

ON THE STRONG CONVERGENCE OF THE FAEDO-GALERKIN APPROXIMATIONS TO A STRONG T-PERIODIC SOLUTION OF THE TORSO-COUPLED BIDOMAIN MODEL

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Abstract. In this paper, we investigate the convergence of the Faedo-Galerkin approximations, in a strong sense, to a strong T -periodic solution of the torso-coupled bidomain model where T is the period of activation of the inner wall of the heart. First, we define the torso-coupled bidomain operator and prove some of its more important properties for our work. After, we define the abstract evolution system of the equations that are associated with torso-coupled bidomain model and give the definition of a strong solution. We prove that the Faedo-Galerkin's approximations have the regularity of a strong solution, and we find that some restrictions can be imposed over the initial conditions, so that this sequence of Faedo-Galerkin fully converges to a strong solution of the Cauchy problem. Finally, these results are used for showing the existence a strong T -periodic solution.

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1. INTRODUCTION

The bidomain model has been shown as one of the most successful models for studying the electrical activity of the heart. This was first formulated and deduced by L. Tung in his Ph.D. thesis, which was presented at MIT in 1978 (see [25]). The bi-domain model has also been shown to be very efficient in simulating the electrical activity of the heart, specifically when it is coupled to the torso through a conducting medium model, allowing computationally accurate electrocardiographic measurements such as ECGs. Even when approximate geometries of the heart are considered, this accuracy seems surprising. In [3], Boulakia *et. al.* obtain realistic simulations of different ECG leads by numerically solving the coupled to the torso bidomain equations. In this work, a simplified geometry of the heart is considered which is based on an ellipsoidal model for the ventricles such that the orientation of the fibers can be parameterized in terms of analytical functions. In more recent works, the bidomain equations are solved, taking into account a more realistic geometry of the heart. For this reason, the Multipatch Isogeometric Analysis is used to approximate the solutions and the geometry of the heart. In addition, is used to assign orientation to myocardial fibers an algorithm based on physiological rules see [2, 5].

Keywords and phrases: Bidomain model, Faedo-Galerkin scheme, variational formulation, weak and strong periodic solutions.

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On the other hand, it is worth noting that the strength of the bi-domain model also lies in the principles that govern its formulation. One of them is the bidomain principle, which assumes that the heart is an average volume, where the intra and extracellular domains occupy the entire region of it. This principle has allowed to consider the microscopic phenomena that determines the generation and propagation of the electrical impulse, in this case, the ion exchange between the intracellular and extracellular domains. The bidomain principle has recently been extended. In the papers [17, 18], Yoichiro Mori formulates what he calls a multidomain model for modeling the ion electro-diffusion and the osmotic flow of water. This model allows the study of wide-range physiological phenomena, for example, the electrochemical activity of the brain [18], the generation and propagation of calcium signals in biological tissues [23], and also the electrical activity of the heart. Indeed, as it was shown by Mori in [17], the bidomain model for the electrical activity of the heart is a particular case of the model proposed by Mori when the electronic scale for time is considered. In this sense, the study of the domain model is an important first step to understand this more general model.

If we take the qualitative study of the solutions of the model as point of view, partial advances have been made. Literature has a few references dealing with the well-posedness of the bidomain model. The most important of these references are Colli-Franzone and Savarés paper [6], Veneroni’s technical report [4, 26]. In [6], the existence and uniqueness of global solutions in the bidomain model time are proven, although its approach applies only to particular cases of ionic models, typically of the form $f(u, w) = k(u) + \alpha w$, $g(u, w) = \beta u + \gamma w$, where $k \in C^1(\mathbb{R})$ satisfies $\inf_{\mathbb{R}} k' > \infty$. The cubic-like FitzHugh-Nagumo model is an ionic model having this form [8], which is important for a qualitative understanding of the action potential propagation. However, its applicability to myocardial excitable cells is limited [14, 19]. Furthermore, from the techniques developed in [6], it is not possible to conclude the existence of a solution for other simple two-variable ionic models widely used in the literature for modeling myocardial cells, such as the Aliev-Panfilov [1] and Rogers-McCulloch [22] models. In Colli-Franzone and Savarés [26], these kind of results have been extended to a more general and realistic ionic model, namely those taking the form of Luo-Rudy model [16]. However, this extension does not include the Aliev-Panfilov and Rogers-McCulloch models. On the other hand, in reference [4], global weak solutions are obtained for ionic models which can be written as a single ODE with polynomial nonlinearities. These last ionic models include the FitzHugh-Nagumo model and other simple models that are more adapted to myocardial cells, such as the Aliev-Panfilov and Rogers-McCulloch models. In Hernandez [12], the authors obtain results about the existence and uniqueness of weak solutions for the Cauchy problem, but the monodomain model is analyzed in an isolated ventricle, and it is activated by the Purkinje fibers. Moreover, the ionic model is the Rogers-McCulloch one. In this regard, homogeneous boundary conditions of Neumann type were considered.

In relation to the existence of periodic solutions, there are some relevant recent papers. For example, in [13], Hieber *et al.* prove the periodic version of the Da Prato-Grisvard theorem as well as its extension to semi-linear evolution equations. From this outcome they proved the existence of strong T -periodic solution of the bidomain model, that is, in the context of the corresponding abstract evolution problem. In their research, they considered T -periodic external sources of intra and extracellular current I_i, I_e . However, the method of [13] is applied to just a few models describing the ionic transport like the FitzHugh-Nagumo, Aliev-Panfilov, or Rogers-McCulloch models. It is worth noting that this methodology is very general. In addition, although it is applied to concrete ionic models, the results of the existence of periodic solutions are not given in terms of relations between the parameters of the model. In other words, the method does not include a study of the ranges of values of the model parameters for which the mathematical results are true. These relations are very important because they explain which are the physiological causes that underlay the generation or lost of cardiac rhythm. We must say that [10] works with the same approach. In [9], we apply compact embedding theorems on Sobolev spaces and fixed point theorems to prove the existence of a weak T -periodic solution of the monodomain model, a simplified version of the bidomain model, considering the activation in the endocardium as a periodic boundary condition. Such periodic solution is obtained as the limit, in a weak sense, of a subsequence of Faedo-Galerkin approximations.

In this work, we are going to study the problem of the existence of a T -periodic solution where T is the period in which the endocardium is activated. Here, as we did in [9], we consider the activation as a boundary condition for extracellular current in the inner wall of the heart. Furthermore, unlike works found in the literature we

consider the heart coupled with the torso, thus it is necessary to define the bidomain operator coupled to the torso in both a weak and a strong context. We formulate the corresponding boundary problem in terms of an abstract evolution problem, involving the torso-coupled bidomain operator (in its strong version), the nonlinear terms describing ionic exchange at the cellular level, and the activation term, which is T -periodic. To obtain the existence result we combine convergence results of the so-called Faedo-Galerkin approximations with theorems of fixed points on Banach spaces, specifically the contracting map theorem. In other words, we find a periodic solution as the limit of certain sequence that lives in a Banach space T -periodic functions defined for all $t \geq 0$. The tools that were used allow us to consider very general nonlinearities, that only satisfy certain Lipschitz conditions, and which are realistic models for the ionic currents. Again, unlike the works found in the literature, the tools that were used in this work allow to explicitly describe relationships between the parameters associated with ionic currents, the parameters associated with the anatomy of the heart and the current conductivity velocities that turn out to be sufficient conditions for the existence a periodic solution. Although these are sufficient conditions, these allow hypotheses about the physiological causes, at the cellular and macroscopic level, that influence the generation or loss of heart rhythm to be formulated. For example, from the relationships found in [9] it can be hypothesized that there are minimum and maximum frequencies such that a heart beating with a lower or higher frequency, will eventually lose its rhythm. These minimum and maximum values depend on the parameters of the model and their normal values are between 60 and 100 beats per minute. However, in a heart with a certain pathology, the minimum value can increase and the maximum decrease, causing a heart beating with a rhythm considered normal to go into fibrillation.

In this paper, we use the formulation of the bidomain equations given in [24], where a parabolic PDE coupled to an elliptic one is considered, and both coupled to a system of ODEs that model the activation variables for the ionic currents. In this case, the state variables are the membrane potential $u_m = u_i - u_e$ and the extra and intra cellular potential u_i, u_e , respectively. We consider a region Ω in \mathbb{R}^3 to represent the heart embedded in the torso. In this regard, it is assumed that $\Omega = \Omega_H \cup \Omega_T$, where Ω_H is the region representing the heart and Ω_T represents the torso. The boundary of Ω_H has two components, that is, $\partial\Omega_H = \Sigma_{endo} \cup \Sigma_{epi}$ whit Σ_{endo} representing the cardiac endocardium, where the periodic activation is given, and Σ_{epi} represents the epicardium which is in contact with the torso. Also, Ω_T has two components, $\partial\Omega_T = \Sigma_{epi} \cup \Sigma_T$, where Σ_T the exterior boundary of the torso.

The equations of bidomain are performed as

$$\begin{aligned} A_m C_m \frac{\partial u_m}{\partial t} + A_m I_{ion}(u_m, \mathbf{w}) - \nabla \cdot (\sigma_i \nabla u_m) - \nabla \cdot (\sigma_e \nabla u_e) &= 0, \quad \text{in } (0, \infty) \times \Omega_H, \\ - \nabla \cdot (\sigma_i \nabla u_m + (\sigma_i + \sigma_e) \nabla u_e) &= 0, \quad \text{in } (0, \infty) \times \Omega_H, \\ \frac{\partial \mathbf{w}}{\partial t} + g(u_m, \mathbf{w}) &= 0, \quad \text{in } (0, \infty) \times \Omega_H, \end{aligned} \quad (1.1)$$

where,

- A_m, C_m are the rate of cellular membrane area per volume unit and the cellular membrane capacitance per area unit, respectively,
- I_{ion} is the ionic current across the cell membrane per area unit which depends on the membrane potential and the vector of activation variables \mathbf{w} ,
- g is a vector field that models the behavior of the ionic channels,
- $\sigma_i(\mathbf{x}), \sigma_e(\mathbf{x})$ are matrices depending on the space variable $\mathbf{x} \in \Omega_H$.

In general, $\mathbf{w} = (w_1, w_2, \dots, w_k)$ is a vector having several components, called gate variables. These variables model the closure or opening of the ionic channels.

These equations are coupled with a model for the torso which are seen as a passive conductor.

$$- \nabla \cdot (\sigma_T \nabla u_T) = 0, \quad \text{in } (0, \infty) \times \Omega_T,$$

$$\sigma_T \nabla u_T \cdot \mathbf{n} = 0, \quad \text{in } (0, \infty) \times \Sigma_T.$$

Here, u_T is the potential in the torso, σ_T is its conductivity which depends on the space variable, and \mathbf{n} is the outward unit normal to Σ_T . The last condition means that no current can flow outside the torso across Σ_T .

