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DECAY IN FULL VON KÁRMÁN BEAM WITH TEMPERATURE AND MICROTEMPERATURES EFFECTS

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Abstract. In this article we derive the equations that constitute the mathematical model of the full von Kármán beam with temperature and microtemperatures effects. The nonlinear governing equations are derived by using Hamilton principle in the framework of Euler–Bernoulli beam theory. Under quite general assumptions on nonlinear damping function acting on the transversal component and based on nonlinear semigroups and the theory of monotone operators, we establish existence and uniqueness of weak and strong solutions to the derived problem. Then using the multiplier method, we show that solutions decay exponentially. Finally we consider the case of zero thermal conductivity and we show that the dissipation given only by the microtemperatures is strong enough to produce exponential stability.

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1. Introduction

The modeling of elastic material models with temperature and microtemperature effects has been increasing interest in recent years due to their nice mathematical and physical properties which are required in several engineering applications and nanofabrication activities. Thus, the theory of thermoelasticity with microtemperatures has been the subject of several studies. Without trying to be exhaustive, let us refer to some studies carried out with microtemperatures applied to different kinds of materials (elastic, plastic, microfluids, hyperthermia, etc.) [1–4, 11, 17, 23–26, 33, 35, 36] among others.

To provide a more realistic formulation of beams which are widely used in several engineering and industrial applications, the influence of temperature and microtemperatures should be taken into account. In addition to microdeformations of the beam, the microelements of the continuum possess microtemperatures which reflect the variation of the temperature within a microelement. The von Kármán model is a second-order approximation of geometrically exact beam theory, accurately reproducing the bending-stretching coupling arising from bending displacements in the order of the thickness. This is due to the fact that the beam is stretchable implying the existence of nonlinear terms in the equations describing the motion. Since von Kármán beams are used frequently in nanotechnologies, we derive in this paper new governing equations for this beam including the coupled effects

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of temperature with microtemperatures. Hence the derived model becomes more adequate to study important problems related to size effects and nanotechnologies and can be employed in various applications, including nonlinear vibratory behavior of nano beams. Moreover, experimental data for the silicone rubber containing spherical aluminium particles and for human blood were found to conform closely to the predicted theoretical thermal conductivity.

Lagnese and Leugering [27] proposed the following coupled two hyperbolic equations to represent the vibration of a one-dimensional von Kármán beam

$$\begin{aligned} \omega_{tt} - d_1[(u_x + \frac{1}{2}\omega_x^2)\omega_x]_x + d_2\omega_{xxxx} = 0, \text{ in } (0,L)\times(0,T), \\ u_{tt} - d_1[u_x + \frac{1}{2}\omega_x^2]_x = 0, \text{ in } (0,L)\times(0,T), \end{aligned} \tag{1.1}$$

where $\omega(x,t)$ is the transverse displacement, $u(x,t)$ is the longitudinal displacement of a generic point, d_1 and d_2 are material constants.

The nonlinear full von Kármán system coupled with temperature has been derived in [6, 7]. Then many results were provided concerning the solutions to von Kármán's systems through different thermal laws (without thermodynamics derivation) in different dimensional setting (see *e.g.* [8–10, 13, 15, 21, 22, 29, 31, 32] and others).

We may think that the thermal conduction is the only realistic phenomenon to take into account in full von Kármán beams. However, the research conducted in the development of high technologies and in nanotechnologies, confirmed that the field of microtemperatures in solids cannot be ignored. So, the obvious question is: what happens when the microtemperatures effect is considered with the thermal effect in full von Kármán beams?

In this paper, we include both the temperature and the microtemperatures effects into the full von Kármán beams governing equations. The concept of microtemperatures has been introduced for the first time by Wozniak in [37]. Grot [20] extended the thermodynamics of a continuum with microstructure to introduce the microtemperatures concept. He developed a theory of thermodynamics for elastic materials with microstructure whose microelements, in addition to microdeformations, possess microtemperatures in the context of the classical theory. The Clausius-Duhem inequality is modified to include microtemperatures, and the first-order moment of the energy equations are added to the usual balance laws of a continuum with microtemperatures. Ieşan and Quintanilla [23–25] were the first to introduce the concept of microtemperatures in thermoelasticity.

The first goal of this paper is to derive the von Kármán beam equation with temperature and microtemperatures effects. We limit our attention to the one-dimensional model otherwise we will encounters difficulties in proving the well-posedness result since the nonlinear terms of the derived model are not bounded in the space of finite energy (due to the lack of appropriate Sobolev's embeddings). In fact, the nonlinear forcing modeled by $[(\omega_x^2)\omega_x]_x$ is neither bounded nor locally Lipschitz on the space $\mathscr{H}$ of finite energy defined here by (3.5). This follows from the fact that the map $\omega \to [(\omega_x^2)\omega_x]_x$ is not bounded from $H^2(\Omega)$ to $L^2(\Omega)$, unless the dimension of the domain Ω is equal to one. Moreover, the restoring force is therefore supercritical on $\mathscr{H}$ if the dimension of Ω is strictly greater than one. This fact will be a source of difficulties in the generation of semigroup.

The second goal of this paper is to show that the C_0-semigroup associated with the nonlinear derived equations in one dimensional setting is well-posed and exponentially stable. Our intention is to show how the dissipation mechanism, due to temperature and microtemperatures effects, implies the exponential stability.

The third objective of this article is to study the case where the dissipation of the derived equations is due only at microtemperatures. It is natural to think in this case that the system lacks exponential stability unless some other damping mechanism is added. However, in this work we establish a contrary result. In this case, we show that the single dissipation due to microtemperatures is strong enough to exponentially stabilize the system without adding damping terms. So far, this case has only been considered for porous thermoelastic materials in [4, 17, 26, 35, 36].

The sections of this paper are organized as follows. In Section 2, we derive the equations that constitute the mathematical model of the one-dimensional full von Kármán beam with temperature and microtemperatures effects. In Section 3, we shall prove that the derived problem is well-posed and in Section 4 is exponentially stable. In Section 5, we show that the single dissipation due to microtemperatures without thermal effects stabilizes the system exponentially. These results are new and improve recent results from the literature.

2. Derivation of the model

In this section, we start by reviewing the main steps involved in the derivation of the governing PDEs for the von Kármán beam. We consider the planar motion of a uniform prismatic beam of length L. Inspired from [6, 27, 28], we consider the planar motion of a beam that occupies, in the reference position, the region

$$\Omega = \Big\{(x, y, z),\ 0 \le x \le L,\ -1 \le y \le 1,\ -\frac{h}{2} \le z \le \frac{h}{2}\Big\},$$

where we suppose that $h \ll L$.

In view of the assumptions of the Euler-Bernoulli beams theory and by following [6, 27, 28], we assume that the motion and the deformation of the beam occurs in the $x - z$ plane. Upon denoting the displacement components along the $x-$, $y-$ and $z-$directions by u_1, u_2, and u_3, respectively, the displacement field can be written in view of the above assumptions

$$u_1(x, z, t) = u(x, t) - z\omega_x(x, t), \quad u_2(x, z, t) = 0, \quad u_3(x, z, t) = \omega(x, t), \quad x \in [0, L], \tag{2.1}$$

where the quantities u and ω will represent, respectively, longitudinal and transversal displacement.

In accordance, the von Kármán type nonlinear strain–displacement relationship takes the following form

$$\varepsilon_{11} = u_x + \frac{1}{2}\omega_x^2 - z\omega_{xx}. \tag{2.2}$$

In order to account for the thermal effects, we suppose that the beam, at a generic point (x, z), is subject to an unknown temperature field $\tau(x, z, t)$. Let τ_0 be the reference-temperature constant value, that is to say, the uniform temperature at which the natural (unstressed) configuration is attained. In the following, we note T the difference between the temperature τ and its reference constant value τ_0, *i.e.*

$$T(x, z, t) = \tau(x, z, t) - \tau_0.$$

In this case, the stress tensor has only one component different from zero, which is expressed in the following way

$$\sigma_{11} = E\varepsilon_{11} - E\alpha_0 T, \tag{2.3}$$

where σ_{11} denotes the normal component of the Cauchy stress, ε_{11} is the normal strain, E is the Young's modulus and α_0 is a positive constant representing the coefficient of thermal expansion. Integrating (2.3) over the beam's cross section area A yields

$$N_{11} = \int_{-h/2}^{h/2} \sigma_{11} b\,\mathrm{d}z = EA(u_x + \frac{1}{2}\omega_x^2 - \alpha_0\theta), \tag{2.4}$$

where b is the width of the beam, $A = bh$ is the area of a cross section and θ defined by

$$\theta(x, t) = \frac{1}{h}\int_{-h/2}^{h/2} T(x, z, t)\,\mathrm{d}z, \tag{2.5}$$

represents the mean value of temperature variation.

Multiplying (2.3) by z and integrating the result over the area of a cross section leads

$$M_{11} = \int_{-h/2}^{h/2} z\sigma_{11} b\mathrm{d}z = -EI(\omega_{xx} + \alpha_0\Theta), \tag{2.6}$$

where $I = \int_{-h/2}^{h/2} bz^2\mathrm{d}z = \frac{bh^3}{12}$ is the moment of inertia, EI is known as the flexural rigidity and Θ defined by

$$\Theta(x,t) = \frac{12}{h^3}\int_{-h/2}^{h/2} zT(x,z,t)\mathrm{d}z \tag{2.7}$$

represents the (normalized) first moment of temperature. Thus, the full deformation energy of the beam takes the following form:

$$U(t) = \frac{1}{2}\int_0^L \int_{-h/2}^{h/2} \boldsymbol{\sigma} \,:\, \boldsymbol{\varepsilon}\; b\mathrm{d}z\mathrm{d}x$$

where in our case $\boldsymbol{\sigma} \,:\, \boldsymbol{\varepsilon} = \sum_{i,j}\sigma_{ij}\varepsilon_{ij} = \sigma_{11}\varepsilon_{11}$. By virtue of (2.2) and (2.3), we have

$$U(t) = \frac{1}{2}\int_0^L \int_{-h/2}^{h/2} \Big[E(u_x + \frac{1}{2}\omega_x^2 - z\omega_{xx})^2 - E\alpha_0(u_x + \frac{1}{2}\omega_x^2 - z\omega_{xx})T\Big] b\mathrm{d}z\mathrm{d}x.$$

After computation of integrals with regard to z and a few simple transformations one gets

$$U(t) = \frac{1}{2}\int_0^L \Big[EI\omega_{xx}^2 + EA(u_x + \frac{1}{2}\omega_x^2)^2 - 2Eb\alpha_0\theta(u_x + \frac{1}{2}\omega_x^2) - 2Eb\alpha_0\Theta\omega_{xx}\Big]\mathrm{d}x.$$

Kinetic beam energy vibration K takes the following form [19]

$$K(t) = \frac{1}{2}\int_0^L (\rho A u_t^2 + \rho A\omega_t^2 + \rho I\omega_{xt}^2)\mathrm{d}x,$$

where ρ is the mass density of the beam in its reference configuration.

On the other hand, we assume that on the boundary, the displacement is only horizontal, which implies

$$\omega(x,\cdot) = \omega_x(x,\cdot) = 0, \quad \text{for } x = 0,\ x = L.$$

External work associated with the forces, moments and stress on the boundary plane is governed by the formula

$$W(t) = \int_0^L \Big(F(x,t)\omega + G(x,t)u\Big)\mathrm{d}x$$

where we have neglected the boundary terms expressing all the resulting forms of normal stress σ_{11}.

