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FORECAST ANALYSIS AND SLIDING MODE CONTROL ON A STOCHASTIC EPIDEMIC MODEL WITH ALERTNESS AND VACCINATION*

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Abstract. In this paper, a stochastic $SEIR$ epidemic model is studied with alertness and vaccination. The goal is to stabilize the infectious disease system quickly. The dynamic behavior of the model is analyzed and an integral sliding mode controller with distributed compensation is designed. By using Lyapunov function method, the sufficient conditions for the existence and uniqueness of global positive solutions and the existence of ergodic stationary distributions are obtained. The stochastic center manifold and stochastic average method are used to simplify the system into a one-dimensional Markov diffusion process. The stochastic stability and Hopf bifurcation are analyzed using singular boundary theory. An integral sliding mode controller with non-parallel distributed compensation is designed by linear matrix inequality (LMI) method, which realizes the stability of system and prevents the outbreak of epidemic disease. The correction of theoretical analysis and the effectiveness of controller are validated using numerical simulation performed in MATLAB/Simulink.

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1. Introduction

The importance of mathematical models is increasing in formulating prevention and control measures. The research results of many scholars have established a solid scientific theoretical basis to fight against epidemics and control the spread of diseases efficiently by proposing various defensive measures, such as, getting vaccination in time, wearing protective masks [17], avoiding crowed places [35] and volunteering quarantine [3]. By exerting public awareness and reducing the infectivity of infected individuals, the spread of disease can be effectively reduced. There have been some research results in this area. Such as, Julien Arino *et al.* [2] proposed a new $SEIAR$ model for influenza controlled by vaccination and antiviral treatment. Subsequently, a model of birth rate equal to death rate was studied by PI and sliding mode control [28]. Regardless of the birth and natural mortality, Abbasi *et al.* [1] presented a class of $SQEIAR$ model and proposed the theory of optimal control to eliminate disease by quarantine and treatment to infected people. In general, the detailed models may more accurately predict the course of an outbreak, but simple models may be more useful for planning the early

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stages of an epidemic. It is notable that all of them are determined models and take into account the infectivity of the latent.

In reality, the growth of organisms and the spread of epidemics are inevitably disturbed by random factors [22, 23, 37], such as temperature and individual differences, which cannot be ignored in the prediction and control of the spread of diseases. Thus, it has great significance to apply stochastic theory to analyse epidemic model, which is more practical. Constructing suitable Lyapunov functions, some authors established sufficient conditions of the existence of global positive solutions [8, 25, 36] and ergodic stationary distribution. Huang *et al.* [9, 10] discussed the stochastic stability and bifurcation according to stochastic center manifold and stationary probability density.

T-S fuzzy control [40] is flexible approximate to a global nonlinear system using several local linear system model by membership functions. Sliding mode control (SMC) [27, 31] based on T-S fuzzy is used extensively in various fields to stabilize the nonlinear system on a desired sliding mode surface. Furthermore, integral sliding mode control ($ISMC$) [33] guarantees the robustness of the $ISMC$ system in the whole trajectory from the initial time by using a new class of nonlinear sliding surfaces [6, 21, 34]. Thus, it is significant to apply sliding mode control to epidemic model to prevent large-scale outbreaks of contagion and eliminate some instability phenomenon. However, there are two limiting assumptions about SMC for stochastic fuzzy T-S systems [13]. Using nonparallel distributed compensation($Non-PDC$) $ISMC$ control [14] can completely eliminate both of these limitations, thereby reducing the conservatism introduced by the choice of the slip surface coefficient matrix.

In this paper, we mainly analyze the stochastic stability and stochastic bifurcation in the vicinity of the representative equilibrium point, and control the epidemic spread by designing a sliding mode controller. The details are as follows: in Section 1, we establish a class of $SEIR$ epidemic model with alertness and vaccination. In Section 2, the existence and uniqueness of the positive solution is discussed, as well as the existence of ergodic stationary distribution. In Section 3, the stochastic center manifold, singular boundary theory and invariant measure are applied to discuss the stochastic stability and stochastic Hopf bifurcation. To control the spreading of the epidemic, we focus on the stability of sliding mode system and the design of integral sliding mode controller with Non-parallel compensation in Section 4. The principal theory results are illustrated via numerical simulations in Section 5. In the last section, this paper ends with conclusions.

2. Model formulation

We assume that one gets lifelong immunity with vaccination and infected individuals first enter the latent period during which they have less infectious [1, 2, 28]. To simplify the model, we ignore individuals who are asymptomatic. The proposed epidemiological model describes four states including $S(t)$(susceptible), $E(t)$(exposed), $I(t)$(infective) and $R(t)$(recovered) as follows.

$$\left\{\begin{array}{l}\dot{S}(t)=A-\beta S(t)(\varepsilon E(t)+(1-q)I(t))-(\mu+m_1)\,S(t),\\ \dot{E}(t)=\beta S(t)(\varepsilon E(t)+(1-q)I(t))-(\mu+k_1+m_2)\,E(t),\\ \dot{I}(t)=k_1E(t)-(\mu+\gamma+k_2)\,I(t),\\ \dot{R}(t)=m_1S(t)+m_2E(t)+k_2I(t)-\mu R(t).\end{array}\right. \tag{2.1}$$

where A denotes the birth rate, μ and γ are the natural death and disease-induced death coefficient, respectively, m_1 and m_2 are the vaccination rates of the susceptible and the exposed, respectively, β denotes the average of contacts between members in the population during infectious period, ε is the decreasing factor of the latent infection rate and q represents a reduction in infectivity owing to the quarantine, isolation and other imperative measures, $\varepsilon E(t)+(1-q)I(t)$ denotes the number of people who are contagious. Latent members proceed to the infective at a rate k_1 and infective members go to the recovered at a rate k_2.

There are stochastic disturbances in death rate among different populations affected by epidemics. Taking the effect of the noise perturbation on death rates μ of $S(t),E(t),I(t)$ and $R(t)$, the death rate coefficient μ is replaced by a random variable $\mu - \sigma_i \xi(t)$ [7, 19, 20] and the following system is obtained

$$\begin{cases} \dot{S}(t) = A - \beta S(t)\left(\varepsilon E(t) + (1-q)\, I(t)\right) - (\mu + m_1)\, S(t) + \sigma_1 S(t)\xi(t), \\ \dot{E}(t) = \beta S(t)\left(\varepsilon E(t) + (1-q)\, I(t)\right) - (\mu + k_1 + m_2)\, E(t) + \sigma_2 E(t)\xi(t), \\ \dot{I}(t) = k_1 E(t) - (\mu + \gamma + k_2)\, I(t) + \sigma_3 I(t)\xi(t), \\ \dot{R}(t) = m_1 S(t) + m_2 E(t) + k_2 I(t) - \mu R(t) + \sigma_4 R(t)\xi(t), \end{cases} \tag{2.2}$$

where $\sigma_i^2 > 0 (i = 1, 2, 3, 4)$ are the intensity of environmental white noise $\xi(t)$ which satisfies $dB(t) = \xi(t)dt$, $B(t)$ denotes mutually independent standard Brownian motion.

Since the state $R(t)$ does not effect the dynamics of $S(t)$, $E(t)$ and $I(t)$, system (2.2) can be reduced to the following system

$$\begin{cases} \dot{S}(t) = A - \beta S(t)(\varepsilon E(t) + (1-q)I(t)) - (\mu + m_1)\, S(t) + \sigma_1 S(t)\xi(t), \\ \dot{E}(t) = \beta S(t)(\varepsilon E(t) + (1-q)I(t)) - (\mu + k_1 + m_2)\, E(t) + \sigma_2 E(t)\xi(t), \\ \dot{I}(t) = k_1 E(t) - (\mu + \gamma + k_2)\, I(t) + \sigma_3 I(t)\xi(t). \end{cases} \tag{2.3}$$

The dynamics properties of system (2.2) is obtained by analyzing the global dynamic behavior of system (2.3). For the corresponding deterministic system to system (2.3), the disease-free equilibrium point is denoted as $P_1(\frac{A}{a}, 0, 0)$. Using the next-generation matrix method [5], we obtain the basic reproduction number R_0 is as follows

$$R_0 = \frac{\beta A(\varepsilon b + k_1(1-q))}{abc},$$

where $a = \mu + m_1$, $b = \mu + \gamma + k_2$, $c = \mu + k_1 + m_2$. When $R_0 > 1$, there exists an endemic equilibrium point denoted as $P^*(S^*, E^*, I^*)$, where

$$S^* = \frac{A}{aR_0}, \quad E^* = \frac{A(1 - \frac{1}{R_0})}{c}, \quad I^* = \frac{Ak_1(1 - \frac{1}{R_0})}{bc}.$$

3. The global dynamic behavior

3.1. Existence and uniqueness of the global positive solution

Given the number of population is non-negative, we will study the existence and uniqueness of the global positive solution to system (2.3) with any positive initial value.

Theorem 3.1. *For any initial value* $(S(0), E(0), I(0)) \in \mathbb{R}^3_+$*, there exists a unique solution* $(S(t), E(t), I(t))$ *to system (2.3) on* $t \geq 0$ *and the solution will remain in* $\mathbb{R}^3_+$ *with probability one, namely,* $(S(t), E(t), I(t)) \in \mathbb{R}^3_+$ *for all* $t \geq 0$ *almost surely* (*a.s.*).

Proof. Since the coefficients of system (2.3) satisfy the local Lipschitz condition, then for any initial value $(S(0), E(0), I(0)) \in \mathbb{R}^3_+$, there exists a unique local solution $(S(t), E(t), I(t)) \in \mathbb{R}^3_+$ on $t \in [0, \tau_e)$ *a.s.*, where τ_e denotes the explosion time [26]. The next step is to prove that it is global, *i.e.*, $\tau_e = \infty$ a.s. Let $\bar{k}_0 \geq 1$ be sufficiently large such that $(S(0), E(0), I(0)) \in \left(\frac{1}{\bar{k}_0}, \bar{k}_0\right)$. For each integer $k > \bar{k}_0$, define the stopping time [26] as follows

$$\tau_k = \inf\left\{ t \in (0, \tau_e): \ \min\left\{S\left(t\right), E\left(t\right), I\left(t\right)\right\} \leq \frac{1}{k} \ or \ \max\left\{S\left(t\right), E\left(t\right), I\left(t\right)\right\} \geq k \right\},$$

where $\inf\emptyset=\infty$ (as usual $\emptyset$ denotes the empty set). τ_k is increasing as $k\to\infty$. Set $\tau_\infty=\lim_{k\to\infty}\tau_k$, where $\tau_\infty\leq\tau_e$ a.s. If $\tau_\infty=\infty$ a.s. is true, then $\tau_e=\infty$ a.s. and $(S(t),E(t),I(t))\in\mathbb{R}^3_+$ for all $t\geq 0$. In order to complete the proof, we just need to prove $\tau_\infty=\infty$ a.s. If not, then there is a pair of constants $T>0$ and $\epsilon_1\in(0,1)$ such that $\mathbb{P}(\tau_k\leq T)>\epsilon_1$ for any $k\geq\bar{k}_0$. Hence, there exists an integer $\bar{k}_1\geq\bar{k}_0$ such that

$$\mathbb{P}(\tau_k\leq T)\geq\epsilon_1,\forall k\geq\bar{k}_1. \tag{3.1}$$

Define a C^2-function $V:\mathbb{R}^3_+\to\mathbb{R}_+$ by

$$V(S(t),E(t),I(t))=S(t)-a_4-a_4\ln\frac{S(t)}{a_4}+E(t)-1-\ln E(t)+I(t)-1-\ln I(t),$$

where $a_4=\min\left\{\frac{c-k_1}{\beta\varepsilon},\frac{b}{\beta(1-q)}\right\}$ is a positive contant. $V(S(t),E(t),I(t))$ is non-negative since inequality $\kappa-1-\ln\kappa>0$ holds for all $\kappa>0$. By Itô's formula, it can be calculated that

$$\begin{aligned}LV(S(t),E(t),I(t))=&-aS(t)-\frac{a_4A}{S(t)}-\beta\varepsilon S(t)+(a_4\beta\varepsilon-(c-k_1))E(t)+(a_4\beta(1-q)-b)I(t)\\&-\frac{\beta(1-q)S(t)}{E(t)}I(t)-\frac{k_1E(t)}{I(t)}+A+aa_4+b+c+\frac{\sigma_1^2a_4}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}.\end{aligned}$$

Since $(a_4\beta\varepsilon-(c-k_1))E(t)+(a_4\beta(1-q)-b)I(t)<0$, then

$$LV(S(t),E(t),I(t))\leq A+aa_4+b+c+\frac{\sigma_1^2a_4}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}\triangleq K,$$

where K is a positive constant. Furthermore,

$$\mathrm{d}V(S(t),E(t),I(t))\leq K\,\mathrm{d}t+\sigma_1S(t)\,\mathrm{d}B(t)+\sigma_2E(t)\,\mathrm{d}B(t)+\sigma_3I(t)\,\mathrm{d}B(t). \tag{3.2}$$

Integrating both sides of (3.2) from 0 to $\tau_k\wedge T=\min\{\tau_k,T\}$ yields

$$\begin{aligned}V(S(\tau_k\wedge T),E(\tau_k\wedge T),I(\tau_k\wedge T))\leq&V(S(0),E(0),I(0))+K(\tau_k\wedge T)+\sigma_1\int_0^{\tau_k\wedge T}S(\tau)\,\mathrm{d}B(\tau)\\&+\sigma_2\int_0^{\tau_k\wedge T}E(\tau)\,\mathrm{d}B(\tau)+\sigma_3\int_0^{\tau_k\wedge T}I(\tau)\,\mathrm{d}B(\tau)\\=&V(S(0),E(0),I(0))+K(\tau_k\wedge T)+M_1(\tau_k\wedge T)\\&+M_2(\tau_k\wedge T)+M_3(\tau_k\wedge T),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}M_1(\tau_k\wedge T)&=\sigma_1\int_0^{\tau_k\wedge T}S(\tau)\,\mathrm{d}B(\tau),\\M_2(\tau_k\wedge T)&=\sigma_2\int_0^{\tau_k\wedge T}E(\tau)\,\mathrm{d}B(\tau),\\M_3(\tau_k\wedge T)&=\sigma_3\int_0^{\tau_k\wedge T}I(\tau)\,\mathrm{d}B(\tau)\end{aligned}$$

are three local matingales.

