta]\right)(\bar{y}) = 0 & \text{if } \bar{y} > 0, \\ K\left(u^\delta(\bar{y}), [v^\delta]\right)(\bar{y}) = 0 & \text{if } \bar{y} \in [-R^3, 0]. \end{cases} \tag{5.25}$$

We will only show the third equality of (5.25) since the others can be treated in the same way and are even simpler. Recalling the definition of ω, and assuming that $\bar{y} + d_0 < 0$, we can write,

$$K\left(u^\delta(\bar{y}), [v^\delta]\right)(\bar{y}) - 1 = \int_{\bar{y}+d_0}^{+\infty} \omega_z\left(\bar{y}, z, |\bar{y}| - |z|\right) F\left(v^\delta(z) - u^\delta(\bar{y})\right) \mathrm{d}z.$$

We claim that $v^\delta(z) - u^\delta(\bar{y}) < -1$ for all $z > \bar{y} + d_0$. If the preceding inequality is true, we get $F\left(v^\delta(z) - u^\delta(\bar{y})\right) = -1$ and then

$$K\left(u^\delta(\bar{y}), [v^\delta]\right)(\bar{y}) = \omega\left(\bar{y}, \bar{y} + d_0, d_0\right) - 1 + 1 = 0.$$

We distinguish two cases:

Case 1: $\bar{x} < z$. Using the supremum, we have

$$u^\delta(\bar{x}) - u^\delta(\bar{y}) < -2.$$

The fact that u^δ is non-increasing implies

$$u^\delta(z) - u^\delta(\bar{y}) < -2. \tag{5.26}$$

Combining (5.26) and Lemma 5.8, we get $v^\delta(z) - u^\delta(\bar{y}) < -1$.

Case 2: $\bar{x} \geq z$. Using the supremum, we have $N \geq u^\delta(z) - u^\delta(\bar{x}) - 2b_0(\bar{x} - z) - 2$, which implies

$$u^\delta(z) - u^\delta(\bar{y}) \leq -2b_0(z - \bar{y}) \leq -2b_0 d_0 \leq -2. \tag{5.27}$$

Combining (5.27) and Lemma 5.8, we get $v^\delta(z) - u^\delta(\bar{y}) < -1$ which gives us the desired result.

If $\bar{y} + d_0 \geq 0$, in that case, we only evaluate the integral for $z \in [0, r]$ since the other one is zero. We proceed as above assuming that $\bar{x} < z$ and $\bar{x} \geq z$. We deduce that $K\left(u^\delta(\bar{y}), [v^\delta]\right)(\bar{y}) = 0$ and similarly (5.25). This ends the proof of the lemma.

□

Step 5: Proof of Proposition 5.4. As in [17], we construct a Lipschitz continuous function m^δ, such that there exists a constant C, (independent of R and δ) such that for $x, y \in [-l_R, l_R]$,

$$\begin{cases} |m^\delta(x) - m^\delta(y)| \leq C|x - y|, \\ |u^\delta(x) - m^\delta(x)| \leq C, \\ |v^\delta(x) - m^\delta(x)| \leq C \end{cases} \tag{5.28}$$

and then using the half relaxed limit method (for $\delta \to 0$) joint to Perron method, we can construct a solution (u^R, v^R) of (5.12) and a lipschitz function m^R satisfying (5.15). The first inequality in (5.15) is a consequence of (5.19). The uniqueness of λ_R is classical so we skip it. This ends the proof of Proposition 5.4.

□

Proposition 5.10 (First definition of the flux limiter)**.** *The following limit exists (up to a subsequence)*

$$\overline{A} = \lim_{R \to +\infty} \lambda_R. \tag{5.29}$$

Moreover, we have

$$H_0 \leq \lambda_R, \overline{A} \leq 0,$$

with H_0 defined in (2.15).

Proof. This results is a direct consequence of the following bound on λ_R which is independent of R (see the barriers in (5.19)),

$$H_0 \leq \lambda_R \leq 0.$$

□

Proposition 5.11 (Control of the slopes)**.** *Assume that R is big enough. Let (u^R, v^R) be the solution of (5.12) given by Proposition 5.4. We also assume that up to a sub-sequence $\overline{A} = \lim_{R\to+\infty} \lambda_R > H_0$. Then there exists $\gamma_0 > 0$ such that for every $\gamma \in (0, \gamma_0)$, there exists a constant $\bar{C}$ such that for $x > 0$ and $h \geq 0$,*

$$\begin{cases} u^R(x+h) - u^R(x) \geq (\overline{p}^3_+ - \gamma)h - \bar{C}, \\ v^R(x+h) - v^R(x) \geq (\overline{p}^3_+ - \gamma)h - \bar{C}. \end{cases} \tag{5.30}$$