Following Hamilton's principle for continuous systems, we have to introduce the dynamical variation

$$\boldsymbol{\delta}\int_0^T \big(K(t) - U(t) + W(t)\big)\mathrm{d}t = 0, \tag{2.8}$$

where $\boldsymbol{\delta}$ is the first variation with respect to u and ω. The first variation of U, K and W are given, respectively, by

$$\begin{aligned}\boldsymbol{\delta}U(t) &= \int_0^L \int_{-h/2}^{h/2} \sigma_{11}\boldsymbol{\delta}\varepsilon_{11} b\mathrm{d}z\mathrm{d}x = \int_0^L \Big(N_{11xx}(\boldsymbol{\delta}u_x + \omega_x\boldsymbol{\delta}\omega_x) - M_{11xx}\boldsymbol{\delta}\omega_{xx}\Big)\mathrm{d}x,\\ \boldsymbol{\delta}K(t) &= \int_0^L \Big(\rho A(u_t\boldsymbol{\delta}u_t + \omega_t\boldsymbol{\delta}\omega_t) + \rho I\omega_{xt}\boldsymbol{\delta}\omega_{xt}\Big)\mathrm{d}x,\\ \boldsymbol{\delta}W(t) &= \int_0^L \rho A(F\boldsymbol{\delta}\omega + G\boldsymbol{\delta}u)\mathrm{d}x.\end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.8), integrating by parts and grouping terms by $\boldsymbol{\delta}\omega$ and $\boldsymbol{\delta}u$ lead to the following partial differential equations

$$\begin{aligned}\int_0^T \int_{-h/2}^{h/2} \Big(\rho A\omega_{tt} - \rho I\omega_{xxtt} - M_{11xx} - (N_{11}\omega_x)_x - F\Big)\boldsymbol{\delta}u\mathrm{d}x\mathrm{d}t = 0,\\ \int_0^T \int_{-h/2}^{h/2} \Big(\rho A u_{tt} - N_{11x} - G\Big)\boldsymbol{\delta}u\mathrm{d}x\mathrm{d}t = 0.\end{aligned} \tag{2.10}$$

Substituting (2.4) and (2.6) into (2.10), yields the following resulting equations of motion

$$\begin{aligned}\rho A\omega_{tt} - \rho I\omega_{xxtt} + EI(\omega_{xx} + \alpha_0\Theta)_{xx} - EA\Big((u_x + \frac{1}{2}\omega_x^2 - \alpha_0\theta)\omega_x\Big)_x = F,\\ \rho A u_{tt} - EA\Big(u_x + \frac{1}{2}\omega_x^2 - \alpha_0\theta\Big)_x = G,\end{aligned} \tag{2.11}$$

with appropriate boundary conditions and initial data.

The equations (2.11) are the same as those derived in [6] and in [27, 28] (except that the effects of temperature are not taken into account).

Now we are interested in the thermodynamic counterpart of the problem.

In the following, the second law of thermodynamics is modified in order to introduce the concept of microtemperatures and the first-order moment of the energy is added to the usual balance law. In one dimensional setting, we have two principles of thermodynamics [20, 23, 37]:

(i) the local form of the second law of thermodynamics

$$\rho S_t - (\frac{1}{T}q + \frac{1}{T}Q\upsilon)_x \geq 0, \tag{2.12}$$

(ii) the balance of first moment of energy

$$\rho\eta_t = Q_x + q - \varsigma, \tag{2.13}$$

where υ denotes the microtemperatures, q is the heat flux, ς is the mean heat flux, Q is the first heat flux moment, η is the first moment of energy vector and S is the entropy deviation from the environmental entropy (per unit mass). Without loss of generality, we have neglected the external heat source and the first moment of the heat source from (2.12) and (2.13).

According to [20, 23, 37], the linear constitutive equation for η is given by

$$\rho\eta = -b_1\,\upsilon, \tag{2.14}$$

where b_1 is a positive constant. According to [12], the linear constitutive equation for S is given by

$$\rho S = \frac{E\alpha_0}{1-\nu}\varepsilon_{11} + \frac{\rho c_v}{\tau_0}T \tag{2.15}$$

where $\nu \in (0, 1/2)$ is the (dimensionless) Poisson ratio and $c_v > 0$ is the beam heat capacity per unit mass at constant strain. By substituting (2.2) into (2.15), and retaining only the linear terms, one can obtain

$$\rho S = \frac{E\alpha_0}{1-\nu}u_x + \frac{\rho c_v}{\tau_0}T. \tag{2.16}$$

In thermoelasticity, the heat equation can be deduced directly from the entropy balance equation by using the Gibbs relation [18]

$$\rho(\mathcal{E}_t - TS_t) - \mathcal{P} = 0,$$

where $\mathcal{E}$ is the internal energy density per unit mass and $\mathcal{P} = \sum_{i,j}\sigma_{ij}\varepsilon_{ijt}$. In fact, this relation allows us to rewrite the internal energy balance equation,

$$\rho\mathcal{E}_t - \mathcal{P} - q_x = 0,$$

in the following form, also known as entropy balance,

$$\rho TS_t = q_x, \tag{2.17}$$

where we have neglected, from the last two equalities, the external heat source supplied within the beam by internal sources or dissipators. The linear form of (2.17) gives the balance equation of the energy

$$\rho\tau_0 S_t = q_x. \tag{2.18}$$

By combining (2.12) and (2.18), we obtain after some manipulations the following linearized dissipation inequality [20]

$$\frac{1}{\tau_0}qT_x - Q\upsilon_x + (q-\varsigma)\upsilon \geq 0. \tag{2.19}$$

In view of (2.19), the linear approximations for q, ς and Q are given by [20]

$$q = kT_x + k_1\upsilon, \quad \varsigma = (k-k_3)T_x + (k_1-k_2)\upsilon, \quad Q = -(k_4+k_5+k_6)\upsilon_x, \tag{2.20}$$

where the constitutive coefficients k and k_i $(i = 1, \dots, 6)$ satisfy the inequalities [23]

$$k \geq 0, \quad 3k_4+k_5+k_6 \geq 0, \quad k_5+k_6 \geq 0, \quad k_6-k_5 \geq 0, \quad \frac{4}{\tau_0}k\,k_2 - \left(\frac{1}{\tau_0}k_1 + k_3\right)^2 \geq 0. \tag{2.21}$$

Substituting (2.16) (resp. (2.14)) into (2.18) (resp. (2.13)) together with (2.20) and integrating the result over the interval $[-\frac{h}{2}, \frac{h}{2}]$, we get the evolution equation of temperature (resp. microtemperatures):

$$\begin{aligned} &\rho c_v\theta_t - k\theta_{xx} - k_1\vartheta_x + \frac{\tau_0 E\alpha_0}{1-\nu}u_{xt} = 0, \\ &b_1\vartheta_t - (k_4+k_5+k_6)\vartheta_{xx} + k_2\vartheta + k_3\theta_x = 0, \end{aligned} \tag{2.22}$$

where θ is defined by (2.5) and

$$\vartheta(x,t)=\frac{1}{h}\int_{-h/2}^{h/2}\upsilon(x,z,t)\mathrm{d}z, \tag{2.23}$$

represents the mean value of microtemperatures variation.

Equations (2.11) together with (2.22), give the governing equations of full von Kármán beam with temperature and microtemperatures

$$\begin{aligned}\rho A\omega_{tt}-\rho I\omega_{xxtt}+EI(\omega_{xxxx}+\alpha_0\Theta_{xx})-EA\Big((u_x+\frac{1}{2}\omega_x^2-\alpha_0\theta)\omega_x\Big)_x&=F,\\ \rho Au_{tt}-EA\Big(u_x+\frac{1}{2}\omega_x^2-\alpha_0\theta\Big)_x&=G,\\ \rho c_v\theta_t-k\theta_{xx}-k_1\vartheta_x+\frac{\tau_0E\alpha_0}{1-\nu}u_{xt}&=0,\\ b_1\vartheta_t-(k_4+k_5+k_6)\vartheta_{xx}+k_2\vartheta+k_3\theta_x&=0.\end{aligned} \tag{2.24}$$

After this complete procedure, we will take into account some considerations to facilitate the mathematical analysis in the following sections:

(i) We neglect the effects of the first moment of temperature in the axial direction, *i.e.* $\Theta_{xx}=0$;

(ii) In the spirit of the linear context used in the previous thermodynamics process (see (2.16) and (2.18)), the nonlinear term $(\theta\omega_x)_x$ in $(2.24)_1$ becomes linear in the form $\theta_0\omega_{xx}$, where θ_0 is the reference constant value of the mean temperature variation;

(iii) We neglect the external body forces, *i.e.* $F=G=0$ and the rotational inertia term, *i.e.* $\rho I\omega_{xxtt}=0$;

(iv) To control the component ω_t, we add to $(2.24)_1$ the damping function $g(\omega_t)$.

Under the above considerations (i)–(iv), we obtain the following von Kármán beam equations with temperature and microtemperatures in $(0,L)\times\mathbb{R}^+$,

$$\begin{aligned}\omega_{tt}-d_1\Big((u_x+\frac{1}{2}\omega_x^2)\omega_x\Big)_x+\alpha\omega_{xx}+d_2\omega_{xxxx}+g(\omega_t)&=0,\\ u_{tt}-d_1[u_x+\frac{1}{2}\omega_x^2]_x+\beta\theta_x&=0,\\ c\theta_t-\kappa\theta_{xx}-\kappa_1\vartheta_x+\beta u_{xt}&=0,\\ b\vartheta_t-\kappa_2\vartheta_{xx}+\kappa_3\theta_x+\kappa_4\vartheta&=0,\end{aligned} \tag{2.25}$$

with the following boundary conditions

$$\omega(x,t)=\omega_x(x,t)=u_x(x,t)=\theta(x,t)=\vartheta_x(x,t)=0 \text{ on } x=0,L,\ t>0, \tag{2.26}$$

and initial conditions

$$\begin{aligned}&\omega(x,0)=\omega_0(x),\quad \omega_t(x,0)=\omega_1(x),\quad u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\\ &\theta(x,0)=\theta_0(x),\quad \vartheta(x,0)=\vartheta_0(x),\quad x\in(0,L).\end{aligned} \tag{2.27}$$

The above physical constants are positive and given by

$$\begin{aligned}&d_1=\frac{E}{\rho},\quad d_2=\frac{EI}{\rho A},\quad \alpha=\alpha_0\theta_0d_1,\quad \beta=\frac{E\alpha_0}{\rho},\quad \varpi=\frac{1-\nu}{\tau_0\rho},\quad c=\rho c_v\varpi,\quad b=b_1\varpi\tau_0,\\ &\kappa=k\varpi,\quad \kappa_1=k_1\varpi,\quad \kappa_2=(k_4+k_5+k_6)\varpi\tau_0,\quad \kappa_3=k_3\varpi\tau_0,\quad \kappa_4=k_2\varpi\tau_0.\end{aligned} \tag{2.28}$$

We assume that the damping function $g:\mathbb{R}\to\mathbb{R}$ is continuous, monotone increasing with $g(0)=0$ and there exist positive constants m and M such that $m\leq M$

$$m\leqslant g'(s)\leqslant M,\quad \forall s\in\mathbb{R} \tag{2.29}$$

which gives as the monotonicity propriety

$$m|u-v|\leqslant|g(u)-g(v)|\leqslant M|u-v|,\quad \forall u,v\in\mathbb{R}, \tag{2.30}$$

and

$$g(u)\leqslant M|u|,\quad \forall u\in\mathbb{R}. \tag{2.31}$$

The dissipation inequalities (2.21) become

$$\kappa\geq 0,\quad 4\kappa\kappa_4-(\kappa_1+\kappa_3)^2\geq 0. \tag{2.32}$$

Note that the last condition in (2.32) implies that

$$\kappa\|\theta_x\|^2+\kappa_4\|\vartheta\|^2+(\kappa_1+\kappa_3)\int_0^L\theta_x\vartheta\mathrm{d}x\geq 0. \tag{2.33}$$

Remark 2.1. (i) Note that the microtemperatures evolution equation is not coupled with any elastic deformation as in the theory of thermoelasticity with microtemperatures (see Eq. (11) of [23]), in the theory of thermo-elastic-plasticity flow with microtemperatures (see Eq. (4.1) of [3]) and in others (see *e.g.* [23–25]).