Since the solution $(S(t), E(t), I(t))$ of system (2.3) is $\mathcal{F}_t$-adapted, taking the expectation on both sides of (3.3), we have

$$\mathbb{E}V(S(\tau_k \wedge T), E(\tau_k \wedge T), I(\tau_k \wedge T)) \leq V(S(0), E(0), I(0)) + K\mathbb{E}(\tau_k \wedge T).$$

Thus,

$$\mathbb{E}V(S(\tau_k \wedge T), E(\tau_k \wedge T), I(\tau_k \wedge T)) \leq V(S(0), E(0), I(0)) + KT. \tag{3.4}$$

Let $\Omega_k = \{\omega \in \Omega : \tau_k = \tau_k(\omega) \leq T\}$ for $k \geq \bar{k}_1$, we obtain $\mathbb{P}(\Omega_k) \geq \epsilon_1$. Note that for every $\omega \in \Omega_k$, there is $S(\tau_k, \omega)$ or $E(\tau_k, \omega)$ or $I(\tau_k, \omega)$ equals either k or $\frac{1}{k}$. Hence, $V(S(\tau_k, \omega), E(\tau_k, \omega), I(\tau_k, \omega))$ is no less than either $(k - a_4 - a_4 \ln \frac{k}{a_4}) \wedge (k - 1 - \ln k)$ or $(\frac{1}{k} - a_4 - a_4 \ln \frac{1}{ka_4}) \wedge (\frac{1}{k} - 1 - \ln \frac{1}{k})$. That is,

$$\begin{aligned} V(S(\tau_k, \omega), E(\tau_k, \omega), I(\tau_k, \omega)) \geq &(k - a_4 - a_4 \ln \frac{k}{a_4}) \wedge (k - 1 - \ln k) \\ &\wedge (\frac{1}{k} - a_4 + a_4 \ln ka_4) \wedge (\frac{1}{k} - 1 + \ln k). \end{aligned} \tag{3.5}$$

It can be obtained from (3.4) that

$$\begin{aligned} V(S(0), E(0), I(0)) + KT \geq &\mathbb{E}(I_{\Omega_k(\omega)} V(S(\tau_k, \omega), E(\tau_k, \omega), I(\tau_k, \omega))) \\ \geq &\epsilon_1 (k - a_4 - a_4 \ln \frac{k}{a_4}) \wedge (k - 1 - \ln k) \\ &\wedge (\frac{1}{k} - a_4 + a_4 \ln ka_4) \wedge (\frac{1}{k} - 1 + \ln k), \end{aligned} \tag{3.6}$$

where $I_{\Omega_k(\omega)}$ denotes the indicator function of Ω_k. Taking $k \to \infty$, we get

$$\infty > V(S(0), E(0), I(0)) + KT = \infty,$$

which leads to the contradiction. Thus we draw the conclusion that $\tau_\infty = \infty$ a.s. □

3.2. Existence of ergodic stationary distribution

To explore the prevalence of epidemic, we talk about the persistence of disease. Based on the theory of Khasminskii [15], there exists a stationary distribution which indicates that the epidemic will prevail if parameters of the system (2.3) are subject to the following condition.

Theorem 3.2. *If* $R_0^s = \frac{k_1 A\beta(1-q)}{\left(a+\frac{\sigma_1^2}{2}\right)\left(b+\frac{\sigma_3^2}{2}\right)\left(c+\frac{\sigma_2^2}{2}\right)} > 1$, *then the system (2.3) has a unique ergodic stationary distribution for any given initial value* $(S(0), E(0), I(0))$.

Proof. Define

$$\begin{aligned} V_1(S, E, I) &= -\ln S - a_5 \ln E - a_6 \ln I + \frac{\beta(1-q)}{b} I, \\ V_2(S, E, I) &= -\ln S, \\ V_3(S, E, I) &= -\ln I, \\ V_4(S, E, I) &= \frac{1}{m+1} (S + E + I)^{m+1}, \end{aligned}$$

where m is a sufficiently small constant satisfying $\mu - \frac{m}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2) > 0$, and a_5, a_6 are positive constants as follows

$$a_5 = \frac{a+\frac{\sigma_1^2}{2}}{c+\frac{\sigma_2^2}{2}}, \qquad a_6 = \frac{a+\frac{\sigma_1^2}{2}}{b+\frac{\sigma_3^2}{2}}.$$

It can be calculated that

$$\begin{aligned}
LV_1 =& -a_5\beta\varepsilon S + \beta\left(\varepsilon + \frac{k_1(1-q)}{b}\right)E - \frac{A}{S} - \frac{a_5\beta(1-q)SI}{E} - \frac{a_6k_1}{I}E \\
&+ a + a_5c + a_6b + \frac{\sigma_1^2}{2} + \frac{a_5\sigma_2^2}{2} + \frac{a_6\sigma_3^2}{2} \\
\leq& \beta\left(\varepsilon + \frac{k_1(1-q)}{b}\right)E + a + \frac{\sigma_1^2}{2} + a_5\left(c+\frac{\sigma_2^2}{2}\right) + a_6\left(b+\frac{\sigma_3^2}{2}\right) - 3\left(a_5a_6k_1\beta A(1-q)\right)^{\frac{1}{3}} \\
=& -\lambda + \beta\left(\varepsilon + \frac{k_1(1-q)}{b}\right)E, \qquad (3.7)
\end{aligned}$$

where

$$\lambda = 3\left(a+\frac{\sigma_1^2}{2}\right)\left((R_0^s)^{\frac{1}{3}} - 1\right) > 0.$$

Similarly, it can be computed that

$$LV_2 = -\frac{A}{S} + \beta\varepsilon E + \beta(1-q)I + a + \frac{\sigma_1^2}{2}, \qquad (3.8)$$

$$LV_3 == -\frac{k_1E}{I} + b + \frac{\sigma_3^2}{2}, \qquad (3.9)$$

$$\begin{aligned}
LV_4 =& (S+E+I)^m\left(A - aS - (c-k_1)E - bI\right) + \frac{m(S+E+I)^{m-1}}{2}\left(\sigma_1^2S^2+\sigma_2^2E^2+\sigma_3^2I^2\right) \\
\leq& A(S+E+I)^m - \mu(S+E+I)^{m+1} + \frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}(S+E+I)^{m+1} \\
\leq& A(S+E+I)^m - \left(\mu - \frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\left(S^{m+1}+E^{m+1}+I^{m+1}\right) \\
\leq& C_1 - \frac{1}{2}\left(\mu - \frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\left(S^{m+1}+E^{m+1}+I^{m+1}\right), \qquad (3.10)
\end{aligned}$$

where

$$C_1 = \sup_{(S,E,I)\in\mathbb{R}_+^3}\left\{A(S+E+I)^m - \frac{1}{2}\left(\mu - \frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)(S+E+I)^{m+1}\right\} < \infty,$$

Define a Lyapunov function $V_5 : \mathbb{R}_+^3 \to \mathbb{R}$ as

$$V_5(S,E,I) = v_1V_1(S,E,I) + V_2(S,E,I) + V_3(S,E,I) + V_4(S,E,I), \qquad (3.11)$$

where v_1 is a positive constant satisfying

$$-v_1\lambda + C_3 \leq -2, \qquad (3.12)$$

$$C_3 = \sup_{(S,E,I)\in\mathbb{R}_+^3} \Bigg\{ -\frac{1}{2}\left(\mu - \frac{m\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right)}{2}\right)\left(S^{m+1}+E^{m+1}+I^{m+1}\right) + \beta\left(1-q\right)I + a + \frac{\sigma_1^2}{2} + b + \frac{\sigma_3^2}{2} + C_1 \Bigg\}.$$

It is easy to verify that

$$\liminf_{k\to\infty,(S,E,I)\in\mathbb{R}_+^3\setminus U_k} V_5(S,E,I) = +\infty$$

where $U_k = \left(\frac{1}{k},k\right)\times\left(\frac{1}{k},k\right)\times\left(\frac{1}{k},k\right)$. Note that $V_5\left(S,E,I\right)$ is continuous, $V_5\left(S,E,I\right)$ has a minimum at point (S_0,E_0,I_0) in the interior of $\mathbb{R}_+^3$. Constructing a nonnegative C^2-function $V_6(S,E,I)$: $\mathbb{R}_+^3 \to \mathbb{R}_+$ by

$$V_6\left(S,E,I\right) = V_5(S,E,I) - V_5(S_0,E_0,I_0). \tag{3.13}$$

Hence, from (3.7) to (3.13), it can be obtained that

$$\begin{aligned} LV_6 \leq & -v_1\lambda - \frac{A}{S} - \frac{k_1 E}{I} + \beta\left(v_1\left(\varepsilon + \frac{k_1\left(1-q\right)}{b}\right)+\varepsilon\right)E + \beta\left(1-q\right)I + a + \frac{\sigma_1^2}{2} \\ & + b + \frac{\sigma_3^2}{2} + C_1 - \frac{1}{2}\left(\mu - \frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\left(S^{m+1}+E^{m+1}+I^{m+1}\right). \end{aligned} \tag{3.14}$$

Define a bounded closed set D_{ϵ_2} as

$$D_{\epsilon_2} = \left\{(S,E,I)\in\mathbb{R}_+^3 : \epsilon_2 \leq S \leq \frac{1}{\epsilon_2}, \epsilon_2 \leq E \leq \frac{1}{\epsilon_2}, \epsilon_2^2 \leq I \leq \frac{1}{\epsilon_2^2}\right\},$$

where $\epsilon_2 > 0$ is a small enough number satisfying the following conditions in the set $\mathbb{R}_+^3 \setminus D_{\epsilon_2}$:

$$-\frac{A}{\epsilon_2} + C_2 \leq -1, \tag{3.15}$$

$$-v_1\lambda + \beta\left(v_1\left(\varepsilon + \frac{k_1\left(1-q\right)}{b}\right)+\varepsilon\right)\epsilon_2 + C_3 \leq -1, \tag{3.16}$$

$$\beta\left(1-q\right)\epsilon_2^2 - \frac{k_1}{\epsilon_2} + C_4 \leq -1, \tag{3.17}$$

$$-\frac{1}{4}\left(\mu - \frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\frac{1}{\epsilon_2^{m+1}} + C_5 \leq -1, \tag{3.18}$$

$$-\frac{1}{4}\left(\mu - \frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\frac{1}{\epsilon_2^{m+1}} + C_6 \leq -1, \tag{3.19}$$

$$-\frac{1}{4}\left(\mu - \frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\frac{1}{\epsilon_2^{2m+2}} + C_7 \leq -1, \tag{3.20}$$

where

$$
\begin{aligned}
C_2 =& \sup_{(S,E,I)\in\mathbb{R}_+^3} \Bigg\{ \left(\beta v_1\left(\varepsilon+\frac{k_1(1-q)}{b}\right)+\beta\varepsilon\right)E+\beta(1-q)I+a+\frac{\sigma_1^2}{2}+b+\frac{\sigma_3^2}{2}+C_1 \\
&-\frac{1}{2}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\left(S^{m+1}+E^{m+1}+I^{m+1}\right)\Bigg\},\\
C_4 =& \sup_{(S,E,I)\in\mathbb{R}_+^3} \Bigg\{ -\frac{1}{2}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\left(S^{m+1}+E^{m+1}+I^{m+1}\right)+a+\frac{\sigma_1^2}{2}+b+\frac{\sigma_3^2}{2} \\
&+C_1+\beta\left(v_1\left(\varepsilon+\frac{k_1(1-q)}{b}\right)+\varepsilon\right)E\Bigg\},\\
C_5 =& \sup_{(S,E,I)\in\mathbb{R}_+^3} \Bigg\{ -\frac{1}{4}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)S^{m+1}-\frac{1}{2}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\left(E^{m+1}+I^{m+1}\right) \\
&+\beta\left(v_1\left(\varepsilon+\frac{k_1(1-q)}{b}\right)+\varepsilon\right)E+\beta(1-q)I+a+\frac{\sigma_1^2}{2}+b+\frac{\sigma_3^2}{2}+C_1\Bigg\},\\
C_6 =& \sup_{(S,E,I)\in\mathbb{R}_+^3} \Bigg\{ -\frac{1}{4}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)E^{m+1}-\frac{1}{2}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\left(S^{m+1}+I^{m+1}\right) \\
&+\beta\left(v_1\left(\varepsilon+\frac{k_1(1-q)}{b}\right)+\varepsilon\right)E+\beta(1-q)I+a+\frac{\sigma_1^2}{2}+b+\frac{\sigma_3^2}{2}+C_1\Bigg\},\\
C_7 =& \sup_{(S,E,I)\in\mathbb{R}_+^3} \Bigg\{ -\frac{1}{4}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)I^{m+1}-\frac{1}{2}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\left(S^{m+1}+E^{m+1}\right) \\
&+\beta\left(v_1\left(\varepsilon+\frac{k_1(1-q)}{b}\right)+\varepsilon\right)E+\beta(1-q)I+a+\frac{\sigma_1^2}{2}+b+\frac{\sigma_3^2}{2}+C_1\Bigg\}.
\end{aligned}
$$

