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# GENERALITIES ON A DELAYED SPATIOTEMPORAL HOST–PATHOGEN INFECTION MODEL WITH DISTINCT DISPERSAL RATES

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**Abstract.** We propose a general model to investigate the effect of the distinct dispersal coefficients infected and susceptible hosts in the pathogen dynamics. The mathematical challenge lies in the fact that the investigated model is partially degenerate and the solution map is not compact. The spatial heterogeneity of the model parameters and the distinct diffusion coefficients induce infection in the low-risk regions. In fact, as infection dispersal increases, the reproduction of the pathogen particles decreases. The dynamics of the investigated model is governed by the value of the basic reproduction number $R_0$. If $R_0 \leq 1$, then the pathogen particles extinct, and for $R_0 > 1$ the pathogen particles persist, and there is at least one positive steady state. The asymptotic profile of the positive steady state is shown in the case when one or both diffusion coefficients for the host tends to zero or infinity.

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## 1. Introduction

The recent studies show that there is an increased focus on the study of the behavior of partially degenerate reaction-diffusion systems, *e.g.* [1–7], and references therein, due to the mathematical challenges and the practical applications. The degenerate reaction-diffusion systems are formed by partial differential equation and ordinary differential equation (ODE) or delayed differential equation (DDE). The degeneration appears in a system of reaction-diffusion equations when some of the dispersal coefficients are equal to zero in some (or all) locations in a bounded domain. In this case, the solution map becomes noncompact, which generates some mathematical challenges in showing the existence of a compact attractor and identifying the basic reproduction number. To overcome these challenges, it is possible to employ the generalized Krein-Ruthman Theorem [8], and Kuratowski measure of non-compactness [9]. There are numerous researches that investigate the reaction-diffusion host–pathogen interaction, and we discuss some of them. In [10], the authors considered a viral infection with virus diffusion only, where the global asymptotic stability of the virus-free steady state (VFSS) for $R_0 < 1$ and the global stability of the unique positive steady state for $R_0 > 1$, similar results proved for more generalized systems, *e.g.* [1, 3, 5], and references therein. Recently, Y. WU and X. Zou [11] provided a mathematical analysis of a degenerate host–pathogen reaction-diffusion system, for the investigated model is given in the following

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structure

$$\begin{cases} \frac{\partial W_1}{\partial t} = d_1\Delta W_1 + a_1(x) - \beta(x)W_1W_3 - b_1(x)W_1, & x\in\Omega, \quad t>0,\\ \frac{\partial W_2}{\partial t} = d_2\Delta W_2 + \beta(x)W_1W_3 - b_2(x)W_2, & x\in\Omega, \quad t>0,\\ \frac{\partial W_3}{\partial t} = a_2(x)W_2 - b_3(x)W_3, & x\in\Omega, \quad t>0, \end{cases} \tag{1.1}$$

with Neumann boundary condition

$$\frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = 0, \quad x\in\partial\Omega, \quad t>0, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ ($n$ is the dimension) is a bounded domain with smooth boundary, and $n$ in the outward normal direction vector to $\partial\Omega$. $W_1(t,x), W_2(t,x), W_3(t,x)$ are respectively the concentration of the susceptible host, infected host, pathogen particles at time $t$ and location $x$. $a_1(x)$ is the recruitment of the susceptible host. $\beta(x)$ is the transmission rate. $b_i(x),\quad i=1,2,3$ are respectively the mortality coefficients for the susceptible host, infected host, and pathogen particles. $a_2(x)$ is the production coefficient of the pathogens particles from the infected hosts. Suppose that all parameters are positive Hölder continuous functions on $\bar{\Omega}$. The Neumann boundary condition represent that the studied population are in an isolated habitat $\Omega$. The authors proved the well-posedness of the solution and the existence of a globally connected attractor. The main difficulty of to show that the semiflow is point dissipative (for definition see [12]), where the distinct diffusion coefficients play a substantial role increasing the difficulty in the model temporal analysis. The basic reproduction number is also identified with its threshold role, where for $R_0 \le 1$ the pathogen constructing a super solution for the pathogen and infected hosts that tend to zero. The uniform persistence is also provided for $R_0 > 1$ by applying the results of [13], Theorem 1.3.2. Moreover, the asymptotic profile of the positive steady state is shown as one of the dispersal rates $d_1$ or $d_2$ tends to zero. Recently, different approaches have been considered in modelling pathogen spread in hosts, we cite a few, [14–16], and references therein.

In this research, we investigate a generalized version of the (1.1), by considering a generalized incidence function of the form $G(x, W_1, W_3)$ with some properties on $G$ that will be fixed later. Indeed, recent work investigate a generalized version of (1.1) as [3] (with the case of the cell-to-cell transmission), where the authors proved the well-posedness of the solution to the threshold dynamics. However, for $R_0 > 1$, the global dynamics of the solution are shown for the homogeneous case only, and no asymptotic profile of the positive steady state, which are difficult to be achieved for the case of the nonlinear incidence function. Based on the best of our knowledge, other than [3], there are no results on a degenerate reaction-diffusion system with generalized nonlinear incidence in the case of distinct dispersal rates. Notice that the newly reproduced infected hosts after a direct contact between a pathogen particle and susceptible host is not instantaneous, this last takes some time (denoted $\tau$) until the pathogen particles reproduce in the susceptible host, and this host becomes a fully infected one. This behavior can be modeled by the presence of time delay in the incidence function in the infected host equation (second equation of (1.1)), see *e.g.* [17, 18], and references therein. The investigated model is structured as

$$\begin{cases} \frac{\partial W_1}{\partial t} = d_1\Delta W_1 + a_1(x) - G(x, W_1, W_3) - b_1(x)W_1, & x\in\Omega, \quad t>0,\\ \frac{\partial W_2}{\partial t} = d_2\Delta W_2 + \mathrm{e}^{-b_4\tau}G(x, W_1(t-\tau,x), W_3(t-\tau,x)) - b_2(x)W_2, & x\in\Omega, \quad t>0,\\ \frac{\partial W_3}{\partial t} = a_2(x)W_2 - b_3(x)W_3, & x\in\Omega, \quad t>0,\\ \frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = 0, & x\in\partial\Omega, \quad t>0, \end{cases} \tag{1.3}$$

where $\mathrm{e}^{-b_4\tau}$ is the survival rate of the infected host from the original transmission $(t-\tau)$ until $t$. Also, we consider that $G \in C^1(\bar{\Omega}\times\mathbb{R}^+\times\mathbb{R}^+, \mathbb{R}^+)$ is the nonlinear incidence function, which satisfies

($\mathbf{H_1}$) $G(\cdot,0,\cdot)\equiv 0$ and $G(\cdot,\cdot,0)\equiv 0$;
($\mathbf{H_2}$) $\partial_2 G(x,x_1,x_2):=\frac{\partial G(x,x_1,x_2)}{\partial x_1}>0,\quad x_1\geq 0,\ x_2>0,\quad x\in\bar{\Omega}$ and $\partial_3 G(x,x_1,x_2):=\frac{\partial G(x,x_1,x_2)}{\partial x_2}>0,\quad x_1>0,\ x_2\geq 0,\quad x\in\bar{\Omega}$;
($\mathbf{H_3}$) $G$ is a concave function with respect to the third variable.

To mention that the assumption ($\mathbf{H_1}$) implies that there is no transmission in the absence of the susceptible particles or pathogen particles. ($\mathbf{H_2}$) implies that the incidence function is increasing in both two last parameters. However, ($\mathbf{H_3}$) implies that $G(x,x_1,x_2)\leq \partial_3 G(x,x_1,0)x_2,\ \forall x_1\geq 0,\ x_2\geq 0,\quad x\in\bar{\Omega}$. To mention that there are numerous well known incidence functionals that fulfill the assumptions *e.g.* ($\mathbf{H_i}$), $\quad i=1,2,3$ as Holling I-III incidence function, Beddington-DeAngelis incidence function, ratio-dependent incidence function.

The presence of the time delay generates an additional challenge in showing the asymptotic compactness of the semiflow, existence of the global compact attractor, and identifying the basic reproduction number (where it will be discussed in the proof of Lem. 3.1) and determining the global dynamics of the steady states. On the other hand, the main challenging point is the asymptotic profile of the positive steady state, where the nonlinearity of the incidence function setting an additional assumption about the incidence function. In literature works, *e.g.* [2, 19, 20], and references therein, many authors considered the asymptotic profile for PSS. However, the bilinear incidence is considered for most part of them, or a specific nonlinear incidence function as in the SIS epidemic model [19], where the authors considered a ratio-dependent type as an incidence function. Our interest is to generalize these results, and provide an additional information on the asymptotic profile of the PSS in the case of the nonlinear incidence. Indeed, we investigate the case when one or both diffusion coefficients $d_1, d_2$ tends to infinity or when $d_1$ goes to zero.

To show the cited goals, we organize our research in the following structure. In the next section, we prove the existence of a global solution and show the existence of a connected global attractor, also, we identify the basic reproduction number, with its susceptibility with respect to the diffusion coefficients. In Section 3, we prove the global asymptotic stability of the pathogen-free steady state (PFSS) for $R_0\leq 1$. In Section 4, we show that the semiflow is strongly uniformly persistent for $R_0>1$, and there exists at least one positive steady state (PSS). Sections 5 and 6 investigate the global prosperities and the uniqueness of the positive steady state. Indeed, in Section 5, and by the Lyaponuv approach and Lasalle invariance principle, we investigate the global attraction of the positive steady state for some particular cases. The asymptotic profile of PSS is shown in Section 6, where the effect of mobility of the hosts on the PSS, large mobility and small mobility rates are investigated. In Section 7, we explore the global dynamics of steady states and the asymptotic profile of PSS. A brief discussion ends the paper.

## 2. Preliminaries

We let $C=C(\bar{\Omega},\mathbb{R})$, and $C_\tau=C([-\tau,0]\times\bar{\Omega},\mathbb{R})$, for all $\tau\geq 0$. Define $\mathbb{X}:=C_\tau\times C\times C_\tau$, equipped with the supreme norm, and $\mathbb{X}^+$ is its positive cone. We let $h\in C$ be a positive function, and denote

$$\bar{h}=\max\{h(x);\ x\in\bar{\Omega}\},\quad \underline{h}=\min\{h(x);\ x\in\bar{\Omega}\}.$$

For any $\tau\geq 0,\ t>0,\ x\in\bar{\Omega}$, we let $W_{i,\tau}(t,x)=W_i(t+\tau,x)$, and for simplicity we write $W_i$ instead $W_i(t,x)$, $i=1,2,3$, and $W_{i,\tau}$ instead $W_{i,\tau}(t,x)\quad i=1,2,3$.

### 2.1. Well-posedness

At first, we shall show the existence and uniqueness of a positive solution of (1.3) through the following theorem

**Theorem 2.1.** *Letting $W(t,x;W_0)$, be the solution of the system (1.3) with the corresponding to the initial condition $W_0=(W_{10},W_{20},W_{30})$. If $W_0\in\mathbb{X}^+$ then the system (1.3) admits a unique positive solution on $[0,T_0)\times\Omega$. Provided that $T_0>0$ and satisfies either $\limsup_{t\to T_0^-}\|W\|_{\mathbb{X}}=\infty$ or $T_0=+\infty$.*

*Proof.* The system (1.3) can be expressed in the following abstract form

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t,\cdot;W_0) = \mathscr{A}W(t,\cdot;W_0) + \mathscr{F}(W(t,\cdot;W_0)), \tag{2.1}$$

with $A$ defined on $D(A) \subset \mathbb{X}$ by

$$\mathscr{A}W := \begin{pmatrix} d_1\Delta W_1(t,\cdot) \\ d_2\Delta W_2(t,\cdot) \\ 0 \end{pmatrix},$$

and $F : \mathbb{X} \to \mathbb{X}$ defined as

$$\mathscr{F}W := \begin{pmatrix} a_1(\cdot) - G(\cdot, W_1(t,\cdot), W_3(t,\cdot)) - b_1(.)W_1(t,\cdot) \\ \mathrm{e}^{-b_4\tau}G(\cdot, W_1(t-\tau,\cdot), W_3(t-\tau,\cdot)) - b_2(\cdot)W_2(t,\cdot) \\ a_2(\cdot)W_2(t,\cdot) - b_3(\cdot)W_3(t,\cdot) \end{pmatrix},$$

with $W \in \mathbb{X}^+$. Clearly, by the assumption on $G$ we deduce that $\mathscr{F}$ is Fréchet differentiable on $\mathbb{X}$. Also, $\mathscr{A}$ is the infinitesimal generator of the strongly continuous semigroup $\{\mathrm{e}^{t\mathscr{A}}\}_{t\geq 0}$ in $\mathbb{X}$. Then, by applying [21], Proposition 4.16, we ensure the existence of a unique solution $W(t,\cdot;W_0)$ on $[0,T_0)$, with either $\limsup_{t\to T_0^-}\|W\|_{\mathbb{X}} = \infty$ or $T_0 = +\infty$.

In the following, we shall show that $W_i(t,x) > 0$, $i = 1,2,3$, for all $t \in (0,T_0)$ and $x \in \bar{\Omega}$. Clearly, $\mathscr{F}$ is twice differential and continuous, therefore by applying Theorem 7.3.1 and Corollary 7.3.2 in [22], we deduce the positivity of the solution. □

The next theorem shows that the solution is globally defined.

**Theorem 2.2.** *For $W_0 = (W_{10}, W_{20}, W_{30}) \in \mathbb{X}^+$, then (1.3) has a unique global solution that defined on $\bar{\Omega} \times [0,+\infty)$.*

*Proof.* Let $W = (W_1, W_2, W_3)$ be the solution of (1.3) for the data $W_0 = (W_{10}, W_{20}, W_{30}) \in \mathbb{X}^+$. The standard comparison principle implies that $W_1(t,x) \leq \tilde{W}_1(x,t)$, and $\tilde{W}_1$ is the unique solution of the problem

$$\begin{cases} \frac{\partial \tilde{W}_1}{\partial t} = d_1\Delta\tilde{W}_1 + a_1(x) - b_1(x)\tilde{W}_1, & x \in \Omega,\ t > 0, \\ \frac{\partial \tilde{W}_1}{\partial n} = 0, & x \in \partial\Omega,\quad t > 0, \\ \tilde{W}_1(0,x) = W_{10}(0,x), & x \in \Omega,\ t > 0. \end{cases} \tag{2.2}$$

Clearly, (2.2) has a unique positive steady state denoted $V(x)$ which is globally asymptotically stable. Therefore, we deduce that

$$\limsup_{t\to+\infty} W_1(t,x) \leq \limsup_{t\to+\infty} \tilde{W}_1(t,x) = V(x), \text{ uniformly for } x \in \bar{\Omega}. \tag{2.3}$$

Thus, the boundedness of $W_1^0$ next to (2.3) implies the existence of a positive constant denoted $D_1$ that depends on the initial data verifying

$$\|W_1\| \leq D_1, \quad t \geq 0. \tag{2.4}$$

Let $\{T_2(t)\}_{t\geq 0}$ be the semigroup generated by the generator $d_2\Delta - b_2(\cdot)$ in $C$ with Neumann boundary condition, hence,

$$\begin{cases} W_2(t,\cdot) = T_2(t)W_{20} + \displaystyle\int_0^t T_2(t-s)\mathrm{e}^{-b_4\tau}G(\cdot, W_1(s-\tau,\cdot), W_3(s-\tau,\cdot))\mathrm{d}s, \\ W_3(t,\cdot) = \mathrm{e}^{-b_3 t}W_{30} + a_2\displaystyle\int_0^t \mathrm{e}^{-b_3(t-s)}W_2(s,\cdot)\mathrm{d}s. \end{cases} \tag{2.5}$$

Let $\lambda > 0$ is the principal eigenvalue of $d_2\Delta - b_2(\cdot)$ with Neumann boundary condition. From $(\mathbf{H_2})$, and using (2.4), we have $||G(x, W_1, W_3)|| \leq ||G(x, D_1, W_3)||$. Using $(\mathbf{H_3})$ on the previous inequality, we obtain $||G(x, W_1, W_3)|| \leq ||\partial_2 G(x, D_1, 0)||||W_3||$. Therefore,

$$\begin{cases} \|W_2(t,\cdot)\| \leq \mathrm{e}^{-\lambda t}\|W_{20}\| + L_1\mathrm{e}^{-b_4\tau}\displaystyle\int_0^t \mathrm{e}^{-\lambda(t-s)}\|W_3(s-\tau,\cdot)\|\mathrm{d}s, \\ \|W_3(t,\cdot)\| \leq \mathrm{e}^{-b_3 t}\|W_{30}\| + \|a_2\|\displaystyle\int_0^t \mathrm{e}^{-\underline{b_3}(t-s)}\|W_2(s,\cdot)\|\mathrm{d}s, \end{cases} \tag{2.6}$$

with $L_1 = ||\partial_2 G(x, D_1, 0)||$. From (2.6), we obtain

$$\begin{aligned} \|W_2(t,\cdot)\| &\leq \mathrm{e}^{-\lambda t}\|W_{20}\| + L_1\mathrm{e}^{-b_4\tau}\int_0^t \mathrm{e}^{-\lambda(t-s)}\left\{\mathrm{e}^{-b_3(s-\tau)}\|W_{30}\| + \|a_2\|\int_0^{s-\tau}\mathrm{e}^{-\underline{b_3}(t-\sigma)}\|W_2(\sigma,\cdot)\|\mathrm{d}\sigma\right\}\mathrm{d}s, \\ &= \mathrm{e}^{-\lambda t}\|W_{20}\| + L_1\mathrm{e}^{-b_4\tau}\|W_{30}\|\int_0^t \mathrm{e}^{-\lambda(t-s)}\mathrm{e}^{-b_3(s-\tau)}\mathrm{d}s \\ &\quad + L_1\|\beta\|\mathrm{e}^{-b_4\tau}\|a_2\|\int_0^t \mathrm{e}^{-\lambda(t-s)}\int_0^{s-\tau}\mathrm{e}^{-\underline{b_3}(t-\sigma)}\|W_2(\sigma,\cdot)\|\mathrm{d}\sigma\mathrm{d}s, \\ &\leq \mathrm{e}^{-\lambda t}\|W_{20}\| + L_1\mathrm{e}^{-b_4\tau}\|W_{30}\|\int_0^t \mathrm{e}^{-\underline{b_3}(s-\tau)}\mathrm{d}s + L_1\mathrm{e}^{-b_4\tau}\|a_2\|\mathrm{e}^{-(\lambda+\underline{b_3})t}\int_0^t\int_{\sigma-\tau}^t \mathrm{e}^{\lambda s}\mathrm{e}^{\underline{b_3}\sigma}\|W_2(\sigma,\cdot)\|\mathrm{d}s\mathrm{d}\sigma, \\ &= \mathrm{e}^{-\lambda t}\|W_{20}\| + L_1\mathrm{e}^{-b_4\tau}\mathrm{e}^{\underline{b_3}\tau}\|W_{30}\|\int_0^t \mathrm{e}^{-\underline{b_3}s}\mathrm{d}s + L_1\mathrm{e}^{-b_4\tau}\|a_2\|\mathrm{e}^{-(\lambda+\underline{b_3})t}\int_0^t \mathrm{e}^{\underline{b_3}\sigma}\|W_2(\sigma,\cdot)\|\int_{\sigma-\tau}^t \mathrm{e}^{\lambda s}\mathrm{d}s\mathrm{d}\sigma, \\ &\leq \|W_{20}\| + \tfrac{L_1\mathrm{e}^{-b_4\tau}\mathrm{e}^{\underline{b_3}\tau}}{\underline{b_3}}\|W_{30}\| + \tfrac{L_1\mathrm{e}^{-b_4\tau}\|a_2\|}{\lambda}\mathrm{e}^{-\underline{b_3}t}\int_0^t \mathrm{e}^{\underline{b_3}\sigma}\|W_2(\sigma,\cdot)\|\mathrm{d}\sigma, \\ &\leq D_1 + D_2\mathrm{e}^{-\underline{b_3}t}\int_0^t \mathrm{e}^{\underline{b_3}\sigma}\|W_2(\sigma,\cdot)\|\mathrm{d}\sigma, \end{aligned}$$

with $D_1 = \|W_{20}\| + \frac{L_1\mathrm{e}^{-b_4\tau}\mathrm{e}^{\underline{b_3}\tau}}{\underline{b_3}}\|W_{30}\|$, and $D_2 = \frac{L_1\mathrm{e}^{-b_4\tau}\|a_2\|}{\lambda}$. Gronwall's inequality implies that

$$\|W_2(t,\cdot)\| \leq D_1\mathrm{e}^{D_2 t}, \quad t \geq 0. \tag{2.7}$$

Substituting (2.7) into the second inequality of (2.4), we get

$$\|W_3(t,\cdot)\| \leq \|W_{30}\| + \frac{\|a_2\|D_1}{D_2}\mathrm{e}^{D_2 t}, \quad t \geq 0. \tag{2.8}$$

Therefore, the solution is globally defined. □

Motivated by [11], we have the following result

**Theorem 2.3.** *We let $W_0 = (W_{10}, W_{20}, W_{30}) \in \mathbb{X}^+$, then the solution of* (1.3) *is point dissipative.*

*Proof.* We show this result step by step as follows

**Step 1:** There exist a positive constant $M_0 > 0$, such that $W_1(t,\cdot)$ satisfy

$$\limsup_{t\to+\infty} ||W_1(t,\cdot)|| \le M_0.$$

From the proof of Theorem 2.2, we have $W_1(t,x) \le \tilde{W}_1(x,t)$, with $\tilde{W}_1$ is the unique solution of the problem (2.2), and therefore $W_1(t,\cdot)$ is ultimately bounded.