In addition to the equations of bidomain and the model for the torso, we should give boundary conditions and coupling conditions with the torso. A condition widely assumed, see [3, 25], is that the intracellular current does not propagate outside the heart:

$$\sigma_i \nabla u_i \cdot \mathbf{n} = 0, \quad \text{in } (0, \infty) \times \partial\Omega_H,$$

which can be rewritten as

$$\sigma_i \nabla u_m \cdot \mathbf{n} + \sigma_i \nabla u_e \cdot \mathbf{n} = 0, \quad \text{in } (0, \infty) \times \partial\Omega_H.$$

For the extracellular current, we give the following boundary condition

$$\sigma_e \nabla u_e \cdot \mathbf{n} = s_e, \quad \text{in } (0, \infty) \times \Sigma_{endo}.$$

where s_e is a T -periodic function in the variable t that represents the activation of endocardium.

On other hand, perfect electric conditions between the heart and the torso are considered

$$\begin{cases} u_e = u_T, \\ \sigma_e \nabla u_e \cdot \mathbf{n} = \sigma_T \nabla u_T \cdot \mathbf{n}, \end{cases} \quad \text{in } (0, \infty) \times \Sigma_{epi}.$$

In summary, the Cauchy problem associated with the torso-coupled bidomain model can be presented as

$$\frac{\partial u_m}{\partial t} + f(u_m, \mathbf{w}) - \nabla \cdot (\sigma_i \nabla u_m) - \nabla \cdot (\sigma_e \nabla u_e) = 0, \quad \text{in } (0, \infty) \times \Omega_H, \quad (1.2)$$

$$- \nabla \cdot (\sigma_i \nabla u_m + (\sigma_i + \sigma_e) \nabla u_e) = 0, \quad \text{in } (0, \infty) \times \Omega_H, \quad (1.3)$$

$$\frac{\partial \mathbf{w}}{\partial t} + g(u_m, \mathbf{w}) = 0, \quad \text{in } (0, \infty) \times \Omega_H, \quad (1.4)$$

$$\sigma_i \nabla u_m \cdot \mathbf{n} + \sigma_i \nabla u_e \cdot \mathbf{n} = 0, \quad \text{in } (0, \infty) \times \partial\Omega_H, \quad (1.5)$$

$$\sigma_i \nabla u_m \cdot \mathbf{n} + (\sigma_i + \sigma_e) \nabla u_e \cdot \mathbf{n} = s_e, \quad \text{in } (0, \infty) \times \Sigma_{endo}, \quad (1.6)$$

$$\begin{cases} u_e = u_T, \\ \sigma_e \nabla u_e \cdot \mathbf{n} = \sigma_T \nabla u_T \cdot \mathbf{n}, \end{cases} \quad \text{in } (0, \infty) \times \Sigma_{epi}, \quad (1.7)$$

$$- \nabla \cdot (\sigma_T \nabla u_T) = 0, \quad \text{in } (0, \infty) \times \Omega_T, \quad (1.8)$$

$$\sigma_T \nabla u_T \cdot \mathbf{n} = 0, \quad \text{in } (0, \infty) \times \Sigma_T, \quad (1.9)$$

$$u_m(0, \mathbf{x}) = u_m^{(0)}(\mathbf{x}), \quad \mathbf{w}(0, \mathbf{x}) = \mathbf{w}^{(0)}(\mathbf{x}), \quad \text{en } \Omega_H. \quad (1.10)$$

Remark 1.1. Note that the constants A_m, C_m do not appear in the equation (1.2). This is owing to the fact that we have divided the equation (1.1) by the term $A_m C_m$ and the terms $I_{ion}/C_m, \sigma_{i,e}/A_m C_m$ have been denoted by $f, \sigma_{i,e}$, respectively.

In this paper, we work with the operational formulation of the bidomain problem. That is, the boundary problem (1.2)–(1.10) is described as an abstract evolution system, where its linear part is defined from the so-called torso coupled bidomain operator. Our goal is to obtain strong solutions associated to this formulation, taking them as limit of a function sequence which have the regularity of a strong solution. These are solutions

are of a non-linear ODEs system sequence. The mentioned sequence of functions is called Faedo-Galerkin approximation.

The results in the above statements, which are proved in this paper, present certain advantages with respect to other results, that were found in the literature about the convergence the Faedo-Galerkin approximations. In general, the Faedo-Galerkin systems arise in the weak or variational setting. In that case, the principal result with respect to the convergence of the Faedo-Galerkin approximations affirms the convergence, in a weak sense, of a subsequence to a weak solution. However, in this paper, we demonstrate the convergence of the entire sequence to a strong solution. This result has important implications. For instance, starting from a result like this one, it is possible to define a method in order to approximate a strong solution. In this sense, the convergence of the entire sequence to the sought strong solution allows us to make sure that the functions of the sequence are every time closer to this solution, when subindex is towards to infinite. This would not be a fact if we only prove that one subsequence of the Faedo-Galerkin sequence converges.

2. SOME GENERAL CONSIDERATIONS

In the strong context, we assume that boundary of Ω , $\partial\Omega$, is $C^{2+\nu}$ and the coefficients of matrices $\sigma_{i,e}, \sigma_T$ are $C^{1+\nu}(\Omega_{H,T})$, for some $\nu > 0$, respectively. Also, the conductivity matrices are uniformly elliptic, that is, there are positive constants M, m such that

$$m|\xi| \leq \xi^t \sigma_{i,e,T}(\mathbf{x}) \xi \leq M|\xi|, \quad \text{for all } \xi \in \mathbb{R}^3 \text{ and a.e } \mathbf{x} \in \Omega_{H,T}.$$

The fiber structure of the heart is another aspect that should be considered. This structure determines the anisotropic properties of the cardiac muscle. It is known that the velocity of conduction of the depolarization wavefront in the longitudinal direction to the fiber is faster than in the transverse direction. This implies the existence of a matrix $M(\mathbf{x})$, whose columns are the orthogonal vectors $\{\mathbf{a}_l(\mathbf{x}), \mathbf{a}_t(\mathbf{x}), \mathbf{a}_n(\mathbf{x})\}$ with \mathbf{a}_l parallel to the local fiber direction, \mathbf{a}_n the vector orthogonal to the fiber lamina and \mathbf{a}_t the vector lying on the lamina and orthogonal to fiber direction, such that

$$\sigma_{i,e}(\mathbf{x}) = M^T(\mathbf{x}) M_{i,e}^* M(\mathbf{x}).$$

Setting

$$M_{i,e}^* = \text{diag} \left[\sigma_l^{i,e}, \sigma_t^{i,e}, \sigma_n^{i,e} \right],$$

where $\sigma_l^{i,e}, \sigma_t^{i,e}, \sigma_n^{i,e}$ are local coefficients of conductivity measured along the corresponding directions, see [7].

The fibers are tangent to the boundary, thus

$$\sigma_{i,e}(\mathbf{x}) \mathbf{n}(\mathbf{x}) = \sigma_n^{i,e} \mathbf{n}(\mathbf{x}), \quad \text{for a.e. } \mathbf{x} \in \partial\Omega_H, \quad (2.1)$$

with $\sigma_n^{i,e} \geq m > 0$.

The above has the following consequence

Lemma 2.1. *If the boundary of Ω_H is C^1 and $M, M_{i,e}^*$ are $C^0(\overline{\Omega_H})$ and $\sigma_{i,e} \in C^0(\overline{\Omega_H})$, then the following statements are true*

i) $\nabla u_i \cdot \mathbf{n} = 0$ and $\nabla u_e \cdot \mathbf{n} = 0 \Leftrightarrow \nabla u_m \cdot \mathbf{n} = 0$.

ii) $\nabla u_{i,e} \cdot \mathbf{n} = 0 \Leftrightarrow (\sigma_{i,e} \nabla u_{i,e}) \cdot \mathbf{n} = 0$ and $(\sigma_i \nabla u_e) \cdot \mathbf{n} = 0 \Leftrightarrow \begin{cases} (\sigma_i \nabla u_m) \cdot \mathbf{n} + (\sigma_i \nabla u_e) \cdot \mathbf{n} = 0, \\ (\sigma_i \nabla u_m) \cdot \mathbf{n} + ((\sigma_i + \sigma_e) \nabla u_e) \cdot \mathbf{n} = 0. \end{cases}$

Proof. The first affirmation **i)** is immediate. Let us prove **ii)**. Note that, due to the symmetry of $\sigma_{i,e}$ and (2.1) we have

$$(\sigma_{i,e} \nabla u_{i,e}) \cdot \mathbf{n} = (\sigma_{i,e}^T \mathbf{n}) \cdot \nabla u_{i,e} = \sigma_n^{i,e} \nabla u_{i,e} \cdot \mathbf{n}.$$

The first equivalence in **ii)** it is immediately obtained from the above equalities. The second equivalence is easy obtained from the first equivalence. \square

3. OPERATIONAL FORMULATION. STRONG PERIODIC SOLUTIONS

In this section, we obtain the bidomain operator coupled to the torso. In that sense, we focus our attention on the diffusion terms of the boundary problem. First, we present the variational formulation of the boundary problem associated with the diffusion terms. This can be written as a system of two equations involving the bilinear forms associated with the diffusion terms. Then, some properties of such bilinear forms are demonstrated, which allows us to ensure the existence of a unique solution to the variational problem, and from this result, adequately define the bidomain weak operator coupled to the torso. Second, a definition subdomain is determined such that the values of the bidomain operator are in a space L_2 . In this step, we obtain the strong version of the operator. Finally, it is shown that this strong bidomain operator is self-adjoint, positive and therefore sectorial, which allows defining the semi-group associated with the exponential of the operator, determining the solutions of the associated abstract evolution problem.

Once it is obtained, the boundary problem associated to the bidomain model is formulated. The key idea is that we can obtain u_e from the membrane potential u_m integrating the equation (1.3) with suitable boundary conditions. This allows us to define a parabolic abstract evolution problem for the pair (u_m, \mathbf{w}) in the form

$$\begin{aligned} \frac{du_m}{dt} + f(u_m, \mathbf{w}) + Au_m &= s(t), \\ \frac{d\mathbf{w}}{dt} + g(u_m, \mathbf{w}) &= 0 \end{aligned}$$

where A is a integro-differential second order elliptic operator and $s(t)$ is a function defined in a real interval with values in certain functional space, that depends on the activation function s_e . From the solution of this problem we can calculate the total extracellular potential that we denote by

$$u = \begin{cases} u_e, & \text{in } \Omega_H, \\ u_T, & \text{in } \Omega_T. \end{cases}$$

We need to give the following notation. Given X , certain space of integrable function on a region $\Theta \subset \mathbb{R}^3$, we define

$$X/\mathbb{R} = \left\{ \phi \in X : \int_{\Theta} \phi = 0 \right\}.$$

We denote by

$$\begin{aligned} L_H &= L_2(\Omega_H), \quad V_H = H_1(\Omega_H), \quad U_H := V_H/\mathbb{R}, \\ L_T &= L_2(\Omega_T), \quad V_T = H_1(\Omega_T), \quad U_T := V_T/\mathbb{R}, \end{aligned}$$

and

$$H = L_2(\Omega), \quad V = \left\{ \phi \in H_1(\Omega) : \int_{\Omega_H} \phi = 0 \right\}, \quad U := V/\mathbb{R},$$

where $\Omega = \Omega_H \cup \Omega_T$.

