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TRAVELING SOLUTIONS FOR A MULTI-ANTICIPATIVE CAR-FOLLOWING TRAFFIC MODEL

N. El Khatib[1,*], A. Ghorbel[2], A. Joumaa[1] and M. Zaydan[1]

Abstract. In this paper, we consider a steady state multi-anticipative traffic model and we provide necessarily and sufficient conditions for the existence of traveling solutions. In our work, the word “traveling” means that the distance between two consecutive vehicles travels continuously between two different states. As application to our result, we show that taking a strictly concave optimal velocity, we can construct a traveling solution such that the distance between two vehicles decreases. The existence, uniqueness and the study of the asymptotic behavior of such solutions is done at the level of the Hamilton-Jacobi equation.

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1. Introduction

In order to reduce traffic’s negative effects, such as emissions and congestion, modeling traffic flows and vehicle trajectories has become essential. The creation of traffic models that effectively depict how vehicles behave on road networks is urgently needed.

Discontinuous change in the traffic kinematic features can be characterized as a macroscopic shock wave. Shocks in traffic might happen as a result of an accident, a lane decrease, an entrance ramp, or abrupt breaking, according to Lighthill and Whitham [13]. Many works, including [11, 14, 15], have addressed the design of traveling shock profiles for finite difference schemes approximating hyperbolic conservation laws.

The authors of [10] constructed shock solutions for a first order microscopic model by utilizing its relationship with a macroscopic model. This work was recently extended in [8] where a second order microscopic model was considered. The existence of traveling waves in traffic at the microscopic scale has apparently only been studied in [19, 20], where authors created discrete traveling wave profiles that serve as local attractors for the solution of a local and non-local follow-the-leader model. To our knowledge, there are very few works on this topic (aside from [8, 10]).

In order to accurately represent the driver’s behaviour, a number of car-following models have been suggested (see [4, 16, 17]). The optimal velocity model (OVM), a simple car-following model that successfully captured many aspects of actual traffic flows was first presented by Bando *et al.* in [2]. In this concept, each vehicle is

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[1] Lebanese American University, Department of computer science and mathematics, Byblos Campus, P.O. Box 36, Byblos, Lebanon.
[2] University of Sfax, Higher Institute of Business Adiministration of Sfax, 3018 Sfax, Tunisia.

[*] Corresponding author: nader.elkhatib@lau.edu.lb

described by the optimal velocity function, which depends on the headway distance, and each driver controls the velocity using this function. The OVM is found to accurately characterize the jamming transition.

Lenz *et al.* noted in [12] that correlations between cars have generally been ignored in the majority of traffic models, despite the fact that it's known from experience that drivers frequently detect two or more nearby vehicles ahead. Multi-vehicle interactions result from this, and they have an impact on the phase separation and the fundamental diagram. For these reasons, authors in [12] introduced the multi-anticipative car-following model in which multivehicle interactions are added to the Bando microscopic car-following model. They demonstrated that responding to more than one vehicle ahead causes the dynamical behavior to stabilize and to increase the stable region.

In this paper, we construct traveling solutions for the steady state of the multi-anticipative car-following model which can be seen as a phase transition between two traffic states. The multi-anticipative car-following model (see [12]) is a second order microscopic model given by

$$U_i''(t) = \sum_{k=1}^{n} s_k \left(V\left(\frac{U_{i+k}(t) - U_i(t)}{k}\right) - U_i'(t)\right) \tag{1.1}$$

where

1) U_i denotes the position of the i-th vehicle, U_i' its velocity, U_i'' its acceleration.
2) $n \geq 1$ is a positive integer number.
3) V represents the optimal velocity function.
4) s_k is a positive parameter such that $\sum_{k=1}^{n} s_k$ is the overall sensitivity and $\dfrac{s_k}{s_1} \leq 1$ for $k \in [2, ..., n]$.

In this model, the driver do not only react on the dynamics of the leading vehicle U_{i+1} but also takes into consideration up to n cars ahead.

2. Main results

In this paper, we consider the steady state multi-anticipative car-following model with $s_k = s$ for all $k \in [1, ..., n]$. The model is given by

$$U_i'(t) = s \sum_{k=1}^{n} V\left(\frac{U_{i+k}(t) - U_i(t)}{k}\right). \tag{2.1}$$

For model (2.1), we construct traveling solutions U_i of (2.1) satisfying

$$\begin{cases} U_{i+1}(t) - U_i(t) \to b & \text{as } i \to -\infty, \\ U_{i+1}(t) - U_i(t) \to a & \text{as } i \to +\infty. \end{cases} \tag{2.2}$$

By traveling solutions we mean that U_i has the following form

$$U_i(t) = u\left(i + \frac{t}{T}\right) + ct$$

with

$$\begin{cases} u(y+1) - u(y) \to b & \text{as } y \to -\infty, \\ u(y+1) - u(y) \to a & \text{as } y \to +\infty. \end{cases}$$

The interpretation of (2.2) is that a shock occurred and the interdistance is b far before the shock, and is a far after it. Moreover, the speed of the shock is c. The reader can also notice that we have

$$U_i(t+T) = U_{i+1}(t) + ct$$

and this means that T is the time period for which each vehicle will replace the one preceding him, up to a shift of a distance cT. We look for particular shock solutions of (2.1) of the form

$$U_i(t) = u\left(i + \frac{t}{T}\right) + ct,$$

where u solves

$$\frac{1}{T}u'(y) + c = s\sum_{k=1}^{n} V\left(\frac{u(y+k)-u(y)}{k}\right) \tag{2.3}$$

and

$$\begin{cases} u(y+1)-u(y) \to b & \text{as } y \to -\infty, \\ u(y+1)-u(y) \to a & \text{as } y \to +\infty. \end{cases}$$

We obtain our results in the framework of viscosity solutions and we refer the reader to reference [3, 5, 6] for a full presentation of this theory. To prove our results, the following assumptions on the function $F(p) = sV(p)$ are imposed.

Assumptions (A)

- (A1) (Regularity) $F \in C^1(\mathbb{R})$, $F' \in L^\infty(\mathbb{R})$.
- (A2) (Monotony) $F' > 0$ on $\mathbb{R}$.
- (A3) ((Strict chord inequality) There exists $a, b \in \mathbb{R}$ such that

$$\begin{cases} \dfrac{p}{T} + c \le nF(p) \;\; \text{for } p \in \mathbb{R} \text{ if and only if } p \in [a,b], \\ \text{with equality if and only if } p \in [a,b], \end{cases}$$

with

$$\frac{1}{T} = \frac{n(F(b)-F(a))}{b-a} \text{ and } c = \frac{n(bF(a)-aF(b))}{b-a}.$$

- (A4) (Non degeneracy) $nF'(b) < \dfrac{1}{T} < nF'(a)$.

Remark 2.1 (Comments on assumptions (A))**.** The regularity assumption (A1) provides regular viscosity solutions. Assumption (A2) allows us to get strong comparison results for equation (2.3) which is crucial to prove many parts of our main results. We will show (see Thm. 2.4) that our traveling solutions exist if and only if assumption (A3) is satisfied. Finally, we will use assumption (A4) to get exponential asymptotics of the solution at infinity.