If $x < -R^1$ and $h \geq 0$, then

$$u^R(x-h) - u^R(x) \geq (-\overline{p}^1_- - \gamma)h - \bar{C}, \tag{5.31}$$

and

$$v^R(x-h) - v^R(x) \geq (-\overline{p}^2_- - \gamma)h - \bar{C}. \tag{5.32}$$

Proof. We only prove (5.31) since the proof for (5.30) and (5.32) is similar. For $\mu > 0$ small enough, we denote by p^1_μ the real number such that

$$\overline{H}_1(p^1_\mu) = \overline{H}^-_1(p^1_\mu) = \lambda_R - \mu.$$

Using that $H_0 < \lambda_R \leq 0$, we deduce that p^1_μ exists, is unique and satisfies $-2k_0 \leq p^1_\mu \leq 0$ for μ small enough. We define for $-l_R \leq x < -R^1$,

$$u^- = p^1_\mu x.$$

We have that

$$\overline{H}^-_1(u^-_x) = \lambda_R - \mu.$$

We also have for $-l_R < x < -R^1$,

$$\begin{aligned} M_1(u^-)(x) + V_{\max} &= \int_{\mathbb{R}} V_1'(z) E_{-2}\left(p^1_\mu z\right) \mathrm{d}z \\ &= \int_{\frac{-2}{p^1_\mu}}^{+\infty} V_1'(z)\mathrm{d}z \\ &= V_{\max} - V_1\left(\frac{-2}{p^1_\mu}\right). \end{aligned}$$

We deduce that u^- satisfies for $-l_R \le x < -R^1$,

$$\psi_R^-(x)M_1(u^-)(x)|u_x^-| + (1-\psi_R^-(x)) \cdot \overline{H}_1^-(u_x^-) = \lambda_R - \mu.$$

Let $x_0 < -R^1$ and $\bar{C} = (2b_0 - p_\mu^1)h_{\max} + 2$. Using Proposition 5.4, we have for $x \in [x_0 - h_{\max}, x_0]$,

$$u^R(x) - u^R(x_0) \ge p_\mu^1(x - x_0) - \bar{C}.$$

Using the comparison principle (Prop. 5.3, point 2), we deduce that the above inequalities are true for all $x \le x_0$. Finally, if we choose $\gamma_0 < |p_1 - \bar{p}_-^1|$ (with p_1 defined in (2.15)), then

$$\overline{H}_1(\bar{p}_-^1 + \gamma) = \overline{H}_1^-(\bar{p}_-^1 + \gamma),$$

and we can choose $\mu > 0$ such that

$$p_1^\mu = \bar{p}_-^1 + \gamma$$

and this ends the proof.

□

Proof of Theorem 5.2. The proof of Theorem 5.2 can be deduced by all the above results. We first pass to the limit as $R \to +\infty$ then we study the uniform local convergence of the rescaled functions, see [17].

□

6. Proof of Convergence

In this section, we will show how to obtain our main result Theorem 2.4. Let $\varepsilon > 0$ and

$$u^\varepsilon(t,x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \qquad v^\varepsilon(t,x) = \varepsilon v\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

with (u,v) the unique viscosity solution of (4.1). Then $(u^\varepsilon, v^\varepsilon)$ is the unique viscosity solution for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ of

$$\begin{cases} u_t^\varepsilon + R_1^\varepsilon\left(\frac{x}{\varepsilon}, \frac{u^\varepsilon(t,\cdot)}{\varepsilon}, \left[\frac{v^\varepsilon(t,\cdot)}{\varepsilon}\right]\right)(x)|u_x^\varepsilon| = 0, \\ v_t^\varepsilon + R_2^\varepsilon\left(\frac{x}{\varepsilon}, \frac{v^\varepsilon(t,\cdot)}{\varepsilon}, \left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right]\right)(x)|v_x^\varepsilon| = 0, \\ u^\varepsilon(0,x) = u_0(x), \\ v^\varepsilon(0,x) = v_0(x). \end{cases} \tag{6.1}$$

First, we will prove the following convergence result.

Theorem 6.1 (Junction condition by homogenization)**.** *Assume (A) and (A0). For $\varepsilon > 0$, let $(u^\varepsilon, v^\varepsilon)$ be the solution of (6.1). Let χ^ε defined by*

$$\chi^\varepsilon(t,x) = \begin{cases} u^\varepsilon(t, -d(0,x)) & \text{if } x \in J_1^*, \\ v^\varepsilon(t, -d(0,x)) & \text{if } x \in J_2^*, \\ u^\varepsilon(t, d(0,x)) & \text{if } x \in J_3. \end{cases} \tag{6.2}$$

Then there exists $\overline{A} \in [H_0, 0]$ *such that* χ^ε *converges locally uniformly to the unique viscosity solution* u^0 *of* (2.17) *with the initial condition* $\overline{u}_0$ *defined by*

$$\overline{u}_0(x) = \begin{cases} u_0(-d(0,x)) & \text{if } x \in J_1^*, \\ v_0(-d(0,x)) & \text{if } x \in J_2^*, \\ u_0(d(0,x)) & \text{if } x \in J_3. \end{cases}$$

Remark 6.2. The same result remains true if we take v^ε on J_3.