In the spaces $U_{H,T}$ and U we use the equivalent norm to the norm in $V_{H,T}$ and V given by

$$|\phi|_{U_{H,T}} = \left(\int_{\Omega_{H,T}} |\nabla \phi|^2 \right)^{1/2},$$

and

$$|\phi|_U = \left(\int_{\Omega} |\nabla \phi|^2 \right)^{1/2},$$

respectively.

Here, each triple $\{U_{H,T}, L_{H,T}/\mathbb{R}, U'_{H,T}\}$, $\{V_{H,T}, L_{H,T}, V'_{H,T}\}$, $\{U, H/\mathbb{R}, U'\}$ and $\{V, H, V'\}$ is a Gelfand triple, see [21]. This means that we have

$$\begin{aligned} U_{H,T} &\subset L_{H,T}/\mathbb{R} \subset U'_{H,T}, \\ V_{H,T} &\subset L_{H,T} \subset V'_{H,T}, \\ U &\subset H/\mathbb{R} \subset U', \\ V &\subset H \subset V', \end{aligned}$$

where the inclusions are continuous and the first inclusion is compact, in each case.

For each duality pair $\{X, X'\}$, in the above Gelfand triplets, by $\langle u, \phi \rangle$, we denote the value of functional $u \in X'$ in the element $\phi \in X$. The spaces X and X' will be recognized from the context. Also, by (\cdot, \cdot) , we denote the inner product in L_H or H .

Furthermore, in that follows we use two operators: a restriction operator and a prolongation operator. By R_H , we denote the operator defined from V into U_H such that to each $\phi \in V \mapsto \phi|_{\Omega_H} \in U_H$. R_H is bounded, in fact $\|R_H\| \leq 1$ and its adjoint operator $R^* : U'_H \rightarrow V'$ is given by the relation

$$\langle R^*_H \phi_H, \phi \rangle = \langle \phi_H, R_H \phi \rangle, \quad \text{for all } (\phi_H, \phi) \in U'_H \times V.$$

If we assume that Ω_H has boundary C^1 from Theorem IX.7 of [21], it is deduced that there is a prolongation operator, which is bounded, $P_H : U_H \rightarrow V$, such that, $R_H P_H = I_d : U_H \rightarrow U_H$. Its adjoint operator $P^*_H : V' \rightarrow U'_H$ defined by

$$\langle P^*_H \phi, \phi_H \rangle_{U'_H \times U_H} = \langle \phi, P_H \phi_H \rangle_{V' \times V}, \quad \text{for all } (\phi, \phi_H) \in V \times U_H.$$

In a similar way, we define the operator $R_T : U \rightarrow U_T$.

In that follows, we obtain the so-called torso-coupled bidomain operator that is the linear part of the abstract evolution problem that we want to define. In this sense, we study the following boundary problem

$$-\nabla \cdot (\sigma_i \nabla u_m) - \nabla \cdot (\sigma_i \nabla u_e) = 0, \quad \text{in } \Omega_H, \quad (3.1)$$

$$-\nabla \cdot (\sigma_i \nabla u_m + (\sigma_i + \sigma_e) \nabla u_e) = 0, \quad \text{in } \Omega_H, \quad (3.2)$$

$$\sigma_i \nabla u_m \cdot \mathbf{n} + \sigma_i \nabla u_e \cdot \mathbf{n} = 0, \quad \text{in } \partial\Omega_H, \quad (3.3)$$

$$\sigma_i \nabla u_m \cdot \mathbf{n} + (\sigma_i + \sigma_e) \nabla u_e \cdot \mathbf{n} = s_e, \quad \text{in } \Sigma_{endo}, \quad (3.4)$$

$$\begin{cases} u_e = u_T, & \text{in } \Sigma_{epi}, \\ \sigma_e \nabla u_e \cdot \mathbf{n} = \sigma_T \nabla u_T \cdot \mathbf{n}, & \end{cases} \quad (3.5)$$

$$-\nabla \cdot (\sigma_T \nabla u_T) = 0, \quad \text{in } (0, \infty) \times \Omega_T, \quad (3.6)$$

$$\sigma_T \nabla u_T \cdot \mathbf{n} = 0, \quad \text{in } (0, \infty) \times \Sigma_T. \quad (3.7)$$

The variational formulation of the problem (3.1)–(3.7) has the following form

$$a_i(u_m, \phi_H) + a_i(R_H u, \phi_H) = 0, \quad \text{for all } \phi_H \in U_H, \quad (3.8)$$

$$(\tilde{a}_e + \tilde{a}_i)(u, \phi) + a_i(u_m, R_H \phi) = \langle \bar{s}_e, R_H \phi \rangle, \quad \text{for all } \phi \in U. \quad (3.9)$$

where

$$u = \begin{cases} u_e, & \text{in } \Omega_H, \\ u_T, & \text{in } \Omega_T, \end{cases}$$

$$a_{i,e}(u_H, \phi_H) = \int_{\Omega_H} \sigma_{i,e} \nabla u_H \nabla \phi_H,$$

are bilinear forms defined in $U_H \times U_H$. Also,

$$\tilde{a}_{i,e}(u, \phi) = \int_{\Omega} \tilde{\sigma}_{i,e} \nabla u \nabla \phi,$$

are bilinear forms defined in $U \times U$, where

$$\tilde{\sigma}_e = \begin{cases} \sigma_e & \text{in } \Omega_H, \\ \sigma_T & \text{in } \Omega_T, \end{cases}$$

and

$$\tilde{\sigma}_i = \begin{cases} \sigma_i & \text{in } \Omega_H, \\ 0 & \text{in } \Omega_T. \end{cases}$$

Finally, $\bar{s}_e \in U'_H$ is defined by

$$\langle \bar{s}_e, \phi_H \rangle := \int_{\Sigma_{endo}} s_e \phi_H, \quad \forall \phi_H \in U_H.$$

The bilinear forms $\tilde{a}_{i,e}$ and $a_{i,e}$ are symmetric, continuous and coerciva. Because of this, by the Lax-Milgram theorem, for each fixed $u_m \in U_H$ there is an only \tilde{u} that satisfies de variational equation (3.9). Besides, we notice that there are operators

$$\begin{aligned} \tilde{A}_{i,e} &: U \rightarrow U', \\ A_{i,e} &: U_H \rightarrow U'_H, \end{aligned}$$

which are one-to-one continuous with continuous inverse, so that

$$\left\langle \tilde{A}_{i,e} \phi_1, \phi_2 \right\rangle_{U' \times U} = \tilde{a}_{i,e}(\phi_1, \phi_2), \quad \text{for all } \phi_1, \phi_2 \in U,$$

$$\left\langle A_{i,e}\phi_H^{(1)}, \phi_H^{(2)} \right\rangle_{U' \times U} = \tilde{a}_{i,e}(\phi_H^{(1)}, \phi_H^{(2)}), \quad \text{for all } \phi_H^{(1)}, \phi_H^{(2)} \in U_H.$$

Equation (3.9) can be rewritten in as follows,

$$\left\langle (\tilde{A}_e + \tilde{A}_i) \tilde{u}, \phi \right\rangle_{U' \times U} = \langle \bar{s}_e - A_i u_m, R_H \phi \rangle_{U'_H \times U_H} = \langle R_H^* \bar{s}_e - R_H^* A_i u_m, \phi \rangle_{U' \times U},$$

thus,

$$\tilde{u} = (\tilde{A}_e + \tilde{A}_i)^{-1} R_H^* \bar{s}_e - (\tilde{A}_e + \tilde{A}_i)^{-1} R_H^* A_i u_m = (\tilde{A}_e + \tilde{A}_i)^{-1} R_H^* (\bar{s}_e - A_i u_m).$$

Substituting this expression for \tilde{u} in (3.8), we obtain:

$$\begin{aligned} & \langle A_i u_m, \phi_H \rangle_{U'_H \times U_H} + \left\langle A_i R_H (\tilde{A}_i + \tilde{A}_e)^{-1} R_H^* \bar{s}_e, \phi_H \right\rangle_{U'_H \times U_H} \\ & - \left\langle A_i R_H (\tilde{A}_i + \tilde{A}_e)^{-1} R_H^* A_i u_m, \phi_H \right\rangle_{U'_H \times U_H} = 0, \quad \text{for all } \phi_H \in U_H. \end{aligned}$$

In terms of the weak operators $\tilde{A}_i, \tilde{A}_e, A_i$ the above equation is written as

$$\bar{A} u_m = \bar{s},$$

where $\bar{A} : U_H \rightarrow U'_H$, with

$$\bar{A} = A_i - A_i R_H (\tilde{A}_i + \tilde{A}_e)^{-1} R_H^* A_i = A_i R_H (\tilde{A}_i + \tilde{A}_e)^{-1} \tilde{A}_e P_H = A_i R_H (\tilde{A}_i + \tilde{A}_e)^{-1} R_H^* A_e,$$

and $\bar{s} \in U'_H$ so that

$$\bar{s} = -A_i R_H (\tilde{A}_i + \tilde{A}_e)^{-1} R_H^* \bar{s}_e.$$

Having the definition and studying of the full bidomain evolution abstract problem as objectives, we must extend the operator \bar{A} and the functional \bar{s} to V_H . Then, we define $J : V_H \rightarrow U_H$ given by relation $\phi_H \mapsto \phi_H - \frac{1}{|\Omega_H|} \int_{\Omega_H} \phi_H$, where $|\Omega_H|$ is the volume of Ω_H . Finally, we define

$$A := J^* \bar{A} J : V_H \rightarrow V'_H,$$

and

$$s = J^* \bar{s} \in V'_H.$$

In order the functions $u \in U$ and $u_m \in U_H$ to be considered as weak solutions of the problems (3.1)–(3.7) in V and V_H it is necessary to assume that $\langle \bar{s}_e, 1 \rangle_{U'_H \times U_H} = 0$, that is,

$$\int_{\Sigma_{endo}} s_e = 0.$$

Let us present the following proposition.

Proposition 3.1. *The bilinear form associated to the operator \bar{A} , denoted by us \bar{a} , is defined in $U_H \times U_H$ and has been given by the expression*

$$\bar{a}(u_H, \phi_H) = a_i \left(u_H - R_H \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* A_i u_H, \phi_H \right),$$

and the bilinear form associated to operator A is $a(u_H, \phi_H) = \bar{a}(Ju_H, J\phi_H)$ for all $(u_H, \phi_H) \in V_H \times V_H$.

Furthermore, a is symmetric, continuous and coercive and there is an increasing sequence $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_n \leq \dots$ in \mathbb{R} and an orthonormal base $(\psi_i)_{i=0}^\infty \subset L_H$, such that $\psi_i \in V_H$ and

$$a(\psi_i, v) = \lambda_i(\psi_i, v) \quad \text{for all } v \in V_H, i \in \mathbb{N}.$$

Proof. Let $u_H, \phi_H \in U_H$ be arbitrary, we have:

$$\begin{aligned} \langle \bar{A}u_H, \phi_H \rangle &= \langle A_i u_H, \phi_H \rangle - \left\langle A_i R_H \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* A_i u_H, \phi_H \right\rangle \\ &= \left\langle A_i \left(u_H - R_H \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* A_i u_H \right), \phi_H \right\rangle = a_i \left(u_H - R_H \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* A_i u_H, \phi_H \right). \end{aligned}$$

From the above, we prove the first part of Proposition.