Remark 2.2. As example of general functions F satisfying (A3) and (A4), we can consider $F \in C^1(\mathbb{R})$ and strictly concave. We recall that

$$\frac{1}{T} = \frac{n(F(b) - F(a))}{b - a}.$$

Using the mean value theorem, there exists $e \in (a, b)$ such that $\frac{1}{T} = nF'(e)$. Using that F is strictly concave, we have

$$F'(a) > F'(e) > F'(b)$$

and the function $p \mapsto nF(p) - \frac{p}{T} - c$ is strictly increasing in $[a, e]$ and strictly decreasing in $(e, b]$. In particular, taking the optimal velocity used in [12] and given by

$$V(p) = \tanh(p - h) + \tanh(h)$$

with $h = constant$, we remark that V (and so F) is strictly concave for $p > h$.

The main result of this paper is given in the following theorem.

Theorem 2.3. *i) (Existence.) Assume that (A) holds for some $a, b, c \in \mathbb{R}$ and $T > 0$. There exists a concave solution $u \in C^2(\mathbb{R})$ of (2.3) satisfying for some constant $C > 0$,*

$$|u(y) - \bar{u}(y)| \leq C \tag{2.4}$$

with

$$\bar{u}(y) = \begin{cases} ay & \text{if } y \geq 0, \\ by & \text{if } y < 0. \end{cases}$$

Moreover, we have

$$u'(+\infty) = a \leq u(y+1) - u(y) \leq b = u'(-\infty). \tag{2.5}$$

ii) (Uniqueness.) The solution u is unique (up to translation and addition of constants) among the solutions v with $v \in \mathcal{C}^1(\mathbb{R})$ such that

$$|v - \bar{u}| \leq C$$

for some constant $C > 0$.

In the following theorem, we provide necessarily conditions to construct traveling solutions for (2.1) and it can be seen as a justification of our choice to impose assumptions (A3) and (A4).

Theorem 2.4. *Let $T > 0$ and $c \in \mathbb{R}$. Assume (A1) and (A2). Let $u \in C^2(\mathbb{R})$ be a solution of (2.3) and assume that $G(y) = u(y+1) - u(y)$ is a bounded function. There exits $\bar{a}, \bar{b} \in \mathbb{R}$ such that*

$$\begin{cases} u'(+\infty) = \bar{a}, \\ u'(-\infty) = \bar{b}. \end{cases}$$

Moreover, if $\bar{a} < \bar{b}$*, then we have* u' *is strictly decreasing on* $\mathbb{R}$ *and*

$$\frac{p}{T} + c \le nF(p) \quad \text{for } p \in [\bar{a}, \bar{b}] \text{ with equality if and only if } p = \bar{a}, \bar{b}. \tag{2.6}$$

If $\bar{a} > \bar{b}$*, then we have* u' *is strictly increasing on* $\mathbb{R}$ *and*

$$\frac{p}{T} + c \ge nF(p) \quad \text{for } p \in [\bar{a}, \bar{b}] \text{ with equality if and only if } p = \bar{a}, \bar{b}. \tag{2.7}$$

If $\bar{a} = \bar{b}$*, then we have* $u'(y)$ *is constant on* $\mathbb{R}$.

As a consequence of Theorem 2.4, we have the following non-existence result:

Corollary 2.5. *Assume (A1) and (A2). We assume that there exists* $\bar{a}, \bar{b}, \bar{d} \in \mathbb{R}$*, and* $T > 0$ *such that*

$$\begin{cases} \dfrac{p}{T} + c \le nF(p) \quad \text{for } p \in [\bar{a}, \bar{b}], \\ \text{with equality at least for } p = \bar{a}, \bar{b}, \bar{d} \text{ with } \bar{d} \in (\bar{a}, \bar{b}) \ . \end{cases}$$

Then there is no solution u *of* (2.3) *satisfying*

$$\begin{cases} u'(+\infty) = \bar{a}, \\ u'(-\infty) = \bar{b}. \end{cases}$$

Remark 2.6 (Application to traffic modeling)**.** Planning and managing the infrastructure for directing traffic on roads require the use of traffic models. To model the traffic at large scale (like a city), traffic engineers use macroscopic models. These models use aggregate quantities such as density, average speed and flow of vehicles by making the analogy between the flow of traffic and the flow of a continuous fluid.

However, a disadvantage of such models is that their assumptions are not easy to justify because, in their formulation, the dynamics of vehicles are not described individually. A way to justify the assumptions of macroscopic models is to derive them from microscopic ones so that the behavior of every single vehicle can be considered with high precision.

As a contribution of our study, the choice of strictly concave flux function in the macroscopic LWR model [18] given by

$$\rho_t + (f(\rho))_x = 0 \tag{2.8}$$

where ρ is the unknown representing the cars' density and f is the flux function is justified.

Shock waves are byproducts of traffic congestion and queuing. They are transition zones between two traffic states that move through a traffic environment like, as their name states, a propagating wave. That is, they form both when a queue is forming and when it is dissipating. In 2008, a team of Japanese researchers [21] demonstrated, through a life-size experiment (see the video of the experiment by hitting the link Shockwave traffic), the reality of a particularly frustrating phenomenon for drivers: at high density, traffic is unstable and traffic waves can arise even in the absence of bottleneck, such as a reduction in the number of lanes, for example following a car accident. These are, therefore, so-called "ghost" traffic waves, with no visible external cause.

It is known in traffic modeling that only upward jumps in the density are admissible which means $a < b$. Actually, one can imagine cars lined up in front of a red light (discontinuous initial data with downward jump). At $t = 0$, the red light turns to green but we do not observe this jump if the cars' density moves forward. Actually, we observe that the cars spread out.

Using the homogenization results of [9] (or [7]), the scalar conservation law (2.8) with $f(p) = npF\left(\frac{1}{p}\right)$ could be derived from the microscopic model (2.1). Hence, shocks that are traveling from a to b are expected to exist at the microscopic level and we prove this fact for F satisfying assumptions (A).

Our justification of the strict concavity of the flux function f could be formulated in the following way: for a strictly concave function F satisfying (A1) and (A2), we can construct discrete shocks traveling from a to b with $a < b$. In particular, F being strictly concave implies that

$$f(p) = npF\left(\frac{1}{p}\right) \quad \text{is strictly concave.}$$

Organization of the paper. In Section 3, we define the viscosity solutions of (2.3) and state the crucial proposition: the strong comparison principle. In Section 4, we prove the exponential behavior of the solution at $\pm\infty$. In Section 5, we first show that the interdistance is strictly monotone and then we prove Theorem 2.4. Finally, in Section 6, we prove Theorem 2.3 by constructing a suitable viscosity subsolution.

Normalization. Up to consider a new force function

$$\tilde{F}(p) = T\left(F(p) - \frac{c}{n}\right),$$

and replacing F by $\tilde{F}$, we can assume that

$$T = 1 \quad \text{and } c = 0.$$

This allows us to present our results in the rest of the paper considering the following equation

$$u'(y) = \sum_{k=1}^{n} F\left(\frac{u(y+k) - u(y)}{k}\right). \tag{2.9}$$

3. Viscosity solution

In this section, we first give the definition of viscosity solutions of (2.9). We then state different comparison principles.

Definition 3.1. Let $u \in L^\infty_{loc}(\mathbb{R})$.