The proof of Theorem 2.4 can be obtained using the convergence result in Theorem 6.1. The detailed proof is an easy adaptation of the proof of Theorem 2.6 in [17].

Theorem 6.1 gives also gradient estimates for the solution of (2.17).

Corollary 6.3. *Assume (A0)-(A). Let* u^0 *be the unique solution of* (2.17) *then* u^0 *is Lipschitz continuous and we have,*

$$\begin{cases} -2k_0 \le u_x^0 \le 0 & \text{if } x \in J_1^*, \\ -2k_0 \le u_x^0 \le 0 & \text{if } x \in J_2^*, \\ -k_0 \le u_x^0 \le 0 & \text{if } x \in J_3^*. \end{cases}$$

Proof of Corollary 6.3. The lipschitz continuity of u^0 was proved in [28].

Let $x, y \in J_1^*$ s.t $x = x_1 \cdot e_1$ and $y = (x_1 + h) \cdot e_1$ with $x_1 + h < 0$ and $h > 0$ small enough. For $\varepsilon > 0$ small enough, we have $x_1 + h < -\varepsilon R^1$. Using Theorem 4.6, we have

$$-2k_0 h - \varepsilon \le u^\varepsilon(t, x_1 + h) - u^\varepsilon(t, x_1) \le 0.$$

Using Theorem 6.1 and passing to the limit as $\varepsilon \to 0$, we obtain the desired result. We proceed similarly on J_2^* and J_3^* by using respectively (4.5) and (4.6).

□

We will do now the Proof of Theorem 6.1.

Proof of Theorem 6.1. We first define for a given point $(t_0, x_0) \in (0,T) \times J$ and for $\overline{r}, \overline{R} > 0$, the set

$$\mathcal{Q}_{\overline{r},\overline{R}}(t_0, x_0) = \begin{cases} \{(t,x) \in (0,T) \times J_i^* \text{ s.t } |t - t_0| \le \overline{r} \text{ and } d(x, x_0) \le \overline{R}\} & \text{if } x_0 \in J_i^*, \\ \{(t,x) \in (0,T) \times J \text{ s.t } |t - t_0| \le \overline{r} \text{ and } d(x, x_0) \le \overline{R}\} & \text{if } x_0 = 0. \end{cases}$$

We introduce

$$\overline{u}(t,x) = \limsup_{(s,y)\to(t,x),\varepsilon\to 0} u^\varepsilon(s,y) \quad \text{and} \quad \underline{u}(t,x) = \liminf_{(s,y)\to(t,x),\varepsilon\to 0} u^\varepsilon(s,y), \tag{6.3}$$

$$\overline{v}(t,x) = \limsup_{(s,y)\to(t,x),\varepsilon\to 0} v^\varepsilon(s,y) \quad \text{and} \quad \underline{v}(t,x) = \liminf_{(s,y)\to(t,x),\varepsilon\to 0} v^\varepsilon(s,y). \tag{6.4}$$

We then define the following functions

$$\overline{z}(t,x) = \begin{cases} \max(\overline{u}(t, d(0,x)), \overline{v}(t, d(0,x))) & \text{if } x \in J_3, \\ \overline{u}(t, -d(0,x)) & \text{if } x \in J_1^*, \\ \overline{v}(t, -d(0,x)) & \text{if } x \in J_2^*, \end{cases} \tag{6.5}$$

and

$$\underline{z}(t,x)=\begin{cases}\min(\underline{u}(t,d(0,x)),\underline{v}(t,d(0,x))) & \text{if } x\in J_3,\\ \underline{u}(t,-d(0,x))) & \text{if } x\in J_1^*,\\ \underline{v}(t,-d(0,x)) & \text{if } x\in J_2^*.\end{cases} \tag{6.6}$$

We will prove that $\overline{z}$ and $\underline{z}$ are respectively sub and super solutions of (2.17). In this case, the comparison principle will imply that $\overline{z}\le\underline{z}$ and by construction, we have that $\underline{z}\le\overline{z}$. Hence, we will get that

$$\overline{z}(t,x)=\underline{z}(t,x)=u^0(t,x)$$

where u^0 is the unique viscosity solution of (2.17). Let us prove that $\overline{z}$ is a sub-solution of (2.17) (the proof for $\underline{z}$ is similar and we skip it). We argue by contradiction and assume that there exists a test function $\varphi\in\mathcal{C}^1((0,T)\times J)$ (in the sense of Thm. 2.7 in [28]), and a point $(\bar t,\bar x)\in(0,+\infty)\times J$ such that

$$\begin{cases}\overline{z}(\bar t,\bar x)=\varphi(\bar t,\bar x)\\ \overline{z}\le\varphi & \text{on } \mathcal{Q}_{\bar r,\bar r}(\bar t,\bar x) \quad \text{with } \bar r>0\\ \overline{z}\le\varphi-2\eta & \text{outside } \mathcal{Q}_{\bar r,\bar r}(\bar t,\bar x) \text{ with } \eta>0\\ \varphi_t(\bar t,\bar x)+\overline{H}(\bar x,\varphi_x(\bar t,\bar x))=\theta \text{ with } \theta>0,\end{cases} \tag{6.7}$$

where

$$\overline{H}(\bar x,\varphi_x(\bar t,\bar x)):=\begin{cases}\overline{H}_k\left(\partial_k\varphi(\bar t,\bar x)\right) & \text{if } \bar x\in J_k^*,\\ \overline{F}_{\overline{A}}\left(\partial_1\varphi(\bar t,0),\partial_2\varphi(\bar t,0),\partial_3\varphi(\bar t,0)\right) & \text{if } \bar x=0.\end{cases}$$