**Step 2:** There exists a positive constant $M_1 > 0$ such that

$$\limsup_{t\to+\infty} (\|W_1\|_1 + \|W_2\|_1) \le M_1.$$

By integrating the two sides of the first equation of (1.3), we get

$$\frac{\partial}{\partial t}\int_\Omega W_1 \mathrm{d}x = \int_\Omega a_1 \mathrm{d}x - \int_\Omega G(x, W_1, W_3)\mathrm{d}x - \int_\Omega b_1(x) W_1 \mathrm{d}x. \tag{2.9}$$

Recall that $W_{2,\tau} = W_2(t+\tau, x)$. Then, the second equation of (1.3) implies

$$\frac{\partial}{\partial t}\int_\Omega W_{2,\tau} \mathrm{d}x = \int_\Omega \mathrm{e}^{-b_4\tau} G(x, W_1, W_2) - \int_\Omega b_2(x) W_{2,\tau} \mathrm{d}x. \tag{2.10}$$

From (2.9) and (2.10), we get

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega (W_1 + \mathrm{e}^{b_4\tau} W_{2,\tau})\mathrm{d}x &= \int_\Omega a_1 \mathrm{d}x - \int_\Omega b_1(x) W_1 \mathrm{d}x - \mathrm{e}^{b_4\tau}\int_\Omega b_2(x) W_{2,\tau}\mathrm{d}x,\\ &\le |\Omega| \|a_1\| - m\int_\Omega (W_1 + \mathrm{e}^{b_4\tau} W_{2,\tau})\mathrm{d}x,\end{aligned}$$

with $m = \min_{x\in\bar{\Omega}}\{b_1(x), \mathrm{e}^{b_4\tau} b_2(x)\}$, and $|\Omega|$ is the volume of $\Omega$. Then, variation of constant formula implies that there exists a positive constant $M_1 > 0$, verifying

$$\limsup_{t\to+\infty} (\|W_1\|_1 + \|W_2\|_1) \le M_1.$$

**Step 3:** For any $p \ge 1$, there exists $M_p > 0$ such that

$$\limsup_{t\to+\infty} (\|W_1\|_p + \|W_2\|_p) \le M_p.$$

At first, for any integer $k \ge 0$, we show the result for $p = 2^k$ by induction. Notice that for $k = 0$ is proved in *Step 2*. Assume that the claim is true for $k-1$. That means there exists $M_{2^{k-1}} > 0$ verifying

$$\limsup_{t\to+\infty} (\|W_2\|_{2^{k-1}} + \|W_3\|_{2^{k-1}}) \le M_{2^{k-1}}.$$

The second equation of (1.3) can be rewritten as

$$\frac{\partial W_{2,\tau}}{\partial t} = d_2 \Delta W_{2,\tau} + \mathrm{e}^{-b_4\tau} G(x, W_1, W_3) - b_2(x) W_{2,\tau}. \tag{2.11}$$

Multiplying (2.11) by $W_{2,\tau}^{2^k-1}$, and integrating the resulting equation, we get

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x = d_2\int_\Omega W_{2,\tau}^{2^k-1}\Delta W_{2,\tau}\mathrm{d}x + \mathrm{e}^{-b_4\tau}\int_\Omega W_{2,\tau}^{2^k-1}G(x,W_1,W_3)\mathrm{d}x - \int_\Omega b_2(x)W_{2,\tau}^{2^k}\mathrm{d}x, \quad (2.12)$$

Using $(\mathbf{H_3})$, we obtain

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x \le d_2\int_\Omega W_{2,\tau}^{2^k-1}\Delta W_{2,\tau}\mathrm{d}x + \mathrm{e}^{-b_4\tau}\int_\Omega W_{2,\tau}^{2^k-1}\partial_3 G(x,W_1,0)W_3\mathrm{d}x - \int_\Omega b_2(x)W_{2,\tau}^{2^k}\mathrm{d}x, \quad (2.13)$$

Note that

$$\begin{aligned}\int_\Omega W_{2,\tau}^{2^k-1}\Delta W_{2,\tau}\mathrm{d}x \le -\int_\Omega \nabla W_{2,\tau}^{2^k-1}\nabla W_{2,\tau}\mathrm{d}x &= -(2^k-1)\int_\Omega (\nabla W_{2,\tau})^2 W_{2,\tau}^{2^k-2}\mathrm{d}x,\\ &= -E_k\int_\Omega |\nabla W_{2,\tau}^{2^{k-1}}|^2\mathrm{d}x,\end{aligned}$$

with $E_k = \frac{2^k-1}{2^{2k-2}}$. Therefore, (2.13) becomes

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x \le -d_2E_k\int_\Omega |\nabla W_{2,\tau}^{2^{k-1}}|^2\mathrm{d}x + \mathrm{e}^{-b_4\tau}\int_\Omega W_{2,\tau}^{2^k-1}\partial_3 G(x,W_1,0)W_3\mathrm{d}x - \int_\Omega b_2(x)W_{2,\tau}^{2^k}\mathrm{d}x. \quad (2.14)$$

From the first step, there exists $t_0 > 0$, and $M_0 > 0$ such that $W_1 \le M_0$ for $t > t_0$. For $t > t_0$, we obtain

$$\int_\Omega W_{2,\tau}^{2^k-1}\partial_3 G(x,W_1,0)W_3\mathrm{d}x \le ||\partial_3 G(x,M_0,0)||\int_\Omega W_{2,\tau}^{2^k-1}W_3\mathrm{d}x, \quad \text{for } t > t_0. \quad (2.15)$$

Applying Young's inequality to separate the term $W_{2,\tau}^{2^k-1}W_3$, we get

$$W_{2,\tau}^{2^k-1}W_3 \le \varepsilon_1(W_{2,\tau}^{2^k-1})^p + C_{\varepsilon_1}(W_3)^q,$$

with $p^{-1}+q^{-1}=1$, and $C_{\varepsilon_1} = (\varepsilon_1 p)^{-\frac{q}{p}}q^{-1}$. Choosing $q = 2^k$, then $p = \frac{2^k}{2^k-1}$, and $\varepsilon_1$ will be determined later. Hence,

$$W_{2,\tau}^{2^k-1}W_3 \le \varepsilon_1 W_{2,\tau}^{2^k} + C_{\varepsilon_1}W_3^{2^k}. \quad (2.16)$$

Substituting (2.15), (2.16) into (2.14), we obtain

$$\begin{aligned}\tfrac{1}{2^k}\tfrac{\partial}{\partial t}\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x \le -d_2E_k\int_\Omega |\nabla W_{2,\tau}^{2^{k-1}}|^2\mathrm{d}x + \mathrm{e}^{-b_4\tau}||\partial_3 G(x,M_0,0)||\varepsilon_1\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x\\ +\mathrm{e}^{-b_4\tau}||\partial_3 G(x,M_0,0)||C_{\varepsilon_1}\int_\Omega W_3^{2^k}\mathrm{d}x - \underline{b_2}\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x.\end{aligned} \quad (2.17)$$

Recall the interpolation inequality: for any $\varepsilon > 0$ (will be determined later), there exists $C_\varepsilon > 0$ satisfying

$$||u||_2^2 \le \varepsilon||\nabla u||_2^2 + C_\varepsilon||u||_1^2, \text{ for any } u \in W^{1,2}(\Omega).$$

We let $u = W_{2,\tau}^{2^{k-1}}$, then

$$-\int_\Omega |\nabla W_{2,\tau}^{2^{k-1}}|^2 \mathrm{d}x \le -\frac{1}{\varepsilon}\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x + \frac{C_\varepsilon}{\varepsilon}\left(\int_\Omega W_{2,\tau}^{2^{k-1}}\mathrm{d}x\right)^2. \tag{2.18}$$

Substituting (2.18) into (2.17), we obtain

$$\begin{aligned}\tfrac{1}{2^k}\tfrac{\partial}{\partial t}\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x \le d_2E_k\left(-\tfrac{1}{\varepsilon}\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x + \frac{C_\varepsilon}{\varepsilon}\left(\int_\Omega W_{2,\tau}^{2^{k-1}}\mathrm{d}x\right)^2\right) + \mathrm{e}^{-b_4\tau}||\partial_3 G(x,M_0,0)||\varepsilon_1\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x \\ +\mathrm{e}^{-b_4\tau}||\partial_3 G(x,M_0,0)||C_{\varepsilon_1}\int_\Omega W_3^{2^k}\mathrm{d}x.\end{aligned}$$

Doing some simplifications and using the induction assumption, we get

$$\begin{aligned}\tfrac{1}{2^k}\tfrac{\partial}{\partial t}\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x \le d_2E_k\tfrac{C_\varepsilon}{\varepsilon}M_{2^{k-1}}^2 + \left(-\tfrac{1}{\varepsilon}d_2E_k + \mathrm{e}^{-b_4\tau}||\partial_3 G(x,M_0,0)||\varepsilon_1 - \underline{b_2}\right)\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x \\ +\mathrm{e}^{-b_4\tau}||\partial_3 G(x,M_0,0)||C_{\varepsilon_1}\int_\Omega W_3^{2^k}\mathrm{d}x.\end{aligned} \tag{2.19}$$

Next, we multiply both sides of the third equation of (1.3) by $W_3^{2^k-1}$, and integrating the resulting equation on $\Omega$, we get

$$\begin{aligned}\tfrac{1}{2^k}\tfrac{\partial}{\partial t}\int_\Omega W_3^{2^k}\mathrm{d}x &= \int_\Omega a_2 W_3^{2^k-1}W_2\mathrm{d}x - \int_\Omega b_3 W_3^{2^k}\mathrm{d}x, \\ &\le ||a_2||\int_\Omega W_3^{2^k-1}W_2\mathrm{d}x - \underline{b_3}\int_\Omega W_3^{2^k}\mathrm{d}x.\end{aligned} \tag{2.20}$$

Again, by using Young's inequality on $W_3^{2^k-1}W_{2,\tau}$, with $q = 2^k$, and $p = \frac{2^k}{2^k-1}$, then for any $\varepsilon_2 > 0$, we have $C_{e_2} = (\varepsilon_2 p)^{-\frac{q}{p}}q^{-1}$, which satisfies the inequality

$$W_3^{2^k-1}W_2 \le \varepsilon_2 W_3^{2^k} + C_{e_2}W_2^{2^k}. \tag{2.21}$$

Substituting (2.21) into (2.20), we get

$$\tfrac{1}{2^k}\tfrac{\partial}{\partial t}\int_\Omega W_3^{2^k}\mathrm{d}x \le \left(||a_2||\varepsilon_2 - \underline{b_3}\right)\int_\Omega W_3^{2^k}\mathrm{d}x + ||a_2||C_{\varepsilon_2}\int_\Omega W_2^{2^k}\mathrm{d}x. \tag{2.22}$$

By the continuity of $W_2$ with respect to $t$, and by the first equation of (2.5), we have $W_2 > 0$ for all $W_{20} > 0$, therefore, there exists a positive constant $\tilde{M}$ that depends only on $\tau$ such that

$$W_2(t,x) \le \tilde{M}W_{2,\tau}(t,x),$$

therefore, (2.22) becomes

$$\tfrac{1}{2^k}\tfrac{\partial}{\partial t}\int_\Omega W_3^{2^k}\mathrm{d}x \le \left(||a_2||\varepsilon_2 - \underline{b_3}\right)\int_\Omega W_3^{2^k}\mathrm{d}x + ||a_2||C_{\varepsilon_2}\tilde{M}\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x. \tag{2.23}$$

Adding (2.20), and (2.23), we get

$$\begin{aligned}\tfrac{1}{2^k}\tfrac{\partial}{\partial t}\int_\Omega (W_{2,\tau}^{2^k}+W_3^{2^k})\mathrm{d}x \leq{}& d_2E_k\tfrac{C_\varepsilon}{\varepsilon}M_{2^{k-1}}^2\\ &+\left(||a_2||C_{\varepsilon_2}\tilde{M}-\tfrac{1}{\varepsilon}d_2E_k+\mathrm{e}^{-b_4\tau}||\partial_3G(x,M_0,0)||\varepsilon_1-\underline{b_2}\right)\int_\Omega W_{2,\tau}^{2^k}\mathrm{d}x\\ &+\left(||a_2||\varepsilon_2-\underline{b_3}+\mathrm{e}^{-b_4\tau}||\partial_3G(x,M_0,0)||C_{\varepsilon_1}\right)\int_\Omega W_3^{2^k}\mathrm{d}x.\end{aligned} \tag{2.24}$$

Choose $\varepsilon$ sufficiently small in such a way

$$||a_2||C_{\varepsilon_2}\tilde{M}-\frac{1}{\varepsilon}d_2E_k+\mathrm{e}^{-b_4\tau}||\partial_3G(x,M_0,0)||\varepsilon_1-\underline{b_2}<0,$$

and fix $\varepsilon_1$, $C_{\varepsilon_2}$ sufficiently small such that

$$||a_2||\varepsilon_2-\underline{b_3}+\mathrm{e}^{-b_4\tau}||\partial_3G(x,M_0,0)||C_{\varepsilon_1}<0.$$

Let

$$C_3=\min\left\{-||a_2||C_{\varepsilon_2}\tilde{M}+\frac{1}{\varepsilon}d_2E_k-\mathrm{e}^{-b_4\tau}||\partial_3G(x,M_0,0)||\varepsilon_1+\underline{b_2},-||a_2||\varepsilon_2+\underline{b_3}-\mathrm{e}^{-b_4\tau}||\partial_3G(x,M_0,0)||C_{\varepsilon_1}\right\}>0.$$

Thus, (2.24) becomes

$$\tfrac{1}{2^k}\tfrac{\partial}{\partial t}\int_\Omega (W_{2,\tau}^{2^k}+W_3^{2^k})\mathrm{d}x \leq d_2E_k\tfrac{C_\varepsilon}{\varepsilon}M_{2^{k-1}}^2-C_3\int_\Omega (W_{2,\tau}^{2^k}+W_3^{2^k})\mathrm{d}x. \tag{2.25}$$

Then, there exists a positive constant $M_{2^k}>0$ satisfies

$$\limsup_{t\to+\infty}(\|W_2\|_{2^k}+\|W_3\|_{2^k})\leq M_{2^k}.$$

For any $p\geq 1$, and from the continuous embedding $L^q(\Omega)\subset L^p(\Omega),\ q\geq p\geq 1$, we have

$$\limsup_{t\to+\infty}(\|W_2\|_p+\|W_3\|_p)\leq M_p,$$

with $M_p$ is a positive constant independent on the initial conditions.

**Step 4:** There exists $M_\infty>0$ such that

$$\limsup_{t\to+\infty}(\|W_1\|+\|W_2\|)\leq M_\infty. \tag{2.26}$$

The proof used in this step is similar to proof of [11], Lemma 2.4. We let $p>\frac{n}{2}$, and $\frac{n}{2p}<a<1$, then $Y_a$ is the fractional power space with graph norm continuously embedded in $C$. Also, there exists $M_a>0$ such that

$$\|A^aT_2(t)\|\leq\frac{M_A}{t^a},$$

for all $t > 0$. To prove (2.26), we need only to show that

$$\limsup_{t\to+\infty}(\|W_1\|_Y + \|W_2\|_Y) \leq M_Y. \tag{2.27}$$

Furthermore, $M_Y$ is a positive constant independent of the initial data. From the previous steps, we fix $\eta > 0$, then there exists $t_m > 0$ sufficiently large such that

$$\|W_1(t,\cdot)\| \leq M_0 + \eta, \quad \|W_2(t,\cdot)\|_p \leq (M_p + \eta)^{\frac{1}{p}}, \quad \|W_3(t,\cdot)\|_p \leq (M_p + \eta)^{\frac{1}{p}},$$

for all $t > t_m + \eta$. By the second equation of (1.3), we have for $t > t_m + \eta$

$$W_2(t,\cdot) = T_2(1)W_2(t-\eta) + \int_{t-\eta}^{t} T_2(t-s)\mathrm{e}^{-b_4\tau}G(W_1(s-\tau,\cdot), W_3(s-\tau,\cdot))\mathrm{d}s.$$

Then, for all $t > t_m + \eta$

$$\begin{aligned}
\|A^a W_2(t,\cdot)\|_p &\leq \|A^a T_2(\eta)W_2(t-\eta)\|_p + \int_{t-\eta}^{t} \|A^a T_2(t-s)\mathrm{e}^{-b_4\tau}G(\cdot, W_1(s-\tau,\cdot), W_3(s-\tau,\cdot))\mathrm{d}s\|_p,\\
&\leq \tfrac{M_a}{\eta^a}\|W_2(t-\eta)\|_p + \int_{t-\eta}^{t} \|A^a T_2(t-s)\mathrm{e}^{-b_4\tau}G(\cdot, M_0+\eta, W_3(s-\tau,\cdot))\mathrm{d}s\|_p,\\
&\leq \tfrac{M_a}{\eta^a}\|W_2(t-\eta)\|_p + \int_{t-\eta}^{t} \|A^a T_2(t-s)\mathrm{e}^{-b_4\tau}W_3(s-\tau,\cdot)\partial_3 G(\cdot, M_0+\eta, 0)\|_p\mathrm{d}s,\\
&\leq \tfrac{M_a}{\eta^a}\|W_2(t-\eta)\|_p + ||\partial_3 G(x, M_0+\eta, 0)||\mathrm{e}^{-b_4\tau}(M_p+\eta)^{\frac{1}{p}}\int_{t-\eta}^{t}\frac{M_a}{(t-s)^a}\mathrm{d}s,\\
&\leq \tfrac{M_a}{\eta^a}(M_p+\eta)^{\frac{1}{p}} + ||\partial_3 G(x, M_0+\eta, 0)||\mathrm{e}^{-b_4\tau}(M_p+\eta)^{\frac{1}{p}}M_a\eta^{1-a}.
\end{aligned}$$

Then, there exists $M_\infty > 0$ such that

$$\limsup_{t\to+\infty}||W_2(t,\cdot)|| \leq M_\infty.$$

Replacing this result into the third equation of (1.3), we get

$$\limsup_{t\to+\infty}||W_3(t,\cdot)|| \leq \frac{M_\infty||a_2||}{\underline{b_3}}.$$

The proof is completed.

□

## 2.2. Compactness

To show this, we apply [12], Theorem 2.4.6. We let $\Psi(t) :\to \mathbb{X}^+,\ t \geq 0$, be the semiflow associated to the system (1.3). This means that $\Psi(t)W_0 := W(t,\cdot,W_0) = (W_1(t,\cdot,W_0), W_2(t,\cdot,W_0), W_3(t,\cdot,W_0)),\ t \geq 0$, with $W(t,\cdot,W_0)$ be the solution of (1.3). The main result is provided in the following theorem

**Theorem 2.4.** *For any $W_0 \in \mathbb{X}^+$, $\Psi(t)$ has a connected global attractor in $\mathbb{X}^+$.*

*Proof.* We prove this result through the following claims.

**Claim 0.** $\Psi(t)$ is point dissipative.
This can be deduce from Theorem 2.3.

**Claim 1.** $\Psi(t)$ is bounded for any bounded set $\mathbf{C} \subset \mathbb{X}^+$.
First, we prove that $W(t,\cdot)$ is bounded for any $W_0 \in \mathbf{C}$. Since $W(t,\cdot) \leq \tilde{W}(t,\cdot)$, with $\tilde{W}(t,\cdot)$ the positive solution of (2.2). Then we deduce that there exists a positive constant $\tilde{M}_1 > 0$ independent of the initial condition such that $\tilde{W} \leq \tilde{M}_1$.
Next, we let $N_1 = W_1 + \mathrm{e}^{b_4\tau} W_{2,\tau}$, and by following the procedures in step 2 in the proof of Theorem 2.3, we obtain

$$\frac{\partial}{\partial t}\int_\Omega N_1 \mathrm{d}x \quad \leq \quad |\Omega| \|a_1\| - m \int_\Omega N_1 \mathrm{d}x.$$

Thus,

$$\int_\Omega N_1 \mathrm{d}x \leq (W_1(0,\cdot) + \mathrm{e}^{b_4\tau} W_{2,\tau}(0,\cdot))\mathrm{e}^{-mt} + \frac{|\Omega|\|a_1\|}{m}\left(1 - \mathrm{e}^{-mt}\right) \leq M_1.$$

with $M_1$ a positive constant. Then we deduce that $\Psi(t)$ is bounded in $L_1(\Omega)$ for any bounded set $\mathbf{C}$. Next, we show that $\Psi(t)$ is bounded in $L_p(\Omega)$ for any bounded set $\mathbf{C}$. We let, $N_{2,k} = W_{2,\tau}^{2^k} + W_3^{2^k}$. Again, by following the calculations performed in **Step 3** in the proof of Theorem 2.3, we obtain

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega N_{2,k} \mathrm{d}x \leq d_2 E_k \frac{C_\varepsilon}{\varepsilon} M_{2^{k-1}}^2 - C_3 \int_\Omega N_{2,k} \mathrm{d}x.$$

Then,

$$\frac{\partial}{\partial t}\int_\Omega N_{2,k} \mathrm{d}x \leq (W_{2,\tau}^{2^k}(0,\cdot) + \mathrm{e}^{b_4\tau} W_3^{2^k}(0,\cdot))\mathrm{e}^{-C_3 2^k t} + \frac{d_2 E_k \frac{C_\varepsilon}{\varepsilon} M_{2^{k-1}}^2}{C_3}\left(1 - \mathrm{e}^{-C_3 2^k t}\right) \leq M_k,$$

with $M_k$ a positive constant. Thus $\Psi(t)$ is bounded in $L_{2^k}(\Omega)$ for any bounded set $\mathbf{C} \subset \mathbb{X}^+$. From the continuous embedding $L^q(\Omega) \subset L^p(\Omega),\ q \geq p \geq 1$, we conclude that $\Psi(t)$ is bounded in $L_p(\Omega)$ $(p \geq 1)$ for any bounded set $\mathbf{C} \subset \mathbb{X}^+$. Similar reasoning can be applied to show that $\Psi(t)$ is bounded in $Y_a$ for any bounded set $\mathbf{C}$ of $Y_a$, with $\frac{n}{2p} < a < 1$, and $p$ and $n$ are positive and satisfy $p > \frac{n}{2}$. By the embedded of $Y_a$ in $C$, we deduce the result. The claim is proved.

