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OUTPUT TRAJECTORY CONTROLLABILITY OF A DISCRETE-TIME SIR EPIDEMIC MODEL

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Abstract. Developing new approaches that help control the spread of infectious diseases is a critical issue for public health. Such approaches must consider the available resources and capacity of the healthcare system. In this paper, we present a new mathematical approach to controlling an epidemic model by investigating the optimal control that aims to bring the output of the epidemic to target a desired disease output $y^d = (y_i^d)_{i\in\{0,\ldots,N\}}$. First, we use the state-space technique to transfer the trajectory controllability to optimal control with constraints on the final state. Then, we use the fixed point theorems to determine the set of admissible controls and solve the output trajectory controllability problem. Finally, we apply our method to the model of a measles epidemic, and we give a numerical simulation to illustrate the findings of our approach.

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1. Introduction

With the world experiencing different types of epidemics, mathematical epidemiology has helped investigate several characteristics of the epidemics and tried to understand in depth the best approaches to contain the spread of an epidemic and reduce its impact on human lives (see, for example [2, 6, 15, 28, 34, 35, 37]). However, these efforts also require using more mathematical tools that give the public health authorities an upper hand in dealing with epidemics. Particularly in terms of how to use the optimal use of human capacities (public health professionals), the management of the treatment and vaccination stockpiles, the hospital and health care facilities

Optimal control theory provides a valuable tool to begin to assess the trade-offs between vaccination and treatment strategies [3, 22–25, 33]. In this paper, we consider the SIR (Susceptible-Infected-Removed) epidemic model. We use an optimal control strategy to control the spread of infectious diseases by setting control to track the desired output. The concept of controllability, introduced by Kalman in 1960, plays an important part

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in many domains of applied mathematics. For example, controllability is one of the properties of dynamical systems, which means it can steer a dynamical system from a given initial state to a desired final state.

There are three types of controllability: approximate controllability, which steers a system's state approximatively to any desired final state; exact controllability, which drives the system precisely to a desired final state; and null controllability, which steers the state to the origin. In recent years, important progress has been made in the controllability of linear and nonlinear systems [8, 9, 19, 20, 26, 30–32, 36]. However, it does not provide any insight into system trajectory control.

A new notion of controllability, namely, trajectory controllability, was introduced by George (1996) for 1-dimensional nonlinear systems. The trajectory controllability ensures steering the dynamical system from an initial state to a final desired state, along a determined trajectory, by using some suitable control function.

Various papers have contributed to the study of trajectory controllability. [1, 4, 5, 7, 14, 16, 21, 27, 29]. Chalishajar *et al.* [7] have studied the trajectory controllability of nonlinear integro-differential systems in finite and infinite-dimensional systems. Maojun Bin and Zhenhai Liu [5] studied the trajectory controllability of semilinear differential evolution equations with impulses and delay by using the fixed point theorem and monotone operator theory. In [27], Malik *et al.* used the strongly continuous family of linear operators and the fixed point methods to study the exact controllability. In [29], Sandilya *et al.* established the trajectory controllability for a semilinear parabolic system in an infinite-dimensional space setting. Diblik [14] has studied the trajectory controllability of discrete-time linear systems with a single delay. Bianchini [4] give high-order conditions for a point of a reference trajectory to be interior to the reachable set.

Beyond engineering problems, trajectory controllability can be applied to other real-world issues such as public health. In fact, in the same way we can steer a dynamical system from a given state, along a determined trajectory, to a desired final state, we can investigate the possibility of driving the outcome of disease spread from its initial situation to a desirable situation. The determined trajectory would be the efforts that the healthcare authorities can bear to handle the disease's spread. Which include the resources such as the number of hospitals' beds, the number of healthcare professionals available, the testing, or laboratory capacity.

In this paper, we provide an epidemic model that aims to demonstrate the importance of trajectory controllability as a mathematical tool for better controlling infectious diseases. More specifically, we investigate the output trajectory controllability of a class of SIR-type epidemic model.

$$\left\{\begin{array}{lcl} S_{i+1} & = & S_i - \beta S_i I_i - dS_i + (d-u_i)T_i \\ I_{i+1} & = & I_i + \beta S_i I_i - dI_i - \gamma I_i \\ R_{i+1} & = & R_i - dR_i + \gamma I_i + u_i T_i,\ 0 \le i \le N-1, \end{array}\right. \tag{1.1}$$

with the corresponding output

$$y_i = (S_i, I_i)^\top, \quad i \in \{0, \dots, N\}, \tag{1.2}$$

subject to initial conditions $S_0 \ge 0$, $I_0 \ge 0$ and $R_0 \ge 0$. $u_i \in U = [0,1]$ is the control variable, $y_i \in \mathbb{R}^2$ is the output function and $T_i = S_i + I_i + R_i$ is the total population.

In such a SIR-model, T_i is the total population, β is the transmission constant (with the total number of infections per unity of time at time t being $\beta S_i I_i$). The parameter d is the rate of deaths from causes unrelated to the infection. In the control strategy introduced in this paper we calculate the proportion of the population at each moment to reach or approach the number of infected and susceptible cases desired and dT_i represents the recruitment rate. γ^{-1} is the average duration of infectious periods. $u_i T_i$ is the proportion of the total populations that are vaccinated (See for example [10, 17]). All the above parameters are assumed to be nonnegative. Schematically, the flow between compartments is represented in Figure 1.

The SIR model (1.1) fulfils the constant population through time constraint, *i.e.*

$$T_{i+1} = T_i = S_i + I_i + R_i = T_0 = T > 0$$

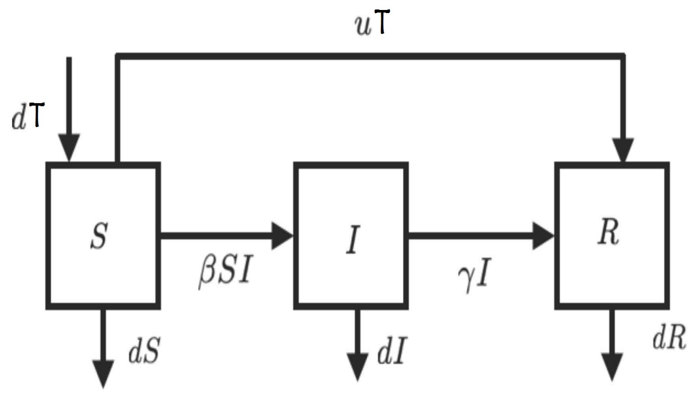

FIGURE 1. Flowchart of the SIR model.

where T is a constant. Note that this assertion proves that the constant population through time is independent of the vaccination strategy.