We can get (3.16) holds from (3.12). Dividing D_{ϵ_2} into six domains as follows:

$$
\begin{aligned}
&D_1=\left\{(S,E,I)\in\mathbb{R}_+^3: 0<S<\epsilon_2\right\}, && D_2=\left\{(S,E,I)\in\mathbb{R}_+^3: 0<E<\epsilon_2\right\},\\
&D_3=\left\{(S,E,I)\in\mathbb{R}_+^3: E>\epsilon_2, 0<I<\epsilon_2^2\right\}, && D_4=\left\{(S,E,I)\in\mathbb{R}_+^3: S>\frac{1}{\epsilon_2}\right\},\\
&D_5=\left\{(S,E,I)\in\mathbb{R}_+^3: E>\frac{1}{\epsilon_2}\right\}, && D_6=\left\{(S,E,I)\in\mathbb{R}_+^3: I>\frac{1}{\epsilon_2^2}\right\}.
\end{aligned}
$$

In order to verify $LV_6<-1$ for any (S,E,I) in $\mathbb{R}_+^3\setminus D_{\epsilon_2}$, we will clarify it by the following six cases.
Case 1. For $(S,E,I)\in D_1$, it follows from (3.14) and (3.15)

$$
\begin{aligned}
LV_6 \le& -\frac{A}{S}+\beta\left(v_1\left(\varepsilon+\frac{k_1(1-q)}{b}\right)+\varepsilon\right)E+\beta(1-q)I+a+\frac{\sigma_1^2}{2}+b+\frac{\sigma_3^2}{2}+C_1\\
&-\frac{1}{2}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\left(S^{m+1}+E^{m+1}+I^{m+1}\right)\\
\le& -\frac{A}{\epsilon_2}+C_2\\
\le& -1.
\end{aligned}
$$

Case 2. For $(S, E, I) \in D_2$, it follows from (3.12), (3.14) and (3.16)

$$\begin{aligned}
LV_6 \leq& -v_1\lambda + \beta\left(v_1\left(\varepsilon + \frac{k_1(1-q)}{b}\right) + \varepsilon\right)E + \beta(1-q)I + a + \frac{\sigma_1^2}{2} + b + \frac{\sigma_3^2}{2} + C_1 \\
&- \frac{1}{2}\left(\mu - \frac{m\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right)}{2}\right)\left(S^{m+1} + E^{m+1} + I^{m+1}\right) \\
\leq& -v_1\lambda + \beta\left(v_1\left(\varepsilon + \frac{k_1(1-q)}{b}\right) + \varepsilon\right)\epsilon_2 + C_3 \\
\leq& -1.
\end{aligned}$$

Case 3. For $(S, E, I) \in D_3$, it follows from (3.14) and (3.17)

$$\begin{aligned}
LV_6 \leq& -\frac{kE}{I} + \beta\left(v_1\left(\varepsilon + \frac{k_1(1-q)}{b}\right) + \varepsilon\right)E + \beta(1-q)I + a + \frac{\sigma_1^2}{2} + b + \frac{\sigma_3^2}{2} + C_1 \\
&- \frac{1}{2}\left(\mu - \frac{m\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right)}{2}\right)\left(S^{m+1} + E^{m+1} + I^{m+1}\right) \\
\leq& \beta(1-q)\epsilon_2^2 - \frac{k_1}{\epsilon_2} + C_4 \\
\leq& -1.
\end{aligned}$$

Case 4. For $(S, E, I) \in D_4$, it follows from (3.14) and (3.18)

$$\begin{aligned}
LV_6 \leq& -\frac{1}{4}\left(\mu - \frac{m\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right)}{2}\right)S^{m+1} - \frac{1}{4}\left(\mu - \frac{m\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right)}{2}\right)S^{m+1} \\
&- \frac{1}{2}\left(\mu - \frac{m\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right)}{2}\right)\left(E^{m+1} + I^{m+1}\right) + \beta\left(v_1\left(\varepsilon + \frac{k_1(1-q)}{b}\right) + \varepsilon\right)E \\
&+ \beta(1-q)I + a + \frac{\sigma_1^2}{2} + b + \frac{\sigma_3^2}{2} + C_1 \\
\leq& -\frac{1}{4}\left(\mu - \frac{m\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2\right)}{2}\right)\frac{1}{\epsilon_2^{m+1}} + C_5 \\
\leq& -1.
\end{aligned}$$

Case 5. For $(S,E,I)\in D_5$, it follows from (3.14) and (3.19)

$$\begin{aligned}
LV_6 \leq & -\frac{1}{4}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)E^{m+1}-\frac{1}{4}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)E^{m+1}\\
& -\frac{1}{2}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\left(S^{m+1}+I^{m+1}\right)+\beta\left(v_1\left(\varepsilon+\frac{k_1(1-q)}{b}\right)+\varepsilon\right)E\\
& +\beta(1-q)I+a+\frac{\sigma_1^2}{2}+b+\frac{\sigma_3^2}{2}+C_1\\
\leq & -\frac{1}{4}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\frac{1}{\epsilon_2^{m+1}}+C_6\\
\leq & -1.
\end{aligned}$$

Case 6. For $(S,E,I)\in D_6$, it follows from (3.14) and (3.20)

$$\begin{aligned}
LV_6 \leq & -\frac{1}{4}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)I^{m+1}-\frac{1}{4}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)I^{m+1}\\
& -\frac{1}{2}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\left(S^{m+1}+E^{m+1}\right)+\beta\left(v_1\left(\varepsilon+\frac{k_1(1-q)}{b}\right)+\varepsilon\right)E\\
& +\beta(1-q)I+a+\frac{\sigma_1^2}{2}+b+\frac{\sigma_3^2}{2}+C_1\\
\leq & -\frac{1}{4}\left(\mu-\frac{m\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)}{2}\right)\frac{1}{\epsilon_2^{2m+2}}+C_7\\
\leq & -1.
\end{aligned}$$

Hence the condition A_2 in Lemma 3.1 [24] is satisfied. Additionally, the condition A_1 is clear. Above all, the system (2.3) has a stable stationary distribution and the solution is ergodic. □

4. Stochastic center manifold and Hopf bifurcation

To further consider the dynamic behavior of system (2.3), we discuss the stochastic stability and stochastic Hopf bifurcation in this section.

Let $x_1(t)=S(t)-S^*, x_2(t)=E(t)-E^*, x_3(t)=I(t)-I^*$, system (2.3) is transformed into the following form:

$$\begin{cases}
\dot{x}_1(t)=(-\beta(\varepsilon E^*+(1-q)I^*)-a)x_1(t)-\beta\varepsilon S^*x_2(t)-\beta S^*(1-q)x_3(t)\\
\qquad -\beta\varepsilon x_1(t)x_2(t)-\beta(1-q)x_1(t)x_3(t)+\sigma_1(x_1(t)+S^*)\xi(t),\\
\dot{x}_2(t)=\beta(\varepsilon E^*+(1-q)I^*)x_1(t)+(\beta\varepsilon S^*-c))x_2(t)+\beta S^*(1-q)x_3(t)\\
\qquad +\beta\varepsilon x_1(t)x_2(t)+\beta(1-q)x_1(t)x_3(t)+\sigma_2(x_2(t)+E^*)\xi(t),\\
\dot{x}_3(t)=k_1x_2(t)-bx_3(t)+\sigma_3(x_3(t)+I^*)\xi(t).
\end{cases} \tag{4.1}$$

To discuss the stability of the system (2.3) near the equilibrium P^*, we only need the stability of the system (4.1) near the origin $O(0,0,0,0)$. The Jacobian J of the deterministic system corresponding to the system (4.1) at the origin $O(0,0,0,0)$ can be expressed as

$$J = \begin{pmatrix} -\beta(\varepsilon E^* + (1-q)I^*) - a & -\beta S^*\varepsilon & -\beta S^*(1-q) \\ \beta(\varepsilon E^* + (1-q)I^*) & \beta S^*\varepsilon - c & \beta S^*(1-q) \\ 0 & k_1 & -b \end{pmatrix}.$$

The characteristic equation of J is that

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{4.2}$$

where

$$\begin{aligned} a_1 &= \beta S^*\varepsilon + a + b + c + \beta(\varepsilon E^* + (1-q)I^*), \\ a_2 &= -\beta S^*(\varepsilon(a+b) + k_1(1-q)) + \beta(b+c)(\varepsilon E^* + (1-q)I^*) + ab + ac + bc, \\ a_3 &= abc + \beta bc(\varepsilon E^* + (1-q)I^*) - \beta S^* a(\varepsilon b + k_1(1-q)). \end{aligned}$$

4.1. Stochastic center manifold

The system (4.1) is projected onto its two-dimensional central manifold applying the theory of stochastic central manifold.

For sufficiently small $\delta > 0$, let

$$b_0 = \frac{a_3}{a_1}, b_1 = b_0 + \frac{2\delta((a_1+2\delta)^2 + b_0)}{a_1 + 2\delta},$$

the eigenvalues of equation (4.2) can be shown as

$$\lambda_{1,2} = \delta + \sqrt{\frac{a_3}{a_1+2\delta} - \delta^2}i, \lambda_3 = -(a_1 + 2\delta).$$

When $\delta = 0$, $b_1 = b_0$. It follows that

$$\lambda_{1,2} = \pm\sqrt{b_1}i, \lambda_3 = -a_1.$$

It implies (0,0,0) is a degenerate critical point of system (4.1), which possesses a two dimensional local center manifold. Owing to $\delta \to 0$, the system (4.1) possesses a local invariant manifold.