1) We say that u is a viscosity subsolution (resp. supersolution) of (2.9) if u is a upper-semi continuous (resp. lower-semi continuous) and if for all test function $\varphi \in \mathcal{C}^1(\mathbb{R})$ such that $u - \varphi$ attains a local maximum (resp. local minimum) at some point x_0, we have

$$\varphi'(x_0) \leq \sum_{k=1}^{n} F\left(\frac{u(x_0+k) - u(x_0)}{k}\right) \quad (\text{resp. } \varphi'(x_0) \geq \sum_{k=1}^{n} F\left(\frac{u(x_0+k) - u(x_0)}{k}\right)).$$

2) We say that u is a viscosity solution of (2.9) if $u \in L^\infty_{loc}(\mathbb{R})$ and u is a viscosity subsolution and viscosity supersolution of (2.9).

Proposition 3.2. *Assume (A1). Let u be a viscosity solution of (2.9). Then $u \in \mathcal{C}^2(\mathbb{R})$.*

Proof. We will first show that u is locally lipschitz. Using (2.9) and that u is locally bounded, for any $R > 0$, there exists $L_R > 0$ such that

$$\left|\sum_{k=1}^{n} F\left(\frac{u(y+k)-u(y)}{k}\right)\right| \leq L_R \text{ if } |y| < R.$$

This implies that u satisfies in the sense of viscosity solutions the following:

$$|u_y| \leq L_R \text{ on } (-R, R).$$

This implies that

$$|u(y+z)-u(y)| \leq L_R z \text{ for } z \geq 0 \text{ and } y, y+z \in (-R, R).$$

Let us now define the function

$$v(y) = u(y) - \int_0^y \sum_{k=1}^{n} F\left(\frac{u(s+k)-u(s)}{k}\right) \mathrm{d}s.$$

Since $v - u \in C^1(\mathbb{R})$, we deduce that in the viscosity sense we have

$$v'(y) = 0 \text{ on } \mathbb{R}.$$

The comparison principle (see [3]) implies that v is constant. This implies $u \in C^1(\mathbb{R})$. Moreover, using $F \in C^1(\mathbb{R})$ and equation (2.9), we deduce that $u \in C^2(\mathbb{R})$.

□

Since we will use the strong comparison principle for two equations ((2.9) and (4.4)), we will state the following proposition for a general function L.

Proposition 3.3 (Strong comparison principle.)**.** *Let $L : \mathbb{R}^{n+1} \to \mathbb{R}$ be a globally lipschitz function such that*

$$L(X_0, X_1, ..., X_n) \text{ is increasing w.r.t each } X_i \text{ for } i \neq 0.$$

We consider the following equation

$$u'(y) = L(u(y), u(y+1), ..., u(y+n)). \tag{3.1}$$

Let u^1 and u^2 be respectively a viscosity subsolution and a viscosity supersolution of (3.1). Assume that for all $y \in \mathbb{R}$,

$$u^1(y) \leq u^2(y).$$

If for some x_0, we have $u^1(x_0) = u^2(x_0)$, then we have for all $y \in \mathbb{R}$,

$$u^1(y) = u^2(y).$$

Proof. See Lemma 6.1 and Lemma 6.2 in [1]. □

Proposition 3.4. *Assume (A1) and (A2). Let u^1 and u^2 be respectively viscosity subsolution and viscosity supersolution of (2.9). We assume that*

$$\lim_{|y|\to+\infty}\left(u^1(y)-u^2(y)\right)\leq 0. \tag{3.2}$$

Then we have,

$$u^1\leq u^2\, on\ \mathbb{R}.$$

Proof. We define

$$M=\sup_{y\in\mathbb{R}}\left\{u^1(y)-u^2(y)\right\}.$$

We want to prove that $M\leq 0$. Assume by contradiction that $M>0$. Using (3.2), we deduce that M is reached at some point y_0. We define

$$\bar{u}^2(y)=u^2(y)+M.$$

For all $y\in\mathbb{R}$, we have $\bar{u}^2(y)\geq u^1(y)$ and $\bar{u}^2(y_0)=u^1(y_0)$. Using that $\bar{u}^2$ is a solution of (2.9) and Proposition 3.3, we get

$$\bar{u}^2(y)=u^1(y)\quad\text{for all } y\in\mathbb{R}.$$

Taking $|y|\to+\infty$, we get $M\leq 0$ which gives a contradiction. □

4. Asymptotic profile

In this section, we study the asymptotic behavior of the solution of (2.9). We have the following proposition.

Proposition 4.1. *Assume (A) (for $T=1$). Let u be a solution of (2.9) and let G be the function defined by $G(y)=u(y+1)-u(y)$. We assume that*

$$\begin{cases} u'(+\infty)=G(+\infty)=a,\\ u'(-\infty)=G(-\infty)=b \end{cases} \tag{4.1}$$

and that for $y\in\mathbb{R}$,

$$a\leq G(y)\leq b. \tag{4.2}$$

Then there exists $K,\gamma>0$ and $c_1,c_2\in\mathbb{R}$ such that

$$\begin{cases} |u(y)-ay-c_1|\leq Ke^{-\gamma y} & \text{for } y\geq 0,\\ |u(y)-by-c_2|\leq Ke^{\gamma y} & \text{for } y\leq 0. \end{cases} \tag{4.3}$$

The proof of this proposition can be derived from the following lemma.

Lemma 4.2. *Assume (A) (for $T = 1$ and $c = 0$). Let u be a solution of (2.9) satisfying (4.1) and G satisfying (4.2). We remark that G satisfies*

$$G'(y) = \sum_{k=1}^{n} F\left(\sum_{l=1}^{k} \frac{G(y+l)}{k}\right) - \sum_{k=1}^{n} F\left(\sum_{l=0}^{k-1} \frac{G(y+l)}{k}\right), \quad y \in \mathbb{R}. \tag{4.4}$$

Recalling that $nF'(a) > 1 > nF'(b)$, let $\varepsilon > 0$ small enough such that

$$\begin{cases} n \min\limits_{[a-e^n\varepsilon, a+e^n\varepsilon]} F'(p) > 1, \\ n \max\limits_{[b-e^n\varepsilon, b+e^n\varepsilon]} F'(p) < 1. \end{cases}$$

We have the following:

1) let $\gamma > 0$ be small enough such that for all $k \in [1, ..., n]$,

$$n \min_{[a-e^n\varepsilon, a+e^n\varepsilon]} F'(p) > \frac{\gamma k}{1 - e^{-\gamma k}}. \tag{4.5}$$

Then there exists a constant $C > 0$ such that for all $y \geq 0$,

$$G(y) \leq a + Ce^{-\gamma y}. \tag{4.6}$$

2) Let $\gamma > 0$ be small enough such that for all $k \in [1, ..., n]$,

$$n \max_{[b-e^n\varepsilon, b+e^n\varepsilon]} F'(p) < \frac{\gamma k}{e^{\gamma k} - 1}. \tag{4.7}$$

Then there exists a constant $C > 0$ such that for all $y \leq 0$,

$$G(y) \geq b - Ce^{\gamma y}. \tag{4.8}$$

Proof of Lemma 4.2. We will only prove part 2) since the proof of part 1) can be done in the same way (even simpler). Using (4.1), let $y_0 < 0$ small enough be such that for all $y \leq y_0$,

$$b - \varepsilon \leq G(y + n) \leq b + \varepsilon. \tag{4.9}$$

We will prove that for $y \leq y_0$,

$$G(y) \geq b - C_1 e^{\gamma(y - y_0)} \tag{4.10}$$

with $C_1 > b - a$.

