ON THE DYNAMICS OF ROTATING RIGID TUBE AND ITS INTERACTION WITH AIR

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Abstract. Rotating an axially symmetric rigid body on a horizontal plane is rather a common and simple experience, but this experience has attracted a great deal of interests due to it exhibiting novel features and containing fairly complicated mechanics. This paper is concerned with the three-dimensional rotational motion of a rigid tube on a plane. We present the governing dynamical equations of this motion and give a numerical treatment, based on which we discuss the nutation of tube and simulate the trajectory of tube end. We also discuss how fast the angular velocity should be in order to initiate an uninterrupted, steady rotational motion. Then the air lift related to such a three-dimensional rotation of tube is modeled by using Kutta-Joukowski law. By employing this model, we show that the air lift indeed “lift” the tube head during rotating.

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1. Introduction

Rotating an axially symmetric rigid body, such as top, disk, ring, ellipsoidal ball, egg-like object, cylinder, tube, and so on, on a horizontal plane and observing the ensuing motion are common experiences to almost everyone. These experiences have also triggered a great deal of researches and revealed abundant phenomena and physics [1–9, 12, 14, 17–21, 23–29, 31–33, 36, 38]. Among all kinds of axially symmetric rigid bodies, the cylinder or tube placed on a horizontal plane is one of the simplest apparatuses, if putting one finger on one end of the cylinder or tube and pressing hard, it will shoot out and then undergo a rotational motion, this phenomenon can be viewed in Veritasium YouTube Channel [39, 40]. Some authors have studied the features of this motion, for example, they explained the phenomenon that only one of the two marks labeled at the ends can be viewed from above during rotating, described a steady state in which the center of mass remains stationary or moves along a horizontal circle, and tried to solve the processing angular velocity for a specific value of the inclination angle, of course, the conditions under which there exist such a state have also been carefully discussed [6, 19, 31, 33, 38]. However, these so-called “steady states” are far from being typical, for example, they do not cover the “nutation” of tube [40], which is one of the most remarkable phenomena. Moreover, the investigations on the interactions between cylinder and air, especially the lift effect during such a three-dimensional rotation are still lacking in previous works.

Keywords and phrases: Rotational motion rigid tube nutation Kutta-Joukowski law air lift.

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In this paper, a rigid tube model is adopted due to that in a real demonstration experiment, a tube-like cylinder, such as a PVC pipe, is the easiest to get and the most widely used. Firstly, we give the dynamical equations of a rotating tube, show their simulation results and discuss the typical features. Then the equations are modified by including the air lift which is modeled based on Kutta-Joukowski law, we present the corresponding results and discuss the effect of air lift on the rotational motion of tube.

2. Governing dynamical equations, results and discussions on rotating rigid tube

We consider a tube in contact with the horizontal plane (inertial frame of reference) at the point P (it is one of points of tube) as schematically illustrated in Figure 1, and suppose it has the mass $M$, gravity $G$, length $l = 6$ cm, outer and inner radius $r_1 = 1$ cm, $r_2 = 0.8$ cm, respectively. Let $O$ be the center of mass to which an orthogonal frame of reference $Oxyz$ (unit vectors: $i$, $j$, $k$) is attached with the $Oz$ along the axis of symmetry, $Ox$ in the plane span by $Oz$ and vertical axis $OZ$ (precession plane), and $Oy$ inward pointed and perpendicular to this plane. Denoting the Euler angles with respect to $OZ$ by $\theta$, $\varphi$, $\psi$ and define, $\Theta=\dot{\theta}$, $\Omega=\dot{\varphi}$, $n=\dot{\psi}+\Omega \cos \theta$, the angular velocity of tube is expressed as

$$\omega = -\Omega \sin \theta i + \Theta j + n k \quad (2.1)$$

(angular velocity in this paper is in rad s$^{-1}$ throughout). The angular moment relative to $O$ is given by

$$L = -A\Omega \sin \theta i + A\Theta j + Cn k \quad (2.2)$$
in which $A$, $A$ and $C$ are the principle moments of inertia about O. In term of the frame of reference $Oxyz$, the equation of rotational motion is written in the form

$$
\left( \frac{dL}{dt} \right)_{Oxyz} + \Omega' \times L = r_{OP} \times F, \quad (2.3)
$$

where $\Omega' = \omega - \dot{\psi}k$, $r_{OP}$ represents the position vector of P in relation to O, which has an angle of $\alpha$ with respect to the -k direction, $r_{OP} = |r_{OP}|$, $F$ is the force acting on the contact point of tube from the horizontal plane. The effect of air force has not been taken into account yet. In another rotating frame $OXYZ$ (unit vectors: $I$, $J$, $K$) in which the horizontal $OX$ axis lies in the precession plane, the momentum equation can be expressed as