Consider the following control problem. Given a desired trajectory $y^d = (y_1^d, \dots, y_N^d)$ with $y_i^d \in \mathbb{R}^2$, $\forall i \in \{1, \dots, N\}$, we try to find the optimal control $u = (u_0, u_1, \dots, u_{N-1})$ which minimizes the functional cost

$$J(u) = \|u\|^2, \tag{1.3}$$

over all controls satisfying

$$y_i = y_i^d, \quad \forall \; i \in \{1, \dots, N\}.$$

To solve this problem and inspired by the results in [13, 18] we use, in Section 2, a state space technique to show that the problem of input retrieval can be seen as a problem of optimal control with constraints on the final state. In Section 3, we give a new statement of the new problem of controllability. In Section 4, we use a technique based on the fixed point theorem (see [11, 12]) to establish the set of admissible controls by the pseudo inverse corresponding to the linear part of the system and the fixed points of an appropriate mapping. Finally, A numerical simulation is given to illustrate the obtained results in Section 5.

2. An adequate state space approach

Let consider the state $X_i = (S_i, I_i, R_i)^\top$ the the system (1.1) can be rewritten as

$$(S) \begin{cases} X_{i+1} & = & AX_i + EX_i + Bu_i \\ X_0 & \in & \mathbb{R}^3 \end{cases}$$

where A and B two matrices defined by

$$A = \begin{pmatrix} 1 & d & d \\ 0 & 1-\gamma-d & 0 \\ 0 & \gamma & 1-d \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} -T \\ 0 \\ T \end{pmatrix}.$$

and E is a nonlinear operator given by

$$\begin{array}{rccc} E: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \longrightarrow & \begin{pmatrix} -\beta xy \\ \beta xy \\ 0 \end{pmatrix} \end{array}$$

The corresponding output is

$$y_i = CX_i$$

with $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

In this subsection, we give some technical results which will be used in the sequel. For a finite subset $\sigma_r^s = \{r, r+1, \ldots, s\}$ of $\mathbb{Z}$, with $s \geq r$. Let $l^2(\sigma_r^s, \mathbb{R}^3)$ be the space of all sequences $(z_i)_{i\in\sigma_r^s}$, $z_i \in \mathbb{R}^3$.

Let L_1 and F be the operators given by

$$\begin{array}{rccc} L_1: & l^2(\sigma_{-N}^{-1};\mathbb{R}^3) & \longrightarrow & l^2(\sigma_{-N}^{-1};\mathbb{R}^3), \\ & (z_{-N},\ldots,z_{-1}) & \longmapsto & (z_{-N+1},\ldots,z_{-1},0), \end{array}$$

$$\begin{array}{rccc} F: & \mathbb{R}^3 & \longrightarrow & l^2(\sigma_{-N}^{-1};\mathbb{R}^3), \\ & x & \longmapsto & (0,\ldots,0,x) \end{array}$$

and define the variables $z^i \in l^2(\sigma_{-N}^{-1};\mathbb{R}^2)$ by

$$\begin{array}{rcl} z^i & = & (z^i_{-N},\ldots,z^i_{-1}), \\ z^i_k & = & \begin{cases} X_{i+k} & , \quad \text{if } i+k \geq 0 \\ X_0 & , \quad \text{else,} \end{cases} \end{array}$$

where $(X_i)_i$ is the solution of system (S). Then the sequence $(z^i)_i$ is the unique solution of the following difference equation

$$\begin{cases} z^{i+1} & = & L_1 z^i + FX_i \ , \quad i \in \sigma_0^{N-1}, \\ z^0 & = & (X_0, X_0, \ldots, X_0). \end{cases}$$

Let $e_i \in \mathbb{R}^3 \times l^2(\sigma_{-N}^{-1};\mathbb{R}^3)$ be the signals defined by $e_i = \begin{pmatrix} X_i \\ z^i \end{pmatrix}$, then we easily establish the following result.

Proposition 2.1. *$(e_i)_{i\in\sigma_0^N}$ is the unique solution of the difference equation described by*

$$(S_1) \begin{cases} e_{i+1} & = & \Psi e_i + \Phi e_i + \bar{B}u_i \ , \ i \in \sigma_0^{N-1}, \\ e_0 & = & \begin{pmatrix} X_0 \\ (X_0,\ldots,X_0) \end{pmatrix}, \end{cases}$$

where $\Psi = \begin{pmatrix} A & 0 \\ F & L_1 \end{pmatrix}$, $\Phi = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\bar{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$.

Remark 2.2. The equality

$$e_N = \begin{pmatrix} X_N \\ z^N \end{pmatrix} = \begin{pmatrix} X_N \\ (X_0, \dots, X_{N-1}) \end{pmatrix},$$

allows us to assimilate the trajectory $(X_0, \dots, X_{N-1}, X_N)$ of system (S) to the final state e_N of (S_1). This implies that our problem of input retrieval is equivalent to a problem of optimal control with constraints on the final state e_N.

3. The optimal control expression

Let's consider the operator Γ defined by

$$\begin{array}{rccc} \Gamma: & \mathbb{R}^3 \times l^2(\sigma_{-N}^{-1}; \mathbb{R}^3) & \longrightarrow & l^2(\sigma_1^N; \mathbb{R}^2) \\ & (x, (\xi_i)_{-N \le i \le 0}) & \longmapsto & (t_1, \dots, t_N) \end{array} \tag{3.1}$$

with $t_i = C\xi_{i-N}, \ \forall i \in \sigma_1^{N-1}$ and $t_N = Cx$.

Definition 3.1. **a)** The system (S) is said to be exactly output controllable on σ_1^N if $\forall X_0 \in \mathbb{R}^3 \times l^2(\sigma_{-N}^{-1}; \mathbb{R}^3)$, $\forall y \in l^2(\sigma_1^N; \mathbb{R}^2)$, $\exists u \in l^2(\sigma_0^{N-1}; U)$ such that $CX_i = y_i, \quad i \in \sigma_1^N$.

b) The system (S) is said to be weakly output controllable on σ_1^N if $\forall \epsilon > 0$, $\forall X_0 \in \mathbb{R}^3 \times l^2(\sigma_{-N}^{-1}; \mathbb{R}^3)$, $\forall y \in l^2(\sigma_1^N; \mathbb{R}^2)$, $\exists u$ such that $\| CX_i - y \|_{\mathbb{R}^2} \le \epsilon$.