**Claim 2.** $\Psi(t)$ is asymptotically smooth.
From [12], Lemma 2.3.4, we need only to prove that the semiflow $\Psi(t)$ is a $\kappa -$ *contraction*. Letting $A \subset \mathbb{X}^+$ be a bounded set, we define the Kuratowski measure of non-compactness $\kappa$ of $A$ by

$$\kappa(A) := \inf\left\{r : A \text{ has a finite cover of diameter} < r\right\},$$

where $A$ is precompact if and only if $\kappa(A) = 0$. Therefore, the $\kappa -$ *contraction* of the semiflow $\Gamma(t)$ implies exists a continuous function $\delta(t) : \mathbb{R}^+ \to \mathbb{R}^+$ verifying $0 \leq \delta(t) < 1$ for any $t > 0$, and if $A$ be a bounded set of $\mathbb{X}^+$, we have $\kappa(\Gamma(s)B)$, $0 \leq s \leq t$ is also bounded, and verifying $\kappa(\Gamma(t)A) \leq \delta(t)\kappa(A)$.

Applying [11], Lemma 2.5, we deduce that for any bounded $A \subset \mathbb{X}^+$ and $t > 0$, $S$ is a precompact set in $C$, where

$$S := \left\{ \int_0^t \mathrm{e}^{-b_3(t-s)} a_2 W_2(s,\cdot,W_0)\mathrm{d}s : \quad W_0 \in A \right\}.$$

Now, we decompose the semiflow as $\Psi(t) = \Psi_1(t) + \Psi_2(t)$, $t \geq 0$, with

$$\Psi_1(t)W_0 = \left( W_1(t,\cdot,W_0), W_2(t,\cdot,W_0), \int_0^t \mathrm{e}^{-b_3(t-s)} a_2 W_2(s,\cdot,W_0)\mathrm{d}s \right), \quad t \geq 0,$$

and

$$\Psi_2(t)W_0 = (0, 0, \mathrm{e}^{-b_3 t} W_{30}), \quad t \geq 0.$$

We let $A \subset \mathbb{X}^+$ is a bounded set. By (2.7)-(2.8), $\{\Psi(s)A, 0 \leq s \leq t\}$ is bounded for any $t > 0$. Since $S$ is a precompact set in $C(\bar{\Omega})$ we deduce that $\Psi_1(t)A$ is also precompact for any $t > 0$. Therefore $\kappa(\Psi_1(t)A) = 0$, $t > 0$. Further,

$$\kappa(\Psi_2(t)A) \leq ||\mathrm{e}^{-b_3 t}||\kappa(A) \leq \mathrm{e}^{-\underline{b_3} t}\kappa(A), \quad t \geq 0,$$

Thus,

$$\kappa(\Psi(t)A) \leq \kappa(\Psi_1(t)A) + \kappa(\Psi_2(t)A) \leq \mathrm{e}^{-\underline{b_3} t}\kappa(A),$$

Thus, $\Psi(t)$ is $\kappa -$ *contracting*.

**Claim 3.** $\Psi(t)$ has a connected global compact attractor.

Using the previous claims and [12], Theorem 2.4.6, we deduce the result.

□

### 2.3. Basic reproduction number

Obviously, system (1.3) has a pathogen-free steady state (PFSS) $E_0 = (V, 0, 0)$, with $V$ the unique positive solution of

$$\begin{cases} d_1 \Delta W_1 + a_1 - b_1 W_1 = 0, \; x \in \Omega, \\ \frac{\partial W_1}{\partial n} = 0, \qquad\qquad\qquad x \in \partial\Omega. \end{cases} \tag{2.28}$$

The basic reproduction number $R_0$ of (1.3) can be determined as the spectral radius of the next generation operator of the model. Linearizing (1.3) at $E_0$, we obtain

$$\begin{cases} \frac{\partial W_1}{\partial t} = d_1 \Delta W_1 - \partial_3 G(x, V, 0) W_3 - b_1(x) W_1, & x \in \Omega, \; t > 0, \\ \frac{\partial W_2}{\partial t} = d_2 \Delta W_2 + \mathrm{e}^{-b_4 \tau} \partial_3 G(x, V, 0) W_{3,-\tau} - b_2(x) W_2, & x \in \Omega, \; t > 0, \\ \frac{\partial W_3}{\partial t} = a_2(x) W_2 - b_3(x) W_3, & x \in \Omega, \; t > 0, \\ \frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = 0, & x \in \partial\Omega, \; t > 0. \end{cases} \tag{2.29}$$

Clearly, the $W_2$ and $W_3$ equations can be separated from the first equation. Let $T(t)$ be the semigroup associated to

$$\begin{cases} \frac{\partial W_2}{\partial t} = d_2\Delta W_2 + \mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)W_3 - b_2W_2, & x\in\Omega,\ t>0,\\ \frac{\partial W_3}{\partial t} = a_2W_2 - b_3W_3, & x\in\Omega,\ t>0,\\ \frac{\partial W_2}{\partial n} = 0, & x\in\partial\Omega,\ t>0. \end{cases} \tag{2.30}$$

Then, $T(t)$ has the generator

$$A := B + F = \begin{pmatrix} d_1\Delta - b_2 & 0\\ a_2 & -b_3 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{e}^{-b_4\tau}\partial_3 G(.,V,0)\\ 0 & 0 \end{pmatrix} \tag{2.31}$$

Clearly, $S(B)<0$, with $S(B)=\sup\{Re\lambda,\ \lambda\in\sigma(B)\}$ is the spectral bound of $B$. By [23], the basic reproduction number for (1.3) is the spectral radius of the operator $L$, and

$$L[\phi](x) = \int_0^\infty F(x)\tilde{T}(t)\phi(x)\mathrm{d}t = F(x)\int_0^\infty \tilde{T}(t)\phi(x)\mathrm{d}t,\quad \phi\in C(\bar{\Omega},\mathbb{R}^2),\quad x\in\bar{\Omega},$$

where $\tilde{T}(t)$ is semigroup associated to $B$. Hence,

$$R_0 := r(L) = \sup\{|\lambda|,\ \lambda\in\sigma(L)\}.$$

**Lemma 2.5.** *The following statements holds*

(i) $r(-F(B)^{-1}) = r(B_1)$, *where for any* $\phi\in C$, $B_1$ *can be defined as*

$$\begin{cases} B_1[\phi] = -(b_2 - d_2\Delta)^{-1}b_3^{-1}a_2\mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\phi, & x\in\Omega,\\ \frac{\partial\phi}{\partial n} = 0, & x\in\partial\Omega. \end{cases}$$

*Moreover,*

$$R_0 = r(B_1) = \sup_{\phi\in H^1(\Omega),\phi\neq 0} \frac{\displaystyle\int_\Omega \mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\frac{a_2}{b_3}\phi^2\mathrm{d}x}{\displaystyle\int_\Omega (d_2|\nabla\phi|^2 + b_2\phi^2)\mathrm{d}x}. \tag{2.32}$$

(ii) *We treat* $d_2$ *as an independent variable in* $(0,\infty)$, *then* $R_0$ *is decreasing function of* $d_2$. *Moreover,* $R_0\to R_0^+$ *as* $d_2\to 0$, *and* $R_0\to R_0^-$ *as* $d_2\to+\infty$, *with*

$$R_0^+ = \overline{\left(\frac{a_2\mathrm{e}^{-b_4\tau}}{b_2b_3}\partial_3 G(x,V,0)\right)},\quad R_0^- = \frac{\int_\Omega \frac{a_2\mathrm{e}^{-b_4\tau}}{b_3}\partial_3 G(x,V,0)\mathrm{d}x}{\int_\Omega b_2\mathrm{d}x}.$$

*Proof.* (i) Let $M = F\phi$ and $\phi = -B^{-1}\psi$, with $\phi = (\phi_1,\phi_2)^T\in C^2$, $\psi=(\psi_1,\psi_2)^T\in C^2$, $M=(M_1,M_2)^T\in C^2$. We consider the following system

$$\begin{cases} \phi_1 = (b_2 - d_2\Delta)^{-1}[\psi_1],\\ \phi_2 = \frac{1}{b_3}\psi_2 - \frac{a_2}{b_3}(b_2 - d_2\Delta)^{-1}[\psi_1]. \end{cases}$$

Thus, we can write

$$\begin{cases} M_1 = \mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\Big(\frac{1}{b_3}\psi_2 - \frac{a_2}{b_3}(b_2-d_2\Delta)^{-1}[\psi_1]\Big), \\ M_2 = 0. \end{cases}$$

Therefore, we have

$$-FB^{-1}[\psi] = \begin{pmatrix} B_1[\psi_1] + B_2[\psi_2] \\ 0 \end{pmatrix},$$

with

$$\begin{cases} B_1[\psi_1] = -\mathrm{e}^{-b_4\tau}\partial_3 G(\cdot,V,0)\frac{a_2}{b_3}(b_2-d_2\Delta)^{-1}[\psi_1], \\ B_2[\psi_2] = \mathrm{e}^{-b_4\tau}\partial_3 G(\cdot,V,0)\frac{1}{b_3}\psi_2. \end{cases}$$

By iterations, we obtain

$$\Big(-FB^{-1}[\psi]\Big)^n = \begin{pmatrix} B_1^n[\psi_1] + B_1^{n-1}B_2[\psi_2] \\ 0 \end{pmatrix}.$$

Hence, $||B_1^n|| \le \|(-FB^{-1}[\psi])^n\| \le ||B_1^{n-1}||(||B_1||+||B_2||)$. Gelfand's formula and squeeze theorem implies $r(-FB^{-1}) = r(B_1)$. Therefore, $R_0$ can be expressed as

$$R_0 = r(B_1) = \sup_{\phi\in H^1(\Omega),\phi\neq 0} \frac{\displaystyle\int_\Omega \mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\frac{a_2}{b_3}\phi^2\mathrm{d}x}{\displaystyle\int_\Omega (d_2|\nabla\phi|^2 + b_2\phi^2)\mathrm{d}x}.$$

(ii) Since $B_1 = -\mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\frac{a_2}{b_3}(b_2-d_2\Delta)^{-1}$ is compact, we have

$$r_e(B_1+B_2) = r_e(B_2) = \overline{\left(\frac{\mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)}{b_3}\right)} = r(B_2) < r(B_1+B_2)),$$

with $r_e$ is the essential spectral radius. The generalized Krein-Rutmam Theorem [8] implies that $R_0 = r(B_1+B_2)$ is the principal eigenvalue of $B_1+B_2$ associated to a positive eigenfunction denoted by $\phi(x)$. We get

$$d_2\Delta\phi - b_2\phi + \frac{1}{R_0}\frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\phi = 0, \tag{2.33}$$

with Neumann boundary condition. We consider $d_2$ as a variable, and denote $\tilde\phi$ the derivative of $\phi$ with respect to $d_2$. Taking the derivative of the two sides of (2.33), we get

$$\Delta\phi + d_2\Delta\tilde\phi - b_2\tilde\phi + \frac{R_0'}{R_0}\frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\phi + \frac{1}{R_0}\frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\tilde\phi = 0, \tag{2.34}$$

where $R_0'$ is the derivative of $R_0$ with respect to $d_2$. Multiplying (2.33) by $\tilde{\phi}$, (2.34) by $\phi$, and subtracting the resulting equations, integrating on $\Omega$, we get

$$R_0' \int_\Omega \frac{R_0'}{R_0}\frac{a_2}{b_3} \mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\phi^2 \mathrm{d}x = \int_\Omega \phi\Delta\phi \mathrm{d}x = -\int_\Omega |\nabla\phi|^2 \mathrm{d}x \leq 0.$$

Using $(\mathbf{H_2})$, we get $R_0' \leq 0$.
Now, we determine the effect of small diffusion on $R_0$. Let $R_0 = R_0(d_2)$, and $m(x) = \mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\frac{a_2(x)}{b_3(x)b_2(x)}$. For any $\psi \in C$, we have

$$\frac{\displaystyle\int_\Omega \mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\frac{a_2}{b_3}\phi^2 \mathrm{d}x}{\displaystyle\int_\Omega (d_2|\nabla\phi|^2 + b_2\phi^2)\mathrm{d}x} \leq \frac{\overline{m}\displaystyle\int_\Omega b_2\phi^2 \mathrm{d}x}{\displaystyle\int_\Omega (d_2|\nabla\phi|^2 + b_2\phi^2)\mathrm{d}x}$$
$$\leq \overline{m}.$$

Hence, $R_0(d_2) \leq \overline{m}$. We claim that $R_0(d_2) \to \overline{m}$ as $d_2 \to 0$, and we need to show that $\liminf_{d_2\to 0} R_0(d_2) \geq \overline{m}$. If not, there exist $\delta > 0$ such that $R_0(d_2) \leq \overline{m} - \delta$, for all $d_2 > 0$. The continuity of the coefficient functions implies the existence of some $x_0 \in \Omega$, and $\delta > 0$ sufficiently small, such that

$$R_0 + \delta < \overline{m} - \delta < m(x),$$

for all $x \in B_\delta(x_0)$, where $B_\delta(x_0)$ is the ball of center $x_0$ and radius $\delta$. By compactness of continuous function on a bounded domain, there exists $\delta_0 > 0$ verifying

$$-b_2 + \frac{1}{R_0}\frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0) > \delta_0,$$

for all $x \in B_\delta(x_0)$, and $d_2 > 0$. Letting $(\mu, \phi_-)$ be the principal eigenpaire of $-\Delta$ on $B_\delta(x_0)$ with Neumann boundary condition, we can normalize $\phi_- \leq 1$ for all $x \in B_\delta(x_0)$. Besides, we consider $d \in (0, \frac{\delta_0}{\mu})$ and normalize the eigenfunction in (2.33) as

$$\phi^+(x) = \frac{\phi(x)}{\inf_{x\in B_\delta(x_0)} \phi(x)}.$$

Clearly, $\phi^-(x) \leq 1 \leq \phi^+(x)$ for all $x \in B_\delta(x_0)$. Moreover, $\phi^+$ satisfies $-\Delta\phi^+ > \frac{\delta_0}{d}\phi^+$, and $\phi^-$ satisfy $-\Delta\phi^- < \frac{\delta_0}{d}\phi^+$. Thus, $\phi^-$ and $\phi^+$ are respectively the lower and upper solution of the operator $-\Delta - \frac{\delta_0}{d}$ with Neumann boundary condition. Then, the problem

$$-\Delta\psi = \frac{\delta_0}{d}\psi,$$

with Neumann boundary condition has a positive solution. Therefore, $\frac{\delta_0}{d} > \mu$ is an eigenvalue of the operator $-\Delta$, which is a contradiction with the assumption that $\mu$ is a principal eigenvalue. Then, we deduce that $R_0 \to R_0^+$ as $d_2 \to 0$.

At last, we show that $R_0 \to R_0^-$ as $d_2 \to +\infty$. We substitute $\phi = 1$ in (2.32), we obtain

$$R_0 \geq R_0^-, \text{ for all } d_2 > 0.$$

From (i), we have $R_0(d_2)$ is uniformly bounded for all $d_2 \geq 1$, which implies that $R_0(d_2)$ has a finite limit $\tilde{R}_0$ as $d_2 \to +\infty$. We claim that $\tilde{R}_0 = R_0^-$. Clearly, $\tilde{R}_0 \geq R_0^-$. It remain to show that $\tilde{R}_0 \leq R_0^-$. We divide both sides of (2.33) by $d_2$ we get

$$\Delta\phi^* - \frac{b_2}{d_2}\phi^* + \frac{1}{d_2 R_0}\frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\partial_3 G(x, V, 0)\phi^* = 0, \tag{2.35}$$

with Neumann boundary condition. The elliptic regularity [24] implies that $\phi^* \to \hat{\phi}$ in $C(\bar{\Omega})$ as $d_2 \to \infty$ for some positive constant $\hat{\phi}$. Integrating both sides of (2.33) on $\Omega$, and get $R_0 = R_0^-$.

□

We let $(\lambda_1, \phi_1(x))$ be the principal eigenpaire of the problem

$$\begin{cases} d_2\Delta\psi - b_2\psi + \frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\partial_3 G(x, V, 0)\psi = \lambda\psi, & x \in \bar{\Omega}, \\ \frac{\partial\psi}{\partial n} = 0, & x \in \bar{\Omega}. \end{cases} \tag{2.36}$$

Notice that $B$ and $F$ satisfy all statements of [25], Theorem 2.3, then we have the following relationship between $R_0, \lambda_1$ and $S(A)$.

**Lemma 2.6.** *$R_0 - 1$, $\lambda_1$ and $S(A)$ have the same sign.*

*Proof.* Clearly, $B$ and $F$ satisfy all statements of [25], Theorem 2.3, then we deduce that $R_0 - 1$ has the same sign as $S(A)$.

By the Krein-Rutmam Theorem [8], we deduce that $\lambda_1$ is simple and $\phi_1(x) > 0$ for all $x \in \bar{\Omega}$. Notice that the eigenpaire $(R_0, \phi)$ satisfies

$$d_2\Delta\phi - b_2\phi + \frac{1}{R_0}\frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\partial_3 G(x, V, 0)\phi = 0, \quad x \in \bar{\Omega}. \tag{2.37}$$

with Neumann boundary condition. Also, the principal eigenpaire $(\lambda_1, \phi_1(x))$ satisfies

$$d_2\Delta\phi_1 - [b_2 - \tfrac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\partial_3 G(x, V, 0)]\phi_1 = \lambda_1\phi_1, \; x \in \bar{\Omega}, \tag{2.38}$$

with Neumann boundary condition. Multiplying both sides of (2.37) by $\phi_1$ and both sides of (2.38) by $\phi$, subtracting the resulting equations, and integrating on $\Omega$, we obtain

$$\left(1 - \frac{1}{R_0}\right)\int_\Omega \frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\partial_3 G(x, V, 0)\phi\phi_1 \mathrm{d}x = \lambda_1 \int_\Omega \phi\phi_1 \mathrm{d}x. \tag{2.39}$$

The positivity of all functions in (2.39) implies that $1 - \frac{1}{R_0}$ and $\lambda_1$ have the same sign, which implies that $R_0 - 1$ has the same sign as $\lambda_1$. From the first part of the proof we deduce that $S(A)$ has also the same sign as $\lambda_1$. The proof is completed. □

**Remark 2.7.** If $R_0 \geq 1$ then $S(A) > 0$ becomes an eigenvalue of the generator $A$ associated to a strictly positive eigenfunction, and hence satisfies the following eigenvalue problem

$$\begin{cases} d_2\Delta\phi_2 + \mathrm{e}^{-b_4\tau}\partial_3 G(x, V, 0)\phi_3 - b_2\phi_2 = \lambda\phi_2, & x \in \Omega, \\ a_2\phi_2 - b_3\phi_3 = \lambda\phi_3, & x \in \Omega, \\ \frac{\partial\phi_2}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.40}$$

The proof can be performed by similar method as in the proof of [11], Lemma 3.7, so we omit the proof.

## 3. Pathogen-free steady state

In this section, we investigate the global stability of the PFSS for $R_0 \leq 1$. The main difficulty is dealing with the nonlinearity of the incidence function $G$ and the delay term. For this, we need some additional results comparing with [11]. We let the following problem

$$\begin{cases} d_2\Delta W + mW - b_2 W = \lambda W, & x \in \Omega, \\ \frac{\partial W_2}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

with $m \in C$, and $m(x) > 0$ for all $x \in \bar{\Omega}$. We let $(\lambda_0(d_2, m), \tilde{\phi}(x))$ be the principal eigenpaire of the eigenvalue problem (3.1). Also, we consider the eigenvalue problem

$$\begin{cases} d_2\Delta W + mW\mathrm{e}^{-\lambda\tau} - b_2 W = \lambda W, & x \in \Omega, \\ \frac{\partial W}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Let $\lambda_0(d_2, m, \tau)$ be the principal eigenvalue of (3.2). The following result is inspired by the proof of [13], Theorem 9.2.1.

**Lemma 3.1.** *There exists a principal eigenpaire $(\hat{\lambda}_0(d_2, m, \tau), \hat{\phi}(x))$ of (3.2), and for any $\tau \geq 0$, $\lambda_0(d_2, m)$ and $\hat{\lambda}_0(d_2, m, \tau)$ have the same sign.*

*Proof.* Define $L : C_\tau \to C$ as

$$L\phi(x) = m(x)\phi(-\tau, x), \quad x \in \Omega, \phi \in C_\tau.$$

Clearly, $C$ is Banach Lattice, and $L$ is positive. For each $\lambda \in \mathbb{R}$, we define $L_\lambda : C \to C$ by

$$L_\lambda(\phi) = L(\mathrm{e}^{\lambda \cdot}\phi), \quad \phi \in C,$$

with $\mathrm{e}^{\lambda \cdot}\phi \in C_\tau$ defined as

$$\mathrm{e}^{\lambda \cdot}\phi(\sigma, x) = \mathrm{e}^{\lambda\sigma}\phi(x), \quad \sigma \in [-\tau, 0], \quad x \in \bar{\Omega}.$$

Letting $Q(t) : C_\tau \to C_\tau,\ t \geq 0$, be the solution of the following parabolic equation

$$\begin{cases} \frac{dv}{\mathrm{d}t} = Av(t) + Lv_{-\tau}, & t \geq 0, \\ v_0 = \phi \in C_\tau, \end{cases} \tag{3.3}$$

with $v_{-\tau} \in C_\tau$, and $A[\phi] = d_2\delta\phi - b_2\phi$. Letting $A_v : D(A_v) \to C_\tau$ be its generator. Thus, $v(t) : C_\tau \to C_\tau$ is positive. From [26], Section 4, we deduce that $S(A_v)$ has the same sign as $S(B + L_0) = \lambda_0(d_2, m)$. Since $L$ is positive, and by applying the strong maximum principle we can prove that $v(t, x) > 0$ for all $x \in \bar{\Omega}$. Hence $Q(t) : C_\tau \to C_\tau$ is strongly positive and compact for each $t > 2\tau$. Fixing some $t > 2\tau$, the Krein-Rutman Theorem implies that $r = r(Q(t))$ is a positive eigenvalue of $Q(t)$. The point spectral mapping Theorem [27], Theorem 2.2.4 implies the existence of a point spectral point $\hat{\lambda}$ of $A_v$ such that $r = \mathrm{e}^{t\hat{\lambda}}$, with $\hat{\lambda} \in \mathbb{R}$, and $\hat{\lambda} \leq S(A_v)$.