$$
M \left( \frac{dV_O}{dt} \right)_{OXYZ} + M\dot{\Omega} \times V_O = F + G, \quad (2.4)
$$

where $\dot{\Omega} = \Omega K$ and $V_O$ denotes the velocity of the center of mass relative to the horizontal plane. In addition, the velocity of P satisfies

$$
V_P = V_O + \omega \times r_{OP}. \quad (2.5)
$$

Let $F_x$, $F_y$, and $F_z$ be the components of $F$ in the frame of reference $Oxyz$, $(x_P, 0, z_P)$ be the coordinates of P in $Oxyz$, the terms in equation (2.3) can be expressed as

$$
\left( \frac{dL}{dt} \right)_{Oxyz} = -A(\hat{\Omega} \sin \theta + \Omega \cos \theta)i + A\hat{\Omega}j + C\hat{n}k \quad (2.6)
$$

$$
\Omega' \times L = (Cn\dot{\theta} - A\dot{\Omega} \cos \theta)i + (-A\dot{\Omega}^2 \sin \theta \cos \theta + Cn \Omega \sin \theta)j, \quad (2.7)
$$

$$
r_{OP} \times F = -z_P F_y i + (z_P F_x - x_P F_z) j + x_P F_y k. \quad (2.8)
$$

Let $X_P$, $Z_P$ be the $X$- and $Z$-component of $r_{OP}$ in $OXYZ$, thus $X_P = -r_{OP} \sin(\theta - \alpha)$, $Z_P = -r_{OP} \cos(\theta - \alpha)$, and let $F_X$, $F_Y$, and $F_Z$ be the components of $F$ with respect to $OXYZ$. Obviously, $F_y = F_Y$, and $z_P F_x - x_P F_z = Z_P F_X - X_P F_Z$ are fulfilled. Therefore, the three components of equation (2.3) in $Oxyz$ are expressed as

$$
A\hat{\Omega} \sin \theta + \Theta(2A\Omega \cos \theta - Cn) = z_P F_Y, \quad (2.9)
$$

$$
A\dot{\Omega} + \Omega \sin \theta(Cn - A\Omega \cos \theta) = Z_P F_X - X_P F_Z, \quad (2.10)
$$

$$
C\hat{n} = x_P F_y. \quad (2.11)
$$

Similarly, equation (2.4) can be decomposed into three components with respect to $OXYZ$

$$
\dot{V}_{OX} - \Omega V_{OY} = F_X/M, \quad (2.12)
$$

$$
\dot{V}_{OY} + \Omega V_{OX} = F_Y/M, \quad (2.13)
$$

$$
\dot{V}_{OZ} = F_Z/M - g, \quad (2.14)
$$
where \( V_{OX}, V_{OY}, \) and \( V_{OZ} \) denote the components of \( \mathbf{V}_O \) in \( OXYZ \). Suppose the tube experiences a pure rolling on the horizontal plane, the contact point of tube, \( P \) is instantaneously at rest, thus \( \mathbf{V}_P = 0 \). Accordingly, equation (2.5) is simplified to \( \mathbf{V}_O + \omega \times \mathbf{r}_{OP} = 0 \), from it we derive

\[
V_{OX} = r_{OP} \theta \cos(\theta - \alpha) = -Z_P \theta, \tag{2.15}
\]
\[
V_{OY} = -nx_P - z_P \Omega \sin \theta, \tag{2.16}
\]
\[
V_{OZ} = -r_{OP} \theta \sin(\theta - \alpha) = X_P \theta, \tag{2.17}
\]

and

\[
\dot{V}_{OX} = r_{OP} [\theta \cos(\theta - \alpha) - \theta^2 \sin(\theta - \alpha)] = -Z_P \dot{\theta} + X_P \theta^2, \tag{2.18}
\]
\[
\dot{V}_{OY} = -\dot{n}x_P - z_P (\dot{\Omega} \sin \theta + \Omega \theta \cos \theta), \tag{2.19}
\]
\[
\dot{V}_{OZ} = -r_{OP} [\dot{\theta} \sin(\theta - \alpha) + \theta^2 \cos(\theta - \alpha)] = X_P \dot{\theta} + Z_P \theta^2, \tag{2.20}
\]