Definition 3.2. **a)** The system (S) is said to be Γ-controllable on σ_1^N if $\forall e_0 \in \mathbb{R}^3 \times l^2(\sigma_{-N}^{-1}; \mathbb{R}^3)$, $\forall y^d \in l^2(\sigma_1^N; \mathbb{R}^2)$, $\exists u \in l^2(\sigma_0^{N-1}; [0, 1])$ such that $\Gamma e_N = y^d$.

b) The system (S) is said to be Γ-weakly controllable on σ_1^N if $\forall \epsilon > 0$, $\forall e_0 \in \mathbb{R}^3 \times l^2(\sigma_{-N}^{-1}; \mathbb{R}^3)$, $\forall y^d \in l^2(\sigma_1^N; \mathbb{R}^2)$, $\exists u$ such that $\| \Gamma e_N - y^d \|_{l^2(\sigma_1^N; \mathbb{R}^2)} \le \epsilon$.

Remark 3.3. From the above definition, we can easily establish the following results

i) (S) is exactly output controllable $\sigma_1^N \Longleftrightarrow (S_1)$ is Γ-controllable on σ_1^N.
ii) (S) is weakly output controllable on $\sigma_1^N \Longleftrightarrow (S_1)$ is Γ-weakly controllable on σ_1^N.

Proposition 3.4. *Given a desired output $y^d = (y_1^d, \dots, y_N^d) \in l^2(\sigma_1^N; \mathbb{R}^2)$, the problem $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ defined as:*

$$(\mathcal{P}_1) \qquad \begin{cases} \textit{Find } u^* \textit{ such that} & \\ CX_i = y_i^d, \quad \forall \ i \in \sigma_1^N & (i) \\ \| u^* \| = \inf\{ \| v \| \ \ / \ v \textit{ verify } \ (i)\} & (ii) \end{cases}$$

$$(\mathcal{P}_2) \qquad \begin{cases} \textit{Find } u^* \textit{ such that} & \\ \Gamma e_N = y^d \ \in \ l^2(\sigma_1^N; \mathbb{R}^2) & (j) \\ \| u^* \| = \inf\{ \| v \| \ \ / \ v \textit{ verify } \ (j)\} & (jj) \end{cases}$$

have the same solution u^*.

By proposition 3.4, the resolution of problem $(\mathcal{P}_1) - (\mathcal{P}_2)$ is equivalent to find the control u^* which ensure the Γ-controllability of system (S_1) and with a minimal cost.

We consider the discrete system described by

$$(S_1) \left\{ \begin{array}{lll} e_{i+1} & = & \Psi e_i + \Phi e_i + \bar{B} u_i \ , \ i \in \sigma_0^{N-1} \\ e_0 & & \text{is given} \end{array} \right. \tag{3.2}$$

where $e_i \in \mathcal{X} = \mathbb{R}^3 \times l^2(\sigma_{-N}^{-1}; \mathbb{R}^3)$ is the state of system (S_1), $u_i \in U = [0,1]$ is the control variable, $\Psi \in \mathcal{L}(\mathcal{X})$ and $\bar{B} \in \mathcal{L}(U, \mathbb{R}^3)$. Consider the following control problem. Given a desired trajectory $y^d = (y_1^d, \ldots, y_N^d)$, we find the control u^* which minimizes the functional cost

$$J(u) = \|u\|^2 \tag{3.3}$$

overall controls satisfying

$$\Gamma e_N = y^d,$$

e_N is the final state of system (S_1) at instant N, and Γ is given by (3.1). We shall call u^* the wanted control and the solution of system (S_1) is

$$e_i = \Psi^i e_0 + \sum_{j=0}^{i-1} \Psi^j \Phi e_{i-1-j} + \sum_{j=0}^{i-1} \Psi^j \bar{B} u_{i-1-j}, \quad i \in \sigma_1^N. \tag{3.4}$$

Let L denote the linear operator defined on $\mathcal{T} = l^2(\sigma_1^N; \mathcal{X})$ by

$$\begin{array}{rrcl} L: & \mathcal{T} = l^2(\sigma_1^N; \mathcal{X}) & \longrightarrow & \mathcal{T} \\ & \xi = (\xi_1, \ldots, \xi_N) & \longmapsto & L\xi = (L\xi)_{1 \le i \le N} \end{array}$$

where

$$\left\{ \begin{array}{lll} (L\xi)_i & = & \Psi^{i-1} \Phi e_0 + \displaystyle\sum_{j=0}^{i-2} \Psi^j \Phi \xi_{i-1-j}; \quad 2 \le i \le N \\ (L\xi)_1 & = & \Phi e_0 \end{array} \right.$$

Let consider $\xi = (\xi_i)_{i \in \sigma_1^N} \in \mathcal{T}$ with $\xi_i = \begin{pmatrix} x_i \\ y_i \\ z_i \\ (v_1^i, \ldots, v_N^i) \end{pmatrix} \in \mathcal{X}$, then we have

$$(L\xi)_1 = \left(\begin{array}{c} \begin{pmatrix} -\beta x_0 y_0 \\ \beta x_0 y_0 \\ 0 \end{pmatrix} \\ (0 \ldots 0) \end{array} \right)$$

and

$$
(L\xi)_i = \begin{pmatrix} \begin{pmatrix} -\beta(1-d)^{i-1}x_0y_0 \\ \beta(1-\gamma-d)^{i-1}x_0y_0 \\ \gamma\beta\displaystyle\sum_{j=0}^{i-2}(1-\gamma-d)^{i-2-j}(1-d)^j x_0y_0 \end{pmatrix} \\ \left(\underbrace{0,\dots,0}_{N-i+1}, \begin{pmatrix} -\beta x_0y_0 \\ \beta x_0y_0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\beta(1-d)x_0y_0 \\ \beta(1-\gamma-d)x_0y_0 \\ \gamma\beta x_0y_0 \end{pmatrix}, \dots, \right. \\ \left. \begin{pmatrix} -\beta(1-d)^{i-2}x_0y_0 \\ \beta(1-\gamma-d)^{i-2}x_0y_0 \\ \gamma\beta\displaystyle\sum_{j=0}^{i-3}(1-\gamma-d)^{i-3-j}(1-d)^j x_0y_0 \end{pmatrix}\right) \end{pmatrix}
$$