Furthermore, since $S(A_v) \in \sigma(A_v)$ next to the spectral mapping Theorem [27], Theorem 2.2.3, we have $\mathrm{e}^{tS(A_v)} \in \sigma(Q(t))$. Thus, $\mathrm{e}^{tS(A_v)} \leq r\mathrm{e}^{t\hat{\lambda}}$, which means that $S(A_v) \leq \hat{\lambda}$. Therefore, we deduce that $S(A_v) = \bar{\lambda}$ is a point spectral value of $A_v$. We let $\psi^+ \in E$ be the positive eigenfunction associated to and by the Krein-Rutman Theorem we deduce that $\psi^+$ is strictly positive. Therefore, $S(A_v)$ is the principal eigenvalue of $A_v$. Therefore, $\hat{\lambda}(d_2, m, \tau)$ exists and it is simple.

It remains to show that $sign\{\hat{\lambda}(d_2,m,\tau)\} = sign\{\lambda(d_2,m)\}$ for all $\tau \geq 0$. We replace $\lambda$ by $\lambda(d_2,m)$ in (3.1), and $\lambda$ by $\hat{\lambda}(d_2,m,\tau)$ in (3.2). Multiplying both sides of (3.1) by $\hat{\phi}$ and the two sides of (3.2) by $\tilde{\phi}$, and integrating the obtain equation, and subtracting the resulting equations, we obtain

$$\int_\Omega m\tilde{\phi}\hat{\phi}(1-\mathrm{e}^{-\tau\hat{\lambda}(d_2,m,\tau)})\mathrm{d}x = (\lambda(d_2,m)-\hat{\lambda}(d_2,m,\tau))\int_\Omega \tilde{\phi}\hat{\phi}\mathrm{d}x. \tag{3.4}$$

We prove the result by contradiction. We assume that $\hat{\lambda}(d_2,m,\tau) > 0$ and $\lambda(d_2,m) \leq 0$. Hence, the left hand side of the equality (3.4) is positive, and then $(\lambda(d_2,m)-\hat{\lambda}(d_2,m,\tau)) > 0$, which is a contradiction. Now, suppose that $\hat{\lambda}(d_2,m,\tau) < 0$ and $\lambda(d_2,m) \geq 0$. Hence, the left hand side of the equality (3.4) is negative, and we get also a contradiction. Then we deduce the result. □

**Remark 3.2.** The choice of the next generation operator $A$ in (2.31) is inspired by the results provided in Lemma 3.1.

Now, we show the global stability of the PFSS for $R_0 \leq 1$. At first, we show the local stability of this steady state through the following theorem:

**Theorem 3.3.** *If $R_0 < 1$ then $E_0$ is locally asymptotically stable, and unstable for $R_0 > 1$.*

*Proof.* At first, we mention that the generator of the linearized system (2.29) (denoted $A_\tau$) and the one for the non-delayed linear system (2.30) (which is $A$) are not the same. Moreover, $S(A_\tau)$ is the principal eigenvalue of the problem

$$\begin{cases} d_2\Delta\phi_2 + \mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\phi_3\mathrm{e}^{-\lambda\tau} - b_2\phi_2 = \lambda\phi_2, & x\in\Omega, \\ a_2\phi_2 - b_3\phi_3 = \lambda\phi_3, & x\in\Omega, \\ \frac{\partial\phi_2}{\partial n} = 0, & x\in\partial\Omega, \end{cases} \tag{3.5}$$

with $A_\tau$ is the generator of the semigroup $T_\tau(t)$ associated to the delayed linear system (2.29). $S(A)$ is the principal eigenvalue of the problem

$$\begin{cases} d_2\Delta\phi_2 + \mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\phi_3 - b_2\phi_2 = \lambda\phi_2, & x\in\Omega, \\ a_2\phi_2 - b_3\phi_3 = \lambda\phi_3, & x\in\Omega, \\ \frac{\partial\phi_2}{\partial n} = 0, & x\in\partial\Omega. \end{cases} \tag{3.6}$$

By applying the results of Lemma 3.1 on the eigenvalue problem formed by the first equation of (3.5) and (3.6), with $m = \mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)$, we deduce that $S(A)$ has the same sign as $S(A_\tau)$. Therefore, the exponential growth rate related to the semigroup associated to the delayed system (2.29) (which is denoted $\omega(T_\tau)$) and the one related to $T(t)$ have the same sign. Now, we determine the exponential growth bound of $T(t)$, which can be defined as

$$\omega(T) = \max\{S(A), \omega_0(T)\},$$

with $\mathrm{e}^{\omega_0(T)t} := \kappa(T(t))$, and $\kappa$ is the measure of non-compactness. To show the local stability of $E_0$, we need to determine the sign of $\omega(T_\tau)$, where if $\omega(T) < 0$ then $E_0$ is locally asymptotically stable, and if $\omega(T) > 0$ then $E_0$ is unstable. Since $\omega(T)$ has the same sign as $\omega(T_\tau)$, the local asymptotic stability of PFSS can be deduced by determining the sign of $\omega(T)$ by applying [28], Theorem 2.1, or [23], Theorem 3.1.

For any $(\tilde{W}_{20}, W_{30}) \in C^2$ (with $\tilde{W}_{20}(\cdot) = W_{20}(0,\cdot)$), and $(W_2(t,\cdot), W_3(t,\cdot)) := T(t)(\tilde{W}_{20}, W_{30})$. Thus, $T(t) = T_1(t) + T_2(t)$, where

$$T_1(t)(\tilde{W}_{20}, W_{30}) = \left(W_1(\cdot,t), \int_0^t \mathrm{e}^{-b_3(t-s)} a_2 W_2(\cdot,\sigma)\mathrm{d}\sigma\right), \quad T_2(t)(\tilde{W}_{20}, W_{30}) = \left(0, \mathrm{e}^{-b_3 t} W_{30}\right).$$

From [11], Lemma 2.5, we have that $T_1(t)$ is compact. Hence,

$$\kappa(T(t)) = \kappa(T_1(t) + T_2(t)) = \kappa(T_2(t)) \leq \|T_2(t)\| \leq \mathrm{e}^{-\underline{b_3} t}.$$

Therefore, $\omega_0 \leq -\underline{b_3} < 0$. Therefore, $S(A)$ determines the stability (resp. instability) of $E_0$. From Lemma 2.6 we have $R_0 - 1$ and $S(A)$ have the same sign. As a result, if $R_0 < 1$ then $S(A) < 0$ and hence $\omega(T) < 0$, which means that $E_0$ is locally asymptotically stable. However, if $R_0 > 1$ then $S(A) > 0$ therefore $\omega(T) = S(A) > 0$, and then $E_0$ is unstable. ☐

**Theorem 3.4.** *If $R_0 < 1$, then $E_0$ is globally asymptotically stable.*

*Proof.* We show the global attractiveness by constructing a Lyapunov function. From Lemma 2.5, $R_0$ satisfies the problem $B_1[\phi] = R_0\phi$, with Neumann boundary condition. Hence,

$$\begin{cases} -d_2\Delta\phi + b_2\phi - \frac{1}{R_0}\mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\frac{a_2}{b_3}\phi = 0, & x \in \Omega, \\ \frac{\partial\phi}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.7}$$

Let $\tilde{\phi} = \frac{a_2}{b_3}\phi > 0$, then (3.7) can be written as the following system

$$\begin{cases} -d_2\Delta\phi + b_2\phi - \frac{1}{R_0}\mathrm{e}^{-b_4\tau}\partial_3 G(x,V,0)\tilde{\phi} = 0, & x \in \Omega, \\ a_2\phi - b_3\tilde{\phi} = 0, & x \in \Omega, \\ \frac{\partial\phi}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.8}$$

Note that $\partial_3 G(x, V+\upsilon, 0)$ is continuous for $\upsilon > 0$. Therefore, for $R_0 < 1$, there exists a small enough $\upsilon$ such that $R_0$ (also denoted as $R_0$) corresponding to the perturbation $\upsilon$ on $\partial_3 G(x, V+\upsilon, 0)$ remains less than 1. The corresponding eigenfunctions will also be denoted by $(\phi, \tilde{\phi})$, which satisfies the perturbed system of (3.8).

By Theorem 3.3, to show the global asymptotic stability of PFSS, it suffice to show the global attraction of $E_0$. Since $\limsup_{t\to\infty} W_1(t,x) \leq V(x)$, then for any $\upsilon > 0$ there exists $t_1 > 0$ such that $W_1(t,\cdot) \leq V(\cdot) + \upsilon$ for all $t \geq t_1$. Notice that by applying $(\mathbf{H_2})$ we get $G(\cdot, W_{1,-\tau}, W_{3,-\tau}) \leq G(\cdot, V+\upsilon, W_{3,-\tau})$, and by using $(\mathbf{H_3})$ we get $G(\cdot, W_{1,-\tau}, W_{3,-\tau}) \leq G(\cdot, V+\upsilon, W_{3,-\tau}) \leq W_{3,-\tau}\partial_3 G(\cdot, V+\upsilon, 0)$. The comparison principle implies that $(W_2(t,x), W_3(x,t)) \leq (\tilde{W}_2(t,x), \tilde{W}_3(x,t))$, on $\bar{\Omega} \times [t_1, +\infty)$, with $(\tilde{W}_2(t,x), \tilde{W}_3(x,t))$ satisfying

$$\begin{cases} \frac{\partial \tilde{W}_2}{\partial t} = d_2\Delta\tilde{W}_2 + \mathrm{e}^{-b_4\tau}\partial_3 G(x, V+\upsilon, 0)W_{3,-\tau} - b_2\tilde{W}_2, & x \in \Omega, \quad t > t_1, \\ \frac{\partial \tilde{W}_3}{\partial t} = a_2\tilde{W}_2 - b_3\tilde{W}_3, & x \in \Omega, \quad t > t_1, \\ \frac{\partial \tilde{W}_2}{\partial n} = 0, & x \in \partial\Omega, \\ \tilde{W}_2(t_1, x) = W_2(t_1, x), \quad \tilde{W}_3(s,x) = W_3(s,x), & x \in \Omega, \quad s \in [t_1 - \tau, t_1]. \end{cases} \tag{3.9}$$

Define $T_\upsilon(t)$ as the linear semigroup generated by (3.9), with the generator $A_\upsilon$. We choose $\upsilon > 0$ sufficiently small such that $\omega(T_\upsilon) < 0$ (for $R_0 < 1$), which means that the steady state $(0,0)$ for (3.9) is locally stable. Next, we show the global attractiveness of $(0,0)$ for $R_0<1$. For this, we let the Lyapunov function

$$L_{VFS}[\tilde{W}_2,\tilde{W}_3](t)=L^{(1)}_{VFS}[\tilde{W}_2,\tilde{W}_3](t)+L^{(2)}_{VFS}[\tilde{W}_2,\tilde{W}_3](t),$$

with

$$L^{(1)}_{VFS}[\tilde{W}_2,\tilde{W}_3](t)=\int_\Omega(\phi\tilde{W}_2+k\tilde{\phi}\tilde{W}_3)\mathrm{d}x,\quad L^{(2)}_{VFS}[\tilde{W}_2,\tilde{W}_3](t)=\int_\Omega \mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)\int_{-\tau}^0\tilde{W}_{3,s}\mathrm{d}s\mathrm{d}x,$$

and $k(x)=\frac{\mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)}{a_2(x)}$. The derivative of $L^{(2)}_{VFS}[\tilde{W}_1,\tilde{W}_2](t)$ along the solution of (3.9) is

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}L^{(2)}_{VFS}[\tilde{W}_2,\tilde{W}_3](t)&=\tfrac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)\int_{-\tau}^0\tilde{W}_{3,s}\mathrm{d}s\mathrm{d}x,\\
&=\int_\Omega \mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)\int_{-\tau}^0\frac{\partial}{\partial t}\tilde{W}_3(t+s,x)\mathrm{d}s\mathrm{d}x,\\
&=\int_\Omega \mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)\int_{-\tau}^0\frac{\partial}{\partial s}\tilde{W}_3(t+s,x)\mathrm{d}s\mathrm{d}x,\\
&=\int_\Omega \mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)\tilde{W}_3(t+s,x)|_{s=-\tau}^{s=0}\mathrm{d}x,\\
&=\int_\Omega \mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)(\tilde{W}_3-\tilde{W}_{3,-\tau})\mathrm{d}x.
\end{aligned}$$

The derivative of $L^{(1)}_{VFS}[\tilde{W}_1,\tilde{W}_2](t)$ along the solution of (3.9) is

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}L^{(1)}_{VFS}[\tilde{W}_2,\tilde{W}_3](t)&=\int_\Omega(\phi\frac{\partial}{\partial t}\tilde{W}_2+k\tilde{\phi}\frac{\partial}{\partial t}\tilde{W}_3)\mathrm{d}x,\\
&=\int_\Omega\left(d_2\Delta\tilde{W}_2+\mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)\tilde{W}_{3,-\tau}-b_2\tilde{W}_2\right)\phi\mathrm{d}x\\
&\quad+\int_\Omega k\tilde{\phi}\left(a_2\tilde{W}_2-b_3\tilde{W}_3\right)\mathrm{d}x,\\
&\le\int_\Omega\left(d_2\Delta\tilde{W}_2+\frac{1}{R_0}\mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)\tilde{W}_{3,-\tau}-b_2\tilde{W}_2\right)\phi\mathrm{d}x\\
&\quad+\int_\Omega k\tilde{\phi}\left(a_2\tilde{W}_2-b_3\tilde{W}_3\right)\mathrm{d}x.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}L_{VFS}[\tilde{W}_2,\tilde{W}_3](t)&\le\int_\Omega\left(d_2\Delta\tilde{W}_2+\frac{1}{R_0}\mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)W_3-b_2\tilde{W}_2\right)\phi\mathrm{d}x\\
&\quad+\int_\Omega k\tilde{\phi}\left(a_2\tilde{W}_2-b_3\tilde{W}_3\right)\mathrm{d}x+\left(\frac{1}{R_0}-1\right)\int_\Omega \mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)\tilde{W}_{3,-\tau}\phi\mathrm{d}x,\\
&=d_2\int_\Omega\phi\Delta\tilde{W}_2\mathrm{d}x+\int_\Omega kW_3\left(-b_3\tilde{\phi}+\frac{1}{kR_0}\mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)\phi\right)\mathrm{d}x\\
&\quad+\int_\Omega w_2\left(k\tilde{\phi}a_2-b_2\phi\right)\mathrm{d}x+\left(\frac{1}{R_0}-1\right)\int_\Omega \mathrm{e}^{-b_4\tau}\partial_3G(x,V+\upsilon,0)\tilde{W}_{3,-\tau}\phi\mathrm{d}x.
\end{aligned}$$

Since that $k = \frac{e^{-b_4\tau}\partial_3 G(x,V+\upsilon,0)}{a_2}$, and using the second equation of (3.8), we deduce that

$$-b_3\tilde{\phi} + \frac{1}{kR_0}\beta e^{-b_4\tau}\partial_2 G(V+\upsilon,0)\phi = 0.$$

Also, by $ka_2 = \frac{e^{-b_4\tau}\partial_3 G(x,V+\upsilon,0)}{a_2}$, and by the first equation of (3.8), we get

$$k\tilde{\phi}a_2 - b_2\phi = -d_2\Delta\phi.$$

Then, $\frac{d}{dt}L_{VFS}[\tilde{W}_2,\tilde{W}_3](t)$ becomes

$$\tfrac{d}{dt}L_{VFS}[\tilde{W}_2,\tilde{W}_3](t) \le d_2\int_\Omega \left[\phi\Delta\tilde{W}_2 - w_2\Delta\phi\right]dx + \left(\frac{1}{R_0}-1\right)\int_\Omega e^{-b_4\tau}\partial_3 G(x,V+\upsilon,0)\tilde{W}_{3,-\tau}\phi dx.$$

By the Green's first identity and the Neumann boundary condition, we obtain

$$\begin{aligned}\tfrac{d}{dt}L_{VFS}[\tilde{W}_2,\tilde{W}_3](t) \le d_2\Big[&-\int_{\partial\Omega} W_2\nabla\phi\cdot n dS + \int_\Omega \nabla W_2\nabla\phi dx + \int_{\partial\Omega}\phi\nabla W_2\cdot n dS - \int_\Omega \nabla W_2\nabla\phi dx\Big]\\ &+\left(\tfrac{1}{R_0}-1\right)\int_\Omega e^{-b_4\tau}\partial_3 G(x,V+\upsilon,0)\tilde{W}_{3,-\tau}\phi dx,\\ &= \left(\tfrac{1}{R_0}-1\right)\int_\Omega e^{-b_4\tau}\partial_3 G(x,V+\upsilon,0)\tilde{W}_{3,-\tau}\phi dx.\end{aligned}$$

Therefore, for $R_0 < 1$, $\frac{d}{dt}L_{VFS}[\tilde{W}_2,\tilde{W}_3](t) \le 0$. Since $\phi(x) > 0$ for all $x \in \bar{\Omega}$, then $\frac{d}{dt}L_{VFS}[\tilde{W}_2,\tilde{W}_3](t) = 0$ if and only if $e^{-b_4\tau}\partial_3 G(x,V+\upsilon,0)\tilde{W}_{3,-\tau} = 0$. Replacing this result into the first equation of (3.9), we $\tilde{W}_2 = 0$, and then $\tilde{W}_3 = 0$. Therefore, $\frac{d}{dt}L_{VFS}[\tilde{W}_2,\tilde{W}_3](t) = 0$ holds if and only if $\tilde{W}_2 = \tilde{W}_3 = 0$. LaSalle's invariance principle implies that the origin is globally asymptotically stable for (3.9). This means that $\tilde{W}_2 \to 0$, and $\tilde{W}_3 \to 0$ as $t \to +\infty$. Replacing this result into the first equation of (1.3), we deduce that $T \to T^0$ as $t \to +\infty$. Moreover, by the first equation of (1.3), and the global stability of $V$ for (2.2), we deduce that $W_1(t,x) \to V(x)$ as $t \to +\infty$. Then, we conclude that $E_0$ is globally asymptotically stable for $R_0 < 1$.

□

Now, we prove the global asymptotic stability of $E_0$ for the critical case $R_0 = 1$.