from equations (2.9)\( \sim \)(2.20) we derive

\[
\dot{\theta} = \frac{\Omega \sin \theta (A \theta \cos(\theta - Cn) + Z_P M \Omega (nx_P + z_P \Omega \sin \theta) - MgX_P)}{A + Mr_{OP}^2}, \tag{2.21}
\]
\[
\dot{\Omega} = \frac{\Omega (Cn - 2A \Omega \cos \theta) - \frac{CM \theta z_P}{C + Mx_P^2} (Z_P + z_P \cos \theta)}{(A + \frac{CM z_P}{C + Mx_P^2}) \sin \theta}, \tag{2.22}
\]
\[
\dot{n} = -\frac{Mx_P}{(C + Mx_P^2)} [2p(\dot{\Omega} \sin \theta + \Omega \cos \theta) + Z_P \theta \Omega], \tag{2.23}
\]
\[
\dot{\varphi} = \varphi(t) = \int_0^t \Omega(\varepsilon) d\varepsilon + \varphi(0), \tag{2.25}
\]
\[
\dot{\psi} = \psi(t) = \int_0^t [n(\varepsilon) - \Omega(\varepsilon) \cos(\theta(\varepsilon))] d\varepsilon + \psi(0). \tag{2.26}
\]

The motion of the center of mass in the inertial frame of reference is calculated by

\[
X_O(t) = \int_0^t [V_{OX}(\varepsilon) \cos \varphi(\varepsilon) - V_{OY}(\varepsilon) \sin \varphi(\varepsilon)] d\varepsilon + X_O(0), \tag{2.27}
\]
\[ Y_O(t) = \int_0^t [V_{OX}(\varepsilon) \sin \varphi(\varepsilon) + V_{OY}(\varepsilon) \cos \varphi(\varepsilon)] d\varepsilon + Y_O(0), \quad (2.28) \]

\[ Z_O(t) = \int_0^t V_{OZ}(\varepsilon) d\varepsilon + Z_O(0), \quad (2.29) \]

where \( X_O(t), Y_O(t), \) and \( Z_O(t) \) denote the coordinates of center of mass in the inertial frame of reference. Equations (2.21)∼(2.29), together with equations (2.18)∼(2.20) constitute a complete description for the motion of tube. No doubt, equations (2.27)∼(2.29) are universally valid for calculating the motion of the center of mass in the inertial frame of reference regardless of whether the tube experiences a pure rolling. Moreover, according to equations (2.12)∼(2.20), \( \mathbf{F} \) with respect to the frame of reference \( OXYZ \) is generated,

\[ F_X = M(-\dot{\Theta}Z_P + \Theta^2X_P) + M\Omega(x_P + z_P \Omega \sin \theta), \quad (2.30) \]

\[ F_Y = -M(x_P \dot{n} + z_P \dot{\Omega} \sin \theta + z_P \Theta \Omega \cos \theta + \Theta \Omega Z_P), \quad (2.31) \]

\[ F_Z = Mg + M(\dot{\Theta}X_P + \Theta^2 Z_P). \quad (2.32) \]

Obviously, \( F_Z \) is the normal reaction and \( \sqrt{F_X^2 + F_Y^2} \) is the static friction. For a pure rolling, the condition \( \mu_s F_Z \geq \sqrt{F_X^2 + F_Y^2} \) must be fulfilled, where \( \mu_s \) is the coefficient of static friction between tube and ground.

However, if the condition of pure rolling is violated, the tube will undergo sliding as well as rotating, in which \( \mathbf{V}_P \) is not zero. Accordingly, the components (in \( OXYZ \)) of \( \mathbf{F} \) in the governing dynamical equations (2.3), (2.4) satisfy

\[ F_X = -\frac{\mu_d F_Z V_P X}{\sqrt{V_P X^2 + V_P Y^2}}, \quad (2.33) \]

\[ F_Y = -\frac{\mu_d F_Z V_P Y}{\sqrt{V_P X^2 + V_P Y^2}}, \quad (2.34) \]

where \( V_P X \) and \( V_P Y \) are the \( X \)- and \( Y \)-component of \( \mathbf{V}_P \) relative to \( OXYZ \), which can be extracted from equation (2.5), \( \mu_d \) represents the coefficient of dynamical friction between tube and ground. Meanwhile, the form of dynamical equations of center of mass (2.12)∼(2.14) keeps unchanged, and the \( Z \)-component of \( \mathbf{V}_P \) satisfies \( V_{PZ} = 0 \), thus, \( V_{OZ} = X_P \Theta, \dot{V}_{OZ} = X_P \dot{\Theta} + Z_P \Theta^2 \), and \( F_Z = Mg + M(\dot{\Theta}X_P + \Theta^2 Z_P) \) still hold. Then we obtain

\[ \dot{\Theta} = \Omega \sin \theta (A\Omega \cos \theta - Cn) - \frac{M(x_P + \mu_d Z_P V_P X)}{\sqrt{V_P X^2 + V_P Y^2}} (g + Z_P \Theta^2) \]

\[ A + M X_P (X_P + \frac{\mu_d Z_P V_P X}{\sqrt{V_P X^2 + V_P Y^2}}) \]

\[ \dot{\Omega} = \frac{z_P F_Y + \Theta (Cn - 2A \Omega \cos \theta)}{A \sin \theta}, \quad (2.36) \]

\[ \dot{n} = \frac{x_P F_Y}{C}, \quad (2.37) \]

\[ \dot{\theta} = \Theta. \quad (2.38) \]
Figure 2. Variation of $\theta$ starting from various initial conditions $(\Omega_0, n_0) = (20, 40)$, $(30, 60)$ and $(40, 80)$.