We denote by $\bar y=sign(\bar x)d(0,\bar x)$ with $sign:J\to\mathbb{R}$, defined by

$$sign(x)=\begin{cases}1 & \text{if } x\in J_3^*,\\ -1 & \text{if } x\in J_1^*\cup J_2^*,\\ 0 & \text{if } x=0.\end{cases}$$

Up to changing φ at infinity, we can assume that for ε small enough, we have

$$\begin{cases}u^\varepsilon(t,sign(x)d(0,x))\le\varphi(t,x)-\eta & \text{for } (t,x)\in(Q_{\bar r,\bar r}(\bar t,\bar x))^c\cap(J_1\cup J_3),\\ v^\varepsilon(t,sign(x)d(0,x))\le\varphi(t,x)-\eta & \text{for } (t,x)\in(Q_{\bar r,\bar r}(\bar t,\bar x))^c\cap(J_2\cup J_3).\end{cases} \tag{6.8}$$

First case: $\bar x\neq 0$**.** We only consider the case where $\bar x\in J_3^*$ since the other cases can be treated in the same way. Let $p=\partial_3\varphi(\bar t,\bar x)$ which according to Corollary 6.3 satisfies $p\le 0$ and

$$\varphi_t(\bar t,\bar x)+\overline{H}_3(p)=\theta.$$

Let $\bar r$ small enough so that $\bar y-\bar r>0$. We define a new test function $\tilde\varphi:[0,T]\times\mathbb{R}\to\mathbb{R}$ given by

$$\tilde\varphi(t,y)=\varphi(t,ye_3)\quad\text{for } y>0.$$

For $(t,y)\in\mathcal{P}_{\bar r,\bar r}(\bar t,\bar y)$ (see (4.2)), ε small enough and by definition of ϕ and ω, we have that

$$\phi\left(\frac{y}{\varepsilon},a,b\right)=b\quad\text{and } \omega\left(\frac{y}{\varepsilon},z,\left|\frac{y}{\varepsilon}\right|-|z|\right)=1. \tag{6.9}$$

Using (6.9), and the definition of the non-local operator K^ε(see (3.11)) implies that on the ball $P_{\bar r,\bar r}(\bar t,\bar y)$, the function $(u^\varepsilon, v^\varepsilon)$ is a solution in the viscosity sense of

$$\begin{cases} u^\varepsilon_t + N^\varepsilon\left(\frac{u^\varepsilon(t,\cdot)}{\varepsilon}, \left[\frac{v^\varepsilon(t,\cdot)}{\varepsilon}\right]\right)(y)|u^\varepsilon_y| = 0,\\ v^\varepsilon_t + N^\varepsilon\left(\frac{v^\varepsilon(t,\cdot)}{\varepsilon}, \left[\frac{u^\varepsilon(t,\cdot)]}{\varepsilon}\right]\right)(y)|v^\varepsilon_y| = 0. \end{cases}$$

We will prove that for $(t,y)\in P_{\bar r,\bar r}(\bar t,\bar y)$, we have

$$\tilde\varphi_t + \tilde N^\varepsilon\left(\frac{\tilde\varphi(t,\cdot)}{\varepsilon}, \left[\frac{\tilde\varphi(t,\cdot)}{\varepsilon}\right]\right)(y)|\varphi_y| \geq 0.$$

For all $(t,y)\in \mathcal{P}_{\bar r,\bar r}(\bar t,\bar y)$, we have for $\bar r$ small enough

$$\begin{aligned} \tilde\varphi_t(t,y) + \tilde N^\varepsilon\left(\frac{\tilde\varphi}{\varepsilon}(t,\cdot), \left[\frac{\tilde\varphi}{\varepsilon}(t,\cdot)\right]\right)(y)\cdot|\tilde\varphi^\varepsilon_y| &= \tilde\varphi_t(\bar t,\bar y) + o_{\bar r}(1)\\ + \tilde N^\varepsilon\left(\frac{\tilde\varphi}{\varepsilon}(t,\cdot), \left[\frac{\tilde\varphi}{\varepsilon}(t,\cdot)\right]\right)(y)\cdot|\tilde\varphi_y|\\ = \theta + o_{\bar r}(1) + \tilde N^\varepsilon\left(\frac{\tilde\varphi}{\varepsilon}(t,\cdot), \left[\frac{\tilde\varphi}{\varepsilon}(t,\cdot)\right]\right)(y)\cdot|p| - \overline{H}_3(p) =: \Delta, \end{aligned} \tag{6.10}$$

where we used that $\tilde\varphi_y(\bar t,\bar y) = \partial_3\varphi(\bar t,\bar x) = p$ and (6.7).