**Theorem 3.5.** *If $R_0 = 1$, $E_0$ remain globally asymptotically stable.*

*Proof.* To overcome this, we need to prove the local stability and the global attraction of $E_0$. First, we investigate the local asymptotic stability of $E_0$. Let $\varepsilon > 0$ be a given positive constant. Assume that $\delta > 0$, and we let $W_0$ be the initial data of (1.3) that satisfies $||W_0 - E_0|| \le \delta$. We define

$$U_1(t,x) = \frac{W_1(t,x)}{V(x)} - 1.$$

It is easy to check that $\nabla U_1 = \frac{1}{V}\nabla W_1 - \frac{\nabla V}{V^2}W_1$. Hence,

$$d_1\Delta U_1 = \frac{1}{V}d_1\Delta W_1 - \frac{W_1}{V^2}d_1\Delta V - 2\frac{d_1}{V}\nabla V\cdot\nabla U_1.$$

Using the fact that $d_1\Delta V = -a_1 + b_1 V$, we obtain

$$\frac{d_1}{V}\Delta W_1 = d_1\Delta U_1 + \frac{W_1}{V^2}(-a_1 + b_1 V) + 2\frac{d_1}{V}\nabla V\cdot\nabla U_1,$$

hence,

$$\begin{aligned}
\tfrac{\partial U_1}{\partial t} &= \tfrac{1}{V}\tfrac{\partial W_1}{\partial t},\\
&= d_1\tfrac{1}{V}\Delta W_1 + \tfrac{a_1}{V} - b_1\tfrac{W_1}{V} - \tfrac{\mathrm{e}^{-b_4\tau}G(x,W_1,W_3)}{V},\\
&= \left[d_1\Delta U_1 + \tfrac{W_1}{V^2}(-a_1+b_1V) + 2\tfrac{d_1}{V}\nabla V\cdot\nabla U_1\right] + \tfrac{a_1}{V} - b_1\tfrac{W_1}{V} - \tfrac{\beta\mathrm{e}^{-b_4\tau}G(x,W_1,W_3)}{V},\\
&= d_1\Delta U_1 - \tfrac{a_1}{V}U_1 + 2\tfrac{d_1}{V}\nabla V\cdot\nabla U_1 - \tfrac{\mathrm{e}^{-b_4\tau}G(x,W_1,W_3)}{V}.
\end{aligned}$$

Hence, we can write

$$\frac{\partial U_1}{\partial t} = d_1\Delta U_1 - \frac{a_1}{V}U_1 + 2\frac{d_1}{V}\nabla V\cdot\nabla U_1 - \frac{\mathrm{e}^{-b_4\tau}G(x,W_1,W_3)}{V}.$$

Let $T_1(t)$ be the semigroup associated to the generator $d_1\Delta - \frac{a_1}{V} + 2\frac{d_1}{V}\nabla V\cdot\nabla$ with Neumann boundary condition. Hence, there exists $\theta > 0$ such that $||T_1(t)|| \leq M\mathrm{e}^{-\theta t}$, for some $\theta > 0$. Then, we have

$$U_1(t,\cdot) = T_1(t)U_{10} - \int_0^t T_1(t-s)\frac{\mathrm{e}^{-b_4\tau}G(\cdot,W_1(s,\cdot),W_3(s,\cdot))}{V}\mathrm{d}s,$$

with $U_1(0,x) = \frac{W_1(0,x)}{V(x)} - 1$. We let $b(t) = \max_{x\in\bar\Omega}\{U_1(t,x),0\}$. From the positivity of $T_1(t)$, we have

$$\begin{aligned}
b(t) &= \max\nolimits_{x\in\bar\Omega}\left\{T_1(t)U_{10} - \int_0^t T_1(t-s)\frac{\mathrm{e}^{-b_4\tau}G(\cdot,W_1(s,\cdot),W_3(s,\cdot))\mathrm{d}s}{V},0\right\},\\
&\leq \max\nolimits_{x\in\bar\Omega}\left\{T_1(t)U_{10},0\right\},\\
&\leq ||T_1(t)U_{10}||,\\
&\leq M\mathrm{e}^{-\theta t}||\tfrac{W_1(0,x)}{V(x)} - 1||,\\
&\leq \tfrac{M\delta}{\underline{V}}\mathrm{e}^{-\theta t}.
\end{aligned}$$

By the assumption $(\mathbf{H_3})$, $W_2, W_3$ satisfy

$$\begin{cases}
\frac{\partial W_2}{\partial t} \leq d_2\Delta W_2 - b_2W_2 + \mathrm{e}^{-b_4\tau}W_{3,-\tau}\partial_3G(x,V,0) + \mathrm{e}^{-b_4\tau}W_{3,-\tau}(\partial_3G(x,W_{1,-\tau},0) - \partial_3G(x,V,0)),\\
\frac{\partial W_3}{\partial t} = a_2(x)W_2 - b_3(x)W_3,
\end{cases}$$

with Neumann boundary conditions. Then, we have

$$\begin{pmatrix} W_2(t,\cdot)\\ W_3(t,\cdot)\end{pmatrix} \leq T_\tau(t)\begin{pmatrix} W_{20}\\ W_{30}\end{pmatrix} + \int_0^t T_\tau(s)\begin{pmatrix} \mathrm{e}^{-b_4\tau}W_{3,-\tau}(\partial_3G(\cdot,W_{1,-\tau},0) - \partial_2G(\cdot,V,0))\\ 0\end{pmatrix}\mathrm{d}s.$$

Since $\partial_3 G(x, x_1, 0)$ is Lipchitz continuous for the second variable, then there exists some $L_1 > 0$ such that

$$\begin{pmatrix} W_2(t,\cdot) \\ W_3(t,\cdot) \end{pmatrix} \leq T_\tau(t) \begin{pmatrix} W_{20} \\ W_{30} \end{pmatrix} + \int_0^t T_\tau(s) \begin{pmatrix} \mathrm{e}^{-b_4\tau} L_1 W_3(s-\tau,\cdot)||W_1(s-\tau,\cdot) - V|| \\ 0 \end{pmatrix} \mathrm{d}s.$$

Hence,

$$\begin{pmatrix} W_2(t,\cdot) \\ W_3(t,\cdot) \end{pmatrix} \leq T_\tau(t) \begin{pmatrix} W_{20} \\ W_{30} \end{pmatrix} + \int_0^t T_\tau(s) \begin{pmatrix} ||V||\mathrm{e}^{-b_4\tau} L_1 W_3(s-\tau,\cdot)||U_1(s-\tau,\cdot)|| \\ 0 \end{pmatrix} \mathrm{d}s.$$

From the last part proof of Theorem 3.3, and for $R_0 = 1$, we have $\omega(T) = \omega(T_\tau) = 0$. Hence $||T_\tau(t)|| \leq M_\tau$ for all $t \geq 0$, for some constant $M_\tau > 0$, which it can be chosen as larger as needed. Notice that $b(t) \leq \frac{M\delta}{\underline{V}}\mathrm{e}^{-\theta t}$, thus

$$\begin{aligned} \max\{||W_2(t,\cdot)||, ||W_3(t,\cdot)||\} &\leq M_\tau \max\{||W_{20}||, ||W_{30}||\} + M_\tau M ||V|| \mathrm{e}^{-b_4\tau} L_1 \int_0^t b(s-\tau)||W_3(s-\tau,\cdot)||\mathrm{d}s, \\ &\leq M_\tau\delta + \delta\tilde{M}_\tau \int_0^t \mathrm{e}^{-\theta(s-\tau)}||W_3(s-\tau,\cdot)||\mathrm{d}s, \end{aligned} \tag{3.10}$$

with $\tilde{M}_\tau = \frac{M_\tau M||V||\mathrm{e}^{-b_4\tau}L_1}{\underline{V}}$. By applying Gronwall's inequality, we get

$$||W_3(t,\cdot)|| \leq M_\tau\delta \mathrm{e}^{\int_0^t \delta\tilde{M}_\tau \mathrm{e}^{-\theta(s-\tau)}\mathrm{d}s} \leq M_\tau\delta\mathrm{e}^{\delta\tilde{M}_\tau \mathrm{e}^{\theta\tau}/\theta}. \tag{3.11}$$

From (3.10), we have

$$||W_2(t,\cdot)|| \leq M_\tau\delta + \delta\tilde{M}_\tau \int_0^t \mathrm{e}^{-\theta(s-\tau)}||W_3(s-\tau,\cdot)||\mathrm{d}s.$$

Using (3.11), we obtain

$$\begin{aligned} ||W_2(t,\cdot)|| &\leq M_\tau\delta + \delta^2\tilde{M}_\tau^2 \mathrm{e}^{\delta\tilde{M}_\tau \mathrm{e}^{\theta\tau}/\theta} \int_0^t \mathrm{e}^{-\theta(s-\tau)}\mathrm{d}s, \\ &\leq M_\tau\delta\left(1 + \frac{\delta\tilde{M}_\tau \mathrm{e}^{\delta\tilde{M}_\tau \mathrm{e}^{\theta\tau}/\theta}\mathrm{e}^{\theta\tau}}{\theta}\right). \end{aligned}$$

By $(\mathbf{H_1})$, we have

$$G(\cdot, W_1, W_3) \leq G\left(\cdot, W_1, M_\tau\delta\mathrm{e}^{\delta\tilde{M}_\tau \mathrm{e}^{\theta\tau}/\theta}\right).$$

Using the fact that $G$ is Lipschitz function for the second and the third variable, we get

$$G\left(\cdot, W_1, M_\tau\delta\mathrm{e}^{\delta\tilde{M}_\tau \mathrm{e}^{\theta\tau}/\theta}\right) - G\left(\cdot, 0, M_\tau\delta\mathrm{e}^{\delta\tilde{M}_\tau \mathrm{e}^{\theta\tau}/\theta}\right) \leq L_\delta W_1,$$

with

$$L_\delta = \max\left\{\partial_2 G\left(x, s, M_\tau\delta\mathrm{e}^{\delta\tilde{M}_\tau \mathrm{e}^{\theta\tau}/\theta}\right),\quad 0 \leq s < M,\quad x \in \bar{\Omega}\right\},$$

where $M < \infty$ is is taken as larger as needed. Notice that $L_\delta \to 0$ as $\delta \to 0$. Thus, from the first equation of (1.3) we have

$$\frac{\partial W_1}{\partial t} > d_1 \Delta W_1 + a_1(x) - \left[ b_1(x) + \mathrm{e}^{-b_4\tau} L_\delta \right] W_1.$$

By comparison principle, we have $\hat{W}_1(t,x) \leq W_1(t,x)$, for all $x \in \bar{\Omega},\ t \geq 0$, with $\hat{W}_1$ be the solution of the problem

$$\begin{cases} \frac{\partial \hat{W}_1}{\partial t} = d_1 \Delta \hat{W}_1 + a_1(x) - \left[ b_1(x) + \mathrm{e}^{-b_4\tau} L_\delta \right] \hat{W}_1, & x \in \Omega, \quad t > 0, \\ \frac{\partial \hat{W}_1}{\partial n} = 0, & x \in \partial\Omega, \quad t > 0, \\ \hat{W}_1(0,x) = W_{10}, & x \in \Omega. \end{cases} \tag{3.12}$$

Let $V_\delta$ be the unique positive steady state of (3.12). We let $\hat{U} = \hat{W}_1 - V_\delta$, then $\hat{U}$ satisfies

$$\begin{cases} \frac{\partial \hat{U}}{\partial t} = d_1 \Delta \hat{U} - \left[ b_1(x) + \mathrm{e}^{-b_4\tau} L_\delta \right] \hat{U}, & x \in \Omega, \quad t > 0, \\ \frac{\partial \hat{U}}{\partial n} = 0, & x \in \partial\Omega, \quad t > 0, \\ \hat{W}_1(0,x) = W_{10} - V_\delta, & x \in \Omega. \end{cases} \tag{3.13}$$

We let $T_1(t)$ be the semigroup of the generator $d_1\Delta - b_1$ with Neumann boundary condition. Then, there exists $M_1 > 0$ sufficiently large such that $||T_1(t)|| \leq M_1 \mathrm{e}^{-\underline{b_1} t}$. From (3.13), we have

$$\hat{U}(t,\cdot) = T_1(t)(W_{10} - V_\delta) - \mathrm{e}^{-b_4\tau} \int_0^t T_1(t-s) L_\delta \hat{U}(s,\cdot) \mathrm{d}s.$$

Therefore,

$$||\hat{U}(t,\cdot)|| \leq ||W_{10} - V_\delta|| M_1 \mathrm{e}^{-\underline{b_1} t} + M_1 L_\delta \mathrm{e}^{-b_4\tau} \int_0^t \mathrm{e}^{-\underline{b_1}(t-s)} ||\hat{U}(s,\cdot)|| \mathrm{d}s.$$

Hence, we can write

$$\left( ||\hat{U}(t,\cdot)|| \mathrm{e}^{\underline{b_1} t} \right) \leq ||W_{10} - V_\delta|| M_1 + M_1 L_\delta \mathrm{e}^{-b_4\tau} \int_0^t \left( ||\hat{U}(s,\cdot)|| \mathrm{e}^{\underline{b_1} s} \right) \mathrm{d}s.$$

Again, by Gronwall's inequality, we obtain

$$||\hat{U}(t,\cdot)|| \leq M_1 ||W_{10} - V_\delta|| \mathrm{e}^{(K(\delta) - \underline{b_1})t},$$

with $K(\delta) = M_1 \mathrm{e}^{-b_4\tau} L_\delta$. Clearly, $K(\delta) \to 0$ as $\delta \to 0$. Therefore, there exists $\delta_m > 0$ sufficiently small such that for $0 < \delta < \delta_m$, and we have $K(\delta) \leq \frac{\underline{b_1}}{2}$. Thus,

$$||\hat{U}(t,\cdot)|| \leq M_1 ||W_{10} - V_\delta|| \mathrm{e}^{-\frac{\underline{b_1} t}{2}}. \tag{3.14}$$

Notice that by the comparison principle we have $V_\delta \leq V$ for all $\delta > 0$ and $x \in \bar{\Omega}$, and $V_\delta \to V$ as $\delta \to 0$. Therefore, (3.14) implies

$$\begin{aligned} W_1(t,\cdot) - V &\geq (U_1(t,\cdot) - V_\delta) + (V_\delta - V), \\ &\geq -M_1 ||W_{10} - V_\delta|| \mathrm{e}^{-\frac{\underline{b_1})t}{2}} + (V_\delta - V), \\ &\geq -M_1 \delta - ||V_\delta - V||. \end{aligned}$$

Besides,

$$W_1(t,\cdot) - V = VU_1(t,\cdot) \leq ||V||b(t) \leq \frac{\delta M}{\underline{V}}.$$

Hence, we get

$$-M_1\delta - ||V_\delta - V|| \leq W_1(t,\cdot) - V \leq \frac{\delta M}{\underline{V}},$$

thus, we can choose $\delta$ sufficiently small such that for all $t > 0$

$$\max\{||W_1(t,\cdot) - V||, ||W_2(t,\cdot)||, ||W_3(t,\cdot)||\} \leq \varepsilon,$$

which is the local stability of $E_0$.

Next, we show the global attractiveness of $E_0$. From Theorem 2.4, the semiflow $\Psi(t)$ has a global compact attractor denoted $\mathbf{D}$. To show the global attractiveness of $E_0$, we need to show that $\mathbf{D} = \{E_0\}$. We define

$$X_0 = \{(\psi_1,\psi_2,\psi_3) \in \mathbb{X}^+ : \quad \psi_2 \not\equiv 0, \text{ and } \psi_3 \not\equiv 0\}, \tag{3.15}$$

$$\partial X_0 = \{(\psi_1,\psi_2,\psi_3) \in \mathbb{X}^+ : \quad \psi_2 \equiv 0, \text{ or } \psi_3 \equiv 0\}, \tag{3.16}$$

$$\partial X_1 = \{(\psi_1,\psi_2,\psi_3) \in \mathbb{X}^+ : \quad \psi_2 \equiv 0, \text{ and } \psi_3 \equiv 0\}. \tag{3.17}$$

We prove the result by showing the following two statements

(i) For any $W_0 \in \mathbf{D}$ the $\omega - limt$ set $\omega(W_0) \subset \partial X_1$.
From (2.3), we must have $W_{10} \leq V$. Clearly, $\partial X_1$ is an invariant set for $\Psi(t)$, hence, if $W_0 \in \partial X_1$, then $\Psi(t)W_0 \in \partial X_1$. Therefore, $E_0$ is attractive in this case. Now, we suppose that $W_0 \in \partial X_0$. From the two last equations of (1.3), we have $W_2(t,\cdot) \geq \tilde{W}_2(t,\cdot)$, and $W_3(t,\cdot) \geq \tilde{W}_3(t,\cdot)$, where $\tilde{W}_2, \tilde{W}_3$ satisfy

$$\begin{cases} \frac{\partial \tilde{W}_2}{\partial t} = d_2\Delta\tilde{W}_2 - b_2(x)\tilde{W}_2, & x \in \Omega, \quad t > 0, \\ \frac{\partial \tilde{W}_3}{\partial t} = a_2(x)\tilde{W}_2 - b_3(x)\tilde{W}_3, & x \in \Omega, \quad t > 0, \\ \frac{\partial \tilde{W}_2}{\partial n} = 0, & x \in \partial\Omega, \quad t > 0, \\ \tilde{W}_2(0,\cdot) = \underline{W_{20}}, \quad \tilde{W}_3(0,\cdot) = \underline{W_{30}}, & x \in \Omega, \end{cases} \tag{3.18}$$

with $\underline{W_{i0}}(\cdot) = \min\{W_{i0}(s,\cdot), \quad s \in [-\tau, 0]\}$, $i = 2, 3$. Hence,

$$\begin{cases} \tilde{W}_2(t,x) = T_2(t)\underline{W_{20}}(x), \\ \tilde{W}_3(t,x) = \mathrm{e}^{-b_3(x)t}W_{30}(x) + \int_0^t \mathrm{e}^{-b_3(x)(t-s)}a_2(x)\tilde{W}_2(s,x)\mathrm{d}s. \end{cases} \tag{3.19}$$

Clearly, if $W_{20} \neq 0$ then $\tilde{W}_2(t,x) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$, and then $\tilde{W}_3(t,x) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$, with $W_{30} = 0$, and therefore $W_i(t,x) > 0, \quad i = 2, 3$ for all $x \in \bar{\Omega}$ and $t > 0$. However, if $W_{20} = 0$ and $W_{30} \neq 0$, then $\tilde{W}_3(t,x) > 0$. That follows $W_3(t,x) > 0$, for all $x \in \bar{\Omega}$ and $t > 0$. Notice that $W_2$ satisfy $W_2(t,\cdot) = \int_0^t T_2(t-s)\beta(\cdot)\mathrm{e}^{-b_4\tau}G(\cdot, W_1(s-\tau,\cdot), W_3(s-\tau,\cdot))\mathrm{d}s > 0$. Then $W_2(t,x) > 0$, for all $x \in \bar{\Omega}$ and

$t > 0$. Therefore, we deduce that if either $W_{20} \equiv 0$ or $W_{30} \equiv 0$, we obtain that $W_i(t,x) > 0, \quad i = 2,3$ for all $x \in \bar{\Omega}$ and $t > 0$. Hence $W_1(t,x)$ satisfies

$$\begin{cases} \frac{\partial W_1}{\partial t} = d_1 \Delta W_1 + a_1(x) - G(x, W_1, W_3) - b_1(x) W_1, \ x \in \Omega, \quad t > 0 \\ \frac{\partial W_1}{\partial n} = 0, & x \in \partial\Omega, \quad t > 0, \\ W_1(0,x) \le V(x), & x \in \Omega. \end{cases} \tag{3.20}$$

The comparison principle implies that $W_1(t,x) < V(x)$ for all $x \in \bar{\Omega}$ and $t > 0$.

Now, motivated by [29], we define

$$c(t, W_0) \inf\{\tilde{c} \in R : \quad W_2(t,\cdot) \le \tilde{c}\psi_2, \text{ and } W_3(t,\cdot) \le \tilde{c}\psi_3\}.$$

Hence, $c(t, W_0) > 0$ for all $t > 0$. Next, we show that $c(t, W_0)$ is strictly decreasing in $t$. To prove this, we fix $t_1 > 0$ and we let $\bar{W}_2(t,\cdot) = c(t; W_0)\phi_2$ and $\bar{W}_3(t,\cdot) = c(t; W_0)\phi_3$ for $t \ge t_1$. Notice that $W_1(t,\cdot) < V$, thus

$$\begin{cases} \frac{\partial \bar{W}_2}{\partial t} > d_2 \Delta \bar{W}_2 - b_2(x) \bar{W}_2 + \beta \mathrm{e}^{-b_4 \tau} G(x, \tilde{W}_{1,-\tau}, \tilde{W}_{3,-\tau}), , & x \in \Omega, \ t > 0, \\ \frac{\partial \tilde{W}_3}{\partial t} = a_2(x) \tilde{W}_2 - b_3(x) \tilde{W}_3, & x \in \Omega, \ t > 0, \\ \frac{\partial \tilde{W}_2}{\partial n} = 0, & x \in \partial\Omega, \ t > 0, \\ \tilde{W}_2(s,\cdot) \ge W_2(s,\cdot), \quad \tilde{W}_3(s,\cdot) \ge W_3(s,\cdot), & s \in [-\tau, 0], \ x \in \bar{\Omega}. \end{cases} \tag{3.21}$$

The comparison principle implies that $(\tilde{W}_2(t,x), \tilde{W}_3(t,x)) \ge (W_2(t,x), W_3(t,x))$ for all $x \in \bar{\Omega}$, and $t \ge t_0$. The first equation of (3.21) and the strong maximum principle imply that $c(t; W_0)\phi_2(s,x) = \tilde{W}_2(t,x) > W_2(t,x)$, for all $x \in \bar{\Omega}$ and $t > t_0$, and $s \in [-\tau, 0]$. Also, the second equation gives all $x \in \bar{\Omega}$ and $t > t_0$, and $s \in [-\tau, 0]$. Since $t_0$ is chosen in arbitrary way, we deduce that $c(t; W_0)$ is strictly decreasing in $t$.

As a result, we deduce that $\lim_{t \to +\infty} c(t; W_0) = c_m$. We claim that $c_m = 0$. We let $\mathbf{W} = (\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3) \in \omega(W_0)$. Hence, there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$ such that $\Psi(t_n)W_0 \to \mathbf{W}$ as $n \to +\infty$. Use the semiflow properties $\lim_{n \to +\infty} \Psi(t + t_n)W_0 = \psi(t) \lim_{n \to +\infty} \Psi(t_n)W_0 = \Psi(t)\mathbf{W}$. Therefore, if $\mathbf{W}_2 \equiv 0$ *or* $\mathbf{W}_3 \equiv 0$, by following the same above reasoning we prove that $c(t; \mathbf{W})$ is strictly decreasing, which is a contradiction with $c(t; \mathbf{W}) = c_m$. Thus, $\mathbf{W}_2 \equiv 0$ *and* $\mathbf{W}_3 \equiv 0$.

(ii) $\mathbf{D} = \{E_0\}$.
If $W_0 \in \partial X_1$ then $\{E_0\}$ is globally attractive for $\Psi(t)$, and $\{E_0\}$ is the only compact invariant set of $\Psi(t) \in \partial X_1$. Now, we let $W_0 \in \mathbf{D}$, since $\omega(W_0)$ is compact invariant set, and satisfies $\omega(W_0) \subset \partial X_1$ (see statement (i)), we deduce that $\omega(W_0) = \{E_0\}$. As $\mathbf{D}$ is compact invariant in $\mathbb{X}^+$, $E_0$ is locally asymptotically stable. By applying [11], Lemma 3.1, we have $\mathbf{D} = \{E_0\}$.

Thus we get the global asymptotic stability of $E_0$ for $R_0 = 1$. □

## 4. Existence of positive steady state

We investigate the uniform persistence of the semiflow $\Psi(t)$. This result is important to show the existence of the solution, where we use [30], Theorem 4.17 to show the existence of the positive steady state. We let $\Psi(t, W_0) = W(t,\cdot)$ for all $t \ge 0$, with $W = (W_1, W_2, W_3)$ be the solution of (1.3), with $W_0$ belonging to $\mathbb{X}^+$. Also, we let $W(t, \cdot; W_0)$ be the solution of (1.3), with $W_0$ belonging to $\mathbb{X}^+$. There are many papers that prove the uniform persistence in literature, and we cite a few [1, 3–5, 11], and references therein.