$$
+ \begin{pmatrix} \begin{pmatrix} -\beta\displaystyle\sum_{j=0}^{i-2}(1-d)^j x_{i-1-j}y_{i-1-j} \\ \beta\displaystyle\sum_{j=0}^{i-2}(1--\gamma-d)^j x_{i-1-j}y_{i-1-j} \\ \gamma\beta\displaystyle\sum_{j=0}^{i-2}\sum_{k=0}^{j-1}(1-\gamma-d)^{j-1-k}(1-d)^k x_{i-1-j}y_{i-1-j} \end{pmatrix} \\ \left(\underbrace{0,\dots,0}_{N-i+2}, \begin{pmatrix} -\beta x_1y_1 \\ \beta x_1y_1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\displaystyle\sum_{k=1}^{2}\beta(1-d)^{2-k}x_{2-k}y_{2-k} \\ \displaystyle\sum_{k=1}^{2}\beta(1-\gamma-d)x_{2-k}y_{2-k} \\ \gamma\beta x_1y_1 \end{pmatrix}, \dots, \right. \\ \left. \begin{pmatrix} -\displaystyle\sum_{j=1}^{i-2}\beta(1-d)^{j-1}x_{i-1-j}y_{i-1-j} \\ \displaystyle\sum_{j=1}^{i-2}\beta(1-\gamma-d)^{j-1}x_{i-1-j}y_{i-1-j} \\ \gamma\beta\displaystyle\sum_{j=2}^{i-2}\sum_{k=0}^{j-2}(1-\gamma-d)^{j-2-k}(1-d)^k x_{i-1-j}y_{i-1-j} \end{pmatrix}\right) \end{pmatrix}, \quad i \in \sigma_1^N.
$$

and let H denote the linear operator defined on $\mathcal{U}$ by

$$
\begin{array}{rrcl} H: & \mathcal{U} = l^2(\sigma_0^{N-1};[0,1]) & \longrightarrow & \mathcal{T} \\ & u = (u_0,\dots,u_{N-1}) & \longmapsto & Hu \end{array}
$$

where

$$
(Hu)_i = \sum_{j=0}^{i-1} \Psi^j \bar{B} u_{i-1-j}, \quad i \in \sigma_1^N.
$$

We have

$$(Hu)_i = \begin{pmatrix} \begin{pmatrix} -\sum_{j=0}^{i-1}(1-d)^j T u_{i-1-j} \\ 0 \\ \sum_{j=0}^{i-1}(1-d)^j T u_{i-1-j} \end{pmatrix} \\ \left(\underbrace{0,\dots,0}_{N-i+1}, \begin{pmatrix} -Tu_0 \\ 0 \\ Tu_0 \end{pmatrix}, \dots, \begin{pmatrix} -\sum_{j=0}^{i-2}(1-d)^j T u_{i-2-j} \\ 0 \\ \sum_{j=0}^{i-2}(1-d)^j T u_{i-2-j} \end{pmatrix} \right) \end{pmatrix} \quad i \in \sigma_1^N.$$

So, the equation (3.4) can be rewritten as

$$e = (e_1, \dots, e_N) = \tilde{\Psi} e_0 + Le + Hu,$$

where

$$\tilde{\Psi} e_0 = (\Psi^i e_0)_{1 \le i \le N}.$$

We have

$$\ker H = \{0\}, \ \ (\ker\ H)^\top = \mathcal{U} = l^2(\sigma_0^{N-1}, \mathbb{R})$$

and

$$\text{Range } H = \left\{ z = (z_i)_{i \in \sigma_1^N} \in l^2(\sigma_0^{N-1}, \mathcal{X}), \ \ z_i = \begin{pmatrix} \begin{pmatrix} \alpha_i \\ 0 \\ \alpha_i \end{pmatrix} \\ \left(\underbrace{0,\dots,0}_{N-i+1}, \begin{pmatrix} \alpha_1 \\ 0 \\ -\alpha_1 \end{pmatrix}, \dots, \begin{pmatrix} \alpha_{i-1} \\ 0 \\ -\alpha_{i-1} \end{pmatrix} \right) \end{pmatrix}, \quad i \in \sigma_1^N, \ \alpha_i \in \mathbb{R}. \right\}.$$

Its inverse is defined

$$\left\{ \begin{array}{cccl} H^{-1} & : & Range(H) & \longrightarrow \ \ \mathcal{U} \\ & & z & \longrightarrow \ \ H(z) = u = \left\{ \begin{array}{l} u_0 = -\frac{\alpha_1}{T} \\ u_i = \frac{-\alpha_{i+1} + (1-d)\alpha_i}{T}, \quad i \in \sigma_1^{N-1}. \end{array} \right. \end{array} \right.$$

We introduce the pseudo inverse operator of H

$$H^\dagger : x + y \in Range(H) \bigoplus Range(H)^\perp \to \tilde{H}^{-1}(x) \in \mathcal{U}.$$

The operator $H^\dagger$ is defined on all the space $\mathcal{T}$ because $Range(H)$ is closed and we have

$$\left\{ \begin{array}{rcl} HH^\dagger x & = & x, \quad \forall x \in Range(H) \\ H^\dagger H y & = & y, \quad \forall y \in \mathcal{U}. \end{array} \right.$$

4. Fixed point approach

4.1. Characterization of the set of admissible controls

Let $y^d = (y_1^d, \dots, y_N^d)$ a predefined output. The aim of this section is to give a characterization of the set of all admissible control in consideration the fixed points of a function appropriately chosen, *i.e.*, We shall characterize the set $\mathcal{U}_{ad}$ of all control which ensure the Γ-controllability.