Let us prove that \bar{a} is symmetric. We have that

$$\begin{aligned} \bar{a}(u_H, \phi_H) &= a_i \left(R_H \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* A_i u_H, \phi_H \right) = a_i \left(\phi_H, R_H \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* A_i u_H \right) \\ &= \left\langle R_H^* \phi_H, \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* A_i u_H \right\rangle = \left\langle R_H^* A_i u_H, \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* \phi_H \right\rangle = \left\langle A_i u_H, R_H \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* \phi_H \right\rangle \\ &= a_i \left(u_H, R_H \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* \phi_H \right) = a_i \left(R_H \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* \phi_H, u_H \right) = \bar{a}(\phi_H, u_H). \end{aligned}$$

In the above equality chain, we have used the fact, which is easy to prove, that

$$\left\langle \phi, \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} u \right\rangle = \left\langle u, \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} \phi \right\rangle, \quad \text{for all } u, \phi \in U'.$$

On the other hand, from a simple calculation we get

$$\bar{a}(u_H, u_H) \geq \frac{m}{3} \left(1 + \frac{m}{2M} \right) |u_H|_{U_H}^2,$$

and

$$|\bar{a}(u_H, \phi_H)| \leq M \left(1 + \frac{M}{2m} \right) |u_H|_{U_H} |\phi_H|_{U_H}.$$

The result about the existence of the sequence $(\lambda_n)_{n=0}^\infty$ and the orthonormal bases $(\psi_n)_{n=0}^\infty \subset V_H$ can be found in [20]. \square

Now, we are going to define the strong operators associated to operators $A_{i,e}, \tilde{A}_{i,e}$ and A .

We define

$$D(A) := \{u_H \in H_2(\Omega_H) : \nabla u_H \cdot \mathbf{n} = 0 \text{ in a.e. } \partial\Omega_H\} \subset L_H,$$

We have

$$D(A_i) = D(A_e) = D(A)/\mathbb{R},$$

and

$$u_H \in D(A_{i,e}) \mapsto A_{i,e}u = \nabla \cdot (\sigma_{i,e} \nabla u_H).$$

On the other hand, if we denote by

$$D(\tilde{A}) = \left\{ u \in H_2(\Omega) : \begin{array}{l} \nabla R_H u \cdot \mathbf{n} = \nabla R_T u \cdot \mathbf{n} = 0, \text{ in a.e. } \partial\Omega_H, \text{ and } \nabla R_T u \cdot \mathbf{n} = 0 \text{ in } \Sigma_T, \\ R_H u = R_T u, \end{array} \right\}$$

we have that

$$D(\tilde{A}_{i,e}) = D(\tilde{A}_i + \tilde{A}_e) = D(\tilde{A})/\mathbb{R}.$$

where for all $u \in D(\tilde{A}_{i,e})$ we have

$$\tilde{A}_{i,e}u = \nabla \cdot (\tilde{\sigma}_{i,e} \nabla u).$$

It is possible to prove that A_i and $\tilde{A}_{i,e}, \tilde{A}_i + \tilde{A}_e$ are maximal monotone, self-adjoint and have compact inverse in L_H/\mathbb{R} and H/\mathbb{R} , respectively.

With respect to the functional $s \in V'_H$, it is known that there is an element $\tilde{s} \in L_H$ such that $\|\tilde{s}\|_{L_H} = \|s\|_{V'_H}$ and

$$\langle s, u_H \rangle = (\tilde{s}, u), \quad \text{for all } u \in V_H.$$

Definition 3.2. We define the strong bidomain operator $A : D(A) \subset H \rightarrow H$ given by the relation

$$Au_H = A_i R_H \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} R_H^* A_e J u_H,$$

for all $u_H \in D(A)$.

Furthermore, by s we denote again the element in L_H that it is identified with the functional $s \in V_H$ given in the above comment.

Note that, we again have denoted by A this operator, which is well-defined due to two facts.

First,

$$R_H^*|_{L_H/\mathbb{R}} : L_H/\mathbb{R} \rightarrow H/\mathbb{R},$$

and it is given by

$$R_H^* u = \begin{cases} u, & \text{in } \Omega_H, \\ 0, & \text{in } \Omega_T. \end{cases}$$

Furthermore,

$$R_H \left(\tilde{A}_i + \tilde{A}_e \right)^{-1} : H/\mathbb{R} \rightarrow D(A)/\mathbb{R}.$$

Theorem 3.3. *The strong operator A of Definition 3.6 is self-adjoint and maximal-monotone, such that,*

$$(Au_H, \phi_H) = a(u_H, \phi_H), \quad \text{for all } u_H, \phi_H \in D(A).$$

The sequence $(\lambda_n)_{n=0}^\infty$ given in Proposition 3.1 satisfies that each λ_i is a eigenvalue of A associated to eigenvector $\psi_i \in D(A)$. Furthermore, one has that

$$D(A) = \left\{ u_H \in L_H : \sum_{n=0}^{\infty} \lambda_n^2 (\psi_n, u_H)^2 < \infty \right\}, \quad \text{and} \quad Au_H = \sum_{n=0}^{\infty} \lambda_n (\psi_n, u_H) \psi_n.$$

Finally, we have that $Au_m = s, u = (\tilde{A}_e + \tilde{A}_i)^{-1} R_H^(s - A_i u_m)$ and $u_e = R_H u \in D(A)$, if and only if (u, u_m) are solutions of problem (3.1)–(3.7) a.e.*

On the other hand, for any $a_1 > 0$, $A + a_1 : D(A) \rightarrow L_H$ is also a sectorial, self-adjoint, maximal-monotone operator such that its spectrum consists of the eigenvalues $\{a_1 + \lambda_i\}_{i=0}^\infty$ with eigenvectors $\{\psi_i\}_{i=0}^\infty$. The family of bounded operators $e^{-t(A+a_1)}$ for all $t \geq 0$, is well defined and $e^{-0(A+a_1)} = I_d$ (the identity operator in L_H).

With respect to this family of operators, it is possible to give several properties, such as:

- for all $t > 0$, the spectrum of operator $e^{-t(A+a_1)}$ is $\{e^{-(\lambda_i+a_1)t}\}_{i=0}^\infty$ thus, $1 \notin \sigma(e^{-t(A+a_1)})$. Furthermore, $(\psi_i)_{i=0}^\infty$ are the eigenvectors associated,
- $(I_d - e^{-T(A+a_1)})^{-1}$ is a linear bounded operator, densely defined. Its spectrum is $\left((1 - e^{-(\lambda_i+a_1)T})^{-1} \right)_{i=0}^\infty$, with eigenvectors $(\psi_i)_{i=0}^\infty$,
- $\|e^{-t(A+a_1)}\| \leq 1$, for all $t \geq 0$.

Definition 3.4. We define $S : [0, \infty) \rightarrow L_H$, Holder continuous function given that $S(t) = J^* \bar{s}(t)$.

3.1. Definition of the strong solution. Uniqueness and local existence

In this section, f, g are locally Lipschitz. That is, for all $(u_0, w_0) \in \mathbb{R}^2$, there is an open $U \ni (u_0, w_0)$ and a constant K such that

$$\begin{aligned} |f(u_1, w_1) - f(u_2, w_2)| &\leq K|(u_1, w_1) - (u_2, w_2)|, \\ |g(u_1, w_1) - g(u_2, w_2)| &\leq K|(u_1, w_1) - (u_2, w_2)|. \end{aligned}$$

Note, the above means that $f + a_1 u$ is also locally Lipschitz continuous, for any $a_1 > 0$.

Consider $\mathcal{Z} := L_H \times L_H$ with the norm $\|(u, w)\|_{\mathcal{Z}} = \max\{\|u\|_{L_H}, \|w\|_{L_H}\}$ which is a Banach space. The operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{Z} \rightarrow \mathcal{Z}$ is defined, given by

$$\mathcal{A}(u, w) = \begin{pmatrix} a_1 + A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix},$$

where $D(\mathcal{A}) = D(A) \times L_H$.

Lemma 3.5. *The operator \mathcal{A} is sectorial, it has compact resolvent and $\mathbf{Re}\{\sigma(\mathcal{A})\} > 0$.*

To deal with the non-linearity, we use the fractional power of the operator A , A^α , and the interpolation spaces $D(A^\alpha)$ with $\alpha \geq 0$. In this sense, we have $A^\alpha : D(A^\alpha) \subset L_H \rightarrow L_H$ is an unbounded operator, such that

$$D(A^\alpha) = \left\{ u \in L_H : \sum_{i=0}^{\infty} \lambda_i^{2\alpha} (u, \psi_i)^2 < \infty \right\}, \quad \text{and} \quad A^\alpha u = \sum_{i=0}^{\infty} \lambda_i^\alpha (u, \psi_i) \psi_i.$$

The space $D(A^\alpha)$ with the norm

$$\|u\|_{D(A^\alpha)} = \|u + A^\alpha u\|_{L_H} = \sqrt{\sum_{i=0}^{\infty} (1 + \lambda_i^{2\alpha})(u, \psi_i)^2},$$

is a Banach space.

We also can define \mathcal{A}^α , and the interpolation spaces $D(\mathcal{A}^\alpha)$ as

$$\mathcal{Z}^\alpha := D(A^\alpha) \times D(A^\alpha), \quad \text{and} \quad \mathcal{A}^\alpha(u, w) = \left(\sum_{i=0}^{\infty} \lambda_i^\alpha (u, \psi_i) \psi_i, 0 \right), \quad \forall (u, w) \in \mathcal{Z}^\alpha.$$

The space \mathcal{Z}^α , equipped with norm $\|(u, w)\|_\alpha = \max \left\{ \|u\|_{D(A^\alpha)}, \|w\|_{D(A^\alpha)} \right\}$ is also a Banach space. It is possible to prove that for all $0 \leq \alpha < \beta \leq 1$, $\mathcal{Z}^\beta \subset \mathcal{Z}^\alpha$ and $\mathcal{Z}^0 = \mathcal{Z}$, $\mathcal{Z}^1 = D(\mathcal{A}) \subset H_2(\Omega) \times H_2(\Omega)$. Since the resolvent of \mathcal{A} is compact, the inclusion $\mathcal{Z}^\beta \subset \mathcal{Z}^\alpha$ is compact, provided that $\alpha < \beta$.

For any $\alpha > 0$, the operator $\mathcal{A}^{-\alpha}$ can also be defined in \mathcal{Z} . It is linear, bounded, injective, and is the inverse of operator \mathcal{A}^α . In this case, \mathcal{Z}^α coincides with the range of $\mathcal{A}^{-\alpha}$. We define $\mathcal{Z}^{-\alpha}$ to be the set \mathcal{Z} with norm

$$\|z\|_{-\alpha} = \|\mathcal{A}^{-\alpha} z\|_{\mathcal{Z}},$$

which is a Banach space. It is easy to prove that the following inclusions are continuous

$$\mathcal{Z}^\alpha \subset \mathcal{Z} \subset \mathcal{Z}^{-\alpha}, \quad \text{for all } \alpha > 0.$$

The proof of these results can be found in [11].