(ii) Meanwhile, from $(2.25)_2$ and the boundary condition $(2.26)_{1-4}$, we easily verify that

$$\frac{d^2}{\mathrm{d}t^2}\int_0^L u(x,t)\mathrm{d}x=0,\quad \forall t\geq 0. \tag{2.34}$$

By solving (2.34) and using the initial data of u, we arrive at

$$\int_0^L u(x,t)\mathrm{d}x=t\int_0^L u_1(x)\mathrm{d}x+\int_0^L u_0(x)\mathrm{d}x,\quad \forall t\geq 0,$$

consequently, if we set

$$\hat{u}(x,t)=u(x,t)-\frac{t}{L}\int_0^L u_1(x)\mathrm{d}x-\frac{1}{L}\int_0^L u_0(x)\mathrm{d}x,\quad \forall t\geq 0,\ x\in[0,L],$$

we find

$$\int_0^L \hat{u}(x,t)\mathrm{d}x=0,\quad \forall t\geq 0.$$

Now, from $(2.25)_4$ and the boundary condition $(2.26)_5$, we easily verify that

$$b\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\vartheta(x,t)\mathrm{d}x+\kappa_4\int_0^L\vartheta(x,t)\mathrm{d}x=0,\quad \forall t\geq 0,$$

thus

$$\int_0^L \vartheta(x,t)\mathrm{d}x = \Big(\int_0^L \vartheta_0(x)\mathrm{d}x\Big)e^{-\frac{\kappa_4}{b}t}.$$

So, if we put

$$\hat{\vartheta}(x,t) = \vartheta(x,t) - \frac{1}{L}\Big(\int_0^L \vartheta_0(x)\mathrm{d}x\Big)e^{-\frac{\kappa_4}{b}t}, \quad \forall t \geq 0,\ x \in [0,L],$$

we arrive at

$$\int_0^L \hat{\vartheta}(x,t)\mathrm{d}x = 0, \quad \forall t \geq 0.$$

Clearly, the use of Poincaré's inequality for $\hat{u}$ and $\hat{\vartheta}$ is justified. In addition, $(\omega, \hat{u}, \omega_t, \hat{u}_t, \theta, \hat{\vartheta})$ satisfies (2.25)–(2.27) with initial data

$$\hat{u}_0 = u_0 - \frac{1}{L}\int_0^L u_0(x)\mathrm{d}x, \quad \hat{u}_1 = u_1 - \frac{1}{L}\int_0^L u_1(x)\mathrm{d}x, \quad \hat{\vartheta}_0 = \vartheta_0 - \frac{1}{L}\int_0^L \vartheta_0(x)\mathrm{d}x.$$

In what follows we will work with $\hat{u}$ and $\hat{\vartheta}$ but, for simplicity of notations, we write u and ϑ instead of $\hat{u}$ and $\hat{\vartheta}$.

3. Well-posedness

In this section, we shall study the well-posedness of the problem (2.25)–(2.27) by using the nonlinear semigroups theory and monotone operators [5, 14]. Throughout this paper, we use the standard Lebesgue space $L^2(0,L)$ and the Sobolev spaces $H^m(0,L) = W^{m,2}(0,L)$ $(1 \leq m \leq \infty)$ with their usual scalar products and norms. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the L^2- inner product and L^2- norm, respectively.

For convenience, we denote by the space $H^{-2}(0,L)$ the dual of $H_0^2(0,L)$, where

$$H_0^2(0,L) = \left\{\omega \in H^2(0,L) \mid \omega = \omega_x = 0 \text{ on } 0, L\right\}.$$

Obviously, $H_0^2(0,L) \hookrightarrow L^2(0,L) \hookrightarrow H^{-2}(0,L)$. Define the operator $A : H_0^2(0,L) \to H^{-2}(0,L)$ by

$$A\omega = \Delta^2\omega = \omega_{xxxx}, \tag{3.1}$$

for every $\omega \in \mathcal{D}(A) = \left\{\omega \in L^2(0,L) \mid \omega_{xxxx} \in L^2(0,L)\right\} = H^4(0,L) \cap H_0^2(0,L)$. The operator A is self-adjoint and strictly positive on $H_0^2(0,L)$. Hence, A is an isomorphism from $H_0^2(0,L)$ onto $H^{-2}(0,L)$ and from $H^2(0,L) \cap H_0^1(0,L)$ onto $L^2(0,L)$. We recall that $\lambda_1 > 0$ is the first eigenvalue of the bi-harmonic operator $A = \Delta^2$ in $H_0^2(0,L)$. Indeed, since λ_1 satisfies

$$\|\omega\|^2 \leq \lambda_1^{-1}\|\omega_x\|^2, \quad \forall \omega \in H_0^1(0,L), \tag{3.2}$$

there exist proper constants $\lambda_2 > 0$ and $\lambda_3 > 0$ will always refer to the following embedding inequalities,

$$\|\omega\|^2 \leq \lambda_2^{-1}\|\omega_{xx}\|^2, \quad \|\omega_x\|^2 \leq \lambda_3^{-1}\|\omega_{xx}\|^2, \tag{3.3}$$

such that

$$\lambda_3 d_2 \geq \alpha. \tag{3.4}$$

Introducing the new variables $\varphi = \omega_t$ and $\psi = u_t$, we define the Hilbert space

$$\mathscr{H} := H_0^2(0,L) \times H_*^1(0,L) \times L^2(0,L) \times L_*^2(0,L) \times L^2(0,L) \times L_*^2(0,L), \tag{3.5}$$

where

$$L_*^2(0,L) = \Big\{ w \in L^2(0,L) \ : \ \int_0^L w(s)ds = 0 \Big\}, \quad H_*^1(0,L) = H^1(0,L) \cap L_*^2(0,L).$$

The Hilbert space is equipped with the inner product

$$\langle U, \widetilde{U} \rangle_{\mathscr{H}} = \langle \varphi, \widetilde{\varphi} \rangle + \langle \psi, \widetilde{\psi} \rangle + d_2 \langle w_{xx}, \widetilde{w}_{xx} \rangle - \alpha \langle w_x, \widetilde{w}_x \rangle + d_1 \langle u_x, \widetilde{u}_x \rangle + c \langle \theta, \widetilde{\theta} \rangle + b \langle \vartheta, \widetilde{\vartheta} \rangle \tag{3.6}$$

for $U = (\omega, u, \varphi, \psi, \theta, \vartheta)$ and $\widetilde{U} = (\widetilde{\omega}, \widetilde{u}, \widetilde{\varphi}, \widetilde{\psi}, \widetilde{\theta}, \widetilde{\vartheta})$ and with the norm

$$\|U\|_{\mathscr{H}}^2 = \|\varphi\|^2 + \|\psi\|^2 + d_2 \|\omega_{xx}\|^2 - \alpha \|\omega_x\|^2 + d_1 \|u_x\|^2 + c \|\theta\|^2 + b \|\vartheta\|^2. \tag{3.7}$$

Thanks to (3.4), we get the above norm is nonnegative.

The system (2.25)–(2.27) can be rewritten as an equivalent Cauchy problem

$$\frac{dU}{dt} = \mathscr{A} U + \mathcal{F}(U), \quad t > 0, \quad U(0) := U_0 = (\omega_0, u_0, \omega_1, u_1, \theta_0, \vartheta_0) \in \mathscr{H}, \tag{3.8}$$

where $\mathscr{A} = \mathcal{A} + \mathcal{B}$, $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathscr{H} \to \mathscr{H}$, $\mathcal{B} : \mathscr{H} \to \mathscr{H}$ and $\mathcal{F}$ are defined by

$$\mathcal{A}U = \begin{pmatrix} \varphi \\ \psi \\ -d_2 \omega_{xxxx} - \alpha \omega_{xx} \\ d_1 u_{xx} - \beta \theta_x \\ c^{-1} \Big(\kappa \theta_{xx} + \kappa_1 \vartheta_x - \beta \psi_x \Big) \\ b^{-1} \Big(\kappa_2 \vartheta_{xx} - \kappa_3 \theta_x - \kappa_4 \vartheta \Big) \end{pmatrix}, \ \mathcal{B}U = \begin{pmatrix} 0 \\ 0 \\ -g(\varphi) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathcal{F}(U) = \begin{pmatrix} 0 \\ 0 \\ d_1 [(u_x + \frac{1}{2}\omega_x^2)\omega_x]_x \\ \frac{d_1}{2}[\omega_x^2]_x \\ 0 \\ 0 \end{pmatrix} \tag{3.9}$$

with the domains

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= [H^4(0,L) \cap H_0^2(0,L)] \times [H^2(0,L) \cap H_*^1(0,L)] \times H_0^2(0,L) \times H_*^1(0,L) \times [H^2(0,L) \cap H_0^1(0,L)] \\ &\quad \times [H^2(0,L) \cap H_*^1(0,L)], \\ \mathcal{D}(\mathcal{B}) &= \mathscr{H} \ \text{ and } \ \mathcal{D}(\mathscr{A}) = \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A}). \end{aligned}$$

Lemma 3.1. *Suppose that assumptions (2.29)–(2.32) and (3.4) hold. Then the operator $-\mathcal{A}$ is maximal monotone in $\mathscr{H}$.*

Proof. For the first step, we prove that the operator $-\mathcal{A}$ is monotone. Let us denote $U = (\omega, u, \varphi, \psi, \theta, \vartheta) \in \mathcal{D}(\mathcal{A})$. Using integration by parts and the boundary conditions, we have

$$\langle -\mathcal{A}U, U \rangle_{\mathscr{H}} = \langle d_2 \omega_{xxxx} + \alpha \omega_{xx}, \varphi \rangle - d_1 \langle u_{xx}, \psi \rangle + \beta \langle \theta_x, \psi \rangle - d_2 \langle \varphi_{xx}, \omega_{xx} \rangle - d_1 \langle \psi_x, u_x \rangle$$

$$
\begin{aligned}
&+\kappa\|\theta_x\|^2+\beta\langle\psi_x,\theta\rangle+\alpha\langle\varphi_x,w_x\rangle+\kappa_2\|\vartheta_x\|^2+\kappa_4\|\vartheta\|^2+(\kappa_1+\kappa_3)\langle\theta_x,\vartheta\rangle\\
&=\kappa_2\|\vartheta_x\|^2+\kappa\|\theta_x\|^2+\kappa_4\|\vartheta\|^2+(\kappa_1+\kappa_3)\int_0^L\theta_x\vartheta\mathrm{d}x.
\end{aligned}
\tag{3.10}
$$

From (2.33), we conclude that $-\mathcal{A}$ is monotone in $\mathscr{H}$.