Take

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ \frac{\sqrt{\frac{a_3}{a_1+2\delta} - \delta^2}}{k_1} & \frac{b-(a_1+2\delta)}{k_1} & \frac{\delta+b}{k_1} \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix},$$

where

$$
\begin{aligned}
b_{11} &= \frac{\sqrt{\frac{a_3}{a_1+2\delta}-\delta^2}}{k_1\beta(\varepsilon E^*+(1-q)I^*)}(b+c-\beta S^*\varepsilon+2\delta),\\
b_{12} &= \frac{(-(a_1+2\delta)-k_1\beta S^*+c)(b-(a_1+2\delta))-k_1\beta S^*(1-q)}{k_1\beta(\varepsilon E^*+(1-q)I^*)},\\
b_{13} &= \frac{-(\frac{a_3}{a_1+2\delta}-\delta^2)-k_1\beta S^*(1-q)+(-\beta\varepsilon S^*+c+\delta)(\delta+b)}{k_1\beta(\varepsilon E^*+(1-q)I^*)}.
\end{aligned}
$$

Then the system (4.1) can be transformed into the following equivalent system

$$
\begin{cases}
\dot{\varphi}_1 =\delta\varphi_1-\sqrt{\dfrac{a_3}{a_1+2\delta}-\delta^2}\varphi_3+F_1(\varphi_1,\varphi_2,\varphi_3)+\sigma_1(d_7\varphi_1+d_8\varphi_2+d_9\varphi_3+d_{10})\xi(t)\\
\qquad +\sigma_2(d_{11}\varphi_1+d_{12}\varphi_2+d_{13}\varphi_3+d_{14})\xi(t)+\sigma_3(d_{15}\varphi_1+d_{16}\varphi_2+d_{17}\varphi_3+d_{18})\xi(t),\\
\dot{\varphi}_2 =-(a_1+2\delta)\varphi_2+F_2(\varphi_1,\varphi_2,\varphi_3)+\sigma_1(e_7\varphi_1+e_8\varphi_2+e_9\varphi_3+e_{10})\xi(t)\\
\qquad +\sigma_2(e_{11}\varphi_1+e_{12}\varphi_2+e_{13}\varphi_3+e_{14})\xi(t)+\sigma_3(e_{15}\varphi_1+e_{16}\varphi_2+e_{17}\varphi_3+e_{18})\xi(t),\\
\dot{\varphi}_3 =\delta\varphi_3+\sqrt{\dfrac{a_3}{a_1+2\delta}-\delta^2}\varphi_1+F_3(\varphi_1,\varphi_2,\varphi_3)+\sigma_1(l_7\varphi_1+l_8\varphi_2+l_9\varphi_3+l_{10})\xi(t)\\
\qquad +\sigma_2(l_{11}\varphi_1+l_{12}\varphi_2+l_{13}\varphi_3+l_{14})\xi(t)+\sigma_3(l_{15}\varphi_1+l_{16}\varphi_2+l_{17}\varphi_3+l_{18})\xi(t),
\end{cases}
\tag{4.3}
$$

where

$$
\begin{aligned}
F_1(\varphi_1,\varphi_2,\varphi_3) &= d_1\varphi_1^2+d_2\varphi_1\varphi_2+d_3\varphi_1\varphi_3+d_4\varphi_2^2+d_5\varphi_2\varphi_3+d_6\varphi_3^2,\\
F_2(\varphi_1,\varphi_2,\varphi_3) &= e_1\varphi_1^2+e_2\varphi_1\varphi_2+e_3\varphi_1\varphi_3+e_4\varphi_2^2+e_5\varphi_2\varphi_3+e_6\varphi_3^2,\\
F_3(\varphi_1,\varphi_2,\varphi_3) &= l_1\varphi_1^2+l_2\varphi_1\varphi_2+l_3\varphi_1\varphi_3+l_4\varphi_2^2+l_5\varphi_2\varphi_3+l_6\varphi_3^2.
\end{aligned}
$$

Here, the lengthy expressions of $d_i, e_i, l_i (i=1,...,6)$ are omitted.

Denote the local invariant manifold of system (4.1) near the origin as follows:

$$
W^c_{loc}(O) =\{(\varphi_1,\varphi_2,\varphi_3)|\varphi_2=h(\varphi_1,\varphi_3),|\varphi_1|+|\varphi_3|\ll 1,h(0,0)=\partial h_{\varphi_1}(0,0)=\partial h_{\varphi_2}(0,0)=0\}.
$$

Assume that the form of $h(\varphi_1,\varphi_3)$ as follows

$$
\begin{aligned}
h(\varphi_1,\varphi_3) =&h_1\varphi_1^2+h_2\varphi_1\varphi_3+h_3\varphi_1\sigma_1\xi(t)+h_4\varphi_1\sigma_2\xi(t)+h_5\varphi_1\sigma_3\xi(t)+h_6\varphi_3^2\\
&+h_7\varphi_3\sigma_1\xi(t)+h_8\varphi_3\sigma_2\xi(t)+h_9\varphi_3\sigma_3\xi(t)+h_{10}\sigma_1^2\xi^2(t)+h_{11}\sigma_1\xi(t)\sigma_2\xi(t)\\
&+h_{12}\sigma_1\xi(t)\sigma_3\xi(t)+h_{13}\sigma_2^2\xi^2(t)+h_{14}\sigma_2\xi\sigma_3\xi(t)+h_{15}\sigma_3^2\xi^2(t)+\cdots.
\end{aligned}
\tag{4.4}
$$

in which h satisfies

$$
\begin{aligned}
\partial h_{\varphi_1}\dot{\varphi}_1+\partial h_{\varphi_3}\dot{\varphi}_3 =&-(a_1+2\delta)h+e_1\varphi_1^2+e_2\varphi_1 h+e_3\varphi_1\varphi_3+e_4h^2+e_5h\varphi_3\\
&+e_6\varphi_3^2+\sigma_1(e_7\varphi_1+e_8h+e_9\varphi_3+e_{10})\xi(t)+\sigma_2(e_{11}\varphi_1+e_{12}h\\
&+e_{13}\varphi_3+e_{14})\xi(t)+\sigma_3(e_{15}\varphi_1+e_{16}h+e_{17}\varphi_3+e_{18})\xi(t),
\end{aligned}
$$

that is

$$
\begin{aligned}
&\partial h_{\varphi_1}\Bigg(\delta\varphi_1-\sqrt{\frac{a_3}{a_1+2\delta}-\delta^2}\varphi_3+d_1\varphi_1^2+d_2\varphi_1 h+d_3\varphi_1\varphi_3+d_4h^2\\
&+d_5h\varphi_3+d_6\varphi_3^2+\sigma_1(d_7\varphi_1+d_8h+d_9\varphi_3+d_{10})\xi(t)\\
&+\sigma_2(d_{11}\varphi_1+d_{12}h+d_{13}\varphi_3+d_{14})\xi(t)+\sigma_3(d_{15}\varphi_1+d_{16}\varphi_2+d_{17}\varphi_3+d_{18})\xi(t)\Bigg)\\
&+\partial h_{\varphi_3}\Bigg(\delta\varphi_3+\sqrt{\frac{a_3}{a_1+2\delta}-\delta^2}\varphi_1+l_1\varphi_1^2+l_2\varphi_1 h+l_3\varphi_1\varphi_3+l_4h^2\\
&+l_5h\varphi_3+l_6\varphi_3^2+\sigma_1\left(l_7\varphi_1+l_8\varphi_2+l_9\varphi_3+l_{10}\right)\xi(t)\\
&+\sigma_2(l_{11}\varphi_1+l_{12}\varphi_2+l_{13}\varphi_3+l_{14})\xi(t)+\sigma_3(l_{15}\varphi_1+l_{16}\varphi_2+l_{17}\varphi_3+l_{18})\xi(t)\Bigg)\\
=&-(a_1+2\delta)h+e_1\varphi_1^2+e_2\varphi_1h+e_3\varphi_1\varphi_3+e_4h^2+e_5h\varphi_3+e_6\varphi_3^2\\
&+\sigma_1(e_7\varphi_1+e_8h+e_9\varphi_3+e_{10})\xi(t)+\sigma_2(e_{11}\varphi_1+e_{12}h+e_{13}\varphi_3+e_{14})\xi(t)\\
&+\sigma_3(e_{15}\varphi_1+e_{16}h+e_{17}\varphi_3+e_{18})\xi(t).
\end{aligned}
\tag{4.5}
$$

Substituting (4.4) into (4.5) and equating coefficients of similar terms, it can be obtained that

$$
\begin{aligned}
h_1&=\frac{(a_1^2+2b_1)e_1-a_1\sqrt{b_1}e_3+2b_1e_6}{a_1(a_1^2+4b_1)},\qquad h_2=\frac{2\sqrt{b_1}e_1+a_1e_3-2\sqrt{b_1}e_6}{a_1^2+4b_1},\\
h_3&=\frac{a_1e_7-\sqrt{b_1}e_9-2a_1d_{10}h_1+(\sqrt{b_1}d_{10}-a_1l_{10})h_2+2l_{10}\sqrt{b_1}h_6}{a_1^2+b_1},\\
h_4&=\frac{a_1e_{11}-\sqrt{b_1}e_{13}-2a_1d_{14}h_1+(\sqrt{b_1}d_{14}-a_1l_{14})h_2+2l_{14}\sqrt{b_1}h_6}{a_1^2+b_1},\\
h_5&=\frac{a_1e_{15}-\sqrt{b_1}e_{17}-2a_1d_{18}h_1+(\sqrt{b_1}d_{18}-a_1l_{18})h_2+2l_{18}\sqrt{b_1}h_6}{a_1^2+b_1},\\
h_6&=\frac{2b_1e_1+a_1\sqrt{b_1}e_3+(a_1^2+2b_1)e_6}{a_1(a_1^2+4b_1)},\\
h_7&=\frac{\sqrt{b_1}e_7+a_1e_9-2\sqrt{b_1}d_{10}h_1-(\sqrt{b_1}l_{10}+a_1d_{10})h_2-2a_1l_{10}h_6}{a_1^2+b_1},\\
h_8&=\frac{\sqrt{b_1}e_{11}+a_1e_{13}-2\sqrt{b_1}d_{14}h_1-(\sqrt{b_1}l_{14}+a_1d_{14})h_2-2a_1l_{14}h_6}{a_1^2+b_1},\\
h_9&=\frac{\sqrt{b_1}e_{15}+a_1e_{17}-2\sqrt{b_1}d_{18}h_1-(\sqrt{b_1}l_{18}+a_1d_{18})h_2-2a_1l_{18}h_6}{a_1^2+b_1}.
\end{aligned}
$$

Substituting them into (4.3) yields

$$\begin{cases} \dot{\varphi}_1 = & \delta\varphi_1 - \sqrt{\dfrac{a_3}{a_1+2\delta} - \delta^2}\varphi_3 + \phi_1(\varphi_1,\varphi_3) + \sigma_1(d_7\varphi_1 + d_9\varphi_3 + d_{10} \\ & + O(|\varphi_1|^2 + |\varphi_3|^2))\xi(t) + \sigma_2(d_{11}\varphi_1 + d_{13}\varphi_3 + d_{14} + O(|\varphi_1|^2 \\ & + |\varphi_3|^2))\xi(t) + \sigma_3(d_{15}\varphi_1 + d_{17}\varphi_3 + d_{18} + O(|\varphi_1|^2 + |\varphi_3|^2))\xi(t), \\ \dot{\varphi}_3 = & \delta\varphi_3 + \sqrt{\dfrac{a_3}{a_1+2\delta} - \delta^2}\varphi_1 + \phi_3(\varphi_1,\varphi_3) + \sigma_1(l_7\varphi_1 + l_9\varphi_3 + l_{10} \\ & + O(|\varphi_1|^2 + |\varphi_3|^2))\xi(t) + \sigma_2(l_{11}\varphi_1 + l_{13}\varphi_3 + l_{14} + O(|\varphi_1|^2 \\ & + |\varphi_3|^2))\xi(t) + \sigma_3(l_{15}\varphi_1 + l_{17}\varphi_3 + l_{18} + O(|\varphi_1|^2 + |\varphi_3|^2))\xi(t), \end{cases}$$

where

$$\begin{aligned} \phi_1(\varphi_1,\varphi_3) = & d_1\varphi_1^2 + d_2\varphi_1^3 h_1 + d_3\varphi_1\varphi_3 + (d_2h_2 + d_5h_1)\varphi_1^2\varphi_3 + (d_2h_6 + d_5h_2)\varphi_1\varphi_3^2 + d_5h_6\varphi_3^2 + O(|\varphi_1|^4 + |\varphi_3|^4), \\ \phi_3(\varphi_1,\varphi_3) = & l_1\varphi_1^2 + l_2h_1\varphi_1^3 + l_3\varphi_1\varphi_3 + (l_2h_2 + l_5h_1)\varphi_1^2\varphi_3 + (l_2h_6 + l_5h_2)\varphi_1\varphi_3^2 + l_5h_6\varphi_3^2 + O(|\varphi_1|^4 + |\varphi_3|^4). \end{aligned}$$

Omit high items, a two dimensional stochastic center manifold of system (4.1) can be shown as

$$\begin{cases} \dot{\varphi}_1 = & -\sqrt{\dfrac{a_3}{a_1}}\varphi_3 + d_1\varphi_1^2 + d_2\varphi_1^3 h_1 + d_3\varphi_1\varphi_3 + (d_2h_2 + d_5h_1)\varphi_1^2\varphi_3 \\ & + (d_2h_6 + d_5h_2)\varphi_1\varphi_3^2 + d_5h_6\varphi_3^2 + \sigma_1(d_7\varphi_1 + d_9\varphi_3 + d_{10})\xi(t) \\ & + \sigma_2(d_{11}\varphi_1 + d_{13}\varphi_3 + d_{14})\xi(t) + \sigma_3(d_{15}\varphi_1 + d_{17}\varphi_3 + d_{18})\xi(t), \\ \dot{\varphi}_3 = & \sqrt{\dfrac{a_3}{a_1}}\varphi_1 + l_1\varphi_1^2 + l_2h_1\varphi_1^3 + l_3\varphi_1\varphi_3 + (l_2h_2 + l_5h_1)\varphi_1^2\varphi_3 \\ & + (l_2h_6 + l_5h_2)\varphi_1\varphi_3^2 + l_5h_6\varphi_3^2 + \sigma_1(l_7\varphi_1 + l_9\varphi_3 + e_{10})\xi(t) \\ & + \sigma_2(l_{11}\varphi_1 + l_{13}\varphi_3 + l_{14})\xi(t) + \sigma_3(l_{15}\varphi_1 + l_{17}\varphi_3 + l_{18})\xi(t). \end{cases} \tag{4.6}$$