If $3/4 < \alpha \leq 1$, then $\mathcal{Z}^\alpha \subset L_\infty(\Omega_H) \times L_\infty(\Omega_H)$ and given that $f + a_1, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ are locally Lipschitz continuous then, $F = (f + a_1, g) : \mathcal{Z}^\alpha \rightarrow \mathcal{Z}$ is locally Lipschitz continuous. That is, given $z_0 \in \mathcal{Z}^\alpha$ there is an open $\mathcal{U} \subset \mathcal{Z}^\alpha$ containing z_0 and a positive constant L , such that

$$\|F(z_1) - F(z_2)\|_{\mathcal{Z}} \leq L \|z_1 - z_2\|_\alpha, \quad \text{for all } z_1, z_2 \in \mathcal{U}.$$

Following [11], it is possible to give a definition of strong solution of the Cauchy problem associated to the bidomain problem (1.2)–(1.10).

Definition 3.6 (Strong solution). Consider the functions $z(t) = (u(t), w(t)) : [0, t_1] \rightarrow \mathcal{Z}$, and $u : [0, t_1] \rightarrow H$ where

$$u(t) = \begin{cases} u_e(t), & \text{in } \Omega_H, \\ u_T(t), & \text{in } \Omega_T. \end{cases}$$

We say that the pair (z, u) is a local strong solution to (1.2)–(1.10) if

1. $z : [0, t_1] \rightarrow \mathcal{Z}$ is continuous such that $z(0) = z_0 = (u_0, w_0)$, for all $t \in (0, t_1)$, $z(t) \in \mathcal{U} \cap D(\mathcal{A})$, and $\frac{dz(t)}{dt}$ exists in the Fréchet sense,
2. for all $t \in (0, t_1)$ the following semilinear evolution equation is satisfied by $z(t)$

$$\frac{dz(t)}{dt} = -\mathcal{A}z(t) + F(z(t)) + S(t), \quad \text{in } \mathcal{Z}, \quad (3.10)$$

$$\text{and } u(t) = \left(\tilde{A}_e + \tilde{A}_i \right)^{-1} R_H^* (s - A_i J u_m(t)).$$

If $t_1 = \infty$, we say that the solution is global. Furthermore, if the pair z, u is a global solution such that $z(t) = z(t+T), u(t) = u(t+T)$, we say that this is a T -periodic solution.

From Lemma 3.3.2 of [11], it can be demonstrated that if $z(t)$ is a local strong solution with initial condition z_0 , then

$$z(t) = e^{-tA} z_0 + \int_0^t e^{-(t-\tau)A} (F(z(\tau)) + S(\tau)) d\tau, \quad t \in (0, t_1). \quad (3.11)$$

Conversely, if $z : [0, t_1) \rightarrow \mathcal{Z}^\alpha$ is a continuous function that satisfies the integral equation (3.11), and

$$\int_0^\rho \|F(z(\tau)) + S(\tau)\|_{\mathcal{Z}} d\tau < \infty, \quad \text{for some } \rho > 0, \quad (3.12)$$

then $z(t)$ is a strong solution in $(0, t_1)$.

The following theorem is a consequence of Theorem 3.2.2 of [11].

Theorem 3.7. *If the initial condition $(u_0, w_0) \in \mathcal{Z}^\alpha$, then the bidomain model has an only strong solution, (u, w) , that satisfies the initial condition $(u(0), w(0)) = (u_0, w_0)$.*

4. FAEDO-GALERKIN SYSTEM ASSOCIATED TO THE OPERATIONAL FORMULATION

In this section, we investigate when the Faedo-Galerkin approximations converge to a strong solution, in the sense of Definition 3.6. We obtain the Faedo-Galerkin systems from the semilinear evolution equation (3.10). Our idea is based in the fact that Faedo-Galerkin approximations have the same regularity of a strong solution. Furthermore, we can achieve that these converge in a suitable sense such that its limit be a strong solution. We believe that it is an important result because, until now, only weak solutions were obtained as the limit of Faedo-Galerkin approximations.

We define $H_m = \langle \psi_0, \psi_1, \dots, \psi_m \rangle$ as the subspace generated by the $m+1$ -first eigenvectors associated to the bidomain operator A , endowed with the norm of L_H . We denote by $\mathcal{Z}_m = H_m \times H_m$.

Note, $\mathcal{Z}_m \subset \mathcal{Z}^\alpha$, for all $\alpha \geq 0$ which is only considered as the inclusion operation among sets. We can define $\mathcal{Z}_m^\alpha = H_m \times H_m$ endowed with the norm in \mathcal{Z}^α , for $\alpha \geq 0$.

We can define $\mathcal{P}_m : \mathcal{Z} \rightarrow \mathcal{Z}_m \subset \mathcal{Z}$ by

$$\mathcal{P}_m z = \left(\sum_{i=0}^m (u, \psi_i) \psi_i, \sum_{i=0}^m (w, \psi_i) \psi_i \right), \quad \forall z = (u, w) \in \mathcal{Z}, \quad \text{for all } m \in \mathbb{N} \cup \{0\}.$$

The operator that was defined in the previous equation is called \mathcal{Z} in \mathcal{Z}_m projection operator, which satisfies that $\|\mathcal{P}_m\|_{\mathcal{L}(\mathcal{Z})} \leq 1$. Similarly, it can be easily proved that

$$\mathcal{P}_m|_{\mathcal{Z}^\alpha} : \mathcal{Z}^\alpha \rightarrow \mathcal{Z}^\alpha,$$

$\|\mathcal{P}_m\|_{\mathcal{L}(\mathcal{Z}^\alpha)} \leq 1$, and

$$\|\mathcal{P}_m z - z\|_{\mathcal{Z}^\alpha} \rightarrow 0, \quad \|\mathcal{P}_m z - z\|_{\mathcal{Z}} \rightarrow 0 \quad \text{when } m \rightarrow \infty, \quad \text{for all } z \in \mathcal{Z}^\alpha.$$

In other words, $\|\mathcal{P}_m - I_d\|_{\mathcal{L}(\mathcal{Z})} \rightarrow 0$ and $\|\mathcal{P}_m - I_d\|_{\mathcal{L}(\mathcal{Z}^\alpha)} \rightarrow 0$. Note that, this implies

$$\|\mathcal{P}_m - \mathcal{P}_n\|_{\mathcal{L}(\mathcal{Z})} \leq \|I_d - \mathcal{P}_n\|_{\mathcal{L}(\mathcal{Z})} + \|I_d - \mathcal{P}_m\|_{\mathcal{L}(\mathcal{Z})} \rightarrow 0,$$

and

$$\|\mathcal{P}_m - \mathcal{P}_n\|_{\mathcal{L}(\mathcal{Z}^\alpha)} \rightarrow 0$$

when $m, n \rightarrow \infty$.

Define the functions

$$u_m(t, x) = \sum_{i=0}^m u_i^{(m)}(t) \psi_i(x), \quad w_m(t, x) = \sum_{i=0}^m w_i^{(m)}(t) \psi_i(x),$$

where the vector functions

$$U_m(t) = \left(u_0^{(m)}(t), u_1^{(m)}(t), \dots, u_m^{(m)}(t) \right) \quad \text{and} \quad W_m(t) = \left(w_0^{(m)}(t), w_1^{(m)}(t), \dots, w_m^{(m)}(t) \right) \quad (4.1)$$

are in $C^1(I, \mathbb{R}^{m+1})$ for certain interval $I \subseteq \mathbb{R}$ containing to 0. Note, $z_m(t) = (u_m(t), w_m(t)) \in \mathcal{Z}_m$, in particular $z_m(t) \in D(\mathcal{A})$ for all $t \in I$. Besides,

$$u'_m(t, x) = \sum_{i=0}^m u_i^{(m)'}(t) \psi_i(x), \quad w'_m(t, x) = \sum_{i=0}^m w_i^{(m)'}(t) \psi_i(x),$$

are the Fréchet derivatives of the function u_m, w_m , respectively, thus, $z'_m(t) = (u'_m(t), w'_m(t)) \in \mathcal{Z}_m$.

In that follows, we are interested in to studying the next problem of initial conditions: to find a function $z_m \in C^1(I; \mathcal{Z}_m^\alpha)$, for certain $0 < \alpha < 1$, which satisfies

$$\begin{cases} \frac{dz_m(t)}{dt} = -\mathcal{A}z_m(t) + \mathcal{P}_m F(z_m(t)) + \mathcal{P}_m S(t), \\ z_m(0) = z_m^{(0)} \in \mathcal{Z}_m^\alpha. \end{cases} \quad (4.2)$$

and converges in a suitable form, which will be specified later, to a strong solution in the sense of Definition 3.6.

Take into account that $z_m^{(0)} = (u_m^{(0)}, w_m^{(0)})$, such that,

$$u_m^{(0)} = \sum_{i=0}^m u_i^{(m,0)} \psi_i, \quad w_m^{(0)} = \sum_{i=0}^m w_i^{(m,0)} \psi_i,$$

thus, to the initial condition $z_m^{(0)}$ we can also associate a pair of vectors of \mathbb{R}^{m+1}

$$U_m^{(0)} = \left(u_0^{(m,0)}, u_1^{(m,0)}, \dots, u_m^{(m,0)} \right), \quad W_m^{(0)} = \left(w_0^{(m,0)}, w_1^{(m,0)}, \dots, w_m^{(m,0)} \right). \quad (4.3)$$

The sequence $\{z_m\}_{m=0}^\infty \subset C^1(I; \mathcal{Z}_m^\alpha)$ built from the solution of problem (4.2) for each $m \in \mathbb{N}$, will be called by us as strong Faedo-Galerkin approximations of the strong solution of the torso-coupled bidomain model. Now, we will show the reason why this terminology is used.

Remark 4.1. In order to achieve this convergence, we suppose that the sequence of initial conditions $\{z_m^{(0)}\}$ converges in \mathcal{Z}^α and its limit is $z_0 \in \mathcal{Z}^\alpha$. In this case, there are constants $m_0 \in \mathbb{N}$, $\delta_{m_0} > 0$ and an open $\mathcal{U}_{m_0} \subset \mathcal{Z}^\alpha$, such that F is Lipschitz continuous in \mathcal{U}_{m_0} with Lipschitz constant L_{m_0} and

$$\{z_m^{(0)}\}_{m=m_0}^\infty \subset \mathcal{B}_\alpha\left(z_0, \frac{\delta_{m_0}}{2}\right) \subset \mathcal{U}_{m_0}.$$

Now, for each of the remaining elements of the sequence, $\{z_k^{(0)}\}_{k=0}^{m_0-1}$, there is an open $\mathcal{U}_k \subset \mathcal{Z}^\alpha$ and a ball $\mathcal{B}_\alpha\left(z_k^{(0)}, \frac{\delta_k}{2}\right)$, such that,

$$\mathcal{B}_\alpha\left(z_k^{(0)}, \frac{\delta_k}{2}\right) \subset \mathcal{U}_k,$$

and F is Lipschitz continuous in \mathcal{U}_k with Lipschitz constant L_k . Then, we have

$$\{z_m^{(0)}\}_{m=0}^\infty \subset \left(\bigcup_{m=0}^{m_0-1} \mathcal{B}_\alpha\left(z_k^{(0)}, \frac{\delta_k}{2}\right)\right) \cup \mathcal{B}_\alpha\left(z_0, \frac{\delta}{2}\right) \subset \mathcal{U} = \bigcup_{m=0}^{m_0} \mathcal{U}_m$$

where $\delta = \min\{\delta_0, \delta_1, \dots, \delta_{m_0}\}$, and F is Lipschitz continuous, with Lipschitz constant $L = \max\{L_0, L_1, \dots, L_{m_0}\}$.