In order to prove that $-\mathcal{A}$ is maximal monotone, we need to prove that Range $(\mathcal{I}-\mathcal{A})=\mathscr{H}$. We must prove that

$$
U-\mathcal{A}U=U^*,
$$

has a solution $U=(\omega,u,\varphi,\psi,\theta,\vartheta)\in\mathcal{D}(\mathcal{A})$ for any $U^*=(\omega^*,u^*,\varphi^*,\psi^*,\theta^*,\vartheta^*)\in\mathscr{H}$. This equation leads to the system

$$
\begin{cases}
\omega-\varphi=\omega^*\in H_0^2(0,L),\\
u-\psi=u^*\in H_*^1(0,L),\\
\varphi+d_2\omega_{xxxx}+\alpha\omega_{xx}=\varphi^*\in L^2(0,L),\\
\psi-d_1u_{xx}+\beta\theta_x=\psi^*\in L_*^2(0,L),\\
\theta-c^{-1}\Big(\kappa\theta_{xx}+\kappa_1\vartheta_x-\beta\psi_x\Big)=\theta^*\in L^2(0,L),\\
\vartheta-b^{-1}\Big(\kappa_2\vartheta_{xx}-\kappa_3\theta_x-\kappa_4\vartheta\Big)=\vartheta^*\in L_*^2(0,L).
\end{cases}
\tag{3.11}
$$

Substituting $(3.11)_{1,2}$ into $(3.11)_{3-5}$ we obtain

$$
\begin{cases}
\omega+d_2\omega_{xxxx}+\alpha\omega_{xx}=\varphi^*+\omega^*,\\
u-d_1u_{xx}+\beta\theta_x=\psi^*+u^*,\\
c\theta-\kappa\theta_{xx}-\kappa_1\vartheta_x+\beta u_x=c\theta^*+\beta u_x^*,\\
b\vartheta-\kappa_2\vartheta_{xx}+\kappa_3\theta_x+\kappa_4\vartheta=b\vartheta^*.
\end{cases}
\tag{3.12}
$$

We denote the Hilbert space $\mathcal{H}=H_0^2(0,L)\times[H_*^1(0,L)]\times[H_0^1(0,L)]\times[H_*^1(0,L)]$, equipped with the norm

$$
\|W\|_{\mathcal{H}}^2=\|\omega\|^2+\|\omega_{xx}\|^2+\|u\|^2+\|u_x\|^2+\|\theta\|^2+\|\theta_x\|^2+\|\vartheta\|^2+\|\vartheta_x\|^2,
$$

for any $W=(\omega,u,\theta,\vartheta)\in\mathcal{H}$. Consequently the problem (3.12) is equivalent to

$$
\mathscr{M}(W,\widetilde{W})=\mathcal{L}(\widetilde{W}),\quad\forall\ \widetilde{W}=(\widetilde{\omega},\widetilde{u},\widetilde{\theta},\widetilde{\vartheta})\in\mathcal{H},
$$

where the bilinear form $\mathscr{M}:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$, and the linear form $\mathcal{L}:\mathcal{H}\to\mathbb{R}$ are defined by

$$
\begin{aligned}
\mathscr{M}(W,\widetilde{W})&=\langle\omega,\widetilde{\omega}\rangle+\langle u,\widetilde{u}\rangle+d_2\langle\omega_{xx},\widetilde{\omega}_{xx}\rangle-\alpha\langle\omega_x,\widetilde{\omega}_x\rangle+d_1\langle u_x,\widetilde{u}_x\rangle+\beta\langle\theta_x,\widetilde{u}\rangle+c\langle\theta,\widetilde{\theta}\rangle\\
&\quad+\kappa\langle\theta_x,\widetilde{\theta}_x\rangle-\kappa_1\langle\vartheta_x,\widetilde{\theta}\rangle+\beta\langle u_x,\widetilde{\theta}\rangle+b\langle\vartheta,\widetilde{\vartheta}\rangle+\kappa_2\langle\vartheta_x,\widetilde{\vartheta}_x\rangle+\kappa_3\langle\theta_x,\widetilde{\vartheta}\rangle+\kappa_4\langle\vartheta,\widetilde{\vartheta}\rangle,\\
\mathcal{L}(\widetilde{W})&=\langle\varphi^*+\omega^*,\widetilde{\omega}\rangle+\langle\psi^*+u^*,\widetilde{u}\rangle+c\langle\theta^*,\widetilde{\theta}\rangle+\beta\langle u_x^*,\widetilde{\theta}\rangle+b\langle\vartheta^*,\widetilde{\vartheta}\rangle.
\end{aligned}
$$

Moreover, thanks to $(3.3)_2$, (3.4) and Young's inequality, we have

$$
\begin{aligned}
\mathscr{M}(W,W)&=\|\omega\|^2+\|u\|^2+d_2\|\omega_{xx}\|^2-\alpha\|\omega_x\|^2+d_1\|u_x\|^2+c\|\theta\|^2+b\|\vartheta\|^2\\
&\quad+\kappa\|\theta_x\|^2+\kappa_4\|\vartheta\|^2+(\kappa_1+\kappa_3)\langle\theta_x,\vartheta\rangle+\kappa_2\|\vartheta_x\|^2\\
&\geq\|\omega\|^2+\|u\|^2+(d_2-\alpha\lambda_3^{-1})\|\omega_{xx}\|^2+d_1\|u_x\|^2+c\|\theta\|^2+b\|\vartheta\|^2+\kappa_2\|\vartheta_x\|^2\\
&\quad+\frac{1}{2}(\kappa-\frac{(\kappa_1+\kappa_3)^2}{4\kappa_4})\|\theta_x\|^2+\frac{1}{2}(\kappa_4-\frac{(\kappa_1+\kappa_3)^2}{4\kappa})\|\vartheta\|^2.
\end{aligned}
$$

By assuming $4\kappa\kappa_4 - (\kappa_1+\kappa_3)^2 > 0$, we get, respectively,

$$\kappa - \frac{(\kappa_1+\kappa_3)^2}{4\kappa_4} > 0, \quad \kappa_4 - \frac{(\kappa_1+\kappa_3)^2}{4\kappa} > 0,$$

then there exists a positive constant C_1 such that for any $W = (\omega, u, \theta, \vartheta) \in \mathcal{H}$, we have

$$\mathscr{M}(W,W) \geq C_1 \|W\|_{\mathcal{H}}^2.$$

Thus, $\mathscr{M}$ is coercive. Similarly, we show that there exist positives constants C_2 and C_3 such that

$$|\mathscr{M}(W,\widetilde{W})| \leq C_2\|W\|_{\mathcal{H}}\|\widetilde{W}\|_{\mathcal{H}} \quad \text{and} \quad |\mathcal{L}(\widetilde{W})| \leq C_3\|\widetilde{W}\|_{\mathcal{H}}.$$

Thus $\mathscr{M}$ is a bilinear and continuous form on $\mathcal{H}\times\mathcal{H}$ and $\mathcal{L}$ is a linear and continuous form on $\mathcal{H}$. So, by applying the Lax-Milgram theorem, we deduce that for all $W = (\omega, u, \theta, \vartheta) \in \mathcal{H}$, the elliptic system (3.12) admits a unique (weak) solution

$$\omega \in H_0^2(0,L), \quad u \in H_*^1(0,L), \quad \theta \in H_0^1(0,L), \quad \vartheta \in H_*^1(0,L). \tag{3.13}$$

Moreover, from system (3.11) we obtain

$$\varphi = \omega - \omega^* = \omega_t \in H_0^2(0,L), \quad \psi = u - u^* = u_t \in H_*^1(0,L).$$

Then, by (3.12), we have

$$\begin{cases} d_2\omega_{xxxx} = -\alpha\omega_{xx} + \varphi^* + \omega^* - \omega \in L^2(0,L), \\ -d_1u_{xx} = -\beta\theta_x + \psi^* + u^* - u \in L^2(0,L), \\ -\kappa\theta_{xx} = \kappa_1\vartheta_x - \beta u_x + c(\theta^* - \theta) + \beta u_x^* \in L^2(0,L), \\ -\kappa_2\vartheta_{xx} = -\kappa_3\theta_x + b(\vartheta^* - \vartheta) - \kappa_4\vartheta \in L^2(0,L). \end{cases} \tag{3.14}$$

Since $d_2 \neq 0$, we conclude from $(3.13)_1$ that $\omega \in H^4(0,L)\cap H_0^2(0,L)$. Using $(3.14)_{2-4}$ and $(3.13)_{2-4}$ we have $u \in H^2(0,L)\cap H_*^1(0,L)$, $\theta \in H^2(0,L)\cap H_0^1(0,L)$ and $\vartheta \in H^2(0,L)\cap H_*^1(0,L)$. Finally, the vector $U = (\omega, u, \varphi, \psi, \theta, \vartheta) \in \mathcal{D}(\mathcal{A})$. Therefore, Range $(\mathcal{I} - \mathcal{A}) = \mathscr{H}$. This complete the proof of the maximal monotonicity of $-\mathcal{A}$. □

Lemma 3.2. *Under the assumptions* (2.29)–(2.32) *and* (3.4)*, the operator* $-\mathcal{B}$ *is monotone, bounded, and hemicontinuous.*

Proof. For any $U_i = (\omega_i, u_i, \varphi_i, \psi_i, \theta_i, \vartheta_i) \in \mathscr{H}, i = 1,2$, we have from (2.30) that

$$\langle -\mathcal{B}U_1 + \mathcal{B}U_2, U_1 - U_2\rangle_{\mathscr{H}} = \langle g(\varphi_1) - g(\varphi_2), \varphi_1 - \varphi_2\rangle \geq m\|\varphi_1 - \varphi_2\|^2 \geq 0.$$

This means that $-\mathcal{B}$ is monotone. In addition, from (2.31), we have

$$\|-\mathcal{B}U\|_{\mathscr{H}} \leq M\|\varphi\|, \tag{3.15}$$

then $-\mathcal{B}$ is bounded.