We consider the spaces $X_0$ and $\partial X_0$ as defined by (3.15), and (3.16), respectively. Also, we let

$$M_\partial = \left\{ W_0 \in \partial X_0 : \Psi(t; W_0) \in \partial X_0 \quad for \quad t \geq 0 \right\}. \tag{4.1}$$

Then, we have the following result

**Theorem 4.1.** *If $R_0 > 1$, then $\Psi(t; U_0)$ is strongly uniformly persistent, that is, there exists a positive constant $a > 0$ such that for any $W_0 \in \mathbb{X}_+$, $U(t; U_0)$ satisfies*

$$\liminf_{t\to\infty} W_i(t,x) \geq a, \quad i = 2, 3, \ \textit{uniformly for all } x \in \bar{\Omega}.$$

*Moreover,* (1.3) *has at least one positive steady state (PSS).*

*Proof.* To show this claim, we check all statements of Theorem [31], Theorem 3. We define $\rho : \mathbb{X}^+ \to \mathbb{R}^+$

$$\rho(\psi) = \min\{\psi_i(x), \quad x \in \bar{\Omega}, \quad i = 1, 2, 3, \quad \psi \in \mathbb{X}^+\}.$$

Clearly, $\rho(\Psi(t; \psi)) > 0$ for all $\psi \in \rho^{-1}(0, \infty) \cup (X_0 \cap \rho^{-1}(0))$. Then $\rho(\psi)$ is a generalized distance for the semiflow $\Gamma(t)$, see [31]. Notice that $\omega(W)$ is the $\omega - limit$ set of the orbit $\gamma^+(W) = \cup_{t\geq 0}\{\Psi(t; W_0)\}$. First, we prove that $X_0$ is positively invariant for $\Psi(t, W_0)$, that is $\Psi(t, X_0) \subseteq X_0$, which also means that for any $W_0 \in X_0$, we have $W > 0$.

By similar reasoning as in the proof of Theorem 3.5, if $W_0 \in X$, we have that $W_i(x, t) > 0$, for all $x \in \bar{\Omega}$, and $t > 0$. Therefore, $\Psi(t, X_0) \subseteq X_0$.

Now, we claim that $\omega(U) = \{E_0\}$, for all $U \in M_\partial$. This is true if we prove that $M_\partial \subseteq \{(\psi_1, 0, 0), \psi_1 \in C_\tau\}$. We prove this claim by contradiction. We suppose that there exists $\psi = (\psi_1, \psi_2, \psi_3) \in M_\partial$ such that $\psi \notin \{(\psi_1, 0, 0), \psi_1 \in C\}$. This means that we have two different cases:

(i) $\psi_2 \not\equiv 0$, and $\psi_3 \equiv 0$.
(ii) $\psi_2 \equiv 0$, and $\psi_3 \not\equiv 0$.

For (i), $W_i, \quad i = 2, 3$ can be expressed as

$$W_2(t, \cdot) = T_1(t)\psi_2 + \int_0^t T_1(t-s)\mathrm{e}^{-b_4\tau} G(\cdot, W_1(s-\tau)), W_3(s-\tau))\mathrm{d}s \geq T_1(t)\psi_2 > 0.$$

Then

$$W_3(t, \cdot) = \int_0^t \mathrm{e}^{-b_3(t-s)} a_2(\cdot) W_2(s, \cdot)\mathrm{d}s > 0.$$

Therefore, $W_i(t, x) > 0, \ \ i = 2, 3$ for all $x \in \bar{\Omega}$, and $t > 0$. which means that $\Psi(t, \psi) \subseteq X_0$, that is a contradiction with the definition of $M_\partial$.

For (ii), $W_3$ is written as

$$W_3(t, \cdot) = \mathrm{e}^{-b_3 t}\psi_3 + \int_0^t \mathrm{e}^{-b_3(t-s)} a_2(\cdot) W_2(s, \cdot)\mathrm{d}s > 0,$$

which means that $W_3(t, x) > 0$ for all $x \in \bar{\Omega}$, and $t > 0$. Moreover, $W_2$ satisfies

$$W_2(t, \cdot) = \int_0^t T_1(t-s)\mathrm{e}^{-b_4\tau} G(\cdot, W_1(s-\tau), W_3(s-\tau))\mathrm{d}s.$$

Hence, $W_i(t,x) > 0,\ i = 2,3$ for all $x \in \bar{\Omega}$, and $t > 0$. Thus, $\Psi(t,\psi) \subseteq X_0$, which is a contradiction with the definition of $M_\partial$. Therefore, $\omega(\psi) = \{E_0\}$ for all $\psi \in M_\partial$. Therefore, $E_0$ is isolated in $\mathbb{X}^+$. Moreover, from the fact that $\omega(\psi) = \{E_0\}$ we deduce that there is no cycle in $M_\partial$ from $\{E_0\}$ to itself.

Now, we let $W^s(E_0)$ is the stable manifold of $E_0$. We claim that $W^s(E_0) \cap \rho^{-1}(0,\infty) = \emptyset$. This means that there exists a positive constant $\delta > 0$ satisfying

$$\limsup_{t\to\infty} \|\Psi(t)\psi - E_0\| \geq \delta \text{ for any } \psi \in \rho^{-1}(0,\infty)$$

with $W_0 \in \rho^{-1}(0,\infty)$ meaning that $W_{i0} > 0,\ \ i = 1,2,3$ for all $x \in \bar{\Omega}$, $s \in [-\tau, 0]$.

We prove the claim by contradiction. We suppose that for any $\delta > 0$ there exists $\psi \in \rho^{-1}(0,\infty)$ satisfying

$$\limsup_{t\to\infty} \|\Psi(t)\psi - E_0\| < \delta \text{ for any } \psi \in \rho^{-1}(0,\infty).$$

Thus, there exists $t_1 > 0$ satisfying $W_i(t,\cdot,\psi_1) - V(\cdot) < \eta$, $W_i(t,\cdot,\psi_i) < \eta,\ \ i = 2,3$, for all $t \geq t_1$. Further, by $(\mathbf{H_3})$, for $t \geq t_1 + \tau$, $(W_2(t,x,\psi_2), W_3(t,x,\psi_2))$ is an upper bound of the solution of the problem

$$\begin{cases} \frac{\partial w_2}{\partial t} = d_2\Delta w_2 + \mathrm{e}^{-b_4\tau}\partial_3 G(x, V-\eta, 0) w_{3,-\tau} - b_2 w_2, \\ \frac{\partial w_3}{\partial t} = a_2 w_2 - b_3 w_3, \\ w_2(\cdot, t_1) = \max\{\psi_2(s,x), s \in [-\tau,0]\} > 0, \quad w_3(x,t_1) = \max\{\psi_3(s,x), s\in[-\tau,0]\} > 0, \quad x \in \bar{\Omega}. \end{cases} \tag{4.2}$$

We let $(\lambda_1(\delta), \phi_{1,\delta}(x))$ be principal eigenpair of the following eigenvalue problem

$$d_2\Delta u - b_2 u + \mathrm{e}^{-b_4\tau}\partial_3 G(x, V-\eta, 0)\frac{a_2}{b_3}u = \lambda u, \quad x \in \Omega.$$

Clearly, $\lambda_1(\eta)$ is continuous in $\eta$. Lemma 2.6 implies that $\lambda_1(0) = \lambda_1 > 0$ for $R_0 > 1$. From the continuity of $\lambda_1(\eta)$ with respect to $\eta$, we choose $\eta > 0$ sufficiently small such that $\lambda_1(\eta) > 0$.

Moreover, Lemma 2.6 implies that the following eigenvalue system

$$\begin{cases} d_2\delta\psi_2 + \mathrm{e}^{-b_4\tau}\partial_3 G(x, V-\eta, 0)\psi_3 - b_2\psi_2 = \lambda\psi_2, & x \in \Omega, \\ a_2\psi_2 - b_3\psi_3 = \lambda\psi_3, & x \in \Omega, \\ \frac{\partial \psi_1}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{4.3}$$

has a principal eigenvalue $\tilde{\lambda}_1(\eta) > 0$ for a sufficiently small constant $\eta > 0$, and $(\tilde{\phi}_2^\eta, \tilde{\phi}_3^\eta)$ is the corresponding eigenfunction. Choose $\alpha > 0$ sufficiently small such that $\alpha\tilde{\phi}_i^\eta \leq W_i(t_1,\cdot),\ \ i = 2,3$. Then, (4.2) has a unique solution

$$(w_1, w_2) = \left(\alpha \mathrm{e}^{\lambda_1(\delta)(t-t_1)}\tilde{\phi}_2^\eta, \alpha\mathrm{e}^{\lambda_1(\delta)(t-t_1)}\tilde{\phi}_3^\eta\right), \quad t \geq t_1.$$

Then, $w_i \to +\infty$, $i = 2,3$ as $t \to +\infty$, which is a contradiction with the results of Theorem 2.3. Hence we deduce that $W^s(E_0) \cap \rho^{-1}(0,\infty) = \emptyset$. Therefore, all statements of Theorem [31], Theorem 3 holds true. $\Psi(t)$ is strongly uniformly persistent. This completes the first part of the proof.

Next, we focus on the second part of the proof. Theorem 2.3 implies that $\Psi(t)$ is point dissipative. Further, $M_\partial$ is a convex set, and $\Psi(t)$ is $\kappa-$condensing, then [30], Theorem 4.7 implies the existence of a PSS denoted by $E^* = (W_1^*, W_3^*, W_3^*)$, that satisfy the system (1.3), and satisfy the persistence result (for $R_0 > 1$). Therefore, this steady state is the positive steady state and belongs to $X_0$, and satisfies

$$\begin{cases} 0 = d_1\Delta W_1(x) + a_1(x) - G(x, W_1, W_3) - b_1(x)W_1(x), & x \in \Omega, \\ 0 = d_2\Delta W_2(x) + \mathrm{e}^{-b_4\tau} G(x, W_1, W_3) - b_2(x)W_2(x), & x \in \Omega, \\ 0 = a_2(x)W_2(x) - b_3(x)W_3(x), & x \in \Omega, \\ \frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{4.4}$$

□

## 5. Global attractivity

Here, we investigate the global attractivity of the PSS. Mention that Theorem 4.1 does not guarantee the uniqueness of the PSS. In the case of the bilinear incidence and one diffusion coefficient, the uniqueness can be performed by applying similar reasoning as in [2], and for applications, *e.g.* [32], Theorem 3.1, and references therein. However, this method cannot be applied in our analysis due to the distinct diffusion coefficients. Notice that $(\mathbf{H_3})$ and $(\mathbf{H_2})$ implies the following

$$\text{For all } w_1, w_3 \in \mathbb{R}^+ \text{ we have } \begin{cases} \frac{G(x,W_1,W_3)}{G(x,W_1,W_3^*)} \geq \frac{W_3}{W_3^*} \text{ for all } W_3 \leq W_3^*, \text{ and } x \in \bar{\Omega}; \\ \frac{G(x,W_1,W_3)}{G(x,W_1,W_3^*)} \leq \frac{W_3}{W_3^*} \text{ for all } W_3 \geq W_3^*, \text{ and } x \in \bar{\Omega}. \end{cases} \tag{5.1}$$

Notice that (5.1) implies that $\frac{G(x,y_1,y_2)}{y_2}$ is nonincreasing in $y_2$. The global attractivness result is provided through the following subsections

### 5.1. Spatially homogeneous case

In this case, we consider that all parameters are space independent, and hence, $G$ is also spatially homogeneous, and then the PSS becomes constant, and satisfies the following system

$$\begin{cases} 0 = a_1 - G(W_1^*, W_3^*) - b_1 W_1^*, \\ 0 = \mathrm{e}^{-b_4\tau} G(W_1^*, W_3^*) - b_2 W_2^*, \\ 0 = a_2 W_2^* - b_3 W_3^*. \end{cases} \tag{5.2}$$

**Remark 5.1.** For the system (5.2), we can prove the existence of the positive steady state by searching on a fixed point, *e.g.* [33], Theorem 1. However, the uniqueness is hard to be achieved by analyzing (5.2). Therefore, we use the global attraction to show it. This reasoning is used in different literature works, we cite a few [1, 33, 34].

We define the Volterra function as

$$p(\rho) = \rho - 1 - \ln\rho, \quad \rho \geq 0. \tag{5.3}$$

It is readily seen that $p(\rho) \geq 0$ for all $\rho > 0$, and $p(\rho) = 0$ if and only if $\rho = 1$.

**Theorem 5.2.** *Assume that* $(\mathbf{H_1}) - (\mathbf{H_3})$ *hold. If* $R_0 > 1$*, then the PSS denoted* $E^* = (W_1^*, W_2^*, W_3^*)$ *of* (5.2) *is unique and globally attractive in* $X_0$.

*Proof.* We construct the following Lyapunov function as

$$V(W_1, W_2, W_3) = V_1(W_1, W_2, W_3) + V_2(W_1, W_2, W_3),$$

with

$$V_1(W_1,W_2,W_3)=\int_\Omega W_1(t,x)-W_1^*\int_{W_1^*}^{W_1(t,x)}\frac{G(W_1^*,W_3^*)}{G(\theta,W_3^*)}\mathrm{d}\theta+\mathrm{e}^{b_4\tau}W_2^*(x)p\left(\frac{W_2(t,x)}{W_2^*}\right)+kW_3^*p\left(\frac{W_3(t,x)}{W_3^*}\right)\mathrm{d}x,$$

and

$$V_2(W_1,W_2,W_3)=G(W_1^*,W_3^*)\int_\Omega\int_{-\tau}^{0}p\left(\frac{G(W_1(t-s,x),W_3(t-s,x))}{G(W_1^*,W_3^*)}\right)\mathrm{d}s\mathrm{d}x,$$

with $k=\frac{G(W_1^*,W_3^*)}{a_2W_2^*}$. From the boundedness of the solution (Thm. 2.3), and uniform persistent (Thm. 4.1), we deduce that the Lyapunov function $V$ is well-defined. The derivative of $V_1$ along the solution of (1.3) is given by

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}V_1(W_1,W_2,W_3)=&\int_\Omega\left(1-\frac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)\left(d_1\Delta W_1+a_1-G(W_1,W_3)-b_1W_1\right)\\&+\left(1-\tfrac{W_2^*}{W_2}\right)\left(\mathrm{e}^{b_4\tau}d_2\Delta W_2+G(W_{1,-\tau},W_{3,-\tau})-\mathrm{e}^{b_4\tau}b_2W_2\right)\\&+k\left(1-\tfrac{W_3^*}{W_3}\right)\left(a_2W_2-b_3W_3\right)\mathrm{d}x.\end{aligned}$$

Adding and subtracting the term $G(W_1,W_3)\left(1-\frac{W_2^*}{W_2}\right)$, then, we obtain

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}V_1(W_1,W_2,W_3)=&\int_\Omega\left(1-\frac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)\left(d_1\Delta W_1+a_1-G(W_1,W_3)-b_1W_1\right)\\&+\left(1-\tfrac{W_2^*}{W_2}\right)\left(\mathrm{e}^{b_4\tau}d_2\Delta W_2+G(W_1,W_3)-\mathrm{e}^{b_4\tau}b_2W_2\right)+k\left(1-\tfrac{W_3^*}{W_3}\right)\left(a_2W_2-b_3W_3\right)\\&+\left(1-\tfrac{W_2^*}{W_2}\right)(G(W_{1,-\tau},W_{3,-\tau})-G(W_1,W_3))\mathrm{d}x.\end{aligned}\tag{5.4}$$

By a simple calculation, we have

$$\begin{aligned}\left(1-\tfrac{W_2^*}{W_2}\right)(G(W_{1,-\tau},W_{3,-\tau})-G(W_1,W_3))=G(W_1^*,W_3^*)\Bigg(&p\left(\tfrac{G(W_{1,-\tau},W_{3,-\tau})}{G(W_1^*,W_3^*)}\right)\\&-p\left(\tfrac{G(W_1,W_3)}{G(W_1^*,W_3^*)}\right)-p\left(\tfrac{W_2^*G(W_{1,-\tau},W_{3,-\tau})}{W_2G(W_1^*,W_3^*)}\right)\\&+p\left(\tfrac{W_2^*G(W_1,W_3)}{W_2G(W_1^*,W_3^*)}\right)\Bigg).\end{aligned}\tag{5.5}$$

Using the steady state equations (5.2), and (5.5), then (5.4) becomes

$$\begin{aligned}&\tfrac{\mathrm{d}}{\mathrm{d}t}V_1(W_1,W_2,W_3)\\&=\int_\Omega\left(1-\frac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)\left(d_1\Delta W_1+G(W_1^*,W_3^*)+b_1W_1^*-G(W_1,W_3)-b_1W_1\right)\\&\quad+\left(1-\tfrac{W_2^*}{W_2}\right)\left(\mathrm{e}^{b_4\tau}d_2\Delta W_2+G(W_1,W_3)-\tfrac{W_2}{W_2^*}G(W_1^*,W_3^*)\right)+a_2W_2^*k\left(1-\tfrac{W_3^*}{W_3}\right)\left(\tfrac{W_2}{W_2^*}-\tfrac{W_3}{W_3^*}\right)\\&\quad+G(W_1^*,W_3^*)\left(p\left(\tfrac{G(W_{1,-\tau},W_{3,-\tau})}{G(W_1^*,W_3^*)}\right)-p\left(\tfrac{G(W_1,W_3)}{G(W_1^*,W_3^*)}\right)-p\left(\tfrac{W_2^*G(W_{1,-\tau},W_{3,-\tau})}{W_2G(W_1^*,W_3^*)}\right)+p\left(\tfrac{W_2^*G(W_1,W_3)}{W_2G(W_1^*,W_3^*)}\right)\right)\mathrm{d}x.\end{aligned}\tag{5.6}$$

By a tedious calculations, (5.6) can be expressed as

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}V_1(W_1,W_2,W_3) =& \int_\Omega \left(1-\frac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)d_1\Delta W_1 + \left(1-\frac{W_2^*}{W_2}\right)\mathrm{e}^{b_4\tau}d_2\Delta W_2 \mathrm{d}x\\ &+\int_\Omega \Bigg[b_1 W_1^*\left(1-\frac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)\left(1-\frac{W_1}{W_1^*}\right)\\ &+G(W_1^*,W_3^*)\tfrac{W_3}{W_3^*}\left(\tfrac{G(W_1,W_3)}{G(W_1,W_3^*)}-1\right)\left(\tfrac{W_3^*}{W_3}-\tfrac{G(W_1,W_3^*)}{G(W_1,W_3)}\right)\\ &-G(W_1^*,W_3^*)\left(p\left(\tfrac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)+p\left(\tfrac{W_2^*}{W_2}\tfrac{g(W_1,W_3)}{G(W_1^*,W_3^*)}\right)+p\left(\tfrac{W_2}{W_2^*}\tfrac{W_3^*}{W_3}\right)+p\left(\tfrac{W_3}{W_3^*}\tfrac{G(W_1,W_3^*)}{G(W_1,W_3)}\right)\right)\\ &+G(W_1^*,W_3^*)\left(p\left(\tfrac{G(W_{1,-\tau},W_{3,-\tau})}{G(W_1^*,W_3^*)}\right)-p\left(\tfrac{G(W_1,W_3)}{G(W_1^*,W_3^*)}\right)-p\left(\tfrac{W_2^*G(W_{1,-\tau},W_{3,-\tau})}{W_2G(W_1^*,W_3^*)}\right)\right.\\ &\left.+p\left(\tfrac{W_2^*G(W_1,W_3)}{W_2G(W_1^*,W_3^*)}\right)\right)\Bigg]\mathrm{d}x.\end{aligned} \tag{5.7}$$

Next, we calculate the derivative of $V_2$ along the solution of the model (1.3), that is

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V_2(W_1,W_2,W_3) = \int_\Omega G(W_1^*,W_3^*)\left(p\left(\frac{G(W_1,W_3)}{G(W_1^*,W_3^*)}\right)-p\left(\frac{G(W_{1,-\tau},W_{3,-\tau})}{G(W_1^*,W_3^*)}\right)\right)\mathrm{d}x.$$

Then, $V$ becomes

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}V(W_1,W_2,W_3) =& \int_\Omega \left(1-\frac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)d_1\Delta W_1 + \left(1-\frac{W_2^*}{W_2}\right)\mathrm{e}^{b_4\tau}d_2\Delta W_2 \mathrm{d}x\\ &+\int_\Omega \Bigg[b_1 W_1^*\left(1-\frac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)\left(1-\frac{W_1}{W_1^*}\right)\\ &+G(W_1^*,W_3^*)\tfrac{W_3}{W_3^*}\left(\tfrac{G(W_1,W_3)}{G(W_1,W_3^*)}-1\right)\left(\tfrac{W_3^*}{W_3}-\tfrac{G(W_1,W_3^*)}{G(W_1,W_3)}\right)\\ &-G(W_1^*,W_3^*)\left(p\left(\tfrac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)+p\left(\tfrac{W_2}{W_2^*}\tfrac{W_3^*}{W_3}\right)+p\left(\tfrac{W_3}{W_3^*}\tfrac{G(W_1,W_3^*)}{G(W_1,W_3)}\right)\right.\\ &\left.+p\left(\tfrac{W_2^*G(W_{1,-\tau},W_{3,-\tau})}{W_2G(W_1^*,W_3^*)}\right)\right)\Bigg]\mathrm{d}x.\end{aligned} \tag{5.8}$$

In the view of $(\mathbf{H_2})$, we deduce that $\left(1-\frac{g(W_1^*,W_3^*}{g(W_1,W_3^*)}\right)\left(1-\frac{W_1}{W_1^*}\right)\leq 0$. From (5.1), we deduce that $\left(\frac{G(W_1,W_3)}{G(W_1,W_3^*)}-1\right)\left(\frac{W_3^*}{W_3}-\frac{G(W_1,W_3^*)}{G(W_1,W_3)}\right)\leq 0$ (for more details see [33], Thm. 1). Next, we apply the Green's first identity, use the Neumann boundary condition to simplify the first term in (5.8), and then

$$\begin{aligned}&\int_\Omega \left(1-\frac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)d_1\Delta W_1 + \left(1-\frac{W_2^*}{W_2}\right)\mathrm{e}^{b_4\tau}d_2\Delta W_2 \mathrm{d}x\\ &=\int_\Omega \left[-d_1\nabla\left(1-\frac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)\nabla W_1 - \mathrm{e}^{b_4\tau}d_2\nabla\left(1-\frac{W_2^*}{W_2}\right)\nabla W_2\right]\mathrm{d}x,\\ &=-\int_\Omega\left(d_1\frac{\partial_1 G(W_1,W_3^*)G(W_1^*,W_3^*)|\nabla W_1|^2}{(G(W_1,W_3^*))^2}+\mathrm{e}^{b_4\tau}d_2\frac{W_2^*|\nabla W_2|^2}{(W_2)^2}\right)\mathrm{d}x\leq 0.\end{aligned}$$

The above estimates next to the definition of $p$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}V(W_1, W_2, W_3) \leq 0, \text{ for all } (W_1, W_2, W_3) \in X_0.$$

We let $M$ be the largest invariant subset of $\Big\{(W_1, W_2, W_3) \in X_0 : V'(t) = 0\Big\}$. Lasalle invariance principle [35] implies that the $\omega - limit$ sets of solution are contained in $M$. Clearly, $V'(t) = 0$ implies that

$$W_1 = W_1^*, \quad \frac{G(W_1^*, W_3^*)}{G(W_1, W_3^*)} = \frac{W_2}{W_2^*}\frac{W_3^*}{W_3} = \frac{W_3}{W_3^*}\frac{G(W_1, W_3^*)}{G(W_1, W_3)} = \frac{W_2^* G(W_{1,-\tau}, W_{3,-\tau})}{W_2 G(W_1^*, W_3^*)} = 1.$$

Substituting the above relations into the third equation of (1.3), we obtain

$$\frac{\partial W_3}{\partial t} = \frac{W_3}{W_3^*}(a_2 W_2^* - b_3 W_3^*) = 0.$$

Hence $W_3 = W_3^*$, and using $\frac{W_2}{W_2^*}\frac{W_3^*}{W_3} = 1$ we obtain that $W_2 = W_2^*$. Therefore, $M = \{E^*\}$. Hence $E^*$ is globally attractive in $X_0$. The uniqueness of the PSS follows immediately from the global attractivity of $E^*$. □

### 5.2. Spatially heterogeneous case

We now establish the global attractivity of the PSS in the case when all parameters are space dependent by combining the method of Lyapunov functionals and Lasalle invariance principle. Indeed, we prove this result when the susceptible tissues diffusion coefficient is zero ($d_1 = 0$).