4.2. Stochastic stability and stochastic Hopf bifurcation

Taking transformation $\varphi_1 = r\cos\theta, \varphi_3 = r\sin\theta$, system (4.6) becomes the following form:

$$\begin{cases} \dot{r} = & g_1(r,\theta) + \sigma_1(d_{10}\cos\theta + l_{10}\sin\theta + rd_7\cos^2\theta + (l_7 + d_9)\sin\theta\cos\theta + l_9\sin^2\theta)\xi(t) \\ & + \sigma_2(d_{14}\cos\theta + l_{14}\sin\theta + rd_{11}\cos^2\theta + (l_{11} + d_{13})\sin\theta\cos\theta + l_{13}\sin^2\theta)\xi(t) \\ & + \sigma_3(d_{18}\cos\theta + l_{18}\sin\theta + rd_{15}\cos^2\theta + (l_{15} + d_{17})\sin\theta\cos\theta + l_{17}\sin^2\theta)\xi(t), \\ \dot{\theta} = & g_2(r,\theta) + \sigma_1\left(\dfrac{1}{r}(l_{10}\cos\theta - d_{10}\sin\theta) + l_7\cos^2\theta + (l_9 - d_7)\sin\theta\cos\theta - d_9\sin^2\theta\right)\xi(t) \\ & + \sigma_2\left(\dfrac{1}{r}(l_{14}\cos\theta - d_{14}\sin\theta) + l_{11}\cos^2\theta + (l_{13} - d_{11})\sin\theta\cos\theta - d_{13}\sin^2\theta\right)\xi(t) \\ & + \sigma_3\left(\dfrac{1}{r}(l_{18}\cos\theta - d_{18}\sin\theta) + l_{15}\cos^2\theta + (l_{17} - d_{15})\sin\theta\cos\theta - d_{17}\sin^2\theta\right)\xi(t), \end{cases} \tag{4.7}$$

where

$$
\begin{aligned}
g_1(r,\theta) =& d_1 r^2\cos^3\theta + d_2h_1r^3\cos^4\theta + l_5h_6r^3\sin^4\theta + (d_3+l_1)r^2\sin\theta\cos^2\theta + q_{11}q_{12}r\sin^3\theta\cos\theta \\
&+ (d_2h_6+d_5h_2+l_2h_2+l_5h_1)r^3\cos^2\theta\sin^2\theta + l_3r^2\sin\theta\cos\theta + q_{21}q_{22}r\sin\theta\cos^3\theta \\
&+ (d_2h_2+d_5h_1+l_2h_1)r^3\cos^3\theta\sin\theta + (d_5h_6+l_2h_6+l_5h_2)r^3\cos\theta\sin^3\theta \\
&+ q_{22}q_{23}\sin\theta\cos^2\theta + q_{12}q_{13}\sin^3\theta + q_{21}q_{23}\cos^3\theta + q_{11}q_{13}\sin^2\theta\cos\theta \\
&+ \frac{1}{2}\left(q_{12}^2r\sin^4\theta + q_{21}^2r\cos^4\theta + \left(q_{11}^2+q_{22}^2\right)r\sin^2\theta\cos^2\theta + \frac{1}{r}\left(q_{13}^2\sin^2\theta + q_{23}^2\cos^2\theta\right)\right), \\
g_2(r,\theta) =& \sqrt{\frac{a_3}{a_1}} - d_3r\sin^2\theta\cos\theta + l_1r\cos^3\theta + l_2h_1r^2\cos^4\theta - d_5h_6r^2\sin^4\theta \\
&+ (l_3-d_1)r\sin\theta\cos^2\theta + (l_2h_2+l_5h_1-d_2h_1)r^2\sin\theta\cos^3\theta \\
&+ (l_2h_6+l_5h_2-d_2h_2-d_5h_1)r^2\sin^2\theta\cos^2\theta + (l_5h_6-d_2h_6-d_5h_2)r^2\sin^3\theta\cos\theta \\
&+ \left(q_{11}^2-q_{21}^2\right)\sin\theta\cos^3\theta + \left(q_{12}^2-q_{22}^2\right)\sin^3\theta\cos\theta + \left(q_{13}^2-q_{23}^2\right)\frac{\sin\theta\cos\theta}{r^2} \\
&+ 2\left(q_{11}q_{12}-q_{21}q_{22}\right)\sin^2\theta\cos^2\theta + \frac{2}{r}\left(q_{11}q_{13}-q_{21}q_{23}\right)\sin\theta\cos^2\theta + \frac{2}{r}\left(q_{12}q_{13}-q_{22}q_{23}\right)\sin^2\theta\cos\theta, \\
q_{11} =& \left(\sigma_1d_7+\sigma_2d_{11}+\sigma_3d_{15}\right), \quad q_{12} = \left(\sigma_1d_9+\sigma_2d_{13}+\sigma_3d_{17}\right), \quad q_{13} = \left(\sigma_1d_{10}+\sigma_2d_{14}+\sigma_3d_{18}\right), \\
q_{21} =& \left(\sigma_1l_7+\sigma_2l_{11}+\sigma_3l_{15}\right), \quad q_{22} = \left(\sigma_1l_9+\sigma_2l_{13}+\sigma_3l_{17}\right), \quad q_{23} = \left(\sigma_1l_{10}+\sigma_2l_{14}+\sigma_3l_{18}\right).
\end{aligned}
$$

According to the Khasminskii limit theorem [16, 18, 39], when the intensities of the white noise $(\sigma_1,\sigma_2,\sigma_3)$ are sufficient small, the response process $\{r(t),\theta(t)\}$ weakly converged to the two-dimensional Markov diffusion process [39]. Using the stochastic averaging method, we obtain the Itô stochastic differential equation

$$
\begin{cases}
\mathrm{d}r = m_r\,\mathrm{d}t + \sigma_{11}\,\mathrm{d}W_r + \sigma_{12}\,\mathrm{d}W_\theta, \\
\mathrm{d}\theta = m_\theta\,\mathrm{d}t + \sigma_{21}\,\mathrm{d}W_r + \sigma_{22}\,\mathrm{d}W_\theta,
\end{cases}
\tag{4.8}
$$

where W_r, W_θ are independent standard Wiener processes, $\begin{pmatrix} m_r \\ m_\theta \end{pmatrix}$ is the drift vector, $\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ is the diffusion coefficient matrix, and

$$
\begin{aligned}
m_i(x) =& < f_i(x,t) + \int_{-\infty}^{0} \frac{\partial g_{ik}(x,t)}{\partial x_j} g_{jl}(x,t+\tau)R_{kl}(\tau)\,\mathrm{d}\tau >_t, \\
\sigma_{il}(x)\sigma_{jl}(x) = b_{ij}(x) =& < \int_{-\infty}^{+\infty} g_{ik}(x,t)g_{jl}(x,t+\tau)R_{kl}(\tau)\,\mathrm{d}\tau >_t,
\end{aligned}
$$

$< \cdot >_t= \lim\limits_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} < \cdot >$ dt is a time averaging operator. The parameters are as follows

$$
\begin{aligned}
\mu_1 =& 3d_2h_1+3l_5h_6+d_2h_6+d_5h_2+l_2h_2+l_5h_1,\\
\mu_2 =& \sigma_1^2(3d_7^2+3l_9^2+2d_7l_9+3(l_7+d_9)^2+2(l_9-d_7)^2)+\frac{1}{2}\left(q_{11}^2+3q_{12}^2+3q_{21}^2+q_{22}^2\right)\\
&+\sigma_2^2(3d_{11}^2+3l_{13}^2+2d_{11}l_{13}+3(l_{11}+d_{13})^2+2(l_{13}-d_{11})^2)\\
&+\sigma_3^2(3d_{15}^2+3l_{17}^2+2d_{15}l_{17}+3(l_{15}+d_{17})^2+2(l_{17}-d_{15})^2),\\
\mu_3 =& \frac{1}{4}\left(q_{13}^2+q_{23}^2+2\sigma_1^2\left(d_{10}^2+l_{10}^2\right)+2\sigma_2^2\left(d_{14}^2+l_{14}^2\right)+2\sigma_3^2\left(d_{18}^2+l_{18}^2\right)\right),\\
\mu_4 =& \sigma_1^2(3d_7^2+(d_9+l_7)^2+3l_9^2+2d_7l_9+4(d_{10}^2+l_{10}^2))+\sigma_2^2(3d_{11}^2+(d_{13}+l_{11})^2+3l_{13}^2+2d_{11}l_{13}+4(d_{14}^2+l_{14}^2))\\
&+\sigma_3^2(3d_{15}^2+(d_{17}+l_{15})^2+3l_{17}^2+2d_{15}l_{17}+4(d_{18}^2+l_{18}^2)),\\
\mu_5 =& \frac{\sigma_1^2}{4}(d_7+l_9)(l_7-d_9)+\frac{\sigma_2^2}{4}(d_{11}+l_{13})(l_{11}-d_{13})+\frac{\sigma_3^2}{4}(d_{15}+l_{17})(l_{15}-d_{17}),\\
\mu_6 =& 3l_2h_1-3d_5h_6+l_2h_6+l_5h_2-d_5h_1-d_2h_2,\\
\mu_7 =& \frac{\sigma_1^2}{8}(3l_7^2+(l_9-d_7^2)+3d_9^2-2l_7d_7)+\frac{\sigma_2^2}{8}(3l_{11}^2+(l_{13}-d_{11})^2+3d_{13}^2-2l_{11}d_{13})\\
&+\frac{\sigma_3^2}{8}(3l_{15}^2+(l_{17}-d_{15})^2+3d_{15}^2-2l_{15}d_{17}),\\
\mu_8 =& \sqrt{\frac{a_3}{a_1}}+\frac{1}{4}\left(q_{11}q_{12}-q_{21}q_{22}\right).
\end{aligned}
$$

When $\sigma_{12}^2=\sigma_{21}^2\neq 0$, rewriting (4.8) as follows

$$
\begin{cases}
\mathrm{d}r=\left(\frac{\mu_1}{8}r^3+\frac{\mu_2}{8}r+\frac{\mu_3}{r}\right)\mathrm{d}t+\left(\mu_3+\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}}\mathrm{d}W_r+(\mu_5 r)^{\frac{1}{2}}\mathrm{d}W_\theta,\\
\mathrm{d}\theta=\left(\mu_8+\frac{\mu_6}{8}r^2\right)\mathrm{d}t+(\mu_5 r)^{\frac{1}{2}}\mathrm{d}W_r+\left(\mu_7+\frac{\mu_3}{r^2}\right)^{\frac{1}{2}}\mathrm{d}W_\theta.
\end{cases}
\tag{4.9}
$$

When $\sigma_{12}^2=\sigma_{21}^2=0$, the averaging amplitude $r(t)$ is a one-dimensional Markov diffusion process, which implies that

$$
\mathrm{d}r=\left(\frac{\mu_1}{8}r^3+\frac{\mu_2}{8}r+\frac{\mu_3}{r}\right)\mathrm{d}t+\left(\mu_3+\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}}\mathrm{d}W_r. \tag{4.10}
$$

In the following, we discuss the stochastic stability of system (4.1) through analyzing the change of stability of the averaging amplitude $r(t)$ in the meaning of probability. According to the singular boundary theory [39], it can be obtained the global stochastic stability of the stochastic averaging system (4.10).

Obviously, $\mu_3\neq 0$. Let $\mu_1=0$, the system (4.10) is rewritten as follows

$$
\mathrm{d}r=\left(\frac{\mu_2}{8}r+\frac{\mu_3}{r}\right)\mathrm{d}t+\left(\mu_3+\frac{\mu_4}{8}r^2\right)^{\frac{1}{2}}\mathrm{d}W_r. \tag{4.11}
$$

Owing to $\sigma_{11}\neq 0$ at $r=0$, $r=0$ is a nonsingular and regular boundary of system (4.11). If $r=+\infty$, then $m_r=+\infty$; so $r=+\infty$ is the second kind of singular boundary. The results are presented as follows:

$$
\begin{aligned}
&\alpha_r=2,\ \beta_r=1,\\
&c_r=-\lim_{r\to+\infty}\frac{2m_r r^{\alpha_r-\beta_r}}{\sigma_{11}^2}=-\lim_{r\to+\infty}\frac{2(\frac{\mu_2}{8}r+\frac{\mu_3}{r})r^{2-1}}{\mu_3+\frac{\mu_4}{8}r^2}=-\frac{2\mu_2}{\mu_4}.
\end{aligned}
$$

So, if $c_r > -1$,*i.e.* $\frac{\mu_2}{\mu_4} < \frac{1}{2}$, the boundary $r = +\infty$ is exclusively natural; if $c_r < -1$,*i.e.* $\frac{\mu_2}{\mu_4} > \frac{1}{2}$, the boundary $r = +\infty$ is attractively natural; if $c_r = -1$,*i.e.* $\frac{\mu_2}{\mu_4} = \frac{1}{2}$, the boundary $r = +\infty$ is strictly natural.