$$\mathcal{U}_{ad} = \{u \in l^2(\sigma_0^{N-1}; [0,1]) \ / \ \Gamma e_N = y^d\}$$

where $(e_0, \dots, e_N)$ is the trajectory which takes system from the initial state e_0. Let $p : T \longrightarrow Range(H)$ be any projection on $Range(H)$ and $\bar{e} \neq 0$ be any fixed element of $Range(H)$, we define

$$\begin{array}{rccl} f_{\bar{e}}: & \mathcal{T} & \longrightarrow & Range(H) \\ & e & \longmapsto & f_{\bar{e}}(e) = \left\{ \begin{array}{ll} 0 & , \ \text{if and only if } \ \Gamma e_N = y^d \\ \bar{e} & , \ \text{else} \end{array} \right. \end{array}$$

and let

$$\begin{array}{rccl} \xi: & \mathcal{T} & \longrightarrow & \mathcal{T} \\ & e & \longmapsto & \xi(e) = e - \tilde{\Psi} e_0 - Le \end{array} \tag{4.1}$$

then

$$(\xi(e))_i = e_i - \left(\begin{array}{c} \begin{pmatrix} x_0 + (1-(1-d)^i)(y_0+z_0) \\ (1-\gamma-d)^i y_0 \\ \gamma \displaystyle\sum_{j=0}^{i-1} (1-\gamma-d)^{i-1-j}(1-d)^j y_0 + (1-d)^i z_0 \end{pmatrix} \\ \left(\underbrace{\begin{pmatrix} x_0 \\ y_0 \\ z_o \end{pmatrix}, \dots, \begin{pmatrix} x_0 \\ y_0 \\ z_o \end{pmatrix}}_{N-i+1}, \begin{pmatrix} x_0 + (1-(1-d))(y_0+z_0) \\ (1-\gamma-d) y_0 \\ \gamma y_0 + (1-d) z_0 \end{pmatrix}, \dots, \right. \\ \left. \begin{pmatrix} x_0 + (1-(1-d)^{i-1})(y_0+z_0) \\ (1-\gamma-d)^{i-1} y_0 \\ \displaystyle\sum_{j=0}^{i-2} (1-\gamma-d)^{i-2-j}(1-d)^j y_0 + (1-d)^{i-1} z_0 \end{pmatrix} \right) \end{array} \right) - (Le)_i$$

and we consider the mapping

$$\begin{array}{rccl} g: & \mathcal{T} & \longrightarrow & \mathcal{T} \\ & e & \longmapsto & g(e) = \tilde{\Psi} e_0 + Le + p\xi(e) + f_{\bar{e}}(e). \end{array} \tag{4.2}$$

Then, we have the following proposition

Proposition 4.1. *Let $P_g = \{e \in \mathcal{T} \ / \ g(e) = e\}$ denotes the set of all fixed points of g. Then*

$$\mathcal{U}_{ad} = \bigcup_{e \in P_g} H^{\dagger} \xi(e).$$

Proof of proposition 4.1. Let $e^* \in P_g$, we have

$$g(e^*) = \tilde{\Psi}e_0 + Le^* + p\xi(e^*) + f_{\bar{e^*}}(e) = e^* \tag{4.3}$$

then

$$e^* - \tilde{\Psi}e_0 - Le^* = p\xi(e^*) + f_{\bar{e}}(e^*)$$

which implies that

$$\xi(e^*) = p\xi(e^*) + f_{\bar{e}}(e^*) \quad \in Range(H)$$

that means

$$p\xi(e^*) = \xi(e^*)$$

and $f_{\bar{e}}(e^*) = 0$ which carries that $\Gamma e_N^* = y^d$.
Consequently, the equation (4.3) become

$$e^* = \tilde{\Psi}e_0 + Le^* + \xi(e^*) = \tilde{\Psi}e_0 + Le^* + HH^\dagger\xi(e^*). \tag{4.4}$$

Let $u^* = H^\dagger\xi(e^*),$ with and $e^* \in P_g$, then

$$Hu^* = HH^\dagger\xi(e^*)$$

and from (4.4), we have

$$Hu^* = HH^\dagger\xi(e^*) = e^* - \tilde{\Psi}e_0^* - Le^*$$

which implies that

$$\begin{cases} e^* & = & \tilde{\Psi}e_0^* + Le^* + Hu^* \\ \Gamma e_N^* & = & y^d \end{cases}$$

thus

$$u^* \in \mathcal{U}_{ad}.$$

Consequently, $\forall e \in P_g$, we have $H^\dagger\xi(e) \subset \mathcal{U}_{ad}$ and

$$\bigcup_{e\in P_g} H^\dagger\xi(e) \subset \mathcal{U}_{ad}.$$

Now, we show that $\mathcal{U}_{ad} \subset \bigcup_{e\in P_g} H^\dagger\xi(e)$. Let $u^* \in \mathcal{U}_{ad}$ and $(e_1^{u^*}, \dots, e_{N-1}^{u^*})$ the trajectory of system (S_1) corresponding to control u^*, then we have

$$\begin{cases} e^{u^*} & = & \tilde{\Psi}e_0 + Le^{u^*} + Hu^* \\ \Gamma e_N^{u^*} & = & y^d \end{cases}$$

and

$$\left\{\begin{array}{lcl}\xi(e^{u^*}) & = & Hu^* \\ \Gamma e_N^{u^*} & = & y^d.\end{array}\right.$$

Consequently

$$\left\{\begin{array}{lcl}\xi(e^{u^*}) & = & Hu^* \in Range(H) \\ f_{\bar{e}}(e^{u^*}) & = & 0\end{array}\right.$$

and

$$e^{u^*} = \tilde{\Psi}e_0 + Le^{u^*} + p\xi(e^{u^*}) + f_{\bar{e}}(e^{u^*}) = g(e^{u^*}).$$

Then e^{u^*} is a fixed point of the mapping of g

and finally we have

$$\mathcal{U}_{ad} \subset \bigcup_{e \in P_g} H^{\dagger}\xi(e).$$

□

Remark 4.2. The fixed points of g are independent of the choice of the projection p and the element $\bar{e}$. Indeed, let p_1 and p_2 two projection on *Range* H and $\bar{e}_1$ and $\bar{e}_2$ two any elements not equal to zero of *Range* H. Let's consider the applications

$$\begin{array}{rccl} g_1: & \mathcal{T} & \longrightarrow & \mathcal{T}, \\ & e & \longrightarrow & g_1(e) = \tilde{\Psi}e_0 + Le + p_1\xi(e) + f_{\bar{e}_1}(e), \end{array}$$

$$\begin{array}{rccl} g_2: & \mathcal{T} & \longrightarrow & \mathcal{T}, \\ & e & \longrightarrow & g_2(e) = \tilde{\Psi}e_0 + Le + p_2\xi(e) + f_{\bar{e}_2}(e). \end{array}$$

Let e a fixed point of g_1, by proof of proposition 3, we have $\Gamma e_N = y^d$ and $\xi(e) \in Range\ H$, he result that $p_2\xi(e) = \xi(e)$ and $f_{\bar{e}_1}(e) = 0$, then

$$g_2(e) = \tilde{\Psi}e_0 + Le + \xi(e) = e.$$

What shows that e is a fixed point of g_2. By symmetry, it clear that the fixed points of g_2 are also the fixed points of g_1.