To prove the hemicontinuity of $-\mathcal{B}$, consider $U_i = (\omega_i, u_i, \varphi_i, \psi_i, \theta_i, \vartheta_i) \in \mathscr{H}, i = 1,2$. So, for any $U = (\omega, u, \varphi, \psi, \theta, \vartheta) \in \mathscr{H}$, we have

$$\langle -\mathcal{B}(U_1 + tU_2), U\rangle_{\mathscr{H}} = \langle g(\varphi_1 + t\varphi_2), \varphi\rangle. \tag{3.16}$$

Taking

$$h(x) = g\left(\varphi_1(x)\right)\varphi(x) \in L^1(0,L) \tag{3.17}$$

and sequence $(h_n) \subset L^1(0,L)$ given by

$$h_n(x) = g\left(\varphi_1(x) + \frac{1}{n}\varphi_2(x)\right)\varphi(x) \in L^1(0,L). \tag{3.18}$$

We have

$$\lim_{n\to\infty} h_n(x) = h(x) \text{ a.e. in } (0,L). \tag{3.19}$$

Using again (2.31), we have

$$|h_n(x)| = \left|g\left(\varphi_1(x) + \frac{1}{n}\varphi_2(x)\right)\varphi(x)\right| \le M\left|\left(\varphi_1(x) + \frac{1}{n}\varphi_2(x)\right)\varphi(x)\right| \le M\left(|\varphi_1(x)| + |\varphi_2(x)|\right)|\varphi(x)|. \tag{3.20}$$

From Lebesgue's dominated convergence theorem, it follows that

$$\lim_{t\to 0}\left\langle -\mathcal{B}\left(U_1 + tU_2\right), U\right\rangle_{\mathscr{H}} = \left\langle g\left(\varphi_1\right), \varphi\right\rangle = \left\langle -\mathcal{B}U_1, U\right\rangle_{\mathscr{H}}. \tag{3.21}$$

Hence, $-\mathcal{B}$ is hemicontinuous. □

Lemma 3.3. *The nonlinear operator $\mathcal{F} : \mathscr{H} \to \mathscr{H}$ defined by $(3.9)_3$ is locally Lipschitz continuous.*

Proof. We denote that $U_1 = (\omega_1, u_1, \varphi_1, \psi_1, \theta_1, \vartheta_1)$ and $U_2 = (\omega_2, u_2, \varphi_2, \psi_2, \theta_2, \vartheta_2)$ such that $\|U_1\|_{\mathscr{H}}$, $\|U_2\|_{\mathscr{H}} \le R$, where $R > 0$. Then, we have

$$\begin{aligned}\Big\|\mathcal{F}(U_1) - \mathcal{F}(U_2)\Big\| &\le d_1\Big\|\Big((u_{1x} + \frac{1}{2}\omega_{1x}^2)\omega_{1x} - (u_{2x} + \frac{1}{2}\omega_{2x}^2)\omega_{2x}\Big)_x\Big\| + d_1\Big\|\frac{1}{2}\Big(\omega_{1x}^2 - \omega_{2x}^2\Big)_x\Big\| \\ &\le d_1\|\mathcal{K}\big((\omega_1, u_1), (\omega_2, u_2)\big)\| + d_1\|\mathcal{G}(\omega_1, \omega_2)\|.\end{aligned} \tag{3.22}$$

where $\mathcal{K}(\cdot,\cdot)$ and $\mathcal{G}(\cdot)$ are given by

$$\mathcal{K}\big((\omega_1, u_1), (\omega_2, u_2)\big) = \Big((u_{1x} + \frac{1}{2}\omega_{1x}^2)\omega_{1x} - (u_{2x} + \frac{1}{2}\omega_{2x}^2)\omega_{2x}\Big)_x \text{ and } \mathcal{G}(\omega_1, \omega_2) = \frac{1}{2}\Big(\omega_{1x}^2 - \omega_{2x}^2\Big)_x. \tag{3.23}$$

The functional $\mathcal{G}$ is estimated as follows:

$$\begin{aligned}\|\mathcal{G}(\omega_1, \omega_2)\| &= \Big\|\frac{1}{2}\Big[(\omega_{1x} - \omega_{2x})(\omega_{1x} + \omega_{2x})\Big]_x\Big\| \\ &= \Big\|\frac{1}{2}(\omega_{1xx} - \omega_{2xx})(\omega_{1x} + \omega_{2x}) + \frac{1}{2}(\omega_{1x} - \omega_{2x})(\omega_{1xx} + \omega_{2xx})\Big\| \\ &\le \frac{1}{2}\Big\|(\omega_{1xx} - \omega_{2xx})(\omega_{1x} + \omega_{2x})\Big\| + \frac{1}{2}\Big\|(\omega_{1x} - \omega_{2x})(\omega_{1xx} + \omega_{2xx})\Big\| \\ &\le \frac{1}{2}\Big(\int_0^L \Big[(\omega_{1xx} - \omega_{2xx})(\omega_{1x} + \omega_{2x})\Big]^2 \mathrm{d}x\Big)^{\frac{1}{2}} + \frac{1}{2}\Big(\int_0^L \Big[(\omega_{1x} - \omega_{2x})(\omega_{1xx} + \omega_{2xx})\Big]^2 \mathrm{d}x\Big)^{\frac{1}{2}} \\ &\le C\Big(\|\omega_{1xx} - \omega_{2xx}\|\left(\|\omega_{1x}\|_\infty + \|\omega_{2x}\|_\infty\right) + \|\omega_{1x} - \omega_{2x}\|_\infty\left(\|\omega_{1xx}\| + \|\omega_{2xx}\|\right)\Big).\end{aligned}$$

From the embedding of $H^1(0,L)$ into $L^\infty(0,L)$, we conclude that there exist positive constants C such that

$$\|\omega_{1x}\|_\infty + \|\omega_{2x}\|_\infty \le C(\|\omega_{1xx}\| + \|\omega_{2xx}\|), \quad \|\omega_{1x} - \omega_{2x}\|_\infty \le C\|\omega_{1xx} - \omega_{2xx}\|. \tag{3.24}$$

Consequently, we obtain

$$\|\mathcal{G}(\omega_1,\omega_2)\| \le C\left(\|U_1\|_{\mathscr{H}} + \|U_2\|_{\mathscr{H}}\right)\|U_1 - U_2\|_{\mathscr{H}}.$$

As $\|U_1\|_{\mathscr{H}}$, $\|U_2\|_{\mathscr{H}} \le R$, where $R > 0$, then it follows that

$$\|\mathcal{G}(\omega_1,\omega_2)\| \le C(R)\|U_1 - U_2\|_{\mathscr{H}}. \tag{3.25}$$

Let's estimate the functional $\mathcal{K}$. Adding and subtracting the term $(u_{1x} + \frac{1}{2}\omega_{1x}^2)\omega_{2x}$ in $\mathcal{K}$, we get

$$\begin{aligned}
\|\mathcal{K}\big((\omega_1,u_1),(\omega_2,u_2)\big)\|_{H^1} &= \Big\|(\omega_{1x} - \omega_{2x})(u_{1x} + \frac{1}{2}\omega_{1x}^2) + \omega_{2x}\Big((u_{1x} - u_{2x}) + \frac{1}{2}(\omega_{1x} + \omega_{2x})(\omega_{1x} - \omega_{2x})\Big)\Big\| \\
&\le \Big(\int_0^L \Big[(\omega_{1x} - \omega_{2x})(u_{1x} + \frac{1}{2}\omega_{1x}^2)\Big]^2 \mathrm{d}x\Big)^{\frac{1}{2}} \\
&+ \Big(\int_0^L \Big[\omega_{2x}\Big((u_{1x} - u_{2x}) + \frac{1}{2}(\omega_{1x} + \omega_{2x})(\omega_{1x} - \omega_{2x})\Big)\Big]^2 \mathrm{d}x\Big)^{\frac{1}{2}} \\
&\le C\,\|\omega_{1x} - \omega_{2x}\|_\infty \Big\|u_{1x} + \frac{1}{2}\omega_{1x}^2\Big\| \\
&+ C\,\|\omega_{2x}\|_\infty \Big(\,\|u_{1x} - u_{2x}\| + \frac{1}{2}\,\|\omega_{1x} + \omega_{2x}\|\,\|\omega_{1x} - \omega_{2x}\|_\infty\Big).
\end{aligned}$$

Using again the embedding of $H^1(0,L)$ into $L^\infty(0,L)$, one has from (3.24) that

$$\begin{aligned}
\|\mathcal{K}\big((\omega_1,u_1),(\omega_2,u_2)\big)\|_{H^1} &\le C\|\omega_{1xx} - \omega_{2xx}\|\big(\|u_{1x}\| + \frac{1}{2}\|\omega_{1xx}\|^2\big) \\
&+ C\,\|\omega_{2xx}\|\Big(\,\|u_{1xx} - u_{2xx}\| + \frac{1}{2}(\|\omega_{1x}\| + \|\omega_{2x}\|)\,\|\omega_{1xx} - \omega_{2xx}\|\Big) \\
&\le C\left(\|U_1\|_{\mathscr{H}} + \|U_2\|_{\mathscr{H}}\right)\|U_1 - U_2\|_{\mathscr{H}} \\
&\le C(R)\|U_1 - U_2\|_{\mathscr{H}}.
\end{aligned} \tag{3.26}$$

By inserting (3.25) and (3.26) into (3.22), we conclude that

$$\|\mathcal{F}(U_1) - \mathcal{F}(U_2)\|_{\mathscr{H}} \le C\|U_1 - U_2\|_{\mathscr{H}}. \tag{3.27}$$

Therefore, $\mathcal{F}$ satisfying the local Lipschitz condition. □

We are now in a position to give the definitions of weak solutions and strong solutions to problem (2.25)–(2.27) (see [34]).

Definition 3.4. (i) A function $(\omega, u, \omega_t, u_t, \theta, \vartheta)$ is called a generalized (or weak) solution to (2.25)–(2.27) if

$$(\omega, u, \omega_t, u_t, \theta, \vartheta) \in C([0,\infty);\mathscr{H}), \quad (\omega(0), u(0), \omega_t(0), u_t(0), \theta(0), \vartheta(0)) = (\omega_0, u_0, \omega_1, u_1, \theta_0, \vartheta_0) \in \mathscr{H},$$

and satisfies the following identity in the sense of distributions

$$\langle \omega_{tt}, \phi_1\rangle + \langle u_{tt}, \phi_2\rangle + d_1\langle (u_x + \frac{1}{2}\omega_x^2)\omega_x, \phi_{1x}\rangle + d_2\langle \omega_{xx}, \phi_{1xx}\rangle - \alpha\langle \omega_x, \phi_{1x}\rangle$$

$$+d_1\langle u_x+\frac{1}{2}\omega_x^2,\phi_{2x}\rangle-\langle g(\omega_t),\phi_1\rangle-\beta\langle\theta,\phi_{2x}\rangle+c\langle\theta_t,\phi_3\rangle+b\langle\vartheta,\phi_4\rangle$$
$$+\kappa\langle\theta_x,\phi_{3x}\rangle+\kappa_2\langle\vartheta_x,\phi_{4x}\rangle-\kappa_1\langle\vartheta_x,\phi_3\rangle+\langle\kappa_3\theta_x+\kappa_4\vartheta,\phi_4\rangle+\beta\langle u_{xt},\phi_3\rangle=0, \tag{3.28}$$

in $[0,L]\times[0,T]$ and for all $(\phi_1,\phi_2)\in H_0^2(0,L)\times H_*^1(0,L)$, $(\phi_3,\phi_4)\in L^2(0,L)\times L_*^2(0,L)$.

(ii) Moreover, if a weak solution further satisfies

$$(\omega,u,\omega_t,u_t,\theta,\vartheta)\in C([0,\infty);\mathcal{D}(\mathcal{A}+\mathcal{B}))\cap C^1([0,\infty);\mathscr{H})$$

then it is called strong solution.

To prove the existence of global solutions, we need first to show an energy estimate.

Lemma 3.5. *We assume that assumptions (2.29)–(2.32) and (3.4) hold. Let $(\omega,u,\omega_t,u_t,\theta,\vartheta)$ be a solution to problem (2.25)–(2.27) on the maximal interval $[0,T)$; then there exists a small parameter ζ satisfying*

$$(\kappa-\zeta)(\kappa_4-\zeta)>(\frac{\kappa_1+\kappa_3}{2})^2 \tag{3.29}$$

such that the energy functional of problem (2.25)–(2.27) defined by

$$E(t)=\frac{1}{2}\left(\|\omega_t\ |^2+\|u_t\|^2+d_2\|\omega_{xx}\|^2-\alpha\|\omega_x\|^2+d_1\|u_x+\frac{1}{2}\omega_x^2\|^2+c\|\theta\|^2+b\|\vartheta\|^2\right) \tag{3.30}$$

satisfies

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}\le-m\|\omega_t\|^2-\kappa_2\|\vartheta_x\|^2-\zeta(\|\theta_x\|^2+\|\vartheta\|^2)<0,\quad\forall t\in[0,T). \tag{3.31}$$

There exists also a positive constant C independent of t such that

$$\|U(t)\|_{\mathscr{H}}^2\le4\big(1+CE(0)\big)E(t),\quad\forall t\in[0,T), \tag{3.32}$$

where $\|U(t)\|_{\mathscr{H}}$ is defined by (3.7).