Consider $\mu_1 \neq 0$, $\sigma_{11} \neq 0$ at $r = 0$, $r = 0$ is a nonsingular and regular boundary of system (4.10). If $r = +\infty$, then $m_r = +\infty$, so $r = +\infty$ is the second kind of singular boundary. The results are presented as follows:

$$\alpha_r = 2,\ \beta_r = 3,$$
$$c_r = -\lim_{r\to+\infty}\frac{2m_r r^{\alpha_r-\beta_r}}{\sigma_{11}^2} = -\lim_{r\to+\infty}\frac{2(\frac{\mu_1}{8}r^3 + \frac{\mu_2}{8}r + \frac{\mu_3}{r})r^{2-3}}{\mu_3 + \frac{\mu_4}{8}r^2} = -\frac{2\mu_1}{\mu_4}.$$

Besides, $\beta_r > \alpha_r - 1$, $m_r(+\infty) = +\infty > 0$, $\beta_r > 1$, so $r = +\infty$ is exist boundary. To sum up, the following conclusion can be drawn.

Theorem 4.1. *The trivial solution $r = 0$ is unstable, i.e., the stochastic system (2.3) is unstable at equilibrium point P^* regardless of whether the deterministic system is stable at P^* or not, a Hopf bifurcation may occur. If $\frac{\mu_2}{\mu_4} > \frac{1}{2}$, the boundary $r = +\infty$ is attractively natural, the equilibrium point P^* is unstable.*

Next, we will discuss the effect of randomness on the stochastic dynamical behavior. According to the Itô equation of amplitude $r(t)$, the FPK equation form of (4.10) can be shown as

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial r}\left(\left(\frac{\mu_1}{8}r^3 + \frac{\mu_2}{8}r + \frac{\mu_3}{r}\right)P(r)\right) + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left(\left(\mu_3 + \frac{\mu_4}{8}r^2\right)P(r)\right)$$

with the initial value condition $P(r,t|r_0,t_0) \to \delta(r \to r_0)$, $t \to t_0$ where $P(r,t|r_0,t_0)$ is the transition probability density of diffusion process $r(t)$. The invariant measure of $r(t)$ is the steady-state probability density $P(r)$ which is the solution of the degenerate system as follows:

$$-\frac{\partial}{\partial r}\left(\left(\frac{\mu_1}{8}r^3 + \frac{\mu_2}{8}r + \frac{\mu_3}{r}\right)P(r)\right) + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left(\left(\mu_3 + \frac{\mu_4}{8}r^2\right)P(r)\right) = 0.$$

The solution is as follows

$$P_{st} = c_1 e^{\frac{\mu_1}{\mu_4}r^2} r^2 (8\mu_3 + \mu_4 r^2)^{-2+\frac{\mu_2}{\mu_4}-\frac{8\mu_1\mu_3}{\mu_4^2}}, \tag{4.12}$$

where c_1 is a normalization constant.

According to Namachivaya's theory [29, 39], the most possible amplitude of system (4.10) is $r^*(t)$,*i.e.* $P_{st}(r)$ maximizes at $r^*(t)$. So we have

$$\frac{\mathrm{d}P_{st}}{\mathrm{d}r}|_{r=r^*} = 0, \quad \frac{d^2P_{st}(r)}{dr^2}|_{r=r^*} < 0,$$

it is calculated that $r^* = \sqrt{\frac{-\mu_2+\mu_4-\sqrt{(\mu_2-\mu_4)^2-32\mu_1\mu_3}}{2\mu_1}}$. Meanwhile, $P_{st}(r)$ is minimal at $r = 0$. It indicates that the system subjected to stochastic excitations is almost unstable at P^* when $r = 0$ in the meaning of probability.

By the singular boundary theory, the stochastic system (4.1) occurs a stochastic Hopf bifurcation at r^*, that is

$$\varphi_1^2 + \varphi_3^2 = \frac{-\mu_2 + \mu_4 - \sqrt{(\mu_2 - \mu_4)^2 - 32\mu_1\mu_3}}{2\mu_1}.$$

5. Non-PDC integral sliding mode control scheme

For the results of the above system analysis, what we expect is to stabilize the system (4.1), in other words, to control the outbreak of disease. Taking transformation $x_1(t) = S(t) - S^*, x_2(t) = E(t) - E^*, x_3(t) = I(t) - I^*, x_4(t) = R(t) - R^*$ for system (2.2) and introducing a sliding mode controller $u(t)$ to $x_3(t)$, the following controlled system can be obtained

$$\begin{cases} \dot{x}_1(t) =(-\beta(\varepsilon E^* + (1-q)I^*) - a)x_1(t) - \beta\varepsilon S^* x_2(t) - \beta S^*(1-q)x_3(t) \\ \qquad\quad - \beta\varepsilon x_1(t)x_2(t) - \beta(1-q)x_1(t)x_3(t) + \sigma_1(x_1(t) + S^*)\xi(t), \\ \dot{x}_2(t) =\beta(\varepsilon E^* + (1-q)I^*)x_1(t) + (\beta\varepsilon S^* - c)x_2(t) + \beta S^*(1-q)x_3(t) \\ \qquad\quad + \beta\varepsilon x_1(t)x_2(t) + \beta(1-q)x_1(t)x_3(t) + \sigma_2(x_2(t) + E^*)\xi(t), \\ \dot{x}_3(t) =k_1x_2(t) - bx_3(t) + u(t) + \sigma_3(x_3(t) + I^*)\xi(t), \\ \dot{x}_4(t) =m_1x_1(t) + m_2x_2(t) + k_2x_3(t) - \mu x_4(t) + \sigma_4(x_4(t) + R^*)\xi(t). \end{cases} \tag{5.1}$$

Constrained by the total population, we take $x_1(t) \in (-c_2, c_2)$. The following T-S fuzzy stochastic system is confined in the probability space $(\Omega, \mathbb{F}, \mathbb{P})$.

Rule 1: if $x_1(t)$ is $\bar{M}_1$, then $\dot{x}(t) = A_1x(t) + Bu(t) + C(x(t) + x^*)\xi(t)$.

Rule 2: if $x_1(t)$ is $\bar{M}_2$, then $\dot{x}(t) = A_2x(t) + Bu(t) + C(x(t) + x^*)\xi(t)$.

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix},\ x^* = \begin{pmatrix} S^* \\ E^* \\ I^* \\ R^* \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ C = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_4 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -\beta(\varepsilon E^* + (1-q)I^*) - a & -\beta\varepsilon(S^* + c_2) & -\beta(S^* + c_2)(1-q) & 0 \\ \beta(\varepsilon E^* + (1-q)I^*) & \beta\varepsilon(S^* + c_2) - c & \beta(S^* + c_2)(1-q) & 0 \\ 0 & k_1 & -b & 0 \\ m_1 & m_2 & k_2 & -\mu \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -\beta(\varepsilon E^* + (1-q)I^*) - a & -\beta\varepsilon(S^* - c_2) & -\beta(S^* - c_2)(1-q) & 0 \\ \beta(\varepsilon E^* + (1-q)I^*) & \beta\varepsilon(S^* - c_2) - c & \beta(S^* - c_2)(1-q) & 0 \\ 0 & k_1 & -b & 0 \\ m_1 & m_2 & k_2 & -\mu \end{pmatrix},$$

$u(t)$ is the input vector, $\lambda_i(x_1(t))$ is the membership function of $x_1(t)$ belonging to fuzzy sets $\bar{M}_i(i = 1, 2)$, and $\lambda_1(x_1(t)) = \frac{1}{2}\left(1 + \frac{x_1(t)}{c_2}\right)$, $\lambda_2(x_1(t)) = \frac{1}{2}\left(1 - \frac{x_1(t)}{c_2}\right)$.

Based on the center-average defuzzifier, product inference, and the singleton fuzzifier, the overall T-S fuzzy stochastic system can be inferred as

$$\mathrm{d}x(t) = \sum_{i=1}^{2} \lambda_i(x_1(t))(A_ix(t) + Bu(t))\,\mathrm{d}t + C(x(t) + x^*)\,\mathrm{d}B(t).$$

To ease the notation, $A(\lambda)$ denotes $\sum_{i=1}^{2} \lambda_i(x_1(t))A_i$, the T-S fuzzy stochastic system is shown as follows:

$$\mathrm{d}x(t) = (A(\lambda)x(t) + Bu(t))\,\mathrm{d}t + C(x(t) + x^*)\,\mathrm{d}B(t). \tag{5.2}$$

5.1. Construction of sliding surface

The sliding surface is defined by $s(t) = 0$, where the sliding variable is constructed as follows:

$$s(t) = Gx(t) - Gx(0) - \int_0^t G(A(\lambda) + BK(\lambda)(Y(\lambda))^{-1})x(\tau)\,\mathrm{d}\tau. \tag{5.3}$$

Here $G = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}$ [4], $K(\lambda) = \sum_{i=1}^{2} \lambda_i(x_1(t))K_i$, $Y(\lambda) = \sum_{i=1}^{2} \lambda_i(x_1(t))Y_i$, $K_i \in \mathbb{R}^{1\times 4}, Y_i \in \mathbb{R}^{4\times 4}$, $i = 1, 2$ are unknown coefficient matrices to be designed later.

5.2. Stability of the sliding motion

Based on (5.2) and (5.3), we obtain that

$$\mathrm{d}s(t) = (GB(u(t) - K(\lambda)(Y(\lambda))^{-1}x(t)))\,\mathrm{d}t + GC(x(t) + x^*)\,\mathrm{d}B(t). \tag{5.4}$$

In the sliding phase, $\mathrm{d}s(t) = 0$ holds. It is necessary to satisfy

$$GB(u(t) - K(\lambda)(Y(\lambda))^{-1}x(t)) = 0$$

when the state trajectories of the system (5.2) reach and are confined to the sliding surface with sliding variable (5.3). Since $det(GB) \neq 0$, the equivalent control is established as

$$u_{eq} = K(\lambda)(Y(\lambda))^{-1}x(t). \tag{5.5}$$

The following sliding mode dynamics is shown by substituting (5.5) into the system (5.2).

$$\mathrm{d}x(t) = (A(\lambda) + BK(\lambda)(Y(\lambda))^{-1})x(t)\,\mathrm{d}t + C(x(t) + x^*)\,\mathrm{d}B(t). \tag{5.6}$$

Assumption 5.1. [30, 32] $\frac{\partial \lambda_i(x_1(t))}{\partial t} \geq \varPhi_i$ ($\varPhi_i \leq 0$) for all $i = 1, 2$ where $\varPhi_i$, $i = 1, 2$ are scalars.

Lemma 5.2 (Finsler's Lemma). *[11] Let $x \in \mathbb{R}^n$, $\varOmega = \varOmega^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{m\times n}$. The followings are equivalent,*
1) $x^{\mathrm{T}}\varOmega x < 0$, $\forall Wx = 0$, $x \neq 0$.
2) exists $X \in \mathbb{R}^{n\times m}$: $\varOmega + He(XW) < 0$, where $He(XW) = XW + W^{\mathrm{T}}X^{\mathrm{T}}$.

Theorem 5.3. *If the following matrix inequalities:*

$$P_i + X \geq 0,\ \ i = 1, 2 \tag{5.7}$$

$$\varTheta_{ii} < 0, \quad i = 1, 2$$

$$\varTheta_{ii} + \frac{1}{2}(\varTheta_{ij} + \varTheta_{ji}) < 0, \quad i, j = 1, 2, \quad i \neq j \tag{5.8}$$

are solvable for $(P_i,\ Y_i,\ K_i,\ X, \eta), i = 1, 2$, where $P_i > 0$, $\eta > 0$, $P_i = P_i^{\mathrm{T}}$, and

$$\varTheta_{ij} = \begin{pmatrix} He(A_iY_j + BK_j) - \sum_{k=1}^{2}\varPhi_k(P_k + X) & * & * \\ P_i - Y_i + \eta(A_iY_j + BK_j)^{\mathrm{T}} & -\eta He(Y_i) & * \\ CP_i & 0 & -P_i \end{pmatrix}.$$

Then, the sliding motion dynamics (5.6) is asymptotically mean square stable when $x_1(t) \geq -\frac{S^*}{2}, x_2(t) \geq -\frac{E^*}{2}, x_3(t) \geq -\frac{I^*}{2}, x_4(t) \geq -\frac{R^*}{2}$.