4.2. Problem of minimization

By the above proposition, we can characterize the set of admissible control $\mathcal{U}_{ad}$, among those controls, we allow to determine those with the minimal norm, *i.e.*, we solve the following problem:

$$\bar{\mathcal{P}}: \min_{u \in \mathcal{U}_{ad}} (J(u) = \| u \|^2).$$

If we suppose that P_g is fini, *i.e.*, $P_g = \{e^1, ..., e^q\}$, we have

$$\mathcal{U}_{ad} = \bigcup_{i=1}^{q} H^{\dagger}\xi(e^i) = \bigcup_{i=1}^{q} \mathcal{U}_{ad}^i$$

where $\mathcal{U}_{ad}^i = H^{\dagger}\xi(e^i)$
then, we obtain

$$\bar{\mathcal{P}} \Longleftrightarrow \min_{1\leq i\leq q}\big(\min_{u\in\ \mathcal{U}_{ad}^i} (J(u) = \parallel u \parallel^2)\big). \tag{4.5}$$

Remark 4.3.
Let $u \in \mathcal{U}_{ad}^i$ then $u = H^{\dagger}\xi(e^i)$. Thus

$$\begin{array}{rcl} \parallel u \parallel^2 & = & < u, u >=< H^{\dagger}\xi(e^i), H^{\dagger}\xi(e_i) > \\ & = & \parallel H^{-1}\xi(e^i) \parallel^2 \end{array}$$

finally we have

$$J(u) = \parallel H^{\dagger}\xi(e^i) \parallel^2,$$

Theorem 4.4. *If we suppose that the set P_g is finite, then the optimal control allows to have the* Γ*-Controllability (then the exactly output controllability of system (S)) is given by*

$$u^* = H^{\dagger}\xi(e^{i_0}),$$

with e^{i_0} a fixed point of application g given by (4.2) and which verified

$$\parallel H^{\dagger}\xi(e^{i_0}) \parallel^2 = \inf_{1\leq i\leq q}\{\parallel H^{\dagger}\xi(e^i) \parallel^2\}.$$

Proof of Theorem 1. Let's consider $P_g = \{e^1, \ldots, e^q\}$, then by lemma 1, we have

$$\min_{u\in\mathcal{U}_{ad}} J(u) = \|u\|^2 \ \text{ with } \ u^* = H^{\dagger}\xi(e^i)$$

However, H is a closed subspace, then the minimum of J is reached and therefore, we have

$$\min_{u\in\mathcal{U}_{ad}} J(u) = \|H^{\dagger}\xi(e^i)\|^2.$$

While using the equivalence (4.5), we deduct that

$$\min_{u\in\mathcal{U}_{ad}} (J(u) = \|u\|^2) = \inf_{1\leq i\leq q}\{\|H^{\dagger}\xi(e^i)\|^2\} = \|H^{\dagger}\xi(e^{i_0})\|^2$$

where $e^{i_0} \in \{e^{i_1}, \ldots, e^{i_q}\}$. $\square$

Let consider the projection on $Range(H)$

$$\begin{array}{rccc} P: & l^2(\sigma_1^N, \mathcal{X}) & \longrightarrow & Range\ (H) \\ & z=(z_1,\dots,z_N) & \longrightarrow & \left(\begin{array}{c} \begin{pmatrix} \overline{G}z_i \\ 0 \\ -\overline{G}z_i \end{pmatrix} \\ \left(\underbrace{0,\dots,0}_{N-i+1}, \begin{pmatrix} \overline{G}z_1 \\ 0 \\ -\overline{G}z_1 \end{pmatrix}, \dots, \begin{pmatrix} \overline{G}z_{i-1} \\ 0 \\ -\overline{G}z_{i-1} \end{pmatrix} \right) \end{array} \right) \end{array}$$

where

$$\begin{array}{rccc} \overline{G}: & \mathcal{X} & \longrightarrow & \mathbb{R} \\ v = & \left(\begin{array}{c} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \\ (v_1,\dots,v_N) \end{array} \right) & \longrightarrow & x_0 \end{array}$$

Theorem 4.5. *For* $y_i^d = \begin{pmatrix} y_i^1 \\ y_i^2 \end{pmatrix}$, $\forall i \in \sigma_1^N$, *a desired output with constraint* $\beta x_0 y_0 + (1-\gamma-d)y_0 = y_1^2$, *the optimal control allow to have the* Γ*-Controllability (then the exactly output controllability of system (S)) is given by*

$$u_i = \frac{-y_{i+1}^1 + (1-d)y_i^1 - (y_{i+1}^2 - (1-\gamma-d)y_i^2) + d(x_0+y_0+z_0)}{T}, \quad \forall i \in \sigma_0^{N-1}.$$

Proof of Theorem 2. The final constraint $\Gamma e_N = (y_1^d, \dots, y_N^d)$ where $y_i^d = \begin{pmatrix} y_i^1 \\ y_i^2 \end{pmatrix}$, $\forall i \in \sigma_1^N$ is equivalent of

$$\begin{cases} x_i = y_i^1, \quad \forall i \in \sigma_1^N \\ \beta x_0 y_0 + (1-\gamma-d)y_0 = y_1^2 \\ \beta(1-\gamma-d)^{i-1}x_0y_0 + \beta\displaystyle\sum_{j=0}^{i-2}(1-\gamma-d)^j x_{i-1-j}y_{i-1-j} + (1-\gamma-d)^i y_0 = y_i^2, \quad \forall i \in \sigma_2^N \end{cases}$$

which implies that

$$y_i = \frac{y_{i+1}^2 - (1-\gamma-d)y_i^2}{\beta y_i^1}, \quad \forall i \in \sigma_1^{N-1}$$

and as $\xi(e^{i_0}) \in Rang(H)$ we find that $\overline{G}\xi(e^{i_0}) = x_i$ and by theorem 1 the optimal control, allow to have the exactly output controllability of system (S)) given by $u^* = H^\dagger \xi(e^{i_0})$, verifie

$$u_i = \frac{-x_{i+1} + (1-d)x_i - \beta x_i y_i + d(x_0+y_0+z_0)}{T}, \quad \forall i \in \sigma_0^{N-1}$$