Proof. Let $(\omega,u,\omega_t,u_t,\theta,\vartheta)$ be a strong solution to problem (2.25)–(2.27). We multiply $(2.25)_1$ by ω_t, $(2.25)_2$ by u_t, $(2.25)_3$ by θ and $(2.25)_4$ by ϑ, using integration by part and the boundary condition, we obtain

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=-\langle g(\omega_t),\omega_t\rangle-\kappa_2\|\vartheta_x\|^2-\kappa\|\theta_x\|^2-\kappa_4\|\vartheta\|^2-(\kappa_1+\kappa_3)\int_0^L\theta_x\vartheta\mathrm{d}x<0,\quad\forall t\in[0,T). \tag{3.33}$$

From (2.30), we have

$$-\langle g(\omega_t),\omega_t\rangle\le-m\|\omega_t\|^2. \tag{3.34}$$

From (3.29) we infer that

$$\begin{aligned}\kappa\|\theta_x\|^2+\kappa_4\|\vartheta\|^2+(\kappa_1+\kappa_3)\int_0^L\theta_x\vartheta\mathrm{d}x&=(\kappa-\zeta)\|\theta_x\|^2+(\kappa_1+\kappa_3)\int_0^L\theta_x\vartheta\mathrm{d}x+(\kappa_4-\zeta)\|\vartheta\|^2\\&\quad+\zeta(\|\theta_x\|^2+\|\vartheta\|^2)\\&\ge\zeta(\|\theta_x\|^2+\|\vartheta\|^2).\end{aligned} \tag{3.35}$$

Substituting (3.34) and (3.35) into (3.33), then (3.31) follows immediately, which gives

$$E(t) \leqslant E(0), \quad \forall t \in [0, T). \tag{3.36}$$

Using Young's inequality, we obtain

$$\begin{aligned}\int_0^L \left(u_x + \frac{1}{2}\omega_x^2\right)^2 \mathrm{d}x &= \int_0^L u_x^2 \mathrm{d}x + \frac{1}{4}\int_0^L \omega_x^4 \mathrm{d}x + \int_0^L u_x \omega_x^2 \mathrm{d}x \\ &\geq \int_0^L u_x^2 \mathrm{d}x + \frac{1}{4}\int_0^L \omega_x^4 \mathrm{d}x - \int_0^L |u_x \omega_x^2| \mathrm{d}x \\ &\geq \int_0^L u_x^2 \mathrm{d}x + \frac{1}{4}\int_0^L \omega_x^4 \mathrm{d}x - \frac{1}{2}\int_0^L u_x^2 \mathrm{d}x - \frac{1}{2}\int_0^L \omega_x^4 \mathrm{d}x \\ &\geq \frac{1}{2}\int_0^L u_x^2 \mathrm{d}x - C\left(\int_0^L \omega_{xx}^2 \mathrm{d}x\right)^2 \\ &\geq \frac{1}{2}\int_0^L u_x^2 \mathrm{d}x - C(E(t))^2, \end{aligned} \tag{3.37}$$

where C is a positive constant independent of t. By (3.36), we obtain

$$\|u_x + \frac{1}{2}\omega_x^2\|^2 \geq \frac{1}{2}\|u_x\|^2 - CE(0)E(t). \tag{3.38}$$

After substituting (3.38) into (3.30) and using $(3.3)_2$, it follows that

$$\begin{aligned} E(t) &\geq \frac{1}{2}\left(\|\omega_t\|^2 + \|u_t\|^2 + d_2\|\omega_{xx}\|^2 - \alpha\|\omega_x\|^2 + \frac{d_1}{2}\|u_x\|^2 + c\|\theta\|^2 + b\|\vartheta\|^2\right) - CE(0)E(t) \\ &\geq \frac{1}{4}\|U(t)\|_{\mathscr{H}}^2 - CE(0)E(t). \end{aligned}$$

Combining this we obtain (3.32). By using a density argument, we obtain that (3.31) and (3.32) are valid for generalized (or weak) solutions. ☐

Now an application of the theory of semigroups [34] gives the following:

Theorem 3.6. *(Existence of global solutions). Suppose that assumptions* (2.29)–(2.32) *and* (3.4) *hold. Then*

(1) for every initial condition $U_0 = U(0) = (\omega_0, u_0, \omega_1, u_1, \theta_0, \vartheta_0) \in \mathscr{H}$, problem (2.25)–(2.27) has a unique generalized (or weak) solution satisfying $U(t) \in C([0,\infty), \mathscr{H})$;

(2) if $U_0 \in \mathcal{D}(\mathscr{A})$, then the above generalized solution is a strong solution; and

(3) for all $T > 0$, if $U_1(t)$ and $U_2(t)$ are two solutions to problem (3.8)*, then there exists a positive constant C depending on $U_{10} = U_1(0)$ and $U_{20} = U_2(0)$ such that*

$$\|U_1(t) - U_2(t)\|_{\mathscr{H}} \leq e^{CT}\|U_1(0) - U_2(0)\|_{\mathscr{H}}, \quad 0 \leq t \leq T. \tag{3.39}$$

Proof. (1)-(2) Since $-\mathcal{A}$ is maximal monotone; $-\mathcal{B}$ is monotone, bounded, and hemicontinuous; and $\mathscr{H}$ is a Hilbert space, it follows from [5, Corollary 2.1] that $-\mathscr{A} = -(\mathcal{A} + \mathcal{B})$ is a maximal monotone operator. In addition, as $\mathcal{F} : \mathscr{H} \to \mathscr{H}$ is locally Lipschitz, it follows from [14, Theorem 7.2] for all $U_0 \in \mathcal{D}(\mathscr{A})$, there exists $t_{\max} \leq \infty$ and a unique strong solution U for (3.8) defined on the interval $[0, t_{\max})$. Moreover, if $U_0 \in \mathscr{H} = \overline{\mathcal{D}(\mathscr{A})}$, then (3.8) has a unique weak solution $U \in C([0, t_{\max}), \mathscr{H})$, and such solutions satisfy $\limsup_{t\to t_{\max}} \|U(t)\|_{\mathscr{H}} = \infty$, provided $t_{\max} < \infty$.

Next we prove that the solution is global, that is, $t_{\max} = \infty$. Let $U = (\omega, u, \omega_t, u_t, \theta, \vartheta)$ be a strong solution in $[0, t_{\max})$. Then, by (3.32), we obtain

$$\|U(t)\|^2_{\mathscr{H}} \leq 4\big(1 + CE(0)\big)E(t) \leq 4\big(1 + CE(0)\big)E(0), \tag{3.40}$$

which yields

$$\|U(t)\|_{\mathscr{H}} < \infty, \quad \forall t \in [0, t_{\max}). \tag{3.41}$$

By a density argument, the inequality (3.41) also holds for weak solutions. Therefore $t_{\max} = \infty$, proving (1) and (2).

(3) Now, we want to prove the continuity dependence of weak solutions on the initial data $U_0 = U(0)$ in $\mathscr{H}$. First we consider the case of strong solutions. Let $U_1 = (\omega_1, u_1, \omega_{1t}, u_{1t}, \theta_1, \vartheta_1)$ and $U_2 = (\omega_2, u_2, \omega_{2t}, u_{2t}, \theta_2, \vartheta_2)$ be strong solutions to problem (2.25)–(2.27) with corresponding initial data U_{10}, $U_{20} \in \mathscr{H}$. Then $U = (\omega, u, \omega_t, u_t, \theta, \vartheta) = U_1 - U_2$ is solution to

$$\begin{aligned} \omega_{tt} - d_1[(u_{1x} + \frac{1}{2}\omega_{1x}^2)\omega_{1x} - (u_{2x} + \frac{1}{2}\omega_{2x}^2)\omega_{2x}]_x + \alpha\omega_{xx} + d_2\omega_{xxxx} + \big(g(\omega_{1t}) - g(\omega_{2t})\big)\omega_t = 0, \\ u_{tt} - d_1[u_x + \frac{1}{2}(\omega_{1x}^2 - \omega_{2x}^2)]_x + \beta\theta_x = 0, \\ c\theta_t - \kappa\theta_{xx} - \kappa_1\vartheta_x + \beta u_{xt} = 0, \\ b\vartheta_t - \kappa_2\vartheta_{xx} + \kappa_3\theta_x + \kappa_4\vartheta = 0, \end{aligned} \tag{3.42}$$

with initial conditions and Dirichlet boundary conditions

$$\begin{aligned} &\omega(x,0) = \omega_1(0) - \omega_2(0), \quad \omega_t(x,0) = \omega_{1t}(0) - \omega_{2t}(0), \quad x \in (0, L), \\ &u(x,0) = u_1(0) - u_2(0), \quad u_t(x,0) = u_{1t}(0) - u_{2t}(0), \quad x \in (0, L), \\ &\theta(x,0) = \theta_1(0) - \theta_2(0), \quad \vartheta(x,0) = \vartheta_1(0) - \vartheta_2(0), \quad x \in (0, L), \\ &\omega(x,t) = \omega_x(x,t) = u_x(x,t) = \theta(x,t) = \vartheta_x(x,t) = 0 \text{ on } x = 0, L, \ t > 0. \end{aligned} \tag{3.43}$$

We multiply the first equation of the system (3.42) by ω_t, the second equation by u_t, the third one by θ and the fourth one by ϑ, integrate the result over $(0, L)$ to conclude that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\|^2_{\mathscr{H}} = d_1\int_0^L \mathcal{K}\big((\omega_1, u_1), (\omega_2, u_2)\big)\omega_t dx + d_1\int_0^L \mathcal{G}(\omega_1, \omega_2)u_t \mathrm{d}x - \int_0^L \big(g(\omega_{1t}) - g(\omega_{2t})\big)\omega_t \mathrm{d}x \\ - \kappa_2\|\vartheta_x\|^2 - \kappa\|\theta_x\|^2 - \kappa_4\|\vartheta\|^2 - (\kappa_1 + \kappa_3)\int_0^L \theta_x\vartheta \mathrm{d}x, \end{aligned}$$

where $\|U\|_{\mathscr{H}} = \|U_1 - U_2\|_{\mathscr{H}}$ is defined by (3.7) and $\mathcal{K}(\cdot, \cdot)$ and $\mathcal{G}(\cdot)$ are defined by (3.23).