The motion can be solved through equations (2.18)∼(2.24) and (2.35)∼(2.38). However, in this paper, we do not intend to investigate the behavior after tube hitting ground or jumping which are thus treated as interruptions of rotational motion, therefore the constraints $\theta < \pi/2$ and $F_Z \geq 0$ must be satisfied anytime for an uninterrupted rotating. Now we specify the time at which $\Theta = \Theta_0$, $\Omega = \Omega_0$, $n = n_0$, $\theta = \theta_0$, $V_{OX} = V_{OX0}$ and $V_{OY} = V_{OY0}$ are fulfilled as the staring moment. In this paper, there are no experimental measurements providing initial conditions and coefficients of friction, we have to specify values for these parameters without loss of representativeness. If we define the direction the tube shoots towards as the Y-axis in OXYZ, the initial $V_{OX}$ is almost zero. In actual operation, one end of tube is pressed and shoots forwards, there is no obvious vertical impulse acting on the tube at the starting moment, so the initial $\theta$ is also close to zero. As a desktop experiment, its operating range is usually within several tens of centimeters. Based on these rough analyses, we assume $\Theta_0 = 0$, $V_{OX0} = 0$, and $V_{OY0} = 0.3 \, \text{m s}^{-1}$. However, the characteristics of friction of various material surfaces and circumstances are extremely complex. For simplicity, we refer to the static frictional coefficient of polyethylene on steel, and assume $\mu_s = 0.2$, $\mu_d = 0.1$. Then let $\theta_0$ be $80^\circ$, which means the tube starts moving with its body fairly close to the ground. As mentioned above, the tube exhibits a nutation and $\theta < \pi/2$ must hold, thus we take $\theta$ as the key parameter describing the motional features. Figure 2 shows the variation of $\theta$ starting from various initial states $(\Omega_0, n_0) = (20, 40)$, $(30, 60)$ and $(40, 80)$, respectively. It is found that from these initial states the tube falls downward and hits the ground ($\theta = \pi/2$) soon after, though the time it takes to fall increases from 0.04s to 0.06s. This implies if the initial angular velocities about OZ and Oz axis are too slow, they are unable to sustain an uninterrupted rotating. Then increasing $\Omega_0$ and $n_0$ to 50 and 100, the corresponding evolution curves of $\theta$ and $\sqrt{V_{Fx}^2 + V_{Fy}^2}$ (denoted by $V_H$) are presented in Figure 3a and b, respectively. According to Figure 3b, the tube has a slide on the ground ($V_H \neq 0$) within the initial 0.39s and then transitions into pure rolling ($V_H = 0$). Meanwhile, $\theta$ enters a steady state after 0.39s. Unlike previous works, in this paper, by the term “steady state” we mean an uninterrupted oscillation of $\theta$ with constant amplitude, period and equilibrium point, by the way, the time it takes to transition to a steady state is referred to as
“transition time” hereafter. In the present steady state the value of $\theta$ is altered between 76.9° and 82.3° with the period being 0.28s (Fig. 3a). This result explicitly predicts the occurrence of a continuous nutation, though the calculated amplitude of $\theta$ under present condition is not so large as that demonstrated in Video [30]. No doubt, $\Omega$ and $n$ also exhibit oscillation as shown in Figure 3c.

Further, we increase the initial angular velocities in the same manner, that is, $n_0 = 2\Omega_0$, Figure 4 shows the dependence of $\theta$ on time for $\Omega_0 = 50, 60, 70$ and 80 respectively. It is found that contrary to the behavior of $\Omega_0 = 50$, $\theta$ decreases rapidly at the beginning when $\Omega_0 = 60$ and 70, indicating that the tube raises its head quickly as soon as it starts to rotate, consequently, the equilibrium position of tube head during nutation is lifted upward. On the other hand, the transition time seemly becomes longer with increasing $\Omega_0$. However, the amplitude of $\theta$ is not monotonic with $\Omega_0$, the largest amplitude appears at $\Omega_0 = 60$ with $\theta$ sweeping within a range of 63.2 ∼ 73.2° at this $\Omega_0$. These features are explicitly illustrated by the simulated trajectories of $Q$, the center of top surface, in Figure 5. As $\Omega_0$ reaches 80, the tube rises more rapidly from initial moment but loses its contact with ground after a short time of only 0.06s as demonstrated by the evolution of $F_Z$ (inset of Fig. 4), this suggests that the tube jumps off the ground almost as soon as it shoots out. Therefore, too fast initial angular velocities should also be avoided in order to achieve an uninterrupted rotating. As shown by Figure 6, the dependence of equilibrium point of $\theta$, amplitude of $\theta$ and transition time on $\Omega_0$ is roughly consistent with that reflected in Figure 4, in addition, the nutation period falls monotonically from 0.28s to 0.14s with $\Omega_0$ changed from 50 to 75.