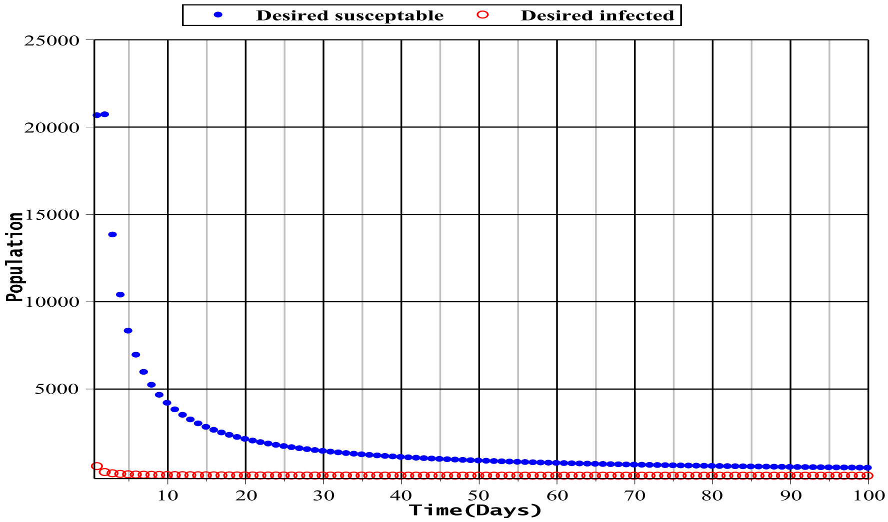

FIGURE 2. The evolution of the desired output

and we replace the expression of x_i and y_i in the equations we deduce that

$$u_i = \frac{-y^1_{i+1} + (1-d)y^1_i - (y^2_{i+1} - (1-\gamma-d)y^2_i) + d(x_0+y_0+z_0)}{T}, \quad \forall i \in \sigma_0^{N-1}.$$

□

Numerical simulation: We consider an example of an epidemic described by the SIR model (1) with parameter values: $\beta = 0.0000079$, $\gamma = 0.0213$, $d = 0.0163$. The initial condition for the individual population are given by: $S_0 = 41\,300$, $I_0 = 430$, $R_0 = 10$, $N = 41\,740$. The desired output signal is

$$\begin{array}{rcl} y^1_1 &=& \frac{S_0}{2}, \\ y^1_i &=& \frac{S_0}{2i} + 2\frac{i-1}{i} + 50, \forall i \in \sigma_2^N \\ y^2_1 &=& \beta S_0 I_0 + (1-\gamma-d)I_0, \\ y^2_i &=& \frac{23 I_0}{0.5i+7.5}, \quad \forall i \in \sigma_2^N, \end{array}$$

In Figure 2, we can observe that the optimal vaccination function has a highly desired impact on the population of susceptible and infected individuals, which is that it declines while the population of recovered individuals grows over almost the whole vaccination campaign. Figures 3 and 4 show the time evolution of the different populations without and with control. The number of people removed with control begins to grow faster than without control. At the end of the vaccination campaign, the number of recovered individuals grew to extremely high levels.

Figure 5 shows the effect of control by indicating that the number of susceptible individuals decreases more rapidly during the vaccination campaign and follows the curve of the desired susceptible.

The evolution of the number of desired infected individuals and infected individuals with control is depicted in Figure 6. We note that the change in the number of infected, which is the number of people who left the susceptible case to get infected. This number rises over time, peaking at 805 people on the ninth day, and then gradually declines as people recover. While the number of infected people under control has changed over time.

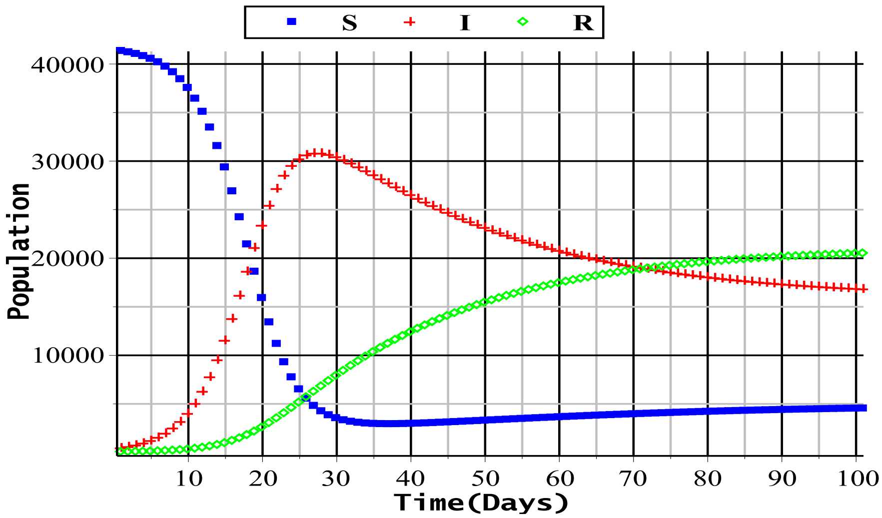

FIGURE 3. The evolution of system SIR without control

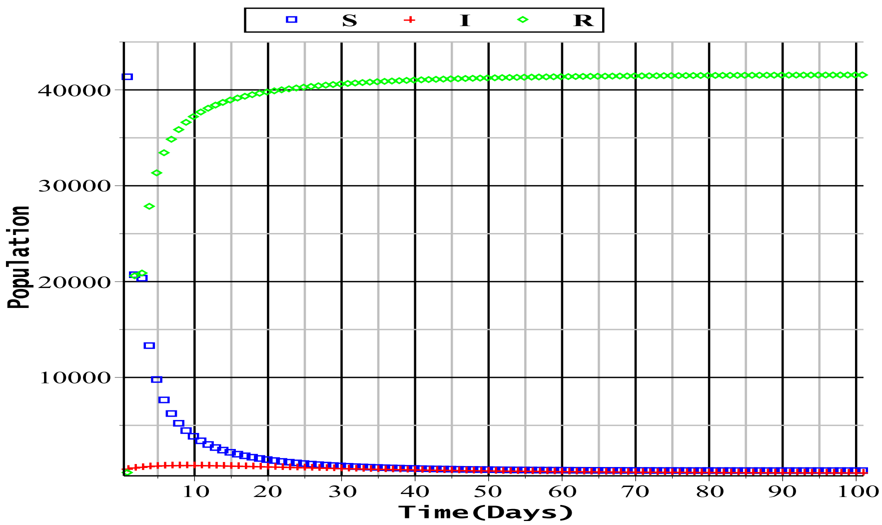

FIGURE 4. The evolution of system SIR with control

In our example, this number began at 430 at the start of the outbreak, dropped to less than 20 after three weeks, and remained consistent at approximately five people until the end of the period.