Summarizing the facts presented above, we assume that $\{z_m^{(0)}\}$ satisfies the following condition

CB There is a finite family of balls $\{\mathcal{B}_\alpha(z_j, \frac{\delta}{2})\}_{j=0}^{m_0}$, where $z_j \in \mathcal{Z}^\alpha$ for $j = 0, 1, \dots, m_0$, such that,

$$\{z_m^{(0)}\} \subset \mathcal{F} = \bigcup_{j=0}^{m_0} \mathcal{B}_\alpha\left(z_j, \frac{\delta}{2}\right),$$

and F is Lipschitz continuous in \mathcal{F} .

If z_m satisfies the problem in (4.2), then the vectors $U_m(t), W_m(t)$ associated with $z_m(t) = (u_m(t), w_m(t))$ given in (4.1), satisfy the following problem of initial conditions associated to a system of $2m + 2$ equations

$$\begin{cases} u_i^{(m)'}(t) = -(a_1 + \lambda_i) u_i^{(m)}(t) + \int_{\Omega_H} f(u_m(t), w_m(t)) \psi_i + \int_{\Omega_H} S(\tau) \psi_i, \\ w_i^{(m)'}(t) = \int_{\Omega_H} g(u_m(t), w_m(t)) \psi_i, \\ U_m(0) = U_m^{(0)}, \quad W_m(0) = W_m^{(0)}. \end{cases} \quad i = 0, \dots, m. \quad (4.4)$$

Thus, the Cauchy problem (4.2) is equivalent to (4.4). That is, we can reduce the problem of finding a unique solution of the system (4.2) to finding one of the system (4.4). The advantage in this case is that the system (4.4) is a system of classical ordinary differential equations to which tools of this theory can be applied.

On the other hand, z_m is a solution of (4.2) if and only if

$$z_m(t) = e^{-tA} z_m^{(0)} + \int_0^t e^{-(t-\tau)A} \mathcal{P}_m [F(z_m(\tau)) + S(\tau)] d\tau. \quad (4.5)$$

In what follows, we set $3/4 < \alpha_0 \leq 1$, and we are going to prove that the Faedo-Galerkin problem (4.2) has an only solution z_m in $L_2(0, t_1; \mathcal{Z}^{\alpha_0})$, for certain $t_1 > 0$, when the sequence of initial conditions satisfies the condition **CB**.

By $C([0, t_1]; \mathcal{Z}_m^{\alpha_0})$ we denote the space of functions $z_m(t)$ continuous in $[0, t_1]$ with values in $\mathcal{Z}_m^{\alpha_0}$, and

$$V_\delta = \{z_m \in C([0, t_1]; \mathcal{Z}^{\alpha_0}) : z_m(t) \in \mathcal{F}, \text{ for } t \in [0, t_1]\}.$$

By $\mathcal{K}_m : V_\delta \subset C([0, t_1]; \mathcal{Z}^{\alpha_0}) \rightarrow C([0, t_1]; \mathcal{Z}^{\alpha_0})$, we denote the operator given by the relation

$$K_m(z_m)(t) = e^{-t\mathcal{A}} z_m^{(0)} + \int_0^t e^{-(t-\tau)\mathcal{A}} \mathcal{P}_m [F(z_m(\tau)) + S(\tau)] d\tau, \quad z_m \in V_\delta.$$

Now, δ and t_1 are chosen, such that $K_m(V_\delta) \subseteq V_\delta$. In fact, if $z_m \in V_\delta$, then, for any $t \in [0, t_1]$, $\|z_m(t) - \tilde{z}_t\|_{\alpha_0} \leq \frac{\delta}{2}$, where $\tilde{z}_t \in \{z_0, z_1, \dots, z_{m_0}\}$.

We have

$$\begin{aligned} \|(K_m z_m)(t) - \tilde{z}_t\|_{\alpha_0} &\leq \|(K_m z_m)(t) - z_m^{(0)}\|_{\alpha_0} + \|z_m^{(0)} - \tilde{z}_t\|_{\alpha_0} \\ &\leq \|(e^{-t\mathcal{A}} - I_d) z_m^{(0)}\|_{\alpha_0} + \left\| \int_0^t e^{-(t-\tau)\mathcal{A}} \mathcal{P}_m [F(z_m(\tau)) + S(\tau)] d\tau \right\|_{\alpha_0} + \frac{\delta}{2}. \end{aligned} \quad (4.6)$$

Let us estimate the first term of the right-side of (4.6). In this sense, we write

$$(I_d + \mathcal{A}^{\alpha_0})(e^{-t\mathcal{A}} - I_d) z_m^{(0)} = - \int_0^t (I_d + \mathcal{A}^{\alpha_0})(e^{-s\mathcal{A}} - I_d) z_m^{(0)} ds = - \int_0^t (e^{-s\mathcal{A}} - I_d)(I_d + \mathcal{A}^{\alpha_0}) z_m^{(0)} ds,$$

thus,

$$\|(e^{-t\mathcal{A}} - I_d) z_m^{(0)}\|_{\alpha_0} \leq \int_0^{t_1} \|(e^{-s\mathcal{A}} - I_d)\|_{\mathcal{L}(\mathcal{Z})} \|z_m^{(0)}\|_{\alpha_0} ds \leq 2 \|z_m^{(0)}\|_{\alpha_0} t_1.$$

We choose $t_1 > 0$, such that

$$\sup_{t \in [0, t_1]} \|(e^{-t\mathcal{A}} - I_d) z_m^{(0)}\|_{\alpha_0} \leq \frac{\delta}{4}.$$

To estimate the second term of the right-side of (4.6), let us see

$$\begin{aligned} \left\| \int_0^t e^{-(t-\tau)\mathcal{A}} \mathcal{P}_m [F(z_m(\tau)) + S(\tau)] d\tau \right\|_{\alpha_0} &\leq \left\| \int_0^{t_1} (I_d + \mathcal{A}^{\alpha_0}) e^{-(t-\tau)\mathcal{A}} \mathcal{P}_m [F(z_m(\tau)) + S(\tau)] d\tau \right\|_{\mathcal{Z}} \\ &\leq \int_0^{t_1} \|(I_d + \mathcal{A}^{\alpha_0}) e^{-(t-\tau)\mathcal{A}} \mathcal{P}_m [F(z_0) + S(\tau)]\|_{\mathcal{Z}} d\tau + \int_0^{t_1} \|(I_d + \mathcal{A}^{\alpha_0}) e^{-(t-\tau)\mathcal{A}} \mathcal{P}_m (F(z_m(\tau)) - F(z_0))\|_{\mathcal{Z}} d\tau \\ &\leq M(B + L\delta) \int_0^{t_1} \tau^{-\alpha_0} e^{a_1\tau} d\tau \leq (B + L\delta) \frac{e^{bt_1} t_1^{1-\alpha_0}}{1 - \alpha_0}, \end{aligned}$$

where we have taken into account that

$$\|F(z_0) + S(\tau)\|_{\mathcal{Z}} \leq B,$$

for certain constant $B > 0$. In summary,

$$\sup_{t \in [0, t_1]} \left\| \int_0^t e^{-(t-\tau)\mathcal{A}} \mathcal{P}_m F(z_m(\tau)) d\tau \right\|_{\alpha_0} \leq M(B + L\delta) \frac{e^{a_1 t_1} t_1^{1-\alpha_0}}{1 - \alpha_0} \leq \frac{\delta}{4},$$

for certain δ, t_1 appropriately chosen. We have proved that, there are positive constants δ, t_1 , such that $\mathcal{K}_m(V_\delta) \subseteq V_\delta$.

In a similar way, it can also be proved that \mathcal{K}_m is a contracting map if we suitably choose t_1, δ . The previously proved results can be stated in the following lemma.

Lemma 4.2. *The Cauchy problem (4.2) has a unique solution $z_m \in L_2(0, t_1; \mathcal{Z}^{\alpha_0})$ for certain $t_1 > 0$ and $3/4 < \alpha_0 < 1$. Besides, the sequence $\{z_m\}_{m=0}^\infty \subset \mathcal{B}_{L_2(0, t_1; \mathcal{Z}^{\alpha_0})}(r_0)$ for certain $r_0 > 0$, where*

$$\mathcal{B}_{L_2(0, t_1; \mathcal{Z}^{\alpha_0})}(r_0) = \left\{ z \in L_2(0, t_1; \mathcal{Z}^{\alpha_0}) : \|z\|_{L_2(0, t_1; \mathcal{Z}^{\alpha_0})} \leq r_0 \right\}.$$

Proof. With the previous explanation, the existence of a solution of problem (4.2) $z_m \in V_\delta \subset C([0, t_1]; \mathcal{Z}_m^{\alpha_0})$, has been demonstrated for each $m \in \mathbb{N} \cup \{0\}$. Note that $C([0, t_1]; \mathcal{Z}_m^{\alpha_0}) \subset L_2(0, t_1; \mathcal{Z}^{\alpha_0})$, continuously. Moreover, $z_m \in V_\delta$, implies that

$$\|z_m(t)\|_{\alpha_0} \leq \max\{\|z_0\|_{\alpha_0}, \|z_1\|_{\alpha_0}, \dots, \|z_{m_0}\|_{\alpha_0}\} + \frac{\delta}{2},$$

thus

$$\left(\int_0^{t_1} \|z_m(\tau)\|_{\alpha_0}^2 d\tau \right)^{\frac{1}{2}} \leq \sqrt{t_1} \left(\max\{\|z_0\|_{\alpha_0}, \|z_1\|_{\alpha_0}, \dots, \|z_{m_0}\|_{\alpha_0}\} + \frac{\delta}{2} \right).$$

That is, $r_0 = \sqrt{t_1} \left(\max\{\|z_0\|_{\alpha_0}, \|z_1\|_{\alpha_0}, \dots, \|z_{m_0}\|_{\alpha_0}\} + \frac{\delta}{2} \right)$. □

Now, we shall demonstrate the following Proposition.