Next, we fix the value of $\Omega_0$ to be 50, Figure 7a shows the dependence of $\theta$ on time for various $n_0$. It is found that when $n_0 = 80$, the tube will fall down within 0.08s. With $n_0$ increased to larger than 100, the tube may enter a steady state which are of course manifested by a nutation, and the faster $n_0$ is, the higher the equilibrium position of tube head becomes. However, as $n_0$ reaches as large as 200, the tube will jump off the ground. Then
Figure 4. Dependence of $\theta$ on time for $\Omega_0 = 50, 60, 70$ and 80 under the constraint $n_0 = 2\Omega_0$. The inset gives the dependence of $F_Z$ on time when $\Omega_0 = 80$.

we fix $n_0$ as 100 and explore the behavior with varying $\Omega_0$, the result is shown in Figure 7b. Similarly, too slow (fast) $\Omega_0$ makes the tube fall (jump) and increasing $\Omega_0$ raises the tube head. It is remarkable that if $\Omega_0$ is larger than 50, the position of tube head strongly climbs before entering the steady state, this can be better viewed by comparing the trajectories of $Q$ of $\Omega_0 = 50$ and 120 (Fig. 8). On the other hand, the transition time is largely prolonged as $\Omega_0$ increases, for example when $\Omega_0 = 120$, not until 1.04s will the tube enter a steady state.

Now let $(\Omega_0, n_0)$ be $(50,120)$ again, changing $\theta_0$ from $60^\circ$ to $10^\circ$, that is to say, the rotational motion starts with the tube closer and closer to vertical axis. From Figure 9, it is found that the tube still transitions into steady state eventually for $\theta_0 = 60^\circ$ but will fall down to ground quickly when $\theta_0 = 40^\circ$, $20^\circ$ or $10^\circ$. Obviously, it is easier to fall down if starting with a more erect posture. In Figure 10 we present a rough range of proper $(\Omega_0, n_0)$s for $\theta_0 = 80^\circ$, $60^\circ$, $40^\circ$ and $20^\circ$, the “proper” means the ensuing motion determined by these initial parameters satisfies $\theta < \pi/2$ and $F_Z \geq 0$ anytime, in other words, the tube will transition into some steady state eventually if starting from these initial states. This range is achieved by examining the behavior of $\theta$ and $F_Z$ resulting from various $(\Omega_0, n_0)$s, these $(\Omega_0, n_0)$s are picked up by scanning the range $\{(\Theta_0, \Omega_0) | \Theta_0 \in [0,200], n_0 \in [0,200]\}$ with the scan step to be 2 both along $\Omega_0$ and $n_0$ axis. Though this is rather a “low pixel” image, it is definite that the proper range is gradually compressed to the upper left corner of graph and a faster $n_0$ is especially needed to sustain an uninterrupted rotation for relatively small $\theta_0$. Obviously, $(\Omega_0, n_0) = (50,120)$ is located outside the proper range when $\theta_0 = 40^\circ$ or $20^\circ$, however, if increasing $n_0$, for example, to 160, this is exactly a proper initial state for $\theta_0 = 20^\circ$, the corresponding revolution curve of $\theta$ and trajectory of $Q$ are shown in Figure 11.

It is needed to be mentioned that the solving of rigid body dynamic problems with friction may generate paradoxical results especially when handling large coefficients of friction [10, 15, 16, 35]. This situation, first presented by Painlevé in solving rigid rod dynamic, is ascribed to the discontinuous behaviors of rigid bodies and the discontinuities of Coulomb friction law which is commonly used in dynamic problems including Painlevé’s problem [10, 15, 16, 35]. In this paper, we find no paradoxes though there exists non-smooth transition between sliding and rolling as reflected in Figure 3. However, this nonexistence of paradoxes is possibly a result under the specified parameters and considered initial conditions in this paper, for the three-dimensionally rotating
Figure 5. Simulated trajectories of $Q$ when $\Omega_0 = (a) 50$, (b) 60, and (c) 70 under the constraint $n_0 = 2\Omega_0$. 
rigid bodies, the effects of discontinuities, non-smooth transitions, and so on are still worthy of study separately and might be considered in further works.