If (4.10) is true, we obtain (4.8) for all $y \in \mathbb{R}$ because we can easily check that for $y > y_0$,

$$G(y) \geq b - C_1 e^{\gamma(y - y_0)}.$$

We define the following function

$$\varphi(y) = b - C_1 e^{\gamma(y-y_0)} - G(y).$$

We then define

$$M = \sup_{y \le y_0} \varphi(y).$$

We will prove that $M \le 0$. Assume by contradiction that $M > 0$. Using that $G(y) \to b$ as $y \to -\infty$, we deduce that M is reached at some point x. If $x = y_0$, we get

$$0 < b - C_1 - G(x) \le b - a - C_1 < 0.$$

We deduce that $x \ne y_0$ and writing the viscosity inequality, we get

$$\begin{aligned}
-C_1\gamma e^{\gamma(x-y_0)} &\ge \sum_{k=1}^{n} F\left(\sum_{l=1}^{k} \frac{G(x+l)}{k}\right) - \sum_{k=1}^{n} F\left(\sum_{l=0}^{k-1} \frac{G(x+l)}{k}\right) \\
&= F\left(G(x+1)\right) - F\left(G(x)\right) \\
&+ \sum_{k=2}^{n} F\left(\sum_{l=1}^{k-1} \frac{G(x+l)}{k} + \frac{G(x+k)}{k}\right) - \sum_{k=2}^{n} F\left(\sum_{l=1}^{k-1} \frac{G(x+l)}{k} + \frac{G(x)}{k}\right).
\end{aligned}$$

We claim that $x + k < y_0$ for all $k \in [1, ..., n]$. If $\exists k \in [1, ..., n]$ such that $x + k \ge y_0$, we get

$$C_1 e^{\gamma(x-y_0)} \ge C_1 e^{-\gamma k} > b - a$$

for C_1 big enough. This implies that

$$G(x) \ge a > b - C_1 e^{\gamma(x-y_0)}$$

which contradicts the fact that $\varphi(x) > 0$. We deduce that $x + k < y_0$ for all $k \in [1, ..., n]$ and using that

$$\varphi(x+k) \le \varphi(x),$$

we obtain that

$$G(x+k) \ge G(x) + C_1 e^{\gamma(x-y_0)}(1 - e^{\gamma k}).$$

Using assumption (A2) and the above inequality, we get

$$\begin{aligned}
-C_1\gamma e^{\gamma(x-y_0)} &\ge F\left(G(x) + f_1(\gamma)\right) - F\left(G(x)\right) \\
&+ \sum_{k=2}^{n} F\left(\left(\sum_{l=0}^{k-1} \frac{G(x+l)}{k}\right) + \frac{f_k(\gamma)}{k}\right) - \sum_{k=2}^{n} F\left(\sum_{l=0}^{k-1} \frac{G(x+l)}{k}\right)
\end{aligned}$$

$$= \sum_{k=1}^{n} F\left(\left(\sum_{l=0}^{k-1}\frac{G(x+l)}{k}\right)+\frac{f_k(\gamma)}{k}\right)-\sum_{k=1}^{n}F\left(\sum_{l=0}^{k-1}\frac{G(x+l)}{k}\right) \tag{4.11}$$

with

$$f_k(\gamma)=C_1e^{\gamma(x-y_0)}(1-e^{\gamma k}). \tag{4.12}$$

Using (4.11), we remark that we will get a contradiction if we prove

$$\sum_{k=1}^{n} F\left(\left(\sum_{l=0}^{k-1}\frac{G(x+l)}{k}\right)+\frac{f_k(\gamma)}{k}\right)-\sum_{k=1}^{n}F\left(\sum_{l=0}^{k-1}\frac{G(x+l)}{k}\right) > -C_1\gamma e^{\gamma(x-y_0)}. \tag{4.13}$$

We have

$$F\left(\left(\sum_{l=0}^{k-1}\frac{G(x+l)}{k}\right)+\frac{f_k(\gamma)}{k}\right)-F\left(\sum_{l=0}^{k-1}\frac{G(x+l)}{k}\right)=F'(p_k)\frac{f_k(\gamma)}{k}$$

with

$$p_k\in\left[\left(\sum_{l=0}^{k-1}\frac{G(x+l)}{k}\right)+\frac{f_k(\gamma)}{k},\sum_{l=0}^{k-1}\frac{G(x+l)}{k}\right]. \tag{4.14}$$

To get (4.13), we have to prove that

$$\sum_{k=1}^{n}\frac{F'(p_k)(e^{\gamma k}-1)}{\gamma k}<1. \tag{4.15}$$

Using (4.9), we have for all $l\in[0,...,k-1]$,

$$G(x+l)\le b+\varepsilon$$

which implies

$$\sum_{l=0}^{k-1}\frac{G(x+l)}{k}\le(b+\varepsilon).$$

Using that $\varphi(x)>0$, we get

$$G(x)<b-C_1e^{\gamma(x-y_0)}.$$

Using that $G(y)\ge b-\varepsilon$ for $y\le y_0$, we deduce that

$$\varepsilon>C_1e^{\gamma(x-y_0)}.$$

This implies that

$$f_k(\gamma)=C_1e^{\gamma(x-y_0)}(1-e^{\gamma k})$$

$$\geq \varepsilon\left(1-e^{\gamma k}\right).$$

Using that $G(x+l) \geq b-\varepsilon$, we deduce that

$$\begin{aligned}\left(\sum_{l=0}^{k-1}\frac{G(x+l)}{k}\right)+\frac{f_k(\gamma)}{k} &\geq (b-\varepsilon)+\frac{\varepsilon(1-e^{\gamma k})}{k}\\ &\geq b-\varepsilon e^{\gamma n}.\end{aligned}$$

Finally, we have that

$$\begin{aligned}\sum_{k=1}^{n}\frac{F'(p_k)(e^{\gamma k}-1)}{\gamma k} &\leq \sum_{k=1}^{n}\frac{\max\limits_{p_k\in[b-\varepsilon e^n, b+\varepsilon e^n]} F'(p_k)(e^{\gamma k}-1)}{\gamma k}\\ &< \sum_{k=1}^{n}\frac{1}{n}=1\end{aligned}$$

where we use the condition on γ in (4.7). We deduce that $M \leq 0$ and in particular, we get (4.8).