Proof. If the matrix inequalities (5.8) hold, then

$$\Theta(\lambda\lambda) = \sum_{i=1}^{2}\sum_{j=1}^{2} \lambda_i(x_1(t))\lambda_j(x_1(t))\Theta_{ij} < 0.$$

Based on $P_i > 0$, it is obtained that $P(\lambda) = \sum_{i=1}^{2} \lambda_i(x_1(t))P_i$ is invertible and its inverse is named $(P(\lambda))^{-1}$. Since $\frac{\partial\lambda_i(x_1(t))}{\partial t} \geq \Phi_i$ and $\Phi_i \leq 0$, we have

$$-\sum_{k=1}^{2}\Phi_k(P_k + X) = -\sum_{k=1}^{2}\Phi_k P_k - \sum_{k=1}^{2}\Phi_k X \geq -\sum_{k=1}^{2}\frac{\partial\lambda_k(x_1(t))}{\partial t}P_k - \sum_{k=1}^{2}\frac{\partial\lambda_k(x_1(t))}{\partial t}X.$$

Note that $\sum_{k=1}^{2}\lambda_k(x_1(t)) = 1$, so $\sum_{k=1}^{2}\frac{\partial\lambda_k(x_1(t))}{\partial t} = 0$, it can be further obtained that

$$-\sum_{k=1}^{2}\Phi_k(P_k + X) \geq -\sum_{k=1}^{2}\frac{\partial\lambda_k(x_1(t))}{\partial t}P_k = -\frac{\partial P(\lambda)}{\partial t}. \tag{5.9}$$

Due to

$$\Theta(\lambda\lambda) = \begin{pmatrix} He(A(\lambda)Y(\lambda) + BK(\lambda)) - \sum_{k=1}^{2}\Phi_k(P_k + X) & * & * \\ P(\lambda) - Y(\lambda) + \eta(A(\lambda)Y(\lambda) + BK(\lambda))^{\mathrm{T}} & -\eta He(Y(\lambda)) & * \\ CP(\lambda) & 0 & -P(\lambda) \end{pmatrix} \tag{5.10}$$

and $\Theta(\lambda\lambda) < 0$, by using the Schur complement lemma, it can be obtained that:

$$\begin{aligned} &He\left(\begin{pmatrix} (Y(\lambda))^{\mathrm{T}} \\ \eta(Y(\lambda))^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} \left(A(\lambda) + BK(\lambda)(Y(\lambda))^{-1}\right)^{\mathrm{T}} & -I \end{pmatrix}\right) \\ &\quad + \begin{pmatrix} -\frac{\partial P(\lambda)}{\partial t} + (CP(\lambda))^{\mathrm{T}}(P(\lambda))^{-1}CP(\lambda) & * \\ P(\lambda) & 0 \end{pmatrix} < 0. \end{aligned} \tag{5.11}$$

By Lemma 5.2, (5.11) holds if

$$w^{\mathrm{T}}\begin{pmatrix} -\frac{\partial P(\lambda)}{\partial t} + (CP(\lambda))^{\mathrm{T}}(P(\lambda))^{-1}CP(\lambda) & * \\ P(\lambda) & 0 \end{pmatrix} w < 0 \tag{5.12}$$

for any $w = \begin{pmatrix} w_1^{\mathrm{T}} & w_2^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} \neq 0$ satisfied $\begin{pmatrix} (A(\lambda) + BK(\lambda)(Y(\lambda))^{-1})^{\mathrm{T}} & -I \end{pmatrix} w = 0$. Furthermore, (5.12) implies that

$$He\left((A(\lambda) + BK(\lambda)(Y(\lambda))^{-1})P(\lambda)\right) + (CP(\lambda))^{\mathrm{T}}(P(\lambda))^{-1}CP(\lambda) - \frac{\partial P(\lambda)}{\partial t} < 0. \tag{5.13}$$

Due to $P(\lambda)(P(\lambda))^{-1} = I$, it can be obtained that

$$\frac{\partial(P(\lambda))^{-1}}{\partial t} = -(P(\lambda))^{-1}\frac{\partial P(\lambda)}{\partial t}(P(\lambda))^{-1}. \tag{5.14}$$

Premultiplying and postmultiplying (5.13) by $(P(\lambda))^{-\mathrm{T}}$ and $(P(\lambda))^{-1}$, it is obtained from (5.14) that:

$$\varDelta(x(t)) \triangleq He\left(\left(A(\lambda)+BK(\lambda)(Y(\lambda))^{-1}\right)^{\mathrm{T}}(P(\lambda))^{-1}\right)+C^{\mathrm{T}}(P(\lambda))^{-1}C+\frac{\partial(P(\lambda))^{-1}}{\partial t}<0. \tag{5.15}$$

Select the Lyapunov function candidate as

$$V(x(t))=(x(t)+x^*)^{\mathrm{T}}(P(\lambda))^{-1}(x(t)+x^*). \tag{5.16}$$

By Itô's formula, it can be obtained that

$$\mathrm{d}V(x(t))=LV(x(t))\,\mathrm{d}t+V_xC(x(t)+x^*)\,\mathrm{d}B(t). \tag{5.17}$$

From (5.15), there exists a positive constant ρ such that

$$LV(x(t))\leq -\rho\|x(t)+x^*\|^2. \tag{5.18}$$

According to (5.16), the following inequality holds

$$\lambda_{\min}((P(\lambda))^{-1})\|x(t)+x^*\|^2\leq V(x(t))\leq \lambda_{\max}((P(\lambda))^{-1})\|x(t)+x^*\|^2. \tag{5.19}$$

Using Itô's formula and (5.17), we obtain

$$\mathrm{d}\left(e^{\alpha t}V(x(t))\right)=\alpha e^{\alpha t}V(x(t))\,\mathrm{d}t+e^{\alpha t}LV(x(t))\,\mathrm{d}t+e^{\alpha t}V_xC(x(t)+x^*)\,\mathrm{d}B(t). \tag{5.20}$$

Integrating and taking expectations on both sides of equation (5.20), it can be obtained that

$$e^{\alpha t}\mathbb{E}\left\{V(x(t))\right\}=\mathbb{E}\left\{V(x(0))\right\}+\mathbb{E}\int_0^t \alpha e^{\alpha\tau}V(x(\tau))\,\mathrm{d}\tau+\mathbb{E}\int_0^t e^{\alpha\tau}LV(x(\tau))\,\mathrm{d}\tau. \tag{5.21}$$

Substituting (5.18)–(5.19) into (5.21), it can be obtained that

$$\mathbb{E}\left\{V(x(t))\right\}\leq e^{-\alpha t}\mathbb{E}\{V(x(0))\}+\mathbb{E}\int_0^t e^{-\alpha(t-\tau)}\left(\alpha\lambda_{\max}((P(\lambda))^{-1})-\rho\right)\|x(t)+x^*\|^2\,\mathrm{d}\tau. \tag{5.22}$$

It is noted that $x_1(t)\geq -\frac{S^*}{2}, x_2(t)\geq -\frac{E^*}{2}, x_3(t)\geq -\frac{I^*}{2}, x_4(t)\geq -\frac{R^*}{2}$, taking expectations on (5.19), we can obtain

$$\begin{aligned}\mathbb{E}\{V\left(x\left(t\right)\right)\}\geq&\mathbb{E}\left\{\lambda_{\min}\left(\left(P\left(\lambda\right)\right)^{-1}\right)\|x\left(t\right)+x^*\|^2\right\}\\ \geq&\mathbb{E}\left\{\lambda_{\min}\left(\left(P\left(\lambda\right)\right)^{-1}\right)\|x\left(t\right)\|^2\right\}.\end{aligned} \tag{5.23}$$

Pick $\alpha>0$ to satisfy $\alpha\lambda_{\max}((P(\lambda))^{-1})-\rho<0$. Substituting α and (5.23) into (5.22), we can obtain

$$\mathbb{E}\{\|x(t)\|^2\}\leq\lambda_{\max}(P(\lambda))\mathbb{E}\{V(x(0))\}e^{-\alpha t}. \tag{5.24}$$

As t tends to ∞, (5.24) yields $\lim_{t\to\infty}\mathbb{E}\{\|x(t)\|^2\}=0$. As a consequence, the sliding motion (5.6) is asymptotically mean square stable. □

Remark 5.4. If LMI (5.7)–(5.8) in Theorem 5.3 are solvable, the sliding mode exists. System (5.6) can be proved to be asymptotically mean square stable. It indicates that epidemics will reach a stable state. Moreover, the matrices $K_i, Y_i (i=1,2)$ are known, which provides convenience for designing controller.

5.3. Design of the sliding mode controller

Based on Theorem 5.3, the controller relative to the sliding variable (5.3) of system (5.2) is designed as follows.

Theorem 5.5. *Assume that matrices G and K_i, Y_i, $i = 1, 2$, satisfy Theorem 5.3. The sliding mode controller*

$$u(t) = K(\lambda)(Y(\lambda))^{-1}x(t) - s(t) - \varrho\frac{s(t)}{\|s(t)\|} \tag{5.25}$$

can confine the state trajectories of the closed-loop system to a sufficiently small band around the sliding mode surface with sliding variable (5.3), where ϱ is a positive constant.

Proof. Select Lyapunov function as $\tilde{V}(s(t)) = \frac{1}{2}s^{\mathrm{T}}(t)s(t)$. By Itô's formula and (5.4), we have

$$\begin{aligned}\mathrm{d}\tilde{V}(s(t)) =& s^{\mathrm{T}}(t)GB(u(t) - K(\lambda)(Y(\lambda))^{-1}x(t))\,\mathrm{d}t + (x(t)+x^*)^{\mathrm{T}}tr\left(\frac{1}{2}C^{\mathrm{T}}G^{\mathrm{T}}GC\right)(x(t)+x^*)\,\mathrm{d}t\\ &+ s^{\mathrm{T}}(t)GC(x(t)+x^*)\,\mathrm{d}B(t)\\ =& L\tilde{V}(s(t))\,\mathrm{d}t + s^{\mathrm{T}}(t)GC(x(t)+x^*)\,\mathrm{d}B(t).\end{aligned}$$

Using (5.25), it can be obtained that

$$L\tilde{V}(s(t)) \leq -\varrho\|s(t)\| - \|s(t)\|^2 + \lambda_m\|x(t)+x^*\|^2, \tag{5.26}$$

where $\lambda_m = \lambda_{\max}\left(\frac{C^{\mathrm{T}}G^{\mathrm{T}}GC}{2}\right)$. To achieve the sliding mode, it should be satisfied that

$$L\tilde{V}(s(t)) \leq -\varrho\|s(t)\|, \tag{5.27}$$

where $\varrho > 0$. Combining (5.26) with (5.27), (5.27) holds if

$$-\|s(t)\|^2 + \lambda_m\|x(t)+x^*\|^2 \leq 0,$$

which means that for $\|s(t)\| \geq \sqrt{\lambda_m\|x(t)+x^*\|^2}$. Inspired by Gao *et al.* [12], we select the following small band around the sliding surface

$$\mathcal{D}(s(t)) = \left\{s(t)|\|s(t)\| \leq \sqrt{\lambda_m\|x(t)+x^*\|^2}\right\}.$$

It can be obtained that the sliding variable remains in the band $\mathcal{D}(s(t))$ from Zhang *et al.* [38] and Gao *et al.* [13]. It concludes that the state trajectories of the closed-loop system are not kept on the sliding surface, but remain in an arbitrarily small band around the sliding surface almost surely since the initial time [12]. □

Remark 5.6. For system (2.2), when the number of susceptible, latent, infected and recovered individuals exceeds half of the endemic equilibrium value, the population of each class can be eventually restricted to a sufficiently small area around the endemic equilibrium point by the sliding mode controller (5.25), which implies that the epidemic is effectively controlled in time.

Remark 5.7. Controller $u(t)$ can be implemented through state and social measures to the confirmed patients. For example, the government formulates rules and regulations for the prevention and treatment of infectious diseases, and all sectors of society actively participate in and strictly implement them to ensure the effective isolation and medical treatment of the infected in the first time. The spread of the epidemic can be largely contained by targeting those who have been diagnosed.

TABLE 1. The possibilities and positions of the Hopf bifurcation occurrence in system (4.10).

Condition	μ_1	μ_2	μ_3	μ_4	$r = r^*$	$\frac{P_{st}(r^*)}{c_1}$
Cond1	-0.6	-0.2	0.1	2	0.5774	0.1413
Cond2	-0.6	-0.2	0.2	2	0.7886	0.0739
Cond3	-0.6	-0.2	0.3	2	0.9380	0.0566
Cond4	-0.6	-0.2	0.4	2	1.0561	0.0515

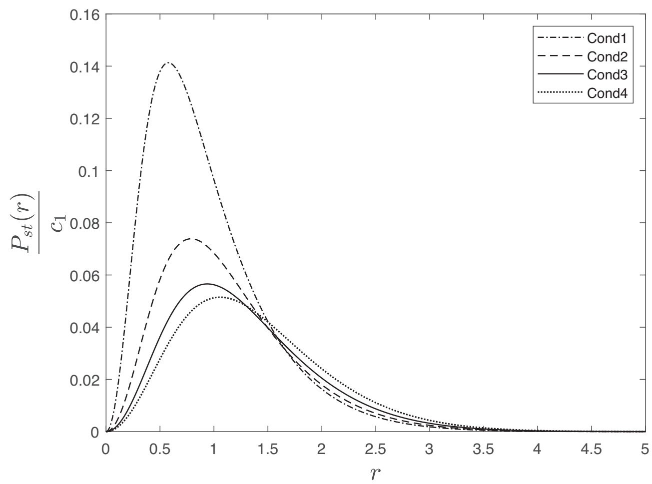


FIGURE 1. The steady-state probability density $P_{st}(t)$ and position r^* of stochastic Hopf bifurcation at $\mu_1 = -0.6$, $\mu_2 = -0.2$, $\mu_3 = 0.1, 0.2, 0.3, 0.4$, $\mu_4 = 2$ in system (4.10).