Up to now, the motion mode of tube we have presented can be summarized as “sliding, then rolling or interrupted (fall or loss of contact)”. Other motion modes are possible and expected to be found by traversing all possible value combinations of $\Omega_0$, $n_0$, and $\theta_0$, such as “sliding, rolling, then interrupted”, “pure rolling at beginning” and so on. However, in this paper we intend to focus mainly on the phenomena which are easy to see and impressive in experiments, such as nutation, trajectory, the steady states are sufficient enough to exhibit these phenomena and characteristics. No doubt, the abovementioned other motion modes are also worthy of attention, but need fairly systematic study, for example traversing all possible value combinations of $\Omega_0$, $n_0$, and $\theta_0$, we believe it is more suitable to leave them to the following works.

In the previous works on rolling disk and ring, the trickiest business is treating the dissipations, such as air viscosity, friction and so on. The features and effects of all kinds of dissipations are extremely complicated and still under debate [2, 4, 9, 12, 18, 20, 21, 23–27, 36]. In the model of this paper, dissipation does not occur unless the tube slides ($V_H \neq 0$). This is of course not true in real experiment, for example, the actual contact “point” is a small contact zone rather than an ideal point because of the shape distortion of tube, the bevelled edge of the bottom of tube, the roughness of supporting plane and so on, therefore, there occurs friction dissipation in this contact zone even when tube undergoes a seemingly “pure” rolling. However, if the roughness of ground surface is low enough and the tube has a fairly ideal cylindrical shape, our treatment could approach the real experimental result better. On the other hand, we do not include the viscous effect of air. In fact, the real features of air viscosity can only be extracted by using computational fluid dynamics (CFD) methods which are beyond the scope of this paper and might be considered in the future work.

**Figure 6.** Dependence of (a) equilibrium point of $\theta$, (b) amplitude of $\theta$, (c) transition time, and (d) nutation period on $\Omega_0$ under the constraint $n_0 = 2\Omega_0$. 
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3. Effect of air lift on rotational motion of tube

As yet another effect has not been included, as is well known, a rotating cylinder (about its axis of symmetry) in uniform stream may suffer a lift force, which is referred to as Magnus effect. In this paper, the tube undergoes a three-dimensional rotation, not only about its own axis of symmetry \( O_z \), but also the vertical \( O_z \) and horizontal \( O_y \), and has a translational motion with the velocity \( \mathbf{V}_O \). Nonetheless, by observing the motion of the two ends of tube relative to air as schematically illustrated in Figure 12, it is reasonable to conjecture that the tube would suffer an air lift during rotating. In order to ground the study of air lift, we consider the Reynolds number which is defined as \( \text{Re} = \frac{l |\mathbf{V}_O|}{\nu} \), where \( \nu \) is the kinematic viscosity of air at 20 °C. According to the specified value of \( l, \nu \), and initial \( |\mathbf{V}_O| \), the typical Reynolds number is on the order of \( 10^3 \). Thus \( \text{Re} \gg 1 \) is satisfied, in which case the air flow surrounding tube is significantly dominated by inertial effect compared to the viscous. In addition, according to the previous results, the typical time during which tube enters steady state or experiences interruption is less than 2 seconds, the viscous would hardly accumulate an observable effect in such a short time. Therefore, it is reasonable to exclude the viscous effect in the governing dynamical equations.

Next, we focus on the formula of lift in the governing dynamical equations. As is known, for the airfoil, cylinder, and any other two-dimensional object translating in an inviscid, uniform fluid, the lift (force per unit length) can be calculated by using Kutta-Joukowski law [13]. Furthermore, in some literatures (e.g., [34]) on the aerodynamics of rotary-wing of rotorcraft, which is similar to the problem in this paper, the authors directly use the Kutta-Joukowski law, \( \tau = \rho|\mathbf{U}_\perp|\Gamma \), to calculate the lift on the blade of rotary-wing, where \( \tau \) is the lift at a given position on the blade, \( \rho \) is the air density, \( \mathbf{U}_\perp \) is the perpendicular (to the length of blade) component of air velocity relative to blade at that position, and \( \Gamma \) is the strength (circulation) of vortex at that position. In this paper, we also directly adopt this law to study the lift acting on the tube. It should be point out that, the real fluids including air are viscous, the conditions on which the Kutta-Joukowski law works should be carefully
Figure 8. Trajectories of Q simulated from $(\Omega_0, n_0) = (a) (50, 100)$, and (b) $(120, 100)$.

examined. As shown by Kutta-Joukowski, for thin airfoil, large Reynolds number, and small angle of attack, the flow can be treated as being inviscid outside the airfoil if Kutta condition is provided [22]. However, for a tube moving through fluid, the geometric characteristics pertaining to a thin airfoil, such as angle of attack, tip, trailing edge and so on, become meaningless, it is the spinning about axis that generate lift. Actually, in many literatures and textbooks, the formula of Kutta-Joukowski law is developed for a cylinder by using potential flow theorem, and then generalized to other two-dimensional objects like airfoil. Of course, the fluid is assumed inviscid, except that in contact with cylinder surface on which the viscous no-slip condition set up a circulation.
Figure 9. Dependence of \( \theta \) on time for various \( \theta_0 \) when \((\Omega_0, n_0) = (50, 120)\).