If $p=0$, then using that $\overline{H}_3(0)=0$, we obtain that $\Delta = \theta + o_{\bar r}(1) > 0$ for $\bar r$ small small enough. Assume now that $p<0$. For all $z\in[h_0,h_{\max}]$, and for ε and $\bar r$ small enough we have that

$$\frac{\tilde\varphi(t,y+\varepsilon z)-\tilde\varphi(t,y)}{\varepsilon} = \frac{\tilde\varphi(t,y+\varepsilon z)-\tilde\varphi(t,y)}{\varepsilon} \leq pz + o_{\bar r}(1) + c\varepsilon,$$

where we used the fact that $\tilde\varphi\in\mathcal{C}^1$ and that $z\in[h_0,h_{\max}]$. Now using the fact that $\tilde E_{-1}$ is decreasing we have

$$\tilde E_{-1}\left(pz + c\varepsilon + o_{\bar r}(1)\right) \leq \tilde E_{-1}\left(\frac{\tilde\varphi(t,y+\varepsilon z)-\tilde\varphi(t,y)}{\varepsilon}\right).$$

We deduce that

$$\begin{aligned} &\tilde N^\varepsilon\left(\frac{\tilde\varphi}{\varepsilon}(t,\cdot), \left[\frac{\tilde\varphi}{\varepsilon}(t,\cdot)\right]\right)(y) + V_{\max} = \int_{z\geq \frac{y}{\varepsilon}} V_3'\left(|z| - \frac{y}{\varepsilon}\right)\tilde E_{-1}\left(\frac{\tilde\varphi(t,\varepsilon z)-\tilde\varphi(t,y)}{\varepsilon}\right)\mathrm{d}z\\ &= \int_0^{+\infty} V_3'(z)\,\tilde E_{-1}\left(\frac{\tilde\varphi(t,y+\varepsilon z)-\tilde\varphi(t,y)}{\varepsilon}\right)\mathrm{d}z && (6.11)\\ &\geq \int_0^{+\infty} V_3'(z)\tilde E_{-1}\left(pz + o_{\bar r}(1) + c\varepsilon\right)\mathrm{d}z && (6.12)\\ &= \int_0^{(-1-c\varepsilon-o_{\bar r}(1))/p} V_3'(z)\tilde E_{-1}\left(pz + o_{\bar r}(1) + c\varepsilon\right)\mathrm{d}z\\ &+ \int_{(-1-c\varepsilon-o_{\bar r}(1))/p}^{+\infty} V_3'(z)\tilde E_{-1}\left(pz + o_{\bar r}(1) + c\varepsilon\right)\mathrm{d}z && (6.13) \end{aligned}$$

$$= 0 + \int_{(-1-c\varepsilon-o_{\bar r}(1))/p}^{+\infty} V_3'(z)\mathrm{d}z = V_{\max} - V_3\left(\frac{-1-c\varepsilon-o_{\bar r}(1)}{p}\right) \tag{6.14}$$

where we used in the last line the fact that $pz + o_{\bar r}(1) + c\varepsilon < -1$ if $z > \frac{-a-c\varepsilon-o_{\bar r}(1)}{p}$. Injecting (6.14) in (6.10) and choosing ε and $\bar r$ small enough, we obtain

$$\begin{aligned}\Delta &\geq \theta + o_{\bar r}(1) + |p|\cdot\left[-V_3\left(\frac{-a-c\varepsilon-o_{\bar r}(1)}{p}\right) + V_3\left(\frac{-1}{p}\right)\right]\\ &\geq \theta + o_{\bar r}(1) - \|V_3'\|_\infty\cdot(c\varepsilon + o_{\bar r}(1))\\ &\geq \frac{\theta}{2},\end{aligned}$$

where we used assumption (A1) for the second line.

Getting a contradiction. By definition, we have for ε small enough and using (6.8),

$$\begin{aligned} u^\varepsilon &\leq \tilde\varphi - \eta \quad \text{outside } \mathcal{P}_{\bar r,\bar r}(\bar t,\bar y),\\ v^\varepsilon &\leq \tilde\varphi - \eta \quad \text{outside } \mathcal{P}_{\bar r,\bar r}(\bar t,\bar y).\end{aligned} \tag{6.15}$$

Using the comparison principle on bounded subsets for (6.1) (Thm. 4.3), we get

$$\begin{aligned} u^\varepsilon &\leq \tilde\varphi - \eta \quad \text{on } \mathcal{P}_{\bar r,\bar r}(\bar t,\bar y),\\ v^\varepsilon &\leq \tilde\varphi - \eta \quad \text{on } \mathcal{P}_{\bar r,\bar r}(\bar t,\bar y).\end{aligned} \tag{6.16}$$

Passing to the limit as $\varepsilon \to 0$, we get $\overline{u} \leq \tilde\varphi - \eta$ and $\overline{v} \leq \tilde\varphi - \eta$ on $\mathcal{P}_{\bar r,\bar r}(\bar t,\bar y)$ and this contradicts the fact that $\overline{z}(\bar t,\bar x) = \varphi(\bar t,\bar x)$.