□

Proof of Proposition 4.1. We will only prove the first line in (4.3). The proof of the second line is similar. We have for $y \geq 0$,

$$\begin{aligned}u(y)-u(0) = \int_0^y u'(s)\mathrm{d}s &= \int_0^y \sum_{k=1}^{n} F\left(\frac{u(s+k)-u(s)}{k}\right)\mathrm{d}s\\ &= \int_0^y \sum_{k=1}^{n} F\left(\sum_{l=0}^{k-1}\frac{G(s+l)}{k}\right)\mathrm{d}s\\ &\leq \int_0^y \sum_{k=1}^{n} F\left(\sum_{l=0}^{k-1}\frac{(a+Ce^{-\gamma s})}{k}\right)\mathrm{d}s\\ &= \int_0^y \sum_{k=1}^{n} F(a+Ce^{-\gamma s})\mathrm{d}s\\ &= \int_0^y \sum_{k=1}^{n}\big(F(a+Ce^{-\gamma s})-F(a)+F(a)\big)\,\mathrm{d}s\\ &= ay+\int_0^y \sum_{k=1}^{n}\big(F(a+Ce^{-\gamma s})-F(a)\big)\,\mathrm{d}s\\ &\leq ay+||F'||_\infty Cn\int_0^y e^{-\gamma s}\mathrm{d}s\\ &= ay+\frac{||F'||_\infty Cn(1-e^{-\gamma y})}{\gamma}\end{aligned}$$

where we use the following:

$$\begin{cases} u(s+k)-u(s)=\sum_{l=0}^{k-1} G(s+l) \text{ in the second line,} \\ \text{Lemma 4.2 in the third line,} \\ nF(a)=a \text{ in the sixth line and} \\ F \text{ is lipschitz in the seventh line.} \end{cases}$$

□

In the next proposition, we show how to construct a particular solution of (2.9).

Proposition 4.3. *Let u be a solution of (2.9) such that u satisfies $|u(y)-\bar{u}(y)| \le C$ with $\bar{u}(y)=\min(\bar{a}y,\bar{b}y)$ with $\bar{a}<\bar{b}$. Assume that u is concave on $\mathbb{R}$ and that*

$$u'(-\infty)=\bar{b} \quad \textit{and} \quad u'(+\infty)=\bar{a}. \tag{4.16}$$

There exists constants c_1, c_2 such that $\tilde{u}$ is solution of (2.9) with

$$\lim_{|y|\to+\infty}(\tilde{u}(y)-\bar{u}(y))=0 \quad \textit{with } \tilde{u}(y)=c_1+u(y+c_2). \tag{4.17}$$

Proof. We define $\tilde{u}(y)=c_1+u(y+c_2)$ where c_1 and c_2 are constants to be chosen later. We can easily verify that $\tilde{u}$ is a solution of (2.9). Let $\phi(y)=u(y)-\bar{u}(y)$ with

$$\bar{u}(y)=\begin{cases} \bar{a}y & y\ge 0, \\ \bar{b}y & y<0. \end{cases}$$

Using that $|\phi(y)|\le C$ and ϕ is concave, we deduce that $\lim_{|y|\to+\infty}\phi(y)$ exists. We denote by

$$c^+=\lim_{y\to+\infty}\phi(y) \quad \text{and} \quad c^-=\lim_{y\to-\infty}\phi(y).$$

Then, choosing (c_1,c_2) as the solution of the following system,

$$\begin{cases} c_1+c^++ac_2=0 \\ c_1+c^-+bc_2=0, \end{cases}$$

we obtain $\lim_{|y|\to+\infty}(\tilde{u}(y)-\bar{u}(y))=0$. □

5. Proof of Theorem 2.4

In this section, we prove our first main result, Theorem 2.4. We will first prove that the bounded interdistance is monotone. Let u be a solution of (2.9). We define the function

$$G(y)=u(y+1)-u(y). \tag{5.1}$$

The function G satisfies

$$G'(y) = \sum_{k=1}^{n} F\left(\sum_{l=1}^{k} \frac{G(y+l)}{k}\right) - \sum_{k=1}^{n} F\left(\sum_{l=0}^{k-1} \frac{G(y+l)}{k}\right), \quad y \in \mathbb{R}. \tag{5.2}$$

We have the following proposition.

Proposition 5.1. *Assume (A1)-(A2). Let $G \in C^1(\mathbb{R})$ be bounded function such that G is a solution of (5.2). Then we have that*

$$\begin{cases} G \text{ is strictly increasing on } \mathbb{R} & \text{or} \\ G \text{ is strictly decreasing on } \mathbb{R} & \text{or} \\ G \text{ is constant on } \mathbb{R}\ . \end{cases} \tag{5.3}$$

Proof. Let $y_0 \in \mathbb{R}$. We have

$$\begin{cases} G(y) \leq G(y_0) \ \text{ for all } y \geq y_0 & \text{or} \\ G(y) \geq G(y_0) \ \text{ for all } y \geq y_0. \end{cases} \tag{5.4}$$

We will prove the first inequality of (5.4), the second one can be done similarly. Let $x_0, x_1 \in \mathbb{R}$ such that $x_1 > x_0$ and

$$G(x_1) \leq G(x_0). \tag{5.5}$$

Let $\eta > 0$ and $m = G(x_0)$. We define

$$M = \sup_{x \geq x_0} \{G(x) - m - \eta\}.$$

By contradiction, we assume that $M > 0$.

Case 1: M is reached for some point $\bar{x} > x_0$. Writing the viscosity inequality, we get

$$\begin{aligned} 0 \leq\ & F(G(\bar{x}+1)) - F(G(\bar{x})) \\ & + \sum_{k=2}^{n} F\left(\left(\sum_{l=1}^{k-1} \frac{G(\bar{x}+l)}{k}\right) + \frac{G(\bar{x}+k)}{k}\right) - \sum_{k=2}^{n} F\left(\left(\sum_{l=1}^{k-1} \frac{G(\bar{x}+l)}{k}\right) + \frac{G(\bar{x})}{k}\right). \end{aligned}$$

Using that $G(\bar{x}) \geq G(\bar{x}+k)$ for all $k \in [1, ..., n]$, we get a contradiction in the above inequality if $G(\bar{x}) > G(\bar{x}+1)$. If $G(\bar{x}) = G(\bar{x}+1)$, we deduce that $M = G(\bar{x}+1) - m - \eta$ and so we can write the viscosity inequality using the function G at the point $\bar{x}+1$. Continuing in the same way, we construct a sequence $x_n = \bar{x} + n$ such that $M = G(x_n) - m - \eta$. We then define the following function

$$G_n(x) = G(x + x_n) - m.$$

Using the fact that G is a bounded Lipshitz continuous function, we have (up to passing to the limit on a subsequence)

$$G_n \to G_\infty.$$

The stability of viscosity solutions implies (see [3]) that G_∞ solves (5.2). In addition, using the definition of M, we also have for $x \in \mathbb{R}$,

$$G_\infty(0) \geq G_\infty(x).$$

Using the strong comparison principle (Prop. 3.3) , we get for all $x \in \mathbb{R}$

$$G_\infty(x) = G_\infty(0) \geq \eta > 0.$$

Equation (5.5) implies that $G_n(x_1 - x_n) \leq 0$. Taking $n \to +\infty$, we get a contradiction. We deduce that $M \leq 0$. Sending η to zero, we get the desired result.