Figure 10. Rough range of proper \((\Omega_0, n_0)\)'s for \( \theta_0 = \) (a) 80\(^\circ\), (b) 60\(^\circ\), (c) 40\(^\circ\), and (d) 20\(^\circ\).
However, the experimental lift is less than 50% of the value resulting from the theorem [30, 37]. It has been demonstrated that in real viscous flow, the flow starts to separate from the cylinder when Reynolds number is about 10, and a steady separation vortex is formed behind it as Reynolds number increases to about $10^2$. Then this vortex is enlarged until it becomes unsteady, turbulent wake as Reynolds number further increases to about $10^3$ [11]. This flow separation is primarily responsible for the deviation between theoretical and experimental lift. In this paper, similar deviation is expected to occur provided the Kutta-Joukowski law, which is essentially inviscid theorem, to calculate the lift and the Reynolds number of the order of $10^3$. Finally, it is noticed that, for
the blade of rotary-wing, the length is much larger than the width, so the air velocity near the blade is almost perpendicular to the blade length. This condition is not fulfilled in our paper, the diameter of the tube is as large as one third of its length, moreover, the tube has an open inner cylindrical space. Therefore, our method studying the lift is still an approximate treatment just without loss of accuracy at the order of magnitude.

Firstly, we inspect the segment WQ solely, it rotates with the angular velocity \( \Omega' \), the position \((0, 0, z)\) has the velocity \( V_z = V_O + \Omega' \times z \), extracting its component perpendicular to \( k \), namely, \( V_{z\perp} \), thus the perpendicular air velocity component relative to this position is \(-V_{z\perp}\). Next, adding a vortex at this position and assuming its strength (circulation) \( \Gamma = 2\pi r^2 \dot{\psi} / (\psi = n - \Omega \cos \theta) \), the lift \( \tau \) can be calculated by Kutta-Joukowski law and is used as an approximation of the true lift at the corresponding position of tube. The resultant lift force is

\[
\mathbf{F} = \int_{-l/2}^{l/2} \tau \, dz = \mathcal{F}_X \mathbf{i} + \mathcal{F}_Y \mathbf{j} + \mathcal{F}_Z \mathbf{k},
\]

where \( \mathcal{F}_X = -2\pi \rho r^2 \dot{\psi} V_{OY} \cos \theta \), \( \mathcal{F}_Y = -2\pi \rho r^2 \dot{\psi} (V_{OZ} \sin \theta - V_{OX} \cos \theta) \), and \( \mathcal{F}_Z = 2\pi \rho r^2 \dot{\psi} V_{OY} \sin \theta \). The resultant lift moment can be calculated by

\[
\mathbf{L} = \int_{-l/2}^{l/2} z \mathbf{k} \times \tau \, dz = \mathcal{L}_x \mathbf{i} + \mathcal{L}_y \mathbf{j},
\]

where \( \mathcal{L}_x = -\pi \rho r^2 l^3 \dot{\psi} \Theta / 6 \), and \( \mathcal{L}_y = -\pi \rho r^2 l^3 \dot{\psi} \Omega \sin \theta / 6 \). The rotational and momentum equations are expressed in the forms

\[
\left( \frac{d \mathbf{L}}{dt} \right)_{Oxyz} + \Omega' \times \mathbf{L} = \mathbf{r}_{OP} \times \mathbf{F} + \mathbf{L},
\]
and

\[ M \left( \frac{dV_O}{dt} \right)_{OYZ} + M\Omega \times V_O = F + G + \mathcal{F}. \]  \hspace{1cm} (3.4)