Proposition 4.3. *Let $\{z_m\}_{m=0}^\infty \subset V_\delta$ be the sequence of solution of the Faedo-Galerkin systems obtained in Lemma 4.2, is a Cauchy sequence in $C(0, t_1; \mathcal{Z}^{\alpha_0})$.*

Proof. We have

$$\begin{aligned} \|z_m(t) - z_n(t)\|_{\alpha_0} &\leq \left\| e^{-t\mathcal{A}} \left(z_m^{(0)} - z_n^{(0)} \right) \right\|_{\alpha_0} + \int_0^t \left\| e^{-(t-\tau)\mathcal{A}} (\mathcal{P}_m F(z_m(\tau)) - \mathcal{P}_n F(z_n(\tau))) \right\|_{\alpha_0} d\tau \\ &+ \int_0^t \left\| e^{-(t-\tau)\mathcal{A}} (\mathcal{P}_m - \mathcal{P}_n) S(\tau) \right\|_{\alpha_0} d\tau \\ &\leq \left\| z_m^{(0)} - z_n^{(0)} \right\|_{\alpha_0} + \|\mathcal{P}_m - \mathcal{P}_n\|_{\mathcal{L}(\mathcal{Z}^{\alpha_0})} \widehat{S} \frac{t_1^{1-\alpha_0}}{1 - \alpha_0} + \int_0^t \left\| e^{-(t-\tau)\mathcal{A}} (\mathcal{P}_m F(z_m(\tau)) - \mathcal{P}_n F(z_n(\tau))) \right\|_{\alpha_0} d\tau. \end{aligned}$$

Now,

$$\begin{aligned} \int_0^t \left\| e^{-(t-\tau)\mathcal{A}} (\mathcal{P}_m F(z_m(\tau)) - \mathcal{P}_n F(z_n(\tau))) \right\|_{\alpha_0} d\tau &\leq \int_0^t e^{-(t-\tau)a_1} (t-\tau)^{-\alpha_0} \|\mathcal{P}_m F(z_m(\tau)) - \mathcal{P}_n F(z_n(\tau))\|_{\mathcal{Z}} d\tau \\ &\leq \int_0^t (t-\tau)^{-\alpha_0} \|\mathcal{P}_m F(z_m(\tau)) - \mathcal{P}_n F(z_n(\tau))\|_{\mathcal{Z}} d\tau \leq \int_0^t (t-\tau)^{-\alpha_0} \|\mathcal{P}_m (F(z_m(\tau)) - F(z_n(\tau)))\|_{\mathcal{Z}} d\tau \end{aligned}$$

$$\begin{aligned}
& + \int_0^t (t-\tau)^{-\alpha_0} \|(\mathcal{P}_m - \mathcal{P}_n) F(z_n(\tau))\|_{\mathcal{Z}} d\tau \leq \left(L \max_{t \in [0, t_1]} \|z_m(t) - z_n(t)\|_{\alpha_0} + \frac{\delta}{2} \|\mathcal{P}_m - \mathcal{P}_n\|_{\mathcal{L}(\mathcal{Z})} + \dots \right. \\
& \left. \|\mathcal{P}_m - \mathcal{P}_n\|_{\mathcal{L}(\mathcal{Z})} \max \{ \|F(z_0)\|_{\mathcal{Z}}, \dots, \|F(z_{m_0})\|_{\mathcal{Z}} \} \right) \frac{t_1^{1-\alpha_0}}{1-\alpha_0}.
\end{aligned}$$

Taking into account the previous estimations, we obtain the following

$$\begin{aligned}
\|z_m - z_n\|_{C([0, t_1; \mathcal{Z}^{\alpha_0}])} & \leq \frac{1}{1 - L \frac{t_1^{1-\alpha_0}}{1-\alpha_0}} \left(\|z_m^{(0)} - z_n^{(0)}\|_{\alpha_0} + \|\mathcal{P}_m - \mathcal{P}_n\|_{\mathcal{L}(\mathcal{Z})} \widehat{S} \frac{t_1^{1-\alpha_0}}{1-\alpha_0} \right. \\
& \left. + \|\mathcal{P}_m - \mathcal{P}_n\|_{\mathcal{L}(\mathcal{Z})} \max \{ \|F(z_0)\|_{\mathcal{Z}}, \dots, \|F(z_{m_0})\|_{\mathcal{Z}} \} \frac{t_1^{1-\alpha_0}}{1-\alpha_0} \right), \tag{4.7}
\end{aligned}$$

where t_1 is taken such that $\frac{t_1^{1-\alpha_0}}{1-\alpha_0} < L$. □

Now, we can give the following Theorem.

Theorem 4.4. *If the sequence of the initial conditions $\{z_m^{(0)}\}_{m=0}^{\infty}$ converges to z_0 in \mathcal{Z}^{α_0} , with $3/4 < \alpha_0 < 1$, then the sequence of solutions of the problem (4.2), $z_m \in V_{\delta}$ converges in $C(0, t_1; \mathcal{Z}^{\alpha_0})$ to function $z \in C(0, t_1; \mathcal{Z}^{\alpha_0})$, which is the only strong solution of the coupled-torso bidomain model, such that $z(0) = z_0$.*

Proof. By Proposition 4.3, we can affirm that the sequence of strong Faedo-Galerkin approximations is a sequence of Cauchy in $C(0, t_1, \mathcal{Z}^{\alpha_0})$. Furthermore, we can choose t_1 and $\delta > 0$, such that $\{z_m\}_{m=0}^{\infty} \subset V_{\delta}$, which is a closed subspace of $C(0, t_1, \mathcal{Z}^{\alpha_0})$, thus there is $z \in V_{\delta}$, such that

$$z_m \rightarrow z, \quad \text{strongly in } C(0, t_1, \mathcal{Z}^{\alpha_0}).$$

To demonstrate that z is a strong solution, we must only prove that it satisfies (3.11) and (3.12). In first place, we have

$$z_m(t) = e^{-t\mathcal{A}} z_m^{(0)} + \int_0^t e^{-(t-\tau)\mathcal{A}} \mathcal{P}_m [F(z_m(\tau)) + S(\tau)] d\tau, \quad \text{for each } m = 0, 1, \dots, \text{ and } t \in [0, t_1].$$

It is very easy to prove that by making the limit on each sum of the equality it is obtained

$$z(t) = e^{-t\mathcal{A}} z_0 + \int_0^t e^{-(t-\tau)\mathcal{A}} \mathcal{P}_m [F(z(\tau)) + S(\tau)] d\tau, \quad t \in [0, t_1].$$

On the other hand, since $z \in V_{\delta}$, it is possible to prove that

$$\int_0^{t_1} \|F(z(\tau)) + S(\tau)\|_{\mathcal{Z}} d\tau < \infty.$$

□

Theorem 4.4 allows us to state that the unique solution of the Cauchy problem associated with the boundary problem of the bidomain model can be obtained as the limit of a succession of strong Faedo-Galerkin approximations, provided that the initial condition z_0 belongs to the space \mathcal{Z}^{α_0} . On the other hand, the succession of Faedo-Galerkin approximations is obtained by solving the problem (4.2), where the initial condition is assumed to be $z_m^{(0)} = \mathcal{P}_m z_0$. If each function z_m of the sequence of approximations is defined for all $t \geq 0$, and in addition,

they are T -periodic functions, taking into account the nature of convergence, it can be affirmed that the limit is also a T -periodic function and is defined for all $t \geq 0$. This existence result allows to define a computational algorithm to calculate the solution of the boundary problem. Basically, a good approximation of the sought solution can be obtained by solving the system (4.2) taking m large enough. The main computational challenge in this method turns out to be obtaining the eigenvalues of the bidomain operator. This problem depends on the geometry considered for the volume that represents the heart. However, once the eigenvalues have been calculated, they can be used to solve the system (4.2). That is, the computational cost of this step only occurs once. However, solving the system (4.2), being a problem that depends on time, it does require speed and computational accuracy in order to solve problems in real time.

5. EXISTENCE OF STRONG PERIODIC SOLUTION

In this section, we demonstrate the existence a strong T -periodic solution, in the sense of Definition 3.6, assuming that S is T -periodic, which is continuous from $[0, T]$ in \mathcal{Z} .

From now on, we will use the following notation

$$\begin{aligned} C_T &= \{z \in C([0, T]; \mathcal{Z}) : z(0) = z(T)\}, \\ C_T^{\alpha_0} &= \{z \in C([0, T]; \mathcal{Z}^{\alpha_0}) : z(0) = z(T)\}, \end{aligned}$$

where, in each case, we consider the following norms

$$\begin{aligned} \|f\|_{C_T} &= \sup_{t \in [0, T]} \|f(t)\|_{\mathcal{Z}}, \\ \|f\|_{C_T^{\alpha_0}} &= \sup_{t \in [0, T]} \|f(t)\|_{\mathcal{Z}^{\alpha_0}}, \end{aligned}$$

for $f \in C_T$ or $f \in C_T^{\alpha_0}$, respectively.

As the previous section, this T -periodic solution is obtained as a limit of a sequence of solutions of system (4.2) which are T -periodic.

Let

$$\frac{dz_m}{dt} = -\mathcal{A}z_m + \mathcal{P}_m S \tag{5.1}$$

be the linear equation obtained from (4.2), and let $z_m \in C([0, \infty); \mathcal{Z}_m^{\alpha_0})$ be a solution (5.1). Then,

$$z_m(t) = e^{-t\mathcal{A}} z_m^{(0)} + \int_0^t e^{-(t-\tau)\mathcal{A}} \mathcal{P}_m S(\tau) d\tau,$$

where $z_m(0) = z_m^{(0)}$. Due to the uniqueness of the solution of the Cauchy problem associated with (5.1) and the periodicity of the function S , it is possible to prove that z_m is T -periodic if and only if $z_m(0) = z_m(T)$. Thus

$$z_m(T) = e^{-T\mathcal{A}} z_m^{(0)} + \int_0^T e^{-(T-\tau)\mathcal{A}} \mathcal{P}_m S(\tau) d\tau = z_m^{(0)},$$

that is,

$$(I_d - e^{-T\mathcal{A}}) z_m^{(0)} = \int_0^T e^{-(T-\tau)\mathcal{A}} \mathcal{P}_m S(\tau) d\tau.$$

Let us remember that $0 \notin \sigma(\mathcal{A}) \Rightarrow 1 \notin \sigma(e^{-T\mathcal{A}})$, thus the operator $I_d - e^{-T\mathcal{A}}$ is invertible and

$$\left\| (I_d - e^{-T\mathcal{A}})^{-1} \right\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{1}{1 - e^{-Ta_1}}.$$

Thus, we obtain

$$z_m^{(0)} = (I_d - e^{-T\mathcal{A}})^{-1} \int_0^T e^{-(T-\tau)\mathcal{A}} \mathcal{P}_m S(\tau) d\tau.$$

Finally, we can affirm that if z_m is T -periodic, then

$$z_m(t) = \int_0^T \mathcal{R}(t, \tau) \mathcal{P}_m S(\tau) d\tau,$$

where

$$\mathcal{R}(t, \tau) = \begin{cases} (I_d - e^{-T\mathcal{A}})^{-1} e^{-(t-\tau)\mathcal{A}}, & \text{if } 0 \leq \tau \leq t \leq T, \\ (I_d - e^{-T\mathcal{A}})^{-1} e^{-(t+T-\tau)\mathcal{A}}, & \text{if } 0 \leq t \leq \tau \leq T. \end{cases}$$

For convenience, for $f \in C_T$ we denote by

$$\mathcal{L}^{(p)}(f)(t) = \int_0^T \mathcal{R}(t, \tau) f(\tau) d\tau.$$

Note that, in this case, z_m is a T -periodic solution of (5.1) if and only if

$$z_m(t) = \mathcal{L}^{(p)}(\mathcal{P}_m S)(t), \quad t \in [0, T].$$

Before continuing with the results that interest us, let us give the following Lemma.