Case 2: M is not reached. In this case, there exists a sequence $x_n \to +\infty$ such that

$$G(x_n) - m - \eta \to M. \tag{5.6}$$

We define

$$G_n(x) = G(x + x_n) - m - \eta.$$

Up to subsequence, we have $G_n \to G_\infty$. This implies that for all $x \in \mathbb{R}$,

$$G_\infty(0) \geq G_\infty(x). \tag{5.7}$$

The strict comparison principle implies

$$G_\infty(x) = G_\infty(0) > 0$$

and this contradicts (5.5) and implies that $M \leq 0$. Since $M \leq 0$, this implies that G is monotone. It remains to show that G is strictly monotone or constant. Assume that G is non-decreasing. We will show that

$$G \text{ is strictly increasing on } \mathbb{R} \text{ or } G \text{ is constant on } \mathbb{R}.$$

Assume that there exists $r > 0$ and $x_0 \in \mathbb{R}$ such that

$$G(x) = G(x_0) \text{ for } x \in (x_0 - r, x_0).$$

We will prove that $G(x) = G(x_0)$ for all $x \geq x_0$. Let $x_1 \in (x_0 - r, x_0)$. We have

$$G(x_0) \leq G(x_1)$$

and this implies (see (5.4)) $G(x) \leq G(x_1)$ for all $x \geq x_1$. We deduce that $G(x) \leq G(x_1) = G(x_0)$ for all $x \geq x_0$. Using that G is non-decreasing, we have $G(x) \geq G(x_0)$ for $x \geq x_0$ and thus $G(x) = G(x_0)$ for $x \geq x_0$. Using all above properties of G , we remark that G is not strictly increasing and G is not constant implies that G has a global maximum at some point x_0 with

$$\begin{cases} G(x) = G(x_0) & \text{if } x \geq x_0, \\ G(x) < G(x_0) & \text{if } x < x_0. \end{cases}$$

Using the strong comparison principle (Prop. 3.3) and that $G(x_0)$ is a solution of (5.2), we get $G(x) = G(x_0)$ for all $x \in \mathbb{R}$ which gives a contradiction. Similarly, we can prove that if G is non-increasing, then G is strictly decreasing or constant.

□

We are now ready to prove Theorem 2.4 and since we are working with $T = 1$ and $c = 0$, we recall the Theorem 2.4 for $T = 1$ and $c = 0$.

Theorem 5.2. *Assume (A1) and (A2). Let $u \in \mathcal{C}^1(\mathbb{R})$ be a solution of (2.9) such that $G(y) = u(y+1) - u(y)$ is a bounded function. There exits $\bar{a}, \bar{b} \in \mathbb{R}$ such that*

$$\begin{cases} u'(+\infty) = G(+\infty) = \bar{a} \\ u'(-\infty) = G(-\infty) = \bar{b}. \end{cases}$$

Moreover, if $\bar{a} < \bar{b}$, then we have u' is strictly decreasing on $\mathbb{R}$ and

$$p \leq nF(p) \quad \textit{for } p \in [\bar{a}, \bar{b}] \textit{ with equality if and only if } p = \bar{a}, \bar{b}.$$

If $\bar{a} > \bar{b}$, then we have u' is strictly increasing on $\mathbb{R}$ and

$$p \geq nF(p) \quad \textit{for } p \in [\bar{a}, \bar{b}] \textit{ with equality if and only if } p = \bar{a}, \bar{b}.$$

If $\bar{a} = \bar{b}$, then we have u' is constant on $\mathbb{R}$.

Proof. Using Proposition 5.1, we deduce that the limit of G at $\pm\infty$ exist,

$$G(+\infty) = \bar{a}, \quad G(-\infty) = \bar{b}.$$

Case 1: if G is strictly decreasing. In this case $\bar{a} < \bar{b}$. Using that

$$u'(y) = \sum_{k=1}^{n} F\left(\sum_{l=0}^{k-1} \frac{G(y+l)}{k}\right), \tag{5.8}$$

we deduce that u' is strictly decreasing. Therefore $u'(\pm\infty)$ exist and

$$\begin{cases} u'(+\infty) = G(+\infty) = \bar{a}, \\ u'(-\infty) = G(-\infty) = \bar{b}. \end{cases}$$

Taking y to $+\infty$ in (5.8), we obtain

$$\bar{a} = nF(\bar{a}).$$

Similarly, we can prove that $\bar{b} = nF(\bar{b})$. It remains to show that $p < nF(p)$ if $p \in (\bar{a}, \bar{b})$. We define the function

$$\hat{u}(t,y) = u(t+y).$$

We remark that $\hat{u}$ is a viscosity solution of

$$\hat{u}_t(t,y) = \sum_{k=1}^{n} F\left(\frac{\hat{u}(t,y+k) - \hat{u}(t,y)}{k}\right).$$

We rescale $\hat{u}$ in the following way

$$\hat{u}^{\varepsilon}(t,y) = \varepsilon \hat{u}\left(\frac{t}{\varepsilon}, \frac{y}{\varepsilon}\right).$$

As ε goes to zero, we have that $\hat{u}^{\varepsilon} \to u^0$ with

$$u^0(t,y) = \begin{cases} \bar{a}(y+t) & \text{if } y+t \geq 0, \\ \bar{b}(y+t) & \text{if } y+t < 0. \end{cases}$$

As in Theorem 1.3 in [9], we can prove that u^0 is a viscosity solution of

$$u^0_t = nF(u^0_y).$$

Testing u^0 from above with any test function of the form

$$\varphi(y+t) \quad \text{where } \bar{a} \leq \varphi'(0) \leq \bar{b},$$

we deduce that

$$p \leq nF(p) \quad \text{for all } p \in [\bar{a}, \bar{b}] \text{ with equality for } p = \bar{a}, \bar{b}.$$

Assume by contradiction that there exists $\bar{d} \in (\bar{a}, \bar{b})$ such that $nF(\bar{d}) = \bar{d}$. Using Proposition 4.3, there exists $\tilde{u}$ solution of (2.9) such that

$$\lim_{|y| \to +\infty} (\tilde{u} - \bar{u}) = 0$$

with $\bar{u}(y) = \min(\bar{a}y, \bar{b}y)$. Using that $nF(\bar{d}) = \bar{d}$, we define the following solution of (2.9),

$$v(y) = \bar{d}y.$$

Moreover, we have

$$\lim_{|y| \to +\infty} (\tilde{u} - v) = -\infty. \tag{5.9}$$

Using Proposition 3.4, we obtain

$$\tilde{u} \leq v \quad \text{on } \mathbb{R}.$$

Using (5.9), we remark that $m = \min_{\mathbb{R}}(v - \tilde{u})$ exists. We have for all $y \in \mathbb{R}$,

$$\tilde{u} + m \leq v$$

with equality at some point x_0. Using Proposition 3.3, we get that

$$\tilde{u}(y) + m = v(y) \quad \text{for all } y \leq x_0.$$

Taking $y \to -\infty$, we get a contradiction and this implies that $nF(\bar{d}) > \bar{d}$.

Case 2: if G is strictly increasing. Similar to the above case.

Case 3: if G is constant. We obtain that u' is constant.

□

6. Proof of Theorem 2.3

In this section, we prove Theorem 2.3. We construct the solution via Perron's method (see [3]). First, the following lemma provides a viscosity supersolution.

Lemma 6.1. *Assume (A1) and (A3). Let $\bar{u}$ defined by*

$$\bar{u}(y) = \min(ay, by) = \begin{cases} ay & \text{if } y > 0 \\ by & \text{if } y \leq 0. \end{cases}$$

Then, $\bar{u}$ is a viscosity supersolution of (2.9).

Proof. Using (A3), we can easily check that $y \to ay$ and $y \to by$ are two solutions of (2.9). Then, using the stability of viscosity solutions (see [3]), we deduce that $\bar{u}$ is a viscosity supersolution of (2.9). □

We turn now to the construction of the subsolution. In order to do this, we need the following lemma.