By Young's inequality and the monotonicity of g, we get after using (2.33) and the fact that $\big(g(\omega_{1t}) - g(\omega_{2t})\big)\omega_t \geq 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\|^2_{\mathscr{H}} \leq \frac{d_1^2}{2}\|\mathcal{K}\big((\omega_1, u_1), (\omega_2, u_2)\big)\|^2 + \frac{1}{2}\|\omega_t\|^2 + \frac{d_1^2}{2}\|\mathcal{G}(\omega_1, \omega_2)\|^2 + \frac{1}{2}\|u_t\|^2. \tag{3.44}$$

Since from (3.41) we have $\|U_i(t)\|_{\mathscr{H}} < \infty$ for $i = 1, 2$, we infer from (3.25) and (3.26), that there exists a positive constant C such that

$$\begin{aligned} \frac{d_1^2}{2}\|\mathcal{K}\big((\omega_1,u_1),(\omega_2,u_2)\big)\|^2 + \frac{d_1^2}{2}\|\mathcal{G}(\omega_1,\omega_2)\|^2 &\le C\left(\|U_1\|_{\mathscr{H}} + \|U_2\|_{\mathscr{H}}\right)\|U\|_{\mathscr{H}}^2 \\ &\le C\|U\|_{\mathscr{H}}^2. \end{aligned} \tag{3.45}$$

Substituting (3.45) into (3.44), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\|_{\mathscr{H}}^2 \le C\|U(t)\|_{\mathscr{H}}^2. \tag{3.46}$$

Applying Gronwall's inequality to (3.46), we conclude that

$$\|U(t)\|_{\mathscr{H}} \le e^{CT}\|U(0)\|_{\mathscr{H}}, \quad \forall t \in [0,T], \tag{3.47}$$

where we obtain the continuous dependence of strong solutions on the initial data. The continuous dependence for weak solutions can be proved by using density arguments. The proof is complete. □

Remark 3.7. (i) The well-posedness of problem (2.25)–(2.27) implies that the solution operator $S(t) : \mathscr{H} \to \mathscr{H}$ given by

$$S(t)U_0 = U(t) = (\omega(t), u(t), \omega_t(t), u_t(t), \theta(t), \vartheta(t)), \qquad t > 0, \tag{3.48}$$

defines a nonlinear C_0-semigroup.

(ii) It is not difficult to show that $S(t)$ satisfies the semigroup property. By using (3.39), one can establish that $S(t)$ is a continuous operator for all $t \ge 0$, and since $t \longmapsto S(t)U_0 = U(t)$ is continuous for all $U_0 \in \mathscr{H}$, it follows that $S(t)$ is a C_0-semigroup [16].

(iii) The continuous dependence result means that for a given $T > 0$ and a bounded set $\mathscr{B}$ of $\mathscr{H}$, there exists a constant $C_{\mathscr{B}T} > 0$ such that

$$\|S(t)U_1 - S(t)U_2\|_{\mathscr{H}} \le C_{\mathscr{B}T}\|U_1 - U_2\|_{\mathscr{H}} \tag{3.49}$$

for all $t \in [0,T]$ and $U_1,\ U_2 \in \mathscr{B}$.

4. Exponential stability

In this section, we prove the stability of the nonlinear system (2.25)–(2.27) using the multiplier technique. We start by introducing some functionals and stating some lemmas.

Lemma 4.1. *Let assumptions of Theorem 3.6 hold, then the functional defined by*

$$\mathcal{S}(t) = \int_0^L (uu_t + \frac{1}{2}\omega\omega_t)\mathrm{d}x, \tag{4.1}$$

satisfies for all $t \ge 0$,

$$\frac{d\mathcal{S}}{\mathrm{d}t}(t) \le -d_1\|u_x + \frac{1}{2}\omega_x^2\|^2 - \frac{d_2}{4}\|\omega_{xx}\|^2 + c_0\|\omega_t\|^2 + \|u_t\|^2 + \frac{d_2}{2L}\|u_x\|^2 + \frac{1}{2}\alpha\|\omega_x\|^2 + \frac{L\beta^2}{2d_2}\|\theta\|^2, \tag{4.2}$$

where $c_0 = \frac{1}{2}(1 + \frac{M}{2\lambda_2 d_2})$.

Proof. Differentiating the functional $\mathcal{S}$ and using the first and the second equation of (2.25) we get

$$\begin{aligned}\frac{d\mathcal{S}}{\mathrm{d}t}(t) &= \|u_t\|^2 + \int_0^L \Big[d_1[u_x + \frac{1}{2}\omega_x^2]_x - \beta\theta_x\Big]u\mathrm{d}x + \frac{1}{2}\|\omega_t\|^2 \\ &+ \frac{1}{2}\int_0^L \Big[d_1\Big((u_x + \frac{1}{2}\omega_x^2)\omega_x\Big)_x - \alpha\omega_{xx} - d_2\omega_{xxxx} - g(\omega_t)\Big]\omega\mathrm{d}x \\ &= \|u_t\|^2 + \frac{1}{2}\|\omega_t\|^2 - d_1\int_0^L [u_x + \frac{1}{2}\omega_x^2]u_x\mathrm{d}x - \frac{d_2}{2}\|\omega_{xx}\|^2 - \frac{d_1}{2}\int_0^L (u_x + \frac{1}{2}\omega_x^2)\omega_x^2\mathrm{d}x \\ &+ \frac{1}{2}\alpha\|\omega_x\|^2 + \beta\int_0^L \theta u_x\mathrm{d}x - \frac{1}{2}\int_0^L g(\omega_t)\omega\mathrm{d}x \\ &= \|u_t\|^2 + \frac{1}{2}\|\omega_t\|^2 - d_1\|u_x + \frac{1}{2}\omega_x^2\|^2 - \frac{d_2}{2}\|\omega_{xx}\|^2 - \frac{1}{2}\int_0^L g(\omega_t)\omega\mathrm{d}x + \frac{1}{2}\alpha\|\omega_x\|^2 \\ &+ \beta\int_0^L \theta u_x\mathrm{d}x. \end{aligned} \tag{4.3}$$

Using (2.31) and $(3.3)_1$, we get

$$-\frac{1}{2}\int_0^L g(\omega_t)\omega\mathrm{d}x \le \frac{M}{4\lambda_2 d_2}\|\omega_t\|^2 + \frac{d_2}{4}\|\omega_{xx}\|^2. \tag{4.4}$$

Applying Young's inequality to the last integral of (4.3) gives

$$\beta\int_0^L \theta u_x\mathrm{d}x \le \frac{d_2}{2L}\|u_x\|^2 + \frac{L\beta^2}{2d_2}\|\theta\|^2. \tag{4.5}$$

After substituting (4.4) and (4.5) into (4.3) we obtain the desired result (4.2). □

We define the functional $\mathscr{K}(t)$

$$\mathscr{K}(t) = -\int_0^L (\frac{c}{\beta}\theta + b\vartheta)\int_0^x u_t(y)\mathrm{d}y\mathrm{d}x. \tag{4.6}$$

Lemma 4.2. *Suppose that assumptions of Theorem 3.6 hold, then there exists a constant $\varepsilon > 0$ such that the functional $\mathscr{K}$ defined by (4.6) satisfies*

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t}(t) \le -\frac{1}{2}\|u_t\|^2 + \varepsilon\|u_x + \frac{1}{2}\omega_x^2\|^2 + C(\varepsilon)(\|\theta_x\|^2 + \|\vartheta\|^2) + C\|\vartheta_x\|^2, \quad \forall t \ge 0. \tag{4.7}$$

Proof. Taking the derivation of (4.6), integrating by parts the result and using the fact that $\int_0^L u(x)\mathrm{d}x = 0$, we arrive at

$$\begin{aligned}\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t}(t) &= -\int_0^L (\frac{c}{\beta}\theta_t + b\vartheta_t)\int_0^x u_t(y)\mathrm{d}y\mathrm{d}x - \int_0^L (\frac{c}{\beta}\theta + b\vartheta)\int_0^x u_{tt}(y)\mathrm{d}y\mathrm{d}x \\ &= -\|u_t\|^2 + \frac{\kappa}{\beta}\int_0^L u_t\theta_x\mathrm{d}x + \frac{\kappa_1}{\beta}\int_0^L u_t\vartheta\mathrm{d}x + \kappa_2\int_0^L u_t\vartheta_x\mathrm{d}x - \kappa_3\int_0^L u_t\theta\mathrm{d}x \\ &+ \kappa_4\int_0^L \vartheta\int_0^x u_t(y)\mathrm{d}y\mathrm{d}x - \int_0^L (\frac{c}{\beta}\theta + b\vartheta)\int_0^x u_{tt}(y)\mathrm{d}y\mathrm{d}x \\ &\leq -\frac{1}{2}\|u_t\|^2 + C(\|\theta_x\|^2 + \|\vartheta\|^2 + \|\vartheta_x\|^2) - \int_0^L (\frac{c}{\beta}\theta + b\vartheta)\int_0^x u_{tt}(y)\mathrm{d}y\mathrm{d}x.\end{aligned} \tag{4.8}$$

It follows that

$$\begin{aligned}-\int_0^L (\frac{c}{\beta}\theta + b\vartheta)\int_0^x u_{tt}(y)\mathrm{d}y\mathrm{d}x &:= -\int_0^L (\frac{c}{\beta}\theta + b\vartheta)\Big(d_1(u_x + \frac{1}{2}\omega_x^2) - \beta\theta\Big)\mathrm{d}x \\ &\leq \varepsilon\|u_x + \frac{1}{2}\omega_x^2\|^2 + C(\varepsilon)(\|\theta\|^2 + \|\vartheta\|^2).\end{aligned} \tag{4.9}$$

Then the estimate (4.7) follows from (4.8)–(4.9). □

Next, we define a Lyapunov functional $\aleph$ and show that it is equivalent to the energy functional E.

Lemma 4.3. *Under the assumptions of Theorem 3.6, there exists a constant $\beta_0 > 0$ such that*

$$(N - \beta_0)E(t) \leq \aleph(t) \leq (N + \beta_0)E(t), \qquad \forall t \geq 0, \tag{4.10}$$

where $\aleph(t)$ is a Lyapunov functional defined by

$$\aleph(t) = NE(t) + \mathcal{S}(t) + 4\mathscr{K}(t), \tag{4.11}$$

where $N > \beta_0$ is a sufficiently large constant.