Second case: $\bar x = 0$: In that case, we have that

$$\varphi_t(\bar t,\bar x) + \overline{F}_{\overline{A}}(\partial_1\varphi(\bar t,0), \partial_2\varphi(\bar t,0), \partial_3\varphi(\bar t,0)) = \theta \tag{6.17}$$

and using Theorem 2.7 in [28], we can assume that φ has a specific form given by

$$\varphi(t,x) = g(t) + \overline{p}_-^1 x 1_{\{x\in J_1^*\}} + \overline{p}_-^2 x 1_{\{x\in J_2^*\}} + \overline{p}_+^3 x 1_{\{x\in J_3^*\}} \tag{6.18}$$

with $g \in \mathcal{C}^1(0,T)$. In particular, (6.17) can be replaced by

$$g_t(\bar t) + \overline{A} = \theta. \tag{6.19}$$

Let us consider the solution $(\mathtt{u},\mathtt{v},\mathtt{w})$ of (5.6) provided by Theorem 5.2, and let us denote by

$$\varphi^\varepsilon(t,y) = \begin{cases} g(t) + \mathtt{u}^\varepsilon(t,y) & \text{on } \mathcal{P}_{\bar r,2\bar r}(\bar t,0),\\ \varphi(t, ye_1 1_{\{y\leq 0\}} + ye_3 1_{\{y\geq 0\}}) & \text{outside } \mathcal{P}_{\bar r,2\bar r}(\bar t,0),\end{cases} \tag{6.20}$$

and

$$\Psi^\varepsilon(t,y) = \begin{cases} g(t) + \mathtt{v}^\varepsilon(t,y) & \text{on } \mathcal{P}_{\bar r,2\bar r}(\bar t,0),\\ \varphi(t, ye_2 1_{\{y\leq 0\}} + ye_3 1_{\{y\geq 0\}}) & \text{outside } \mathcal{P}_{\bar r,2\bar r}(\bar t,0).\end{cases} \tag{6.21}$$

We will prove that $(\varphi^\varepsilon, \Psi^\varepsilon)$ is a viscosity super-solution on $P_{\bar r,\bar r}(\bar t, 0)$ of (6.1). Let h be a test function touching φ^ε from below at $(t_1, x_1) \in \mathcal{P}_{\bar r,\bar r}(\bar t, 0)$, so we have

$$\mathtt{u}\left(\frac{x_1}{\varepsilon}\right) = \frac{1}{\varepsilon}\left(h(t_1, x_1) - g(t_1)\right),$$

and

$$\mathtt{u}(y) \geq \frac{1}{\varepsilon}\left(h(\varepsilon s, \varepsilon y) - g(\varepsilon s)\right),$$

for (s, y) in a neighbourhood of $\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}\right)$. Therefore, we have

$$h_t(t_1, x_1) - g'(t_1) + \tilde R_1\left(\frac{x_1}{\varepsilon}, \mathtt{u}\left(\frac{x_1}{\varepsilon}\right), [\mathtt{v}]\right)(x_1) \cdot |h_x(t_1, x_1)| \geq \overline{A}.$$

This implies that (using (6.19) and taking $\bar r$ small enough)

$$h_t(t_1, x_1) + \tilde R_1\left(\frac{x_1}{\varepsilon}, \mathtt{u}\left(\frac{x_1}{\varepsilon}\right), [\mathtt{v}]\right)(x_1) \cdot |h_x(t_1, x_1)| \geq \overline{A} + g'(t_1) \geq \frac{\theta}{2}.$$

Now for ε small enough such that $\varepsilon h_{\max} \leq \bar r$, we deduce from the previous inequality and using the fact that the non-local operators are of bounded support, that

$$h_t(t_1, x_1) + \tilde R_1^\varepsilon\left(\frac{x_1}{\varepsilon}, \frac{\varphi^\varepsilon(t,x)}{\varepsilon}, \left[\frac{\psi^\varepsilon(t,\cdot)}{\varepsilon}\right]\right)(x) \cdot |h_x(t_1, x_1)| \geq \frac{\theta}{2}.$$

The two last super-solution inequalities are obtained in same way using the correctors and (6.19).