Lemma 5.1. *Let f be a function in C_T , then $\mathcal{L}_m^{(p)}(f) \in C_T^{\alpha_0}$, and*

$$\left\| \mathcal{L}^{(p)}(f) \right\|_{C_T^{\alpha_0}} \leq 2 \left(\frac{1}{a_1} + \frac{T^{1-\alpha_0}}{(1-\alpha_0)(1-e^{-a_1 T})} \right) \|f\|_{C_T}.$$

Proof. If $f \in C_T^{\alpha_0}$

$$\left\| \mathcal{L}^{(p)}(f)(t) \right\|_{\alpha_0} \leq \int_0^t \left\| (I_d - e^{-T\mathcal{A}})^{-1} e^{-(t-\tau)\mathcal{A}} f(\tau) \right\|_{\alpha_0} d\tau + \int_t^T \left\| (I_d - e^{-T\mathcal{A}})^{-1} e^{-(t+T-\tau)\mathcal{A}} f(\tau) \right\|_{\alpha_0} d\tau.$$

Now,

$$\begin{aligned} & \int_0^t \left\| (I_d - e^{-T\mathcal{A}})^{-1} e^{-(t-\tau)\mathcal{A}} f(\tau) \right\|_{\alpha_0} d\tau = \int_0^t \left\| (I_d - e^{-T\mathcal{A}})^{-1} (I_d + A^{\alpha_0}) e^{-(t-\tau)\mathcal{A}} f(\tau) \right\|_{\mathcal{Z}} d\tau \\ & \leq \int_0^t \left\| (I_d - e^{-T\mathcal{A}})^{-1} e^{-(t-\tau)\mathcal{A}} f(\tau) \right\|_{\mathcal{Z}} d\tau + \int_0^t \left\| (I_d - e^{-T\mathcal{A}})^{-1} A^{\alpha_0} e^{-(t-\tau)\mathcal{A}} f(\tau) \right\|_{\mathcal{Z}} d\tau \\ & \leq \frac{\|f\|_{C_T^{\alpha_0}}}{1 - e^{-a_1 T}} \int_0^t e^{-a_1(t-\tau)} d\tau + \frac{\|f\|_{C_T^{\alpha_0}}}{1 - e^{-a_1 T}} \int_0^t (t-\tau)^{-\alpha_0} d\tau \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|f\|_{C_T^{\alpha_0}}}{1 - e^{-a_1 T}} \int_0^T e^{-a_1(t-\tau)} d\tau + \frac{\|f\|_{C_T^{\alpha_0}}}{1 - e^{-a_1 T}} \int_0^T (t-\tau)^{-\alpha_0} d\tau \\
&\leq \left(\frac{1}{a_1} + \frac{T^{1-\alpha_0}}{(1-\alpha_0)(1-e^{-a_1 T})} \right) \|f\|_{C_T^{\alpha_0}}.
\end{aligned}$$

Similarly, it is possible to demonstrate

$$\int_t^T \left\| (I_d - e^{-T\mathcal{A}})^{-1} e^{-(t+T-\tau)\mathcal{A}} f(\tau) \right\|_{\alpha_0} d\tau \leq \left(\frac{1}{a_1} + \frac{T^{1-\alpha_0}}{(1-\alpha_0)(1-e^{-a_1 T})} \right) \|f\|_{C_T^{\alpha_0}}.$$

□

Now, consider the equation in (4.2)

$$\frac{dz_m}{dt} = -\mathcal{A}z_m + \mathcal{P}_m F(z_m) + \mathcal{P}_m S. \quad (5.2)$$

Following the same line of thought, it shows that z_m is a T -periodic solution of (5.2) if it satisfies the integral equation

$$z_m(t) = \mathcal{L}^{(p)} \mathcal{P}_m (F(z_m) + S)(t), \quad t \in [0, T].$$

This means that we will search these periodic solutions as the fixed points of the operator

$$\mathcal{K}_m^{(p)}(z_m)(t) := \mathcal{L}^{(p)} \mathcal{P}_m (F(z_m) + S)(t),$$

defined from $C_T^{\alpha_0}$ in itself.

We consider a ball $\mathcal{B}_{\alpha_0}(r_0) \subset \mathcal{Z}^{\alpha_0}$, such that $F : \mathcal{B}_{\alpha_0}(r_0) \rightarrow \mathcal{Z}$ be Lipschitz continuous with Lipschitz constant L . Next, we shall prove it is possible to choose T, r_0 , suitably, such that the operator $\mathcal{K}_m^{(p)}$ has an only fixed point in

$$C_T^{(\alpha_0, r_0)} = \{z_m \in C([0, T]; \mathcal{B}_{\alpha_0}(r_0)) : z_m(0) = z_m(T)\},$$

which is endowed with the norm in $C_T^{\alpha_0}$.

We suppose that $z_m \in C_T^{(\alpha_0, r_0)}$, taking into account Lemma 5.1 we have

$$\left\| \mathcal{K}_m^{(p)}(z_m) \right\|_{C_T^{\alpha_0}} \leq 2 \left(\frac{1}{a_1} + \frac{T^{1-\alpha_0}}{(1-\alpha_0)(1-e^{-a_1 T})} \right) \|F(z_m) + S\|_{C_T}.$$

On the other hand, we have

$$\|F(z_m(t)) + S(t)\|_{\mathcal{Z}} \leq \|F(z_m(t)) - F(0)\|_{\mathcal{Z}} + \|F(0)\|_{\mathcal{Z}} + \widehat{S} \leq L \|z_m(t)\|_{\alpha_0} + \|F(0)\|_{\mathcal{Z}} + \widehat{S},$$

where $\widehat{S} = \|S\|_{C_T}$, from which it is deduced that

$$\|F(z_m) + S\|_{C_T} \leq Lr_0 + \|F(0)\|_{\mathcal{Z}} + \widehat{S}$$

From the above, we obtain

$$\left\| \mathcal{K}_m^{(p)}(z_m) \right\|_{C_T^{\alpha_0}} \leq 2 \left(Lr_0 + \|F(0)\|_{\mathcal{Z}} + \widehat{S} \right) \left(\frac{1}{a_1} + \frac{T^{1-\alpha_0}}{(1-\alpha_0)(1-e^{-a_1T})} \right). \quad (5.3)$$

From a similar analysis, we have

$$\left\| \mathcal{K}_m^{(p)}(z_m^{(1)}) - \mathcal{K}_m^{(p)}(z_m^{(2)}) \right\|_{C_T^{\alpha_0}} \leq 2 \left(\frac{1}{a_1} + \frac{T^{1-\alpha_0}}{(1-\alpha_0)(1-e^{-a_1T})} \right) Lr_0 \left\| z_m^{(1)} - z_m^{(2)} \right\|_{C_T^{\alpha_0}}. \quad (5.4)$$

By the inequalities (5.3) and (5.4), we can affirm that $\mathcal{K}_m^{(p)}(C_T^{(\alpha_0, r_0)}) \subseteq C_T^{(\alpha_0, r_0)}$ if

$$\left(Lr_0 + \|F(0)\|_{\mathcal{Z}} + \widehat{S} \right) \left(\frac{1}{a_1} + \frac{T^{1-\alpha_0}}{(1-\alpha_0)(1-e^{-a_1T})} \right) \leq \frac{r_0}{2}, \quad (5.5)$$

and the map $\mathcal{K}_m^{(p)}$ is a contraction when

$$\left(\frac{1}{a_1} + \frac{T^{1-\alpha_0}}{(1-\alpha_0)(1-e^{-a_1T})} \right) Lr_0 < \frac{1}{2}. \quad (5.6)$$

Theorem 5.2. *If conditions (5.5), (5.6) are assumed, and besides*

$$\frac{1}{a_1} + \frac{T^{1-\alpha_0}}{(1-\alpha_0)(1-e^{-a_1T})} < \frac{1}{2},$$

then, there is a sequence of strong Faedo-Galerkin approximations $\{z_m^{(p)}\}_{m=0}^{\infty} \subset C_T^{(\alpha_0, r_0)}$, whose elements are T -periodic functions. Furthermore, the sequence converges to a T -periodic function $z^{(p)} \in C_T^{(\alpha_0, r_0)}$, which is a strong solution of the coupled-torso bidomian model in the sense of Definition 3.6.

Proof. The inequalities (5.5) and (5.6), imply the existence of an only fixed point of $\mathcal{K}_m^{(p)}$ in $C_T^{(\alpha_0, r_0)}$, denoted by $z_m^{(p)}$, which is a T -periodic solution of (5.2).

Now, we prove that $\{z_m^{(p)}\}_{m=0}^{\infty}$ is a Cauchy sequence in $C_T^{\alpha_0}$. In fact, we have

$$\begin{aligned} \|z_m - z_n\|_{C_T^{\alpha_0}} &= \left\| \mathcal{K}_m^{(p)}(z_m) - \mathcal{K}_n^{(p)}(z_n) \right\|_{C_T^{\alpha_0}} = \left\| \mathcal{L}^{(p)}(\mathcal{P}_m F(z_m) - \mathcal{P}_n F(z_n)) \right\|_{C_T^{\alpha_0}} \\ &\leq \left\| \mathcal{L}^{(p)} \mathcal{P}_m (F(z_m) - F(z_n)) \right\|_{C_T^{\alpha_0}} + \left\| \mathcal{L}^{(p)} (\mathcal{P}_m - \mathcal{P}_n) (F(z_n) - F(0)) \right\|_{C_T^{\alpha_0}} + \left\| \mathcal{L}^{(p)} (\mathcal{P}_m - \mathcal{P}_n) F(0) \right\|_{C_T^{\alpha_0}} \\ &\leq 2 \left(\frac{1}{a_1} + \frac{T^{1-\alpha_0}}{(1-\alpha_0)(1-e^{-a_1T})} \right) \left(L \|z_m - z_n\|_{C_T^{\alpha_0}} + \|\mathcal{P}_m - \mathcal{P}_n\|_{\mathcal{L}(\mathcal{Z})} (Lr_0 + \|F(0)\|_{\mathcal{Z}}) \right), \end{aligned}$$

thus

$$\|z_m - z_n\|_{C_T^{\alpha_0}} \leq \frac{Lr_0 + \|F(0)\|_{\mathcal{Z}}}{1 - 2 \left(\frac{1}{a_1} + \frac{T^{1-\alpha_0}}{(1-\alpha_0)(1-e^{-a_1T})} \right)} \|\mathcal{P}_m - \mathcal{P}_n\|_{\mathcal{L}(\mathcal{Z})}.$$

The right-side of the above inequality converges towards zero, thus, it is demonstrated that the sequence $\left\{z_m^{(p)}\right\}_{m=0}^{\infty}$ is a Cauchy sequence in $C_T^{\alpha_0}$. Furthermore, since $C_T^{(\alpha_0, r_0)}$ is closed in $C_T^{\alpha_0}$, this last being a Banach space, we obtain that

$$z_m^{(p)} \rightarrow z^{(p)} \in C_T^{(\alpha_0, r_0)}.$$

It is easy to see that $z^{(p)}$ is a T -periodic strong solution of the torso-coupled bidomain model. \square

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