**Theorem 5.3.** *Suppose that* $(\mathbf{H_1}) - (\mathbf{H_4})$ *hold. If* $R_0 > 1$*, then the PSS* $E^* = (W_1^*, W_2^*, W_3^*)$ *of* (5.2) *is unique and globally attractive in* $X_0$*.*

*Proof.* We construct the following Lyapunov function as

$$\begin{aligned} V(W_1, W_2, W_3) = &\int_\Omega W_2^*\Bigg[W_1(t,x) - W_1^*(x) + \int_{W_1^*(x)}^{W_1(t,x)} \frac{G(x, W_1^*, W_3^*)}{G(x, \theta, W_3^*)}\mathrm{d}\theta + \mathrm{e}^{b_4\tau}W_2^*(x)p\left(\frac{W_2(t,x)}{W_2^*}\right) \\ &+k(x)W_3^* p\left(\tfrac{W_3(t,x)}{W_3^*}\right)\mathrm{d}x \\ &\int_\Omega G(x, W_1^*, W_3^*)\int_{-\tau}^0 p\left(\frac{G(x, W_1(t-s,x), W_3(t-s,x))}{G(x, W_1^*, W_3^*)}\right)\mathrm{d}s\Bigg]\mathrm{d}x, \end{aligned}$$

with $k(x) = \frac{G(x, W_1^*(x), W_3^*(x))}{a_2(x) W_2^*(x)}$. Clearly, $V$ is well-defined. The derivative of $V$ along the solution of (1.3) is given by

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}V_1(W_1, W_2, W_3) = &\int_\Omega W_2^*\Bigg[\left(1 - \frac{G(x, W_1^*, W_3^*)}{G(x, W_1, W_3^*)}\right)\Big(a_1 - G(x, W_1, W_3) - b_1 W_1\Big) \\ &+\left(1 - \tfrac{W_2^*}{W_2}\right)\Big(\mathrm{e}^{b_4\tau}d_2\Delta W_2 + G(x, W_{1,-\tau}, W_{3,-\tau}) - \mathrm{e}^{b_4\tau}b_2 W_2\Big) \\ &+k\left(1 - \tfrac{W_3^*}{W_3}\right)\Big(a_2 W_2 - b_3 W_3\Big)\Bigg]\mathrm{d}x \\ &+\int_\Omega W_2^*\Bigg[G(x, W_1^*, W_3^*)\left(p\left(\frac{G(x, W_1, W_3)}{G(x, W_1^*, W_3^*)}\right) - p\left(\frac{G(x, W_{1,-\tau}, W_{3,-\tau})}{G(x, W_1^*, W_3^*)}\right)\right)\Bigg]\mathrm{d}x. \end{aligned}$$

Using the steady state equations (6.1), and performing similar calculation to the homogeneous case, we obtain

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}V(W_1,W_2,W_3) = &\int_\Omega W_2^*\left[\left(1-\frac{W_2^*}{W_2}\right)\mathrm{e}^{b_4\tau}d_2\Delta W_2 + \left(1-\frac{W_2}{W_2^*}\right)\mathrm{e}^{b_4\tau}d_2\Delta W_2^*\right]\mathrm{d}x \\ &+\int_\Omega W_2^*\Bigg[b_1W_1^*\left(1-\frac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)\left(1-\frac{W_1}{W_1^*}\right) \\ &+G(W_1^*,W_3^*)\tfrac{W_3}{W_3^*}\left(\tfrac{G(W_1,W_3)}{G(W_1,W_3^*)}-1\right)\left(\tfrac{W_3^*}{W_3}-\tfrac{G(W_1,W_3^*)}{G(W_1,W_3)}\right) \\ &-G(W_1^*,W_3^*)\Bigg(p\left(\tfrac{G(W_1^*,W_3^*)}{G(W_1,W_3^*)}\right)+p\left(\tfrac{W_2}{W_2^*}\tfrac{W_3^*}{W_3}\right)+p\left(\tfrac{W_3}{W_3^*}\tfrac{G(W_1,W_3^*)}{G(W_1,W_3)}\right) \\ &+p\left(\tfrac{W_2^*G(W_{1,-\tau},W_{3,-\tau})}{W_2G(W_1^*,W_3^*)}\right)\Bigg)\Bigg]\mathrm{d}x.\end{aligned} \tag{5.9}$$

Now, we simplify the first term (5.9). By applying Green's first identity, and the Neumann boundary condition, we obtain

$$\begin{aligned}&\int_\Omega W_2^*\left(1-\frac{W_2^*}{W_2}\right)\mathrm{e}^{b_4\tau}d_2\Delta W_2\mathrm{d}x + W_2^*\left(1-\frac{W_2}{W_2^*}\right)\mathrm{e}^{b_4\tau}d_2\Delta W_2^*\mathrm{d}x \\ &= \mathrm{e}^{b_4\tau}d_2\int_\Omega W_2^*\left(-\nabla\left(W_2^*-\frac{(W_2^*)^2}{W_2}\right)\nabla W_2 - \nabla(W_2^*-W_2)\nabla W_2^*\right)\mathrm{d}x, \\ &= -\mathrm{e}^{b_4\tau}d_2\int_\Omega \Sigma_{i=1}^n\left(\frac{W_3^*}{W_3}\frac{\partial W_3}{\partial x_i}-\frac{\partial W_3^*}{\partial x_i}\right)\mathrm{d}x \le 0.\end{aligned}$$

Therefore, $\frac{\mathrm{d}}{\mathrm{d}t}V \le 0$, and equality holds if and only if $W_1 = W_1^*,\ W_2 = W_2^*,\ W_3 = W_3^*$. Therefore, $M = \{E^*\}$. Hence $E^*$ is globally attractive in $X_0$. The global attractivity of $E^*$ implies the uniqueness of the PSS. □

**Remark 5.4.** Notice that the global attractiveness of the PSS is done only in the case $d_1 = 0$ and $d_2 \neq 0$. However, if $d_1 \neq 0$ and $d_2 = 0$, we couldn't construct a Lyapunov function even in the case of the bilinear incidence.

## 6. Asymptotic profiles

In this section, we investigate the asymptotic profile of PSS. From Theorem 4.1, the system (5.2) has at least one positive steady state for $R_0 > 1$, but no information has been provided on the uniqueness of this steady state. Indeed, in the spatially homogeneous case, the uniqueness of PSS is shown using the global attraction of $E^*$ and by employing Lyapunov function and Lasalle invariance principle, similar result is proved for the heterogeneous case with $d_2 \neq 0,\ \ d_1 = 0$. The asymptotic profiles of PSS when one or both dispersal rates goes to infinity or zero is the subject of interest in this section. Also, to generalize the results established in [11]. However, due to the nonlinearity of the incidence function, we are required to consider additional assumption on the nonlinear incidence to establish the asymptotic profile of PSS, that is

($\mathbf{H_4}$) Suppose that $\frac{G(x,x_1,y_1)}{G(x,x_1,y_2)},\ \ x \in \Omega,\ x_1,\ y_1,\ y_2 > 0$ is independent of $x_1$.

Notice that ($\mathbf{H_4}$) implies that there exists $f, g \in C^1(\bar{\Omega}\times\mathbb{R}^+,\mathbb{R}^+)$, such that for all $x \in \bar{\Omega},\ x_1, y_1 > 0$, we have $G(x,x_1,y_1) = f(x,x_1)g(x,y_1)$. From ($\mathbf{H_2}$), we get $\partial_2 f(x,x_1) = \frac{\partial f(x_1)}{\partial x_1} > 0$, and $\partial_2 g(x,y_1) = \frac{\partial g(x,y_1)}{\partial y_1} > 0$ for all $x \in \bar{\Omega},\ x_1, y_1 > 0$. By ($\mathbf{H_3}$), we have $g(x,y) \le \partial_2 g(x,0)y$ for all $x \in \bar{\Omega}$, $y > 0$, and $g$ is a concave function with respect to the second variable.

From the third equation of the system (4.4), we have $W_3 = \frac{a_2 W_2}{b_3}$. Then, the PSS satisfies the following system

$$\begin{cases} 0 = d_1\Delta W_1 + a_1 - f(x, W_1)g\left(x, \frac{a_2}{b_3}W_2\right) - b_1 W_1, \ x \in \Omega, \\ 0 = d_2\Delta W_2 + \mathrm{e}^{-b_4\tau} f(x, W_1)g\left(x, \frac{a_2}{b_3}W_2\right) - b_2 W_2, \ x \in \Omega, \\ \frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = 0, \qquad x \in \partial\Omega. \end{cases} \tag{6.1}$$

Motivated by [11, 19], for any $d > 0$ and $m \in C$, we let $\lambda_1(d, m)$ be the principal eigenvalue of the problem

$$\begin{cases} d\Delta\psi + m\psi = \lambda\psi, \quad x \in \Omega, \\ \frac{\partial \psi}{\partial n} = 0. \end{cases} \tag{6.2}$$

$\lambda_1(d, m)$ depends continuously on $d$ and $m$, and satisfies

$$\lambda_1(d, m) = \sup_{\psi \in H^2(\Omega),\ \psi \neq 0} \left\{ \int_\Omega (-d|\nabla\psi|^2 + m\psi^2)\mathrm{d}x, \quad \text{with } \int_\Omega \psi^2 = 1 \right\}. \tag{6.3}$$

Clearly, $\lambda_1(d, m)$ is decreasing in $d$, and by Lemma 2.5, we have $\lambda_1(d, m) \to \overline{m}$ as $d \to 0$, and $\lambda_1(d, m) \to \int_\Omega h\mathrm{d}x$ as $d \to +\infty$. Moreover, $\lambda_1(d, m)$ is increasing in $m$, with $\lambda_1(d, m_1) > \lambda_1(d, m_2)$ if $m_1 \geq m_2$, and $m_1(x) > m_2(x)$ for some $x \in \bar{\Omega}$, with $m_i \in C,\ i = 1, 2$.

### 6.1. Profile as $d_1 \to +\infty$

In this subsection, we treat $d_1$ as independent parameter in $(0, +\infty)$. We denote

$$\lambda_1^* = \lambda_1\left(d_2, f\left(x, \frac{\int_\Omega a_1\mathrm{d}x}{\int_\Omega b_1\mathrm{d}x}\right)\mathrm{e}^{-b_4\tau}\partial_2 g(x, 0)\frac{a_2}{b_3} - b_2\right).$$

**Lemma 6.1.** *$\lambda_1 \to \lambda_1^*$, as $d_1 \to +\infty$, with $(\lambda_1, \phi_1(x))$ is principal eigenpaire of (2.38). Moreover, $R_0 \to R_0^*$, as $d_1 \to +\infty$, and satisfies $\lambda_1^* > 0$ if and only if $R_0^* > 1$.*

*Furthermore, $R_0^*$ can be expressed in the following variational form*

$$R_0^* = \sup_{\phi \in H^1(\Omega), \phi \neq 0} \frac{\displaystyle\int_\Omega \mathrm{e}^{-b_4\tau} f\left(x, \frac{\int_\Omega a_1\mathrm{d}x}{\int_\Omega b_1\mathrm{d}x}\right)\frac{a_2}{b_3}\partial_2 g(x, 0)\phi^2\mathrm{d}x}{\displaystyle\int_\Omega (d_2|\nabla\phi|^2 + b_2\phi^2)\mathrm{d}x}. \tag{6.4}$$

*Proof.* Notice that $V(x)$ satisfy (2.28). We define the sequence $\{d_{1,n}\}$ such that $d_{1,n} \to +\infty$ as $n \to +\infty$, and $V_n$ is the corresponding solution of (2.28), and $(\lambda_{1,n}, \phi_{1,n}(x))$ is the corresponding principal eigenpaire of (2.38). We divide both sides of (2.28) over $d_{1,n}$, we obtain

$$\begin{cases} \Delta V_n + \frac{a_1}{d_{1,n}} - \frac{b_1}{d_{1,n}}V_n = 0, \ x \in \Omega, \\ \frac{\partial V_n}{\partial n} = 0, \qquad x \in \partial\Omega. \end{cases} \tag{6.5}$$

Letting $n \to +\infty$, and by the elliptic regularity, we obtain that $\Delta V^* = 0$, hence $V^*$ is constant. Then, we integrate both sides of (2.28), and we get $V^* = \frac{\int_\Omega a_1\mathrm{d}x}{\int_\Omega b_1\mathrm{d}x}$.

Letting $n \to +\infty$ for (2.38), we obtain that $(\lambda_{1,n}, \phi_{1,n}(x)) \to (\lambda_1^*, \phi_1^*(x))$. Using $(\mathbf{H_5})$, and the value of $V^*$, we obtain that the principal eigenpaire $(\lambda_1^*, \phi_1^*(x))$ satisfy

$$\begin{cases} d_2\Delta\psi - b_2\psi + \frac{a_2}{b_3}\mathrm{e}^{-b_4\tau} f\left(x, \frac{\int_\Omega a_1 \mathrm{d}x}{\int_\Omega b_1 \mathrm{d}x}\right)\partial_2 g(x,0)\psi = \lambda\psi, & x \in \bar{\Omega}, \\ \frac{\partial\psi}{\partial n} = 0, & x \in \bar{\Omega}. \end{cases} \tag{6.6}$$

For the second part of the lemma, again, we let a sequence $\{d_{1,n}\}$ that satisfy $d_{1,n} \to +\infty$ as $n \to +\infty$, where $V_n$ and $(R_{0,n}, \phi_n(x))$ satisfy

$$\begin{cases} \Delta V_n + \frac{a_1}{d_{1,n}} - \frac{b_1}{d_{1,n}} V_n = 0, & x \in \Omega, \\ d_2\Delta\phi_n - b_2\phi_n + \frac{1}{R_{0,n}}\frac{a_2}{b_3}\mathrm{e}^{-b_4\tau} f(x, V_n)\partial_2 g(x,0)\phi_n = 0, & x \in \Omega, \end{cases}$$

with Neumann boundary condition. Letting $n$ tends to $+\infty$, then we get $V_n \to V^*$ and $R_{0,n} \to R_0^*$, with $R_0^*$ is defined by the variational form (6.4). From Lemma 2.6, we deduce that $R_0^* - 1$ has the same sign as $\lambda_1^*$. Hence, we deduce the result. ☐

Next, we prove the main result of this subsection through the following theorem

**Theorem 6.2.** *Assume that $R_0^* > 1$, and $(\mathbf{H_1}) - (\mathbf{H_4})$ holds. Then, for any fixed $d_2 > 0$ there exists a sequence $\{d_{1,n}\}$ that verify $d_{1,n} \to +\infty$ as $n \to +\infty$, such that the corresponding PSS $(W_{1,n}, W_{2,n})$ of (6.1) satisfy $(W_{1,n}, W_{2,n}) \to (W_1^*, W_2^*)$ in $C$, where $(W_1^*, W_2^*)$ is unique, and $W_1^*$ is constant.*

*Proof.* The existence of of $PSS$ can be deduced from Theorem 4.1. It remains to show the convergence of $(W_{1,n}, W_{2,n})$ as $n \to +\infty$. By the first equation of (6.1), one has

$$-d_1\Delta W_1 \le a_1 - b_1 W_1,$$

with Neumann boundary condition. Hence, the maximum principle implies that $||W_1|| \le \frac{\overline{a_1}}{\underline{b_1}}$ for all $d_1 > 0$. Next, we integrate both sides of the two equations of (6.1), and we multiply both sides of the first resulting equation by $\mathrm{e}^{-b_4\tau}$, then, we add the resulting equations, we obtain

$$\int_\Omega b_2 W_2 \mathrm{d}x = \mathrm{e}^{-b_4\tau}\int_\Omega (a_1 - b_1 W_1)\mathrm{d}x \le \overline{a_1}|\Omega|.$$

Therefore, $||W_2||_1 \le \frac{\overline{a_1}|\Omega|}{\underline{b_2}}$. Next, for any $p > 0$, and by the second equation of (6.1), the uniform boundedness of $W_1$, and the assumptions $(\mathbf{H_1}) - (\mathbf{H_3})$, the elliptic estimate, and bootstrapping argument, there exists $C > 0$ such that

$$||W_2||_{2,p} \le C, \text{ for all } d_1 > 0. \tag{6.7}$$

Fixing $p > n$, by the boundedness of $W_1$ in $C$, and $W_2$ in $W^{2,p}(\Omega)$, there exists a sequence $\{d_{1,n}\}$, with $d_{1,n} \to +\infty$ such that the corresponding solution $(W_{1,n}, W_{2,n})$ of (6.1) satisfy $W_{1,n} \to W_1^*$ weakly in $L^p(\Omega)$, and $W_{2,n} \to W_2^*$ weakly in $W^{2,p}(\Omega)$ and by the embedding theorem of $W^{2,p}(\Omega)$ in $C$, we deduce that $W_{2,n} \to W_2^*$ strongly in $C$ as $n \to +\infty$, for some positive $W_1^*$ in $L^p(\Omega)$, and nonnegative $W_2^*$ in $W^{2,p}(\Omega)$ as $n \to \infty$. We divide both sides of the first equation of (6.1) over $d_{1,n}$, and we obtain

$$\Delta W_{1,n} + \frac{a_1}{d_{1,n}} - \frac{f(x, W_1) g\left(x, \frac{a_2}{b_3} W_2\right)}{d_{1,n}} - \frac{b_1}{d_{1,n}} W_1 = 0, \quad x \in \Omega,$$

with Neumann boundary condition. Letting $n \to \infty$, and by the elliptic regularity, and the assumptions $(\mathbf{H_1})-(\mathbf{H_3})$, we get $\Delta W_1^* = 0$, thus $W_1^*$ is constant. We integrate both sides of the first equation of (6.1), and we obtain

$$F(W_1^*, W_2^*(x)) = 0, \tag{6.8}$$

with

$$F(W_1^*, W_2^*(x)) = \int_\Omega a_1 \mathrm{d}x - \int_\Omega f(x, W_1^*) g\left(x, \frac{a_2}{b_3} W_2^*\right) \mathrm{d}x - W_1^* \int_\Omega b_1 \mathrm{d}x.$$

We consider the equation $F(l, W_2^*(x)) = 0$, with a fixed function $W_2^*(x)$. We show that $F(l, W_2^*(x)) = 0$ has a unique positive solution denoted $l^*$. Clearly, $F$ is strictly decreasing in $l$, and $F(0, W_2^*(x)) = \int_\Omega a_1 \mathrm{d}x$, and $\lim_{l\to+\infty} F(l, W_2^*(x)) = -\infty$. Therefore, $F(l, W_2^*(x)) = 0$ has a unique positive solution in $[0, +\infty)$ denoted $W_1^*$. Obviously, there exists a positive function denoted $h_1 : C(\bar\Omega) \to \mathbb{R}$, such that $W_1^* = h_1(W_2^*(x))$.