If there occurs the pure rolling, from equations (2.5), (2.21) and (2.22), we derive

\[ \dot{\Theta} = \frac{\Omega \sin \theta (A\Omega \cos \theta - Cn) + ZpM\Omega (nxp + zp\Omega \sin \theta) - Zp\mathcal{F}_X + Xp\mathcal{F}_Z - Mgxp + L_y}{A + Mr_O^2}, \]  \hspace{1cm} (3.5)

\[ \dot{\Omega} = \frac{\Theta(Cn - 2A\Omega \cos \theta) - \frac{C}{C + Mr_O^2}[M\Theta zp(Xp + zp\cos \theta)] - L_x}{(A + \frac{CMr_O^2}{C + Mr_O^2}) \sin \theta}, \]  \hspace{1cm} (3.6)

\[ \dot{n} = -\frac{Mxp[zp(\dot{\Omega} \sin \theta + \Theta \Omega \cos \theta) + Zp\Theta\Omega] + xp\mathcal{F}_Y}{C + Mr_O^2}, \]  \hspace{1cm} (3.7)

\[ \dot{\theta} = \Theta, \]  \hspace{1cm} (3.8)

and

\[ F_X = M \left( -\dot{\Theta}Zp + \Theta^2Xp \right) + M\Omega (nxp + zp\Omega \sin \theta) - \mathcal{F}_X, \]  \hspace{1cm} (3.9)

\[ F_Y = -M \left( xpn + zp\dot{\Theta} \Omega \sin \theta + zp\Theta \Omega \cos \theta + \Theta Zp \right) - \mathcal{F}_Y, \]  \hspace{1cm} (3.10)

\[ F_Z = Mg + M \left( \dot{\Theta}Xp + \Theta^2Zp \right) - \mathcal{F}_Z. \]  \hspace{1cm} (3.11)

Contrarily, if the tube undergoes sliding, \( F_X \) and \( F_Y \) should have the form of (2.33) and (2.34) again, meanwhile \( V_{pz} = 0, V_{oz} = Xp\Theta, \dot{V}_{oz} = Xp\dot{\Theta} + Zp\Theta^2, \) and \( F_Z = Mg + M \left( \dot{\Theta}Xp + \Theta^2Zp \right) - \mathcal{F}_Z \) are also satisfied, we achieve

\[ \dot{\Theta} = \frac{\Omega \sin \theta (A\Omega \cos \theta - Cn) - (Xp + \frac{\mu_d zp V_{px} V_{py}}{\sqrt{V_{px}^2 + V_{py}^2}})(Mg + Mzpz \Theta^2 - \mathcal{F}_z) + L_y}{A + MXp(Xp + \frac{\mu_d zp V_{px} V_{py}}{\sqrt{V_{px}^2 + V_{py}^2}})}, \]  \hspace{1cm} (3.12)

\[ \dot{\Omega} = \frac{zpF_Y + \Theta(Cn - 2A\Omega \cos \theta) - L_x}{A \sin \theta}, \]  \hspace{1cm} (3.13)

\[ \dot{n} = \frac{xpF_Y}{C}, \]  \hspace{1cm} (3.14)
Let \( \rho = 1.2 \text{ kg m}^{-3} \), \( \theta_0 = 80^\circ \) and initial states \((\Omega_0, n_0)\) be \((40, 90)\), \((40, 100)\) and \((40, 110)\), Figure 13 shows the dependence of \( \theta \) on time with and without including air lift. It is found that for \( n_0 = 90 \) (Fig. 13a), the tube quickly falls down in these two cases, though the duration time before hitting ground is prolonged due to the presence of air lift. If \( n_0 = 100 \) (Fig. 13b), the \((\Omega_0, n_0)\) is located very near the proper range according to Figure 10a but still outside it, no doubt, the tube would fall down if air lift does not work. However, once the air lift takes effect, the behavior of tube will be very different, after experiencing a threatening but not dangerous falling, tube rises back from the lowest position which is extremely close to the ground \((\theta = 89.7^\circ)\), and then enters a steady state at 0.36s (see inset of Fig. 13b), indicating that the air indeed “lift” the tube, thus, saving the rotational motion from being interrupted. When \( n_0 = 110 \) (Fig. 13c), the tube will not be interrupted in both cases, thus transitions to steady state, but the presence of air lift makes the tube rise slightly, the height of Q, which is calculated by \((l \cos \theta + r_1 \sin \theta)\), with air lift is around 0~2 mm higher than that without air lift. Though the vertical distance between these two cases is rather small, it is definite that the air force lifts the tube position and is in favor of sustaining an uninterrupted motion.
In summary, the features of rotational motion of tube are investigated theoretically, the effect of lift of air are also discussed. It should point out that, the models concerning the friction and air lift in this paper are far from being precise though we believe reflect the physical essentials and are of accuracy of order of magnitude. Maybe it is impossible to precisely extract the features of friction because of the extreme complexity of the real state of contact zone. However, the effect of air, including viscosity, lift, even the influences of the air inner the tube space, may be precisely analyzed by using CFD methods, which might be the topics of our future works.

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References


[39] https://www.youtube.com/watch?v=wQTVcaA3PQw.

[40] https://www.youtube.com/watch?v=7rAIZR_zasg.

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