Getting the contradiction. Using (6.8) and (6.18), we have for ε small enough on $\mathcal{P}_{\bar r,2\bar r}(\bar t, 0)\backslash\mathcal{P}_{\bar r,\bar r}(\bar t, 0)$,

$$\begin{aligned} u^\varepsilon &\leq g(t) + \bar p_-^1 y 1_{\{y<0\}} + \bar p_+^3 y 1_{\{y>0\}} - \eta, \\ v^\varepsilon &\leq g(t) + \bar p_-^2 y 1_{\{y<0\}} + \bar p_+^3 y 1_{\{y>0\}} - \eta. \end{aligned}$$

Using the fact that $\mathtt{u}^\varepsilon \to U$ and $\mathtt{v}^\varepsilon \to V$ with U, V defined in (5.11) we have for ε small enough on $\mathcal{P}_{\bar r,2\bar r}(\bar t, 0)\backslash\mathcal{P}_{\bar r,\bar r}(\bar t, 0)$

$$u^\varepsilon \leq \varphi^\varepsilon - \frac{\eta}{2} \quad \text{and } v^\varepsilon \leq \Psi^\varepsilon - \frac{\eta}{2}.$$

Combining this with the particular form of the test function given in (6.18), we get that outside $\mathcal{P}_{\bar r,\bar r}(\bar t, 0)$,

$$u^\varepsilon \leq \varphi^\varepsilon - \frac{\eta}{2} \quad \text{and } v^\varepsilon \leq \Psi^\varepsilon - \frac{\eta}{2}.$$

By the comparison principle on bounded subsets (Thm. 4.3) the previous inequality holds in $\mathcal{P}_{\bar r,\bar r}(\bar t, 0)$. Passing to the limit as $\varepsilon \to 0$ and evaluating the inequality in $(\bar t, 0)$, we obtain

$$\overline{u}(\bar t, 0) \leq \varphi(\bar t, 0) - \frac{\eta}{2} \quad \text{and} \quad \overline{v}(\bar t, 0) \leq \varphi(\bar t, 0) - \frac{\eta}{2}$$

which is a contradiction.

□

Appendix A. Appendix: extension

Our result (micro to macro) can be extended to the case of a convergent junction with N incoming roads with an imposed rule near the junction point. The strategy is always the same: inject using the "cumulative distribution functions" the ODE into a non-local PDE and then obtain the homogenization result in the framework of viscosity solutions. The number of these functions depends on the number of branches and the different type of leaders far from the junction point. To be more precise, we consider a junction J with N incoming roads and one outgoing road as in Figure A.1.

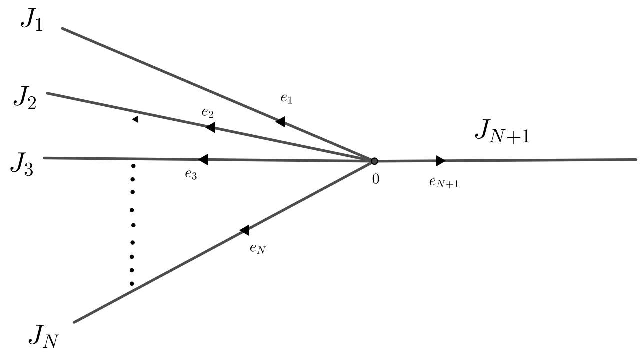

Figure A.1. Junction with $N+1$ branches.

We assume that the imposed rule is the following: n_1 pedestrians enter from J_1 then n_2 pedestrians from J_2,..., then n_N pedestrians from J_N and this phenomena will be repeated. We write a microscopic model following the same ideas of (2.10). Far before the junction point, the velocity of each pedestrian depends on the distance to its leader at time $t=0$. We also add the function ω which models pedestrian's respect to the imposed rule. After the junction point, each pedestrian W_i will follow W_{i+1}. We write the model using the variables U_i defined by

$$W_i(t) \in J \Longleftrightarrow W_i(t) = U_i(t) \cdot e_k, \quad k = 1, ..., N+1.$$

In this case, the scheme to obtain our micro-macro result is the following:

1) Let $i = 1, .., N$. Using the pedestrians located at $t=0$ on $J_i \cup J_{N+1}$, we define $n_i \in \mathbb{N}^*$ distribution functions $\rho_i^1, \rho_i^2, ..., \rho_i^{n_i}$.
2) We inject these function into a system of non-local PDE. The number of equations of this system is $n_1 + n_2 + ... + n_N$.
3) Following the ideas of our work, we can prove the following theorem.

Theorem A.1. *We denote by lead(i) the leader of pedestrian i at time $t=0$. We assume that*

$$U_i(0) \le U_{lead(i)}(0) - h_0.$$

We define $h_1, h_2, ..., h_{N+1} \ge h_0$. We also assume that there exists $R > 0$ such that

$$\begin{cases} U_{lead(i)}(0) - U_i(0) = h_k & \text{if } W_i(0) \in J_k, U_i(0) < -R \text{ with } k = 1, .., N \\ U_{i+1}(0) - U_i(0) = h_{k+1} & \text{if } W_i(0) \in J_{N+1}, U_i(0) > R. \end{cases}$$

We define N functions $u_0^1, u_0^2, ..., u_0^N$ by

$$u_0^i(x) = -\frac{\frac{\sum_{j=1}^N n_j}{n_i}}{h_i} x 1_{\{x<0\}} - \frac{1}{h_{N+1}} x 1_{\{x \geq 0\}}.$$