Lemma 6.2. *Assume (A1). Let us consider $h_0 \in C([0, n])$. Then, there exists a unique $h \in C^1(-\infty, 0) \cap C((-\infty, n])$ solution of*

$$\begin{cases} h'(y) = \displaystyle\sum_{k=1}^{n} F\left(\frac{h(y+k) - h(y)}{k}\right) & \text{on } (-\infty, 0), \\ h(y) = h_0(y) & \text{on } [0, n]. \end{cases} \tag{6.1}$$

Proof. For $\delta > 0$ small, we define

$$\Phi(h)(y) = \begin{cases} h_0(y) & \text{on } [0, n], \\ h_0(y) - \displaystyle\int_y^0 \sum_{k=1}^{n} F\left(\frac{h(z+k) - h(z)}{k}\right) \mathrm{d}z & \text{on } [-\delta, 0). \end{cases}$$

The operator Φ is defined on the set

$$X = \{h \in C([-\delta, n]) \ \text{ with } \ h = h_0 \ \text{ on } \ [0, n]\}$$

which is a closed subset of the Banach space $C([-\delta, n])$. We can easily check that we have

$$||\Phi(h) - \Phi(g)||_{L^\infty(-\delta,n)} \leq 2n\delta ||F'||_{L^\infty(\mathbb{R})} ||h-g||_{L^\infty(-\delta,n)}$$

which shows that Φ is a contraction on X for small δ. Using the fixed point theorem, there exists $h \in X$ such that $h = \Phi(h)$. This gives a solution of (6.1) on $[-\delta, 0)$. To construct a solution on $(-\infty, 0)$, we use an iteration argument by proceeding as above on the intervals $\left[-(k+2)\frac{\delta}{2}, -k\frac{\delta}{2}\right)$ for $k \in \mathbb{N}$.

□

Proposition 6.3. *Assume (A). For $y \geq 0$, we define $g(y) = ay - \delta e^{-\gamma y}$ for $y \geq 0$. Using Lemma 6.2, we extend g by continuity on $y < 0$ as the solution of*

$$g'(y) = \sum_{k=1}^{n} F\left(\frac{g(y+k) - g(y)}{k}\right).$$

Then, g is a viscosity subsolution of (2.9).

Proof. By construction (see Lem. 6.2), we need to show that g is a viscosity subsolution for $y \geq 0$.

Case 1: $y > 0$: We will prove that

$$g'(y) < \sum_{k=1}^{n} F\left(\frac{g(y+k) - g(y)}{k}\right).$$

On the one hand, we have

$$g'(y) = a + \delta\gamma e^{-\gamma y}.$$

On the other hand, we have

$$\begin{aligned}
\sum_{k=1}^{n} F\left(\frac{g(y+k) - g(y)}{k}\right) &= \sum_{k=1}^{n} F\left(a + \frac{\delta e^{-\gamma y}(1 - e^{-k\gamma})}{k}\right) \\
&= \sum_{k=1}^{n} \left(F(a) + F'(\sigma_k)\frac{\delta e^{-\gamma y}(1 - e^{-k\gamma})}{k}\right) \\
&= a + \sum_{k=1}^{n} F'(\sigma_k)\frac{\delta e^{-\gamma y}(1 - e^{-k\gamma})}{k}
\end{aligned}$$

where $\sigma_k \in \left[a, a + \frac{\delta e^{-\gamma y}(1 - e^{-k\gamma})}{k}\right] \subset \left[a, a + \frac{\delta}{k}\right]$. We claim that

$$\delta\gamma e^{-\gamma y} < \delta e^{-\gamma y}\left(\sum_{k=1}^{n} F'(\sigma_k)\frac{(1 - e^{-k\gamma})}{k}\right)$$

which is equivalent to

$$1 < \sum_{k=1}^{n} \frac{F'(\sigma_k)(1-e^{-k\gamma})}{\gamma k}. \tag{6.2}$$

Using (A4), we obtain (6.2) for δ, γ small enough.

Case 2: $y = 0$**:** Let $\varphi \in C^1(\mathbb{R})$ be a test function such that

$$\begin{aligned} &g(y) \le \varphi(y) \quad \text{on } \mathbb{R} \\ &g(0) = \varphi(0). \end{aligned}$$

Then, we have

$$\varphi'(0) \le g'(0^-) = \sum_{k=1}^{n} F\left(\frac{g(k)-g(0)}{k}\right)$$

which is the desired result.

□

Proposition 6.4. *Let $\bar{F}$ be a truncation of F defined as follows:*

$$\bar{F}(p) = \begin{cases} F(p) & \text{if } p \in [a,b] \\ F(a) & \text{if } p < a \\ F(b) & \text{if } p > b. \end{cases}$$

Let g be the viscosity subsolution of (2.9) provided in Proposition 6.3 replacing F by $\bar{F}$. We define for $y \in \mathbb{R}$ the function ,

$$H(y) = g(y+1) - g(y).$$

Then, the function H is non-increasing on $\mathbb{R}$. Moreover, we have $b \ge H(y) \ge a$ for all $y < -1$.

Proof. If $y < 0$, we have that

$$g'(y) = \sum_{k=1}^{n} \bar{F}\left(\frac{g(y+k)-g(y)}{k}\right).$$

Using that $F(b) \ge \bar{F} \ge F(a)$, we get

$$nF(b) = b \ge g'(y) \ge nF(a) = a$$

which implies $b \ge H(y) \ge a$ for $y < -1$. Let us now prove that H is non-increasing on $(-\infty, 0)$. First, let us recall that the following holds:

$$\begin{cases} g'(x) = \sum_{k=1}^{n} \bar{F}\left(\sum_{l=0}^{k-1} \dfrac{H(x+l)}{k}\right) & \text{if } x < 0, \\ H'(x) = \sum_{k=1}^{n} \bar{F}\left(\sum_{l=1}^{k} \dfrac{H(x+l)}{k}\right) - \sum_{k=1}^{n} \bar{F}\left(\sum_{l=0}^{k-1} \dfrac{H(x+l)}{k}\right) & \text{if } x < -1. \end{cases} \tag{6.3}$$

Let $x_0 < 0$. We want to prove that

$$M = \sup_{x \geq x_0} \{H(x) - H(x_0)\} \leq 0.$$

By contradiction, assume that $M > 0$. We recall that we have the following:

$$\begin{cases} H \text{ is continuous,} \\ H(x) \geq a \text{ for all } x < -1, \\ H(x) = a + \delta e^{-\gamma x}(1 - e^{-\gamma}) \text{ if } x \geq 0\ . \end{cases} \tag{6.4}$$

From (6.4), we remark that H is non-increasing for $x \geq 0$. Hence, M is reached at some point $\bar{x} \leq 0$. Moreover, if $\bar{x} = x_0$, we get $M = 0$. We deduce that $\bar{x} > x_0$.

Case 1: $\bar{x} = 0$. Using that H reaches its maximum on $(x_0, +\infty)$ at the point 0, we deduce that

$$\begin{cases} H'(0^+) \leq 0, \\ H'(0^-) \geq 0. \end{cases}$$

This implies

$$g'(0^+) \geq g'(1) \geq g'(0^-) = \sum_{k=1}^{n} \bar{F}\left(\frac{g(k) - g(0)}{k}\right).$$

From the proof of Proposition 6.3 (case 1), we have

$$g'(0^+) < \sum_{k=1}^{n} \bar{F}\left(\frac{g(k) - g(0)}{k}\right)$$

which gives a contradiction.