Proof. It follows from Young's, Poincaré, Cauchy-Schwarz inequalities that

$$|\mathcal{S}(t)| \leq C\Big(\|u\|^2 + \|u_t\|^2 + \|\omega\|^2 + \|\omega_t\|^2\Big), \quad |\mathscr{K}(t)| \leq C_1\Big(\|u_t\|^2 + \|\theta\|^2 + \|\vartheta\|^2\Big).$$

Thus there exists a constant $\beta_0 > 0$ such that

$$|\aleph(t) - NE(t)| = |\mathcal{S}(t) + 4\mathscr{K}(t)| \leq \beta_0 E(t)$$

and therefore (4.10) holds. □

Theorem 4.4. *Suppose that assumptions of Theorem 3.6 hold. Then, there exist positive constants c_0, c_1 such that the energy functional defined by (3.30) satisfies*

$$E(t) \leq c_1 E(0)e^{-c_0 t}, \qquad \forall t > 0. \tag{4.12}$$

Proof. By following the same approach as in (3.37), one can show that (see [15])

$$\|u_x\|^2 \le 2\|u_x + \frac{1}{2}\omega_x^2\|^2 + \frac{L}{4}\|\omega_{xx}\|^2. \tag{4.13}$$

Combining (3.31), (4.2), (4.7) and (4.13), we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\aleph(t) \le &-(mN - c_0)\|\omega_t\|^2 - \|u_t\|^2 - (N\kappa_2 - 4C)\|\vartheta_x\|^2 - \frac{d_2}{8}\|\omega_{xx}\|^2 + \frac{1}{2}\alpha\|\omega_x\|^2 \\ &- \Big(N\zeta - C'(\varepsilon)\Big)(\|\theta_x\|^2 + \|\vartheta\|^2) - \Big(d_1 - \frac{d_2}{L} - 4\varepsilon\Big)\|u_x + \frac{1}{2}\omega_x^2\|^2. \end{aligned} \tag{4.14}$$

We need to select our coefficients in an appropriate way. First, we pick N large enough such that

$$N > \sup\Big\{\frac{c_0}{m}, \frac{4C}{\kappa_2}, \frac{C'(\varepsilon)}{\zeta}\Big\}.$$

From $(2.28)_1$ and $(2.28)_2$ we pick ε small enough such that

$$\varepsilon \le \frac{d_1}{4}(1 - \frac{h^2}{12L}). \tag{4.15}$$

Since $h \ll L$, we have necessarily $\frac{h^2}{12L} \ll 1$. Thus, there exists a positive constant ς such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\aleph(t) \le -\varsigma E(t)$$

which yields, by using (4.10),

$$\frac{\mathrm{d}}{\mathrm{d}t}\aleph(t) \le -c_0\aleph(t) \tag{4.16}$$

for some positive constant c_0. Then (4.12) follows by using (4.10) again, which completes the proof. □

5. Case without thermal conductivity

In this section, we consider the thermal effects, but without the thermal conductivity. Apalara [4] considered this case for porous-elastic system and proved that the dissipation given only with the microtemperatures is sufficient to get an exponential stability for the case of equal speeds of wave propagation. In [17, 26, 35, 36], the authors studied the porous thermoelastic system in case of zero thermal conductivity with temperatures and microtemperatures effects. They proved that the unique dissipation due to the microtemperatures is strong enough to make the energy of the considered systems decay to zero in an exponential manner without any condition on the coefficients of the system. Without the thermal conductivity and under the condition $\kappa_1 + \kappa_3 = 0$, *i.e.*

$$\kappa = 0 \quad \text{and} \quad \kappa_3 = -\kappa_1 \ge 0, \tag{5.1}$$

the system (2.25) becomes

$$\omega_{tt} - d_1\Big((u_x + \frac{1}{2}\omega_x^2)\omega_x\Big)_x + \alpha\omega_{xx} + d_2\omega_{xxxx} + g(\omega_t) = 0,$$

$$\begin{aligned} u_{tt} - d_1[u_x + \frac{1}{2}\omega_x^2]_x + \beta\theta_x &= 0,\\ c\theta_t + \kappa_3\vartheta_x + \beta u_{xt} &= 0,\\ b\vartheta_t - \kappa_2\vartheta_{xx} + \kappa_3\theta_x + \kappa_4\vartheta &= 0, \end{aligned} \tag{5.2}$$

subject to the boundary conditions (2.26) and the initial data (2.27).

By following the same steps under the condition (5.1), it is easy to check that the problem (5.2), (2.26) and (2.27) is global well posed with respect to strong and weak solutions over $\mathscr{H}$ and the estimations (4.2) and (4.7) keep the same form. In this case, the energy functional of problem (5.2), (2.26) and (2.27) defined by (3.30) satisfies

$$\frac{\mathrm{d}E}{\mathrm{d}t}(t) \le -m\|\omega_t\|^2 - \kappa_2\|\vartheta_x\|^2 - \kappa_4\|\vartheta\|^2 < 0, \quad \forall t \ge 0. \tag{5.3}$$

We omit the details here for the sake of brevity.

To control the term $\|\theta\|^2$ in (4.2) and (4.7), we define the functional

$$\mathscr{I}(t) = cb\int_0^L \theta\Big(\int_0^x \vartheta(y)\mathrm{d}y\Big)\mathrm{d}x, \quad \forall t \ge 0.$$

Lemma 5.1. *Let $(\omega, u, \omega_t, u_t, \theta, \vartheta)$ be a solution of (5.2), (2.26) and (2.27). Then the functional $\mathscr{I}$ satisfies, for any $\varepsilon_1 > 0$, the following estimate*

$$\frac{\mathrm{d}\mathscr{I}}{\mathrm{d}t}(t) \le -\frac{c\kappa_3}{2}\|\theta\|^2 + \frac{c\kappa_2^2}{\kappa_3}\|\vartheta_x\|^2 + \varepsilon_1\|u_t\|^2 + C(\varepsilon_1)\|\vartheta\|^2, \quad \forall t \ge 0. \tag{5.4}$$

Proof. Differentiating $\mathscr{I}(t)$ and integrating by parts, we get

$$\begin{aligned} \frac{\mathrm{d}\mathscr{I}}{\mathrm{d}t}(t) = {}& c\kappa_2\int_0^L \theta\Big(\int_0^x \vartheta_{yy}(y)\mathrm{d}y\Big)\mathrm{d}x - c\kappa_4\int_0^L \theta\Big(\int_0^x \vartheta(y)\mathrm{d}y\Big)\mathrm{d}x - c\kappa_3\int_0^L \theta\Big(\int_0^x \theta_y(y)\mathrm{d}y\Big)\mathrm{d}x\\ &- b\beta\int_0^L u_{xt}\Big(\int_0^x \vartheta(y)\mathrm{d}y\Big)\mathrm{d}x - b\kappa_3\int_0^L \vartheta_x\Big(\int_0^x \vartheta(y)\mathrm{d}y\Big)\mathrm{d}x. \end{aligned}$$

By integrating by parts and using the fact that $\int_0^L \vartheta(x)\mathrm{d}x = 0$, we arrive at

$$\frac{\mathrm{d}\mathscr{I}}{\mathrm{d}t}(t) \le -c\kappa_3\|\theta\|^2 + b\kappa_3\|\vartheta\|^2 + c\kappa_2\int_0^L \theta\vartheta_x\mathrm{d}x - c\kappa_4\int_0^L \theta\Big(\int_0^x \vartheta(y)\mathrm{d}y\Big)\mathrm{d}x + b\beta\int_0^L u_t\vartheta\mathrm{d}x. \tag{5.5}$$

By virtue of Young's inequality, we find

$$\begin{aligned} c\kappa_2\int_0^L \theta\vartheta_x\mathrm{d}x &\le \frac{c\kappa_2^2}{\kappa_3}\|\vartheta_x\|^2 + \frac{c\kappa_3}{4}\|\theta\|^2,\\ -c\kappa_4\int_0^L \theta\Big(\int_0^x \vartheta(y)\mathrm{d}y\Big)\mathrm{d}x &\le C\|\vartheta\|^2 + \frac{c\kappa_3}{4}\|\theta\|^2,\\ b\beta\int_0^L u_t\vartheta\mathrm{d}x &\le \varepsilon_1\|u_t\|^2 + C(\varepsilon_1)\|\vartheta\|^2. \end{aligned} \tag{5.6}$$

Estimate (5.4) follows by substituting (5.6) into (5.5). □

We are now ready to state and prove the following exponential stability result.

Theorem 5.2. *Let $(\omega, u, \omega_t, u_t, \theta, \vartheta)$ be a solution of (5.2), (2.26) and (2.27). Then, there exist positive constants c_2, c_3 such that the energy functional of the problem defined by (3.30) satisfies*

$$E(t) \leq c_3 E(0) e^{-c_2 t}, \qquad \forall t > 0. \tag{5.7}$$

Proof. As previously, we define the following Lyapunov functional

$$\aleph(t) = NE(t) + \mathcal{S}(t) + 4\mathscr{K}(t) + \delta\mathscr{I}(t), \tag{5.8}$$

where $\delta > 0$ and $N > 0$ is a sufficiently large constant. Similarly, one can prove that there exists a positive constant $\beta_0 < N$ such that

$$(N - \beta_0)E(t) \leq \aleph(t) \leq (N + \beta_0)E(t), \qquad \forall t \geq 0. \tag{5.9}$$

Using (4.2), (4.7), (5.4), (5.8) and (4.13), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\aleph(t) \leq & -(mN - c_0)\|\omega_t\|^2 - \Big(\kappa_2 N - 4C - \delta\frac{c\kappa_2^2}{\kappa_3}\Big)\|\vartheta_x\|^2 - \Big(\kappa_4 N - 4C(\varepsilon) - \delta C(\varepsilon_1)\Big)\|\vartheta\|^2 \\ & - \Big(\frac{c\kappa_3\delta}{2} - \frac{L\beta^2}{2d_2} - 4C(\varepsilon)\Big)\|\theta\|^2 - \Big(1 - \varepsilon_1\delta\Big)\|u_t\|^2 - \frac{d_2}{8}\|\omega_{xx}\|^2 + \frac{1}{2}\alpha\|\omega_x\|^2 \\ & - \Big(d_1 - \frac{d_2}{L} - 4\varepsilon\Big)\|u_x + \frac{1}{2}\omega_x^2\|^2.\end{aligned} \tag{5.10}$$

We need to select our coefficients in an appropriate way. First, we pick ε_1, δ and N such that

$$\varepsilon_1 = \frac{1}{2\delta}, \qquad \delta > \frac{2}{c\kappa_3}\Big(\frac{L\beta^2}{2d_2} + 4C(\varepsilon)\Big), \qquad N > \sup\Big\{\frac{c_0}{m}, \frac{4C + \delta\frac{c\kappa_2^2}{\kappa_3}}{\kappa_2}, \frac{4C(\varepsilon) + \delta C(\varepsilon_1)}{\kappa_4}\Big\}$$

and we pick ε small enough such that (4.15) holds, *i.e.* $\varepsilon \leq \frac{d_1}{4}(1 - \frac{h^2}{12L})$. Thus, there exists a positive constant ς such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\aleph(t) \leq -\varsigma E(t)$$

which yields, by using (5.9),

$$\frac{\mathrm{d}}{\mathrm{d}t}\aleph(t) \leq -c_2\aleph(t) \tag{5.11}$$

for some positive constant c_2. Then (5.7) follows by using (5.9) again, which completes the proof. □

6. Conclusion

(a) The equations that constitute the mathematical model of the von Kármán beam have been derived for the first time by including the microtemperatures concept into the second law of thermodynamics. The proposed model in this paper is more reasonable in predicting the propagation of thermal, microtemperatures and elastic waves in the von Kármán beam. This work, which has not been obtained in any reference yet, represents a first step towards understanding the fundamental limits of intrinsic thermal and microtemperatures dissipations in von Kármán beams.

(b) We have proved the well-posedness and the exponential decay of solutions of the derived model with a damping function. This dapping function is added to the derived equations in order to guarantee the exponential decay of solutions. Moreover, the temperature effects introduce into von Kármán beam equations the term $(\theta\omega_x)_x$ in $(2.24)_1$ which becomes in the linear context in the form $\theta_0\omega_{xx}$. This term requires the condition (3.4) to ensure the well-posedness and the exponential stability of the model. This is contrary to the case of linear thermoelastic systems with microtemperatures where the thermal and microtemperatures dissipations provides the exponential stability in the one dimensional setting without imposing conditions on the physical parameters.

(c) We have proved that the unique dissipation due to the microtemperatures is strong enough to exponentially stabilize the system. This result is new for von Kármán beam equations and improve some recent results in the literature.

(d) It should be pointed out that the extension of the results from the one-dimensional case to higher dimensional case is nontrivial. Indeed, in addition to considerably higher complexity of the equations involved, the nonlinear term is not bounded on the space determined by the finite energy solutions. This is due to critical exponents occuring in Sobolev's embeddings and, in particular, the fact that $H^1(\Omega)$ does not embed, in the two-dimensional case, in $L^\infty(\Omega)$.

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