Figure 7 displays the time evolution of the optimal vaccination effort u_i to be applied to street the output signal to a desired one. The cost function shows that controlling the output state requires a plus effort at the beginning of the epidemic to prevent the disease at the end. This effort demands time and energy (healthcare resources), and it cannot be performed without those resources. The cost of the control reaches its peak on the first day with a value of 49.15%, then goes to a low value of 1.46% on the second day, until it reaches a steady level of 1.65% on day 40.

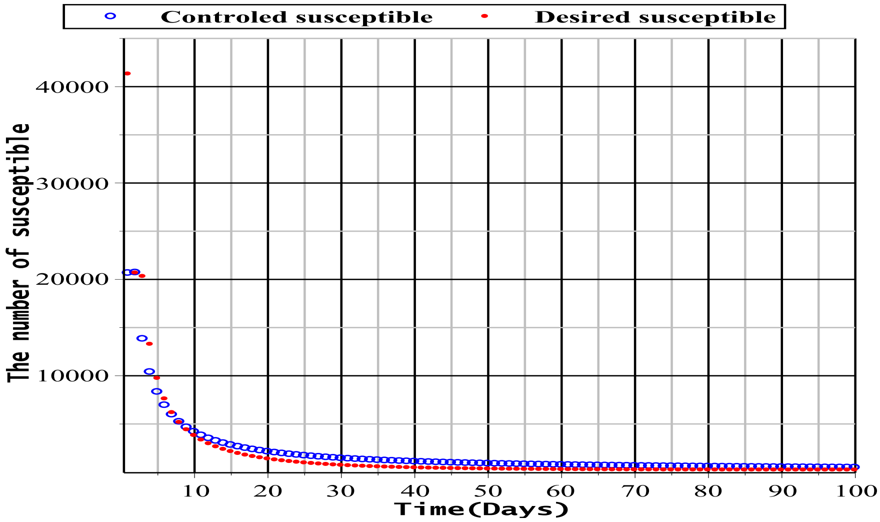

FIGURE 5. The number of infected individuals desired and the number of infected individuals with control.

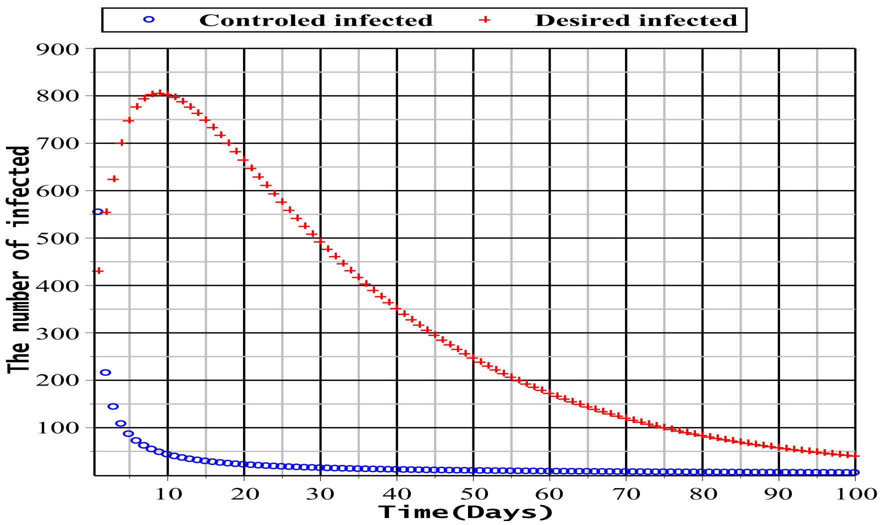

FIGURE 6. The number of susceptible individuals desired and those with control

5. Conclusion

This paper investigated the trajectory controllability problem for a discrete SIR epidemic model with energy constraints. Our approach aims to give an alternative approach to controlling pandemics by considering the desirable output that aligns with the control strategy that public health authorities put in place to contain an infectious disease. We used a technique based on the fixed point theorem, and we established that the set of admissible controls could be characterized by the set of the fixed point of an appropriate mapping. We showed, via numerical simulation, how we can obtain full control of pandemic and disease vanishes as some diseases dictate the use of other types of epidemic models. In future work, we want to extend this method to models with compartments of type SEIR and to systems continuous over time.

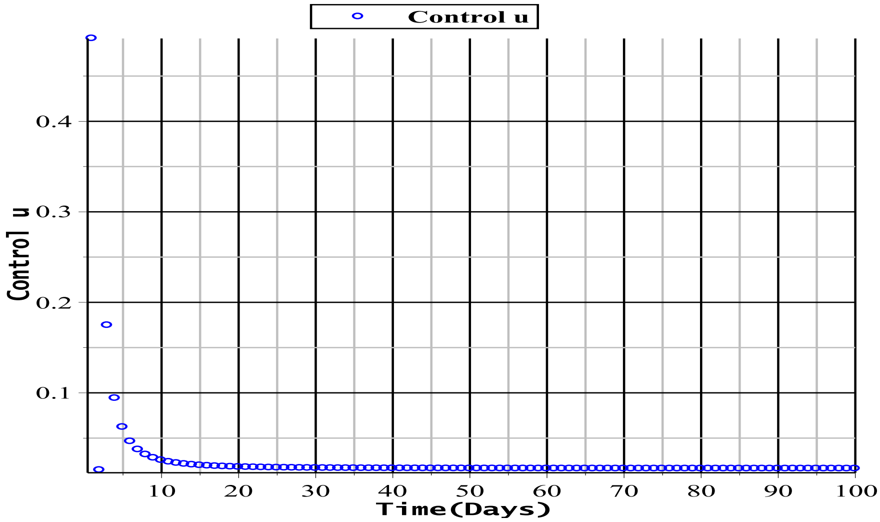

FIGURE 7. Graph of the function control u_i.

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