6. NUMERICAL SIMULATION

To indicate the validity above, some parameters are taken to reflect the relationship among the probability density and position of stochastic Hopf bifurcation with the different value of μ_3. Furthermore, the effect of the controller $u(t)$ in Section 4 is demonstrated as far as possible.

For system (4.12), selecting $\mu_1 = -0.6, \mu_2 = -0.2, \mu_4 = 2$, taking μ_3 with different values listed in Table 1, the relevant results are shown in Figure 1. It is not difficult to find that the positions of the Hopf bifurcation occurrence increase as the increase of the value of μ_3.

Let $\beta = 0.0139$, $\varepsilon = 0.001$, $q = 0.5$, $A = 400$, $\mu = 0.1$, $m_1 = 0.5$, $k_1 = 0.54$, $m_2 = 0.5$, $\gamma = 0.02$, $k_2 = 0.5$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\sigma_3 = 0.4$, $\sigma_4 = 0.2$, under the initial condition $x(0) = (3, 2, 5, 4)$, the time responses of the system (5.1) without controller are given in Figure 2. It implies that the stochastic system (2.2) is unstable and oscillatory around P^*. By calculating, we can obtain

$$A_1 = \begin{pmatrix} -2.1288 & -0.0082 & -4.0859 & 0 \\ 1.5288 & -1.1318 & 4.0859 & 0 \\ 0 & 0.5400 & -0.6200 & 0 \\ 0.5000 & 0.5000 & 0.5000 & -0.1000 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$A_2 = \begin{pmatrix} -2.1288 & -0.0029 & 1.4741 & 0 \\ 1.5288 & -1.1429 & -1.4741 & 0 \\ 0 & 0.5400 & -0.6200 & 0 \\ 0.5000 & 0.5000 & 0.5000 & -0.1000 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.2 & & & \\ & 0.3 & & \\ & & 0.4 & \\ & & & 0.2 \end{pmatrix}.$$

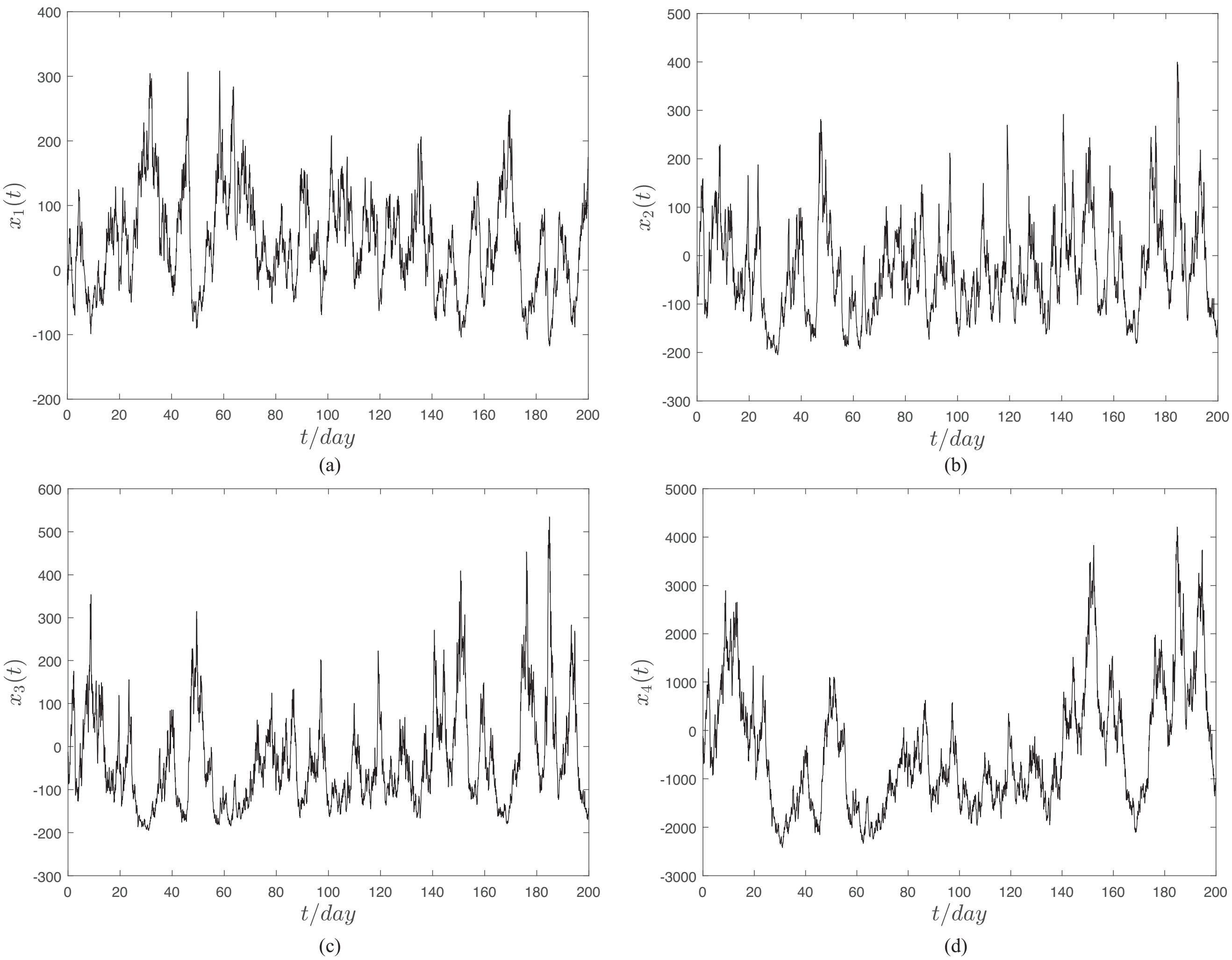
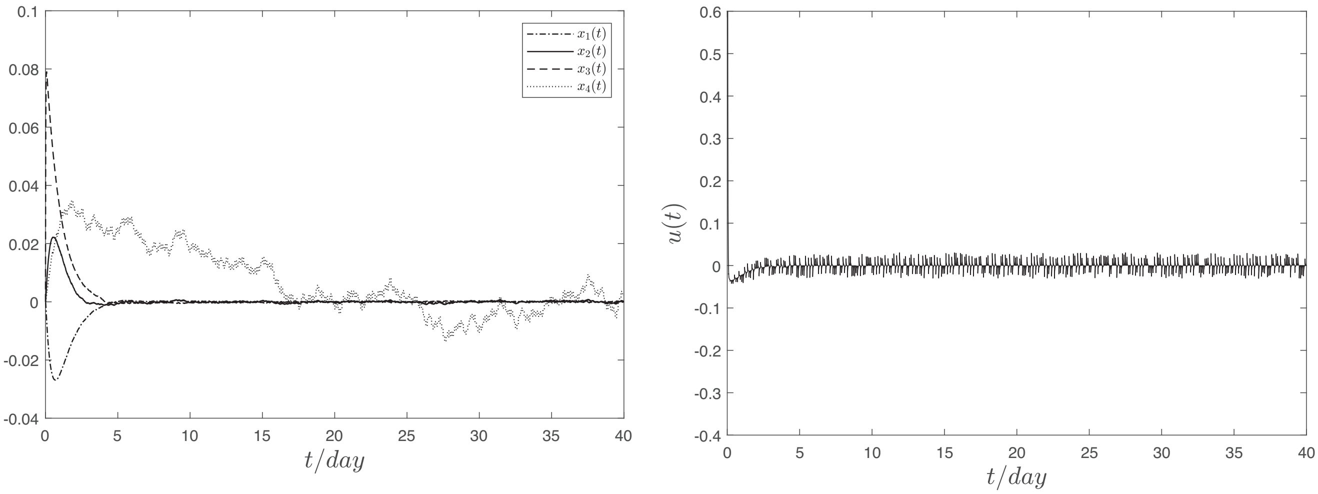


FIGURE 2. Time responses of the system (5.1) without controller $u(t)$ with $\beta = 0.0139$, $\varepsilon = 0.001$, $q = 0.5$, $A = 400$, $\mu = 0.1$, $m_1 = 0.5$, $k_1 = 0.54$, $m_2 = 0.5$, $\gamma = 0.02$, $k_2 = 0.5$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\sigma_3 = 0.4$, $\sigma_4 = 0.2$ with the initial value $x(0) = (3, 2, 5, 4)$.

Take $\phi_1 = \phi_2 = -5000$ and $\eta = 0.01$. Utilizing LMI toolbox in Matlab software, it can be computed from Theorem 5.3 as follows

$$K_1 = (2.3524,\ -3.9257,\ -73.6791,\ 0.0295),$$
$$K_2 = (-4.5078,\ 2.9692,\ -73.6642,\ 0.0087),$$
$$Y_1 = \begin{pmatrix} 1.8488 & -0.3187 & -0.0056 & -0.6358 \\ -0.3187 & 2.2261 & -0.0208 & -0.3819 \\ -0.0056 & -0.0208 & 1.2390 & -0.0253 \\ -0.6358 & -0.3819 & -0.0253 & 3.0068 \end{pmatrix},$$
$$Y_2 = \begin{pmatrix} 1.8475 & -0.3246 & -0.0106 & -0.6497 \\ -0.3246 & 2.2324 & -0.0227 & -0.3719 \\ -0.0106 & -0.0227 & 1.2419 & -0.0346 \\ -0.6497 & -0.3719 & -0.0346 & 3.0123 \end{pmatrix}.$$



(a) (b)

FIGURE 3. Time responses of the system (5.1) using integral sliding mode controller (5.25) with $\beta = 0.0139$, $\varepsilon = 0.001$, $q = 0.5$, $A = 400$, $\mu = 0.1$, $m_1 = 0.5$, $k_1 = 0.54$, $m_2 = 0.5$, $\gamma = 0.02$, $k_2 = 0.5$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\sigma_3 = 0.4$, $\sigma_4 = 0.2$ and the initial value $x(0) = (3, 2, 5, 4)$.

The sliding variable $s(t)$ is shown as

$$s(t) = x_3(t) - 5 - \int_0^t G \sum_{i=1}^{2} \lambda_i(x_1(\tau)) \left(A_i + BK_i \left(\sum_{i=1}^{2} \lambda_i(x_1(\tau)) Y_i \right)^{-1} \right) x(\tau)\, d\tau.$$

With $\varrho = 0.1$, the sliding mode controller $u(t)$ is shown as

$$u(t) = (\lambda_1(t)K_1 + \lambda_2(t)K_2)(\lambda_1(t)Y_1 + \lambda_2(t)Y_2)^{-1}x(t) - s(t) - 0.1\frac{s(t)}{\|s(t)\| + 0.005}.$$

It is obvious that the state variables $(x_1(t), x_2(t), x_3(t), x_4(t))$ of the closed-loop system gradually tend to zero as time changes in Figure 3(a). That is to say, it is asymptotically mean square stable. Figure 3(b) describes the time response of sliding mode controller. From biological point of view, epidemics can be contained around the endemic equilibrium point in short order and prevented with taking effect advantage of this controller in theory. This fully demonstrates the feasibility of the sliding mode controller (5.25) in Theorem 5.5.

7. CONCLUSION

In this paper, it is studied that a stochastic epidemic model with alert factors and varying population size. The dynamic behavior and the method of epidemic control are analyzed. The existence and uniqueness of global positive solution, and the existence of condition of ergodic distribution are proved. Especially, at the endemic equilibrium P^* , the dimension reduction is realized using stochastic central manifold and stochastic average method. The results show that system (2.2) is unstable at P^* when $\frac{\mu_2}{\mu_4} > \frac{1}{2}$. Meanwhile, a Hopf bifurcation will occur at r^*. Taking μ_3 as a bifurcation parameter, it is further obtained that the position where Hopf-bifurcation occurs increases with the increase of μ_3. Next, T-S fuzzy control is adopted to make the system stable. We construct a sliding surface and prove that the sliding motion is asymptotically mean square stable. More than anything, an integral sliding mode controller with Non-parallel compensation is designed to stabilize the system to a sufficient small neighborhood around P^*, which implies it is theoretically effective in preventing outbreaks.

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