Let $\varepsilon > 0$ and $\chi^\varepsilon : \mathbb{R}^+ \times J \to \mathbb{R}$ be the function defined by

$$\chi^\varepsilon(t,x) = \begin{cases} (\rho_i^1)^\varepsilon(t, -d(0,x)) & \text{if } x \in J_i^*, i = 1, ..., N, \\ (\rho_1^1)^\varepsilon(t, d(0,x)) & \text{if } x \in J_{N+1} \end{cases}$$

with

$$(\rho_i^1)^\varepsilon(t,x) = \varepsilon \rho_i^1 \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right).$$

We define

$$\overline{u}_0(x) = \begin{cases} u_0^i(-d(0,x)) & \text{if } x \in J_i^*, i = 1, ..., N, \\ u_0^1(d(0,x)) & \text{if } x \in J_{N+1}. \end{cases}$$

Then there exists a unique $\overline{A} \in [H_0, 0]$ such that the function χ^ε converges towards the unique solution u^0 of

$$\begin{cases} u_t^0 + \overline{H}_i(u_x^0) = 0 & \text{for } (t,x) \in (0, +\infty) \times J_i^*, \\ u_t^0 + F_{\overline{A}}\left(u_x^0\right) = 0 & \text{for } (t,x) \in (0,+\infty) \times \{0\}, \\ u^0(0,x) = \overline{u}_0(x), \end{cases}$$

where for $i = 1, ...N$, the Hamiltonian H_i is given by

$$\overline{H}_i(p) = \begin{cases} -p - -\frac{\sum_{j=1}^N n_j}{n_i} k_0 & \text{for } p < - - \frac{\sum_{j=1}^N n_j}{n_i} k_0, \\ -V_i \left(\frac{-\frac{\sum_{j=1}^N n_j}{n_i}}{p} \right) |p| & \text{for } - - \frac{\sum_{j=1}^N n_j}{n_i} k_0 \leq p \leq 0, \\ p & \text{for } p > 0, \end{cases}$$

and

$$\overline{H}_{N+1}(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V_{N+1}\left(\frac{-1}{p} \right) |p| & \text{for } -k_0 \leq p \leq 0. \\ p & \text{for } p > 0. \end{cases}$$

For $p \in \mathbb{R}^{N+1}$, the function $F_{\overline{A}}(p)$ is defined by

$$F_{\overline{A}}(p) = \max\left(\overline{A}, \max_{i=1,..,N} \overline{H}_i^+(p_i), \overline{H}_{N+1}^-(p_{N+1}) \right).$$

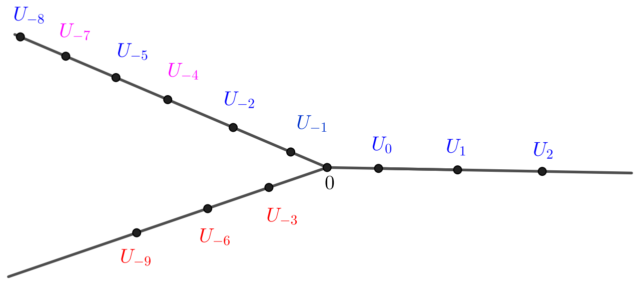

FIGURE A.2. In blue (resp. pink and red): pedestrians U_i whose leader's label at the initial time is $i+1$ (resp. $i+2$ and $i+3$). The definition of the function ρ_1^1 (resp. ρ_1^2 and ρ_2^1) depends on pedestrians colored in blue (resp. pink and red).

Example if $N = 2$ and $n_1 = 2, n_2 = 1$, see Figure A.2. We give the definition of the cumulative functions. We need to introduce three cumulative functions ρ_1^1, ρ_1^2 and ρ_2^1. The need of three functions arises from the different type of leaders: far before zero, on branch J_1, the leader can be U_{i+1} or U_{i+2} and on branch J_2, the leader is U_{i+3}.

$$\begin{aligned}\rho_1^1(t,y) = &- \left(\sum_{i\geq 0} H(y-U_i(t)) + (-1+H(y-U_{-1}(t))) + (-1+H(y-U_{-2}(t))\right) \\ &- \sum_{i=-2-3k, k\in\mathbb{N}^*} 3\left(-1+H\left(y-U_i(t)\right)\right),\end{aligned}$$

$$\begin{aligned}\rho_1^2(t,y) = &- \left(\sum_{i\geq 0} H\left(y-U_i(t)\right) + 4(-1+H\left(y-U_{-4}(t)\right))\right) \\ &- \sum_{i=-4-3k, k\in\mathbb{N}^*} 3(-1+H\left(y-U_i(t)\right)),\end{aligned}$$

and

$$\rho_2^1(t,y) = -\left(\sum_{i\geq 0} H\left(y-U_i(t)\right) + \sum_{i=-3k, k\in\mathbb{N}^*} 3(-1+H\left(y-U_i(t)\right))\right),$$

with

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

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