Case 2: $-1 < \bar{x} < 0$. Since g is derivable for $x \neq 0$, we obtain

$$H'(\bar{x}) = g'(\bar{x} + 1) - g'(\bar{x}) = 0.$$

Using that $\bar{x} + 1 > 0$, we have

$$\sum_{k=1}^{n} \bar{F}\left(\frac{g(\bar{x} + 1 + k) - g(\bar{x} + 1)}{k}\right) > g'(\bar{x} + 1) = g'(\bar{x}) = \sum_{k=1}^{n} \bar{F}\left(\frac{g(\bar{x} + k) - g(\bar{x})}{k}\right). \tag{6.5}$$

Using that $H(\bar{x}) \geq H(\bar{x} + k)$, we obtain

$$g(\bar{x} + k) - g(\bar{x}) \geq g(\bar{x} + 1 + k) - g(\bar{x} + 1).$$

Finally, using that $\bar{F}$ is non-decreasing, we get a contradiction in (6.5).

Case 3: $\bar{x} = -1$**.** Using that H reaches its maximum on $(x_0, +\infty)$ at the point -1, we deduce that

$$H'(-1^-) \geq 0.$$

This implies

$$g'(-1) \leq g'(0^-)$$

which gives (using the first line in (6.3)),

$$\bar{F}(H(-1)) + \sum_{k=2}^{n} \bar{F}\left(\left(\sum_{l=1}^{k-1} \frac{H(-1+l)}{k}\right) + \frac{H(-1)}{k}\right) \leq \bar{F}(H(0)) + \sum_{k=2}^{n} \bar{F}\left(\left(\sum_{l=1}^{k-1} \frac{H(-1+l)}{k}\right) + \frac{H(-1+k)}{k}\right).$$

Using that $H(-1) \geq H(-1+k)$, and that $\bar{F}$ is non-decreasing, we deduce that

$$H(-1) \leq H(0).$$

But we have $H(-1) \geq H(0)$ which implies that H reaches its maximum at 0. This gives a contradiction thanks to case 1.

Case 4: $\bar{x} < -1$**.** Using the second line in (6.3), we have

$$\begin{aligned} 0 = {} & \bar{F}(H(\bar{x}+1)) - \bar{F}(H(\bar{x})) \\ & + \sum_{k=2}^{n} \bar{F}\left(\left(\sum_{l=1}^{k-1} \frac{H(\bar{x}+l)}{k}\right) + \frac{H(\bar{x}+k)}{k}\right) - \sum_{k=2}^{n} \bar{F}\left(\left(\sum_{l=1}^{k-1} \frac{H(\bar{x}+l)}{k}\right) + \frac{H(\bar{x})}{k}\right). \end{aligned}$$

We recall that the function $\bar{F}$ is non-decreasing. Using that $H(\bar{x}) \geq H(\bar{x}+k)$ for all $k \in [1, ..., n]$, we get

$$0 \leq \bar{F}(H(\bar{x}+1)) - \bar{F}(H(\bar{x})).$$

If $H(\bar{x}) > H(\bar{x}+1)$, we obtain a contradiction using that

$$\bar{F}(H(\bar{x})) > \bar{F}(H(\bar{x}+1)). \tag{6.6}$$

In fact, we know that $H(x) \in [a, b]$ for all $x < -1$ which implies

$$H(\bar{x}+1) < H(\bar{x}) \leq b.$$

If $H(\bar{x}+1) < a$ then (6.6) is true since $H(\bar{x}) \geq a$. If $H(\bar{x}+1) \geq a$, we use the strict monotony of the function $\bar{F}$ on $[a, b]$.

If $H(\bar{x}) = H(\bar{x}+1)$, we deduce that $M = H(\bar{x}+1) - H(x_0)$ and so we can write the viscosity inequality using the function H at the point $\bar{x}+1$. Continuing in the same way, we obtain a contradiction once we reach $x_n = \bar{x} + n$ with $n \in \mathbb{N}$ such that $x_n \geq -1$.

□

As a consequence of the preceding proposition, we have the following corollary.

Corollary 6.5. *Assume (A). Let g be the viscosity subsolution provided by Proposition 6.4. Then, we have*

$$\begin{cases} g \text{ is a viscosity subsolution of } (2.9) \\ g \text{ is concave on } \mathbb{R} \\ g'(+\infty) = H(+\infty) = a \\ g'(-\infty) = H(-\infty) = b. \end{cases} \tag{6.7}$$

Proof. The proof of this corollary is divided into the following steps:

Step 1: $H'(-\infty)$ and $g'(-\infty)$ exist. Using that H is non-increasing and $a \le H(y) \le b$ on $(-\infty, -1)$, we deduce that $H'(-\infty)$ exists. Moreover, for $x < 0$, we have

$$g''(y) = \sum_{k=1}^{n} \left(\left(\sum_{l=0}^{k-1} \frac{H'(y+l)}{k} \right) \bar{F}' \left(\sum_{l=0}^{k-1} \frac{H(y+l)}{k} \right) \right) \le 0 \tag{6.8}$$

and $a \le g'(y) \le b$ which implies that $g'(-\infty)$ exists. We denote

$$d = \lim_{y \to -\infty} g'(y) = \lim_{y \to -\infty} H(y).$$

Step 2: $d = b$. Passing to the limit, we have that

$$d = \sum_{k=1}^{n} \bar{F}(d) = nF(d)$$

which implies (thanks to (A3)) that

$$d = a \quad \text{or} \quad d = b.$$

Using that H is non-increasing and its definition for $x \ge 0$, we deduce that $d = b$.

Step 3: g is a subsolution of (2.9). Since H is non-increasing, $H(-\infty) = b$ and $H(+\infty) = a$, we deduce that $H(y) \in [a, b]$ for all $y \in \mathbb{R}$. Using that $\bar{F}(p) = F(p)$ in $[a, b]$, we get the desired result.

Step 4: g is concave. We remark that for $y > 0$, we have

$$g''(y) = -\delta\gamma^2 e^{-\gamma y} < 0$$

and for $y < 0$, thanks to (6.8), we also have $g''(y) \le 0$. Therefore, g is concave on $\mathbb{R}$.

□

We can now otain the proof of the main result.

Proof of Theorem 2.3 . Let g be the viscosity subsolution in Corollary 6.5. Applying Proposition 4.1 for $y < 0$ and using the definition of g for $y \geq 0$, we deduce that there exists $C > 0$ such that

$$|g(y) - \bar{u}(y)| \leq C$$

with $\bar{u}(y) = \min(ay, by)$. Using Proposition 4.3, we construct a new subsolution $\underline{u}(y)$ such that

$$\lim_{|y| \to +\infty} (\underline{u}(y) - \bar{u}(y)) = 0.$$

Finally, from Proposition 3.4, we know that $\underline{u}(y) \leq \bar{u}(y)$. Applying Perron method (see [3]), we construct a viscosity solution u of (2.9) such that

$$\underline{u}(y) \leq u(y) \leq \bar{u}(y).$$

Using Proposition 3.2, we have that $u \in C^2(\mathbb{R})$. Applying the results of Theorem 2.4, we deduce that u is concave and satisfies (2.4) and (2.5). To prove the uniqueness result (point ii) in Theorem 2.3, we follow the same path: Proposition 4.3 then Proposition 3.4.

□

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