Now, we claim that $h_1(\phi)$ is decreasing in $\phi \in C$, that is, for all $\phi_1, \phi_2 \in C$ satisfying $\phi_1(x) \le \phi_2(x)$ for all $x \in \bar\Omega$ and $\phi_1(x) < \phi_2(x)$ for some $x \in \bar\Omega$, we have $h_1(\phi_1(x)) \ge h_1(\phi_2(x))$. We show this claim by contradiction, and we suppose that $h_1$ is non-decreasing in $W_2^*(x)$, that is, for all $\phi_1, \phi_2 \in C$ satisfying $\phi_1(x) \le \phi_2(x)$ for all $x \in \bar\Omega$ and $\phi_1(x) < \phi_2(x)$ for some $x \in \bar\Omega$ we have $h_1(\phi_1(x)) \ge h_1(\phi_2(x))$, and we let the constants $W_1^1, W_1^2$ that satisfy $W_1^1 = h_1(\phi_1(x))$, and $W_1^2 = h_1(\phi_2(x))$, respectively. Notice that by the assumption put on $h_1$, we have $W_1^1 \ge W_1^2$. Therefore, they satisfy the equations

$$\begin{cases} F(W_1^1, \phi_1(x)) = 0, \\ F(W_1^2, \phi_2(x)) = 0. \end{cases} \tag{6.9}$$

Subtracting the two equations of (6.9), and we obtain

$$\int_\Omega \left[ f(x, W_1^2) g\left(x, \frac{a_2}{b_3}\phi_2\right) - f(x, W_1^1) g\left(x, \frac{a_2}{b_3}\phi_1\right)\right] \mathrm{d}x = (W_1^1 - W_1^2) \int_\Omega b_1 \mathrm{d}x. \tag{6.10}$$

The monotonicity of $f$ with respect to the second variable gives $f(x, W_1^2) \ge f(x, W_1^1)$. We substitute this result into (6.10), we obtain

$$0 > \int_\Omega f(x, W_1^1) \left[ g\left(x, \frac{a_2}{b_3}\phi_2\right) - g\left(x, \frac{a_2}{b_3}\phi_1\right)\right] \mathrm{d}x \ge (W_1^1 - W_1^2) \int_\Omega b_1 \mathrm{d}x \ge 0, \tag{6.11}$$

which is a contradiction with the monotonicity of $g$. Then, $h_1$ is decreasing in $W_2^*(x)$.

By the monotonicity of $f$ with respect to the second parameter, we deduce that $f(x, h_1(y))$ is also decreasing in $y$, $y \in C$. For simplicity, we let $H_1(x, y)$ defined by

$$H_1(x, y) = f(x, h_1(y)),$$

which is decreasing in $y$, $x \in \bar\Omega$, $y \in C$. By the definition of $H_1$, and the second equation of (6.1), $W_2^*$ is the solution of the following problem

$$\begin{cases} d_2 \Delta U + \mathrm{e}^{-b_4\tau} H_1(x, U) g\left(x, \frac{a_2}{b_3} U\right) - b_2 U = 0, & x \in \Omega, \\ \frac{\partial U}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{6.12}$$

Now, for $R_0 > 1$, we claim that (6.12) has a unique positive solution. We prove this claim by constructing an upper and lower solution of (6.12). We denote

$$Q(u) = d_2\Delta u + H_1(x,u)g\left(x, \frac{a_2}{b_3}u\right) - b_2 u.$$

We let $\tilde{u} = \varepsilon\phi_1^*$ with $\varepsilon > 0$ sufficiently small. Then, we have

$$\begin{aligned} Q(\tilde{u}) &= \varepsilon d_2\Delta\phi_1^* + \mathrm{e}^{-b_4\tau}H_1(x,\varepsilon\phi_1^*)g\left(x, \tfrac{a_2}{b_3}\varepsilon\phi_1^*\right) - b_2\varepsilon\phi_1^*, \\ &= \varepsilon\left(d_2\Delta\phi_1^* + \frac{\mathrm{e}^{-b_4\tau}H_1(x,\varepsilon\phi_1^*)g\left(x,\frac{a_2}{b_3}\varepsilon\phi_1^*\right)}{\varepsilon} - b_2\phi_1^*\right). \end{aligned}$$

By Lemma 6.1, we have

$$Q(\tilde{u}) = \varepsilon\left[\lambda_1^* + \mathrm{e}^{-b_4\tau}\left(\frac{H_1(x,\varepsilon\phi_1^*)g\left(x,\frac{a_2}{b_3}\varepsilon\phi_1^*\right)}{\varepsilon\phi_1^*} - \frac{a_2}{b_3}f\left(x, \frac{\int_\Omega a_1\mathrm{d}x}{\int_\Omega b_1\mathrm{d}x}\right)\partial_2 g(x,0)\right)\right]\phi_1^*.$$

Notice that $H_1(x,\varepsilon\phi_1^*) \to f\left(x, \frac{\int_\Omega a_1\mathrm{d}x}{\int_\Omega b_1\mathrm{d}x}\right)$, and $\frac{g\left(x,\frac{a_2}{b_3}\varepsilon\phi_1^*\right)}{\varepsilon\phi_1^*} \to \frac{a_2}{b_3}\partial_2 g(x,0)$ as $\varepsilon \to 0$. Since $H(x,y)$ and $\frac{g\left(x,y\right)}{y}$ are both decreasing in $y$, for all $y > 0$, we deduce that $H_1(x,\varepsilon\phi_1^*)\frac{g\left(x,\frac{a_2}{b_3}\varepsilon\phi_1^*\right)}{\varepsilon\phi_1^*} - \frac{a_2}{b_3}f\left(\frac{\int_\Omega a_1\mathrm{d}x}{\int_\Omega b_1\mathrm{d}x}\right)\partial_2 g(x,0) < 0$ if $\varepsilon > 0$ sufficiently small. Moreover, for $R_0 > 1$ (more precisely $R_0^* > 1$) we have $\lambda^* > 0$ (Lem. 6.1), then $Q(\tilde{u}) > 0$ if $\varepsilon > 0$ is small. Thus, $\tilde{u}$ is a lower solution of (6.12) if $\varepsilon$ is small. Next, we let $\hat{U} = A$, with $A$ is a positive constant. Then, by ($\mathbf{H_3}$), and the fact that $H(A) < H(0)$, we obtain

$$Q(\hat{U}) \leq \left[H_1(x,A)\frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}f\left(x, \frac{\int_\Omega a_1\mathrm{d}x}{\int_\Omega b_1\mathrm{d}x}\right)\partial_2 g(x,0) - b_1\right]A.$$

By the definition of $H_1$, we have $\lim_{A\to\infty} H_1(x,A) = 0$ for all $x \in \bar{\Omega}$, then, there exists a positive constant $A > 0$ sufficiently large such that $Q(\hat{U}) < 0$. Hence, $\hat{U}$ is an upper solution of (6.12). Thus, the upper-lower solution method implies that (6.12) has at least one positive solution.

Next, we show the uniqueness of the solution for (6.12). We suppose by contradiction that (6.12) has two positive solutions $U_1, U_2$, we choose $\varepsilon$ sufficiently small, and $A$ sufficiently large such that $U_i \in [\tilde{u}, \hat{U}]$, $\;i = 1, 2$. The lower-upper solution method implies the existence of a minimal solution $U_m$, and maximal solution $U_M$, such that $\tilde{u} \leq U_m \leq U_1, U_2 \leq U_M \leq \hat{U}$. We multiply both sides of (6.12) with $U = U_m$ by $U_M$, and both sides of (6.12) with $U = U_M$ by $U_m$, and subtracting the resulting equations, and we integrate both sides of the obtained equation on $\Omega$ we get

$$\int_\Omega \frac{a_2}{b_3}U_m U_M\left[H_1(x,U_m)\frac{g\left(x,\frac{a_2}{b_3}U_m\right)}{\frac{a_2}{b_3}U_m} - H_1(x,U_M)\frac{g\left(x,\frac{a_2}{b_3}U_M\right)}{\frac{a_2}{b_3}U_M}\right]\mathrm{d}x = 0.$$

By ($\mathbf{H_4}$), we have $\frac{g(x,y)}{y}$ is non-increasing in $y$, $y \in C$, with $\frac{g(x,y_1)}{y_1} > \frac{g(x,y_2)}{y_2}$ if $y_1 \leq y_2$, and $y_1(x) < y_2(x)$ for some $x \in \bar{\Omega}$, and $y_i$, $\;i = 1, 2 \in C$. The monotonicity of $H_1(x,y)$, and $\frac{g(x,y)}{y}$ with respect to $y$, for all $x \in \bar{\Omega}$, which implies that $U_m = U_M$ as $U_M \geq U_m$, thus, (6.12) has a unique positive solution denoted $W_2^*$. The proof is completed. □

### 6.2. Profile as $d_2 \to +\infty$

Here, we consider $d_1$ as independent parameter in $(0, +\infty)$. We define $h_2 : \mathbb{R}^+ \to \mathbb{R}^+$, with $h_2(y) = \int_\Omega f(x,y)\mathrm{d}x$, $y > 0$, since $\partial_2 f(x,y) > 0$ for all $x \in \bar{\Omega}$, $y > 0$, then $h_2$ is invertible, and its inverse is denoted $h_2^{-1}$.

**Theorem 6.3.** *Assume that $R_0^- > 1$, and $(\mathbf{H_1}) - (\mathbf{H_5})$, holds, and $\frac{f(x,y)}{y}$ is non-increasing in $y$, $x \in \bar{\Omega}$, and $y > 0$. Then, for any fixed $d_1 > 0$, there exists a sequence $\{d_{2,n}\}$ satisfy $d_{2,n} \to +\infty$ as $n \to +\infty$ such that the PSS $(W_{1,n}, W_{2,n})$ of (6.1) verify $(W_{1,n}, W_{2,n}) \to (W_1^*, W_2^*)$ in $C$, with $W_2^*$ is a constant and satisfying*

$$W_2^* = \mathrm{e}^{-b_4\tau} \frac{\displaystyle\int_\Omega a_1 \mathrm{d}x - \int_\Omega b_1 W_1^* \mathrm{d}x}{\displaystyle\int_\Omega b_2 \mathrm{d}x},$$

*and $W_1^*$ is the unique positive solution of the problem*

$$\begin{cases} d_1 \Delta U + a_1 - f(x,U) g\left(x, \frac{a_2}{b_3} \mathrm{e}^{-b_4\tau} \frac{\int_\Omega a_1 \mathrm{d}x - \int_\Omega b_1 U \mathrm{d}x}{\int_\Omega b_2 \mathrm{d}x}\right) - b_1 U = 0, & x \in \Omega, \\ \frac{\partial U}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{6.13}$$

*Proof.* Theorem 4.1 implies the existence of PSS. Then, we check the convergence of $(W_{1,n}, W_{2,n})$ as $n \to +\infty$. By the first equation of (6.1), and performing similar reasoning to the proof of Theorem 6.2, we obtain that $\{W_1\}_{d_2>0}$ is uniformly bounded in $C$, and $\{W_2\}_{d_2>0}$ is uniformly bounded in $W^{2,p}(\Omega)$. $(\mathbf{H_1}) - (\mathbf{H_5})$ and the elliptic estimate implies that $(W_{1,n}, W_{2,n})$ the solution of (6.1) satisfies $(W_{1,n}, W_{2,n}) \to (W_1^*, W_2^*)$ weakly in $W^{2,p}(\Omega) \times W^{2,p}(\Omega)$ as $n \to +\infty$. By the second equation of (6.1), we obtain that $\Delta W_2^* = 0$, which means $W_2^*$ is constant. By integrating both sides of the second equation of (6.1), we obtain

$$\mathrm{e}^{-b_4\tau} \int_\Omega f(x, W_1^*) g\big(x, \frac{a_2}{b_3} W_2^*\big) \mathrm{d}x - W_2^* \int_\Omega b_2 \mathrm{d}x = 0. \tag{6.14}$$

Then, two cases appear

(i) $W_2^* = 0$ and $\mathrm{e}^{-b_4\tau} \int_\Omega f(x, W_1^*) \frac{a_2}{b_3} \partial_2 g(x,0) \mathrm{d}x - \int_\Omega b_2 \mathrm{d}x \neq 0$;

(ii) $W_2^* \neq 0$ and $\mathrm{e}^{-b_4\tau} \int_\Omega f(x, W_1^*) \frac{g\left(x, \frac{a_2}{b_3} W_2^*\right)}{W_2^*} \mathrm{d}x - \int_\Omega b_2 \mathrm{d}x = 0$.

For (i), we let $\widetilde{W}_{2,n} = \frac{W_{2,n}}{\|W_{2,n}\|}$. Notice that as $n \to +\infty$, $W_{2,n} \to 0$, and the definition of the derivative (and $(\mathbf{H_1})$) gives $\lim_{y\to 0} \frac{G(x,ay)}{y} = a\partial_2 g(x,0)$ for all $x \in \bar{\Omega}$. Then, $\widetilde{W}_{2,n}$ satisfy

$$d_{2,n} \Delta \widetilde{W}_{2,n} + \mathrm{e}^{-b_4\tau} f(x, W_{1,n}) \frac{g\big(x, \frac{a_2}{b_3} W_{2,n}\big)}{\|W_{2,n}\|} - b_2 \widetilde{W}_{2,n} = 0, \tag{6.15}$$

with Neumann boundary condition. By a similar reasoning as above we have $\widetilde{W}_{2,n} \to 1$ as $n \to +\infty$. Integrating the two sides of (6.15), we obtain $\mathrm{e}^{-b_4\tau} \int_\Omega f(x, W_1^*) \frac{a_2}{b_3} \partial_2 g(x,0) \mathrm{d}x - \int_\Omega b_2 \mathrm{d}x = 0$, which is a contradiction.

For (ii), $W_1^*$ and $W_2^*$ satisfy

$$\begin{cases} d_1\Delta W_1^* + a_1 - f(x, W_1^*)g(x, \frac{a_2}{b_3}W_2^*) - b_1W_1^* = 0, \; x \in \Omega, \\ \mathrm{e}^{-b_4\tau}\displaystyle\int_\Omega f(x, W_1^*)\frac{g(x, \frac{a_2}{b_3}W_2^*)}{W_2^*}\mathrm{d}x - \int_\Omega b_2\mathrm{d}x = 0, \\ \frac{\partial W_1^*}{\partial n} = 0, \qquad x \in \partial\Omega. \end{cases} \tag{6.16}$$

Integrating both sides of the first equation of (6.16) on $\Omega$, we obtain

$$\int_\Omega a_1\mathrm{d}x - W_2^*\int_\Omega f(x, W_1^*)\frac{g(x, \frac{a_2}{b_3}W_2^*)}{W_2^*}\mathrm{d}x - \int_\Omega b_1W_1^*\mathrm{d}x = 0.$$

Using the second equation of (6.16), we obtain

$$\int_\Omega a_1\mathrm{d}x - \mathrm{e}^{b_4\tau}W_2^*\int_\Omega b_2\mathrm{d}x - \int_\Omega b_1W_1^*\mathrm{d}x = 0.$$

Hence,

$$W_2^* = \mathrm{e}^{-b_4\tau}\frac{\displaystyle\int_\Omega a_1\mathrm{d}x - \int_\Omega b_1W_1^*\mathrm{d}x}{\displaystyle\int_\Omega b_2\mathrm{d}x}.$$

Substituting this result into the first equation of (6.16), we get (6.13). It remains to show the existence and the uniqueness of the solution for (6.13). By similar reasoning to the proof of Theorem 6.2, we can define the lower solution $\tilde{u} = \varepsilon$ with $\varepsilon > 0$ is a small constant, and $\hat{U} = A$, with $A > 0$ is a sufficiently large constant. Then, we guarantee the existence of at least one positive solution of (6.13). For uniqueness, we assume by contradiction that (6.16) has two positive solutions $U_1, U_2$, and the existence of a minimal solution $U_m$, and maximal solution $U_M$, such that $\tilde{u} \le U_m \le U_1, U_2 \le U_M \le \hat{U}$. We multiply both sides of (6.13) with $U = U_m$ by $U_M$, and both sides of (6.13) with $U = U_M$ by $U_m$, and subtracting the resulting equations, and we integrate both sides of the obtained equation. Then, we have

$$\int_\Omega U_mU_M\left(\frac{f(x, U_M)}{U_M}g\left(x, \frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\frac{\int_\Omega a_1\mathrm{d}x - \int_\Omega b_1U_M\mathrm{d}x}{\int_\Omega b_2\mathrm{d}x}\right) - \frac{f(x, U_m)}{U_m}g\left(x, \frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\frac{\int_\Omega a_1\mathrm{d}x - \int_\Omega b_1U_m\mathrm{d}x}{\int_\Omega b_2\mathrm{d}x}\right)\right)\mathrm{d}x + \int_\Omega a_1(U_m - U_M)\mathrm{d}x = 0,$$

By $(\mathbf{H_3})$, we have $g\left(x, \frac{a_2}{b_3}\mathrm{e}^{-b_4\tau}\frac{\int_\Omega a_1\mathrm{d}x - \int_\Omega b_1y\mathrm{d}x}{\int_\Omega b_2\mathrm{d}x}\right)$ is decreasing in $y$, with $g(x, y_1) > g(x, y_2)$ if $y_1 \le y_2$, and $y_1(x) < y_2(x)$ for some $x \in \bar{\Omega}$, and $y_i$, $i = 1, 2 \in C$, and, since that $\frac{f(x,y)}{y}$ is non-increasing in $y$, $x \in \bar{\Omega}$, and $y > 0$, we deduce that that $U_m = U_M$ as $U_M \ge U_m$. □

### 6.3. Profile as $d_1, d_2 \to +\infty$

Here, we treat both $d_1, d_2$ as independent parameters in $(0, +\infty)$. We let

$$R_0^{**} = \frac{\int_\Omega \frac{a_2\mathrm{e}^{-b_4\tau}}{b_3}f\big(x, \frac{\int_\Omega a_1\mathrm{d}x}{\int_\Omega b_1\mathrm{d}x}\big)\partial_2 g(x, 0)\mathrm{d}x}{\int_\Omega b_2\mathrm{d}x}$$

By similar prove to Lemma 6.1, and Lemma 2.5, we have the following result

**Lemma 6.4.** *$R_0 \to R_0^{**}$, as $d_1, d_2 \to +\infty$.*

Then we have the principal result of this subsection

**Theorem 6.5.** *Assume that $R_0^{**} > 1$. Then there exists sequences $\{d_{1,n}\}$, $\{d_{2,n}\}$ that satisfy $d_{1,n} \to +\infty$, and $d_{1,n} \to +\infty$ as $n \to +\infty$, such that the corresponding PSS $(W_{1,n}, W_{2,n})$ of (6.1) satisfy $(W_{1,n}, W_{2,n}) \to (W_1^*, W_2^*)$ in $C$, with $W_1^*, W_2^*$ is unique and both are constants, and satisfy*

$$\begin{cases} W_1^* = h_1(W_2^*), \\ \mathrm{e}^{-b_4\tau} \int_\Omega \frac{a_2}{b_3} H_1(x, W_2^*) \frac{g\left(x, \frac{a_2}{b_3} W_2^*\right)}{\frac{a_2}{b_3} W_2^*} \mathrm{d}x - \int_\Omega b_2 \mathrm{d}x = 0. \end{cases} \tag{6.17}$$

*Proof.* We check the convergence of $(W_{1,n}, W_{2,n})$ as $n \to +\infty$. By a routine calculation as in the proof of Theorem 4.1, and we get the uniform the boundedness of $W_1$ in $C$ and $W_2$ in $W^{2,p}(\Omega)$ for all $d_1 > 0$ and $d_2 > 0$. The elliptic estimate implies that $(W_{1,n}, W_{2,n}) \to (W_1^*, W_2^*)$ weakly in $W^{2,p}(\Omega) \times W^{2,p}(\Omega)$ as $n \to +\infty$. By the equations of (6.1), we have $\Delta W_1^* = 0$, and $\Delta W_2^* = 0$, hence, both $W_1^*$ and $W_1^*$ are constant. We integrate both sides of the two equation of (6.1), we get

$$\begin{cases} W_1^* = h_1(W_2^*), \\ \mathrm{e}^{-b_4\tau} \int_\Omega \frac{a_2}{b_3} H_1(x, W_2^*) \frac{g\left(x, \frac{a_2}{b_3} W_2^*\right)}{\frac{a_2}{b_3} W_2^*} \mathrm{d}x - \int_\Omega b_2 \mathrm{d}x = 0. \end{cases} \tag{6.18}$$

We define $H_3 : \mathbb{R}^+ \to \mathbb{R}^+$, by

$$H_3(y) = \mathrm{e}^{-b_4\tau} \int_\Omega \frac{a_2}{b_3} H_1(x, y) \frac{g\left(x, \frac{a_2}{b_3} y\right)}{\frac{a_2}{b_3} y} \mathrm{d}x - \int_\Omega b_2 \mathrm{d}x.$$

Clearly, $\lim_{y\to 0} H_3(y) = R_0^{**} - 1 > 0$, and $\lim_{y\to+\infty} H_3(y) = -\int_\Omega b_2 \mathrm{d}x < 0$. Note that the second limit is obtained from the fact that $\frac{g(x,y)}{y}$ is bounded for all $x \in \bar{\Omega}$ and $y > 0$, and $H_1(x, y)$ is decreasing in $y$ for all $x \in \bar{\Omega}$, and satisfy $H_1(x, y) \to 0$ as $y \to 0$ for all $x \in \bar{\Omega}$. Since $H_1$ is decreasing with respect to the second variable, and $\frac{g(x,y)}{y}$ is also decreasing in the second variable, we obtain that $H_3'(y) < 0$, then $H_3(y) = 0$ has a unique positive solution $W_2^*$, which guarantee the existence and the uniqueness of PSS as $d_1, d_2 \to +\infty$. The proof is completed.

□

### 6.4. Profile as $d_1 \to 0$

In this subsection, we consider that $d_1$ is an independent parameter in $(0, +\infty)$, and investigate the asymptotic profile of PSS as $d_1 \to 0$. The result of this subsection generalizes the obtained result in [11], Lemme 4.1, and Theorem 4.2. We define $h_2 : \overline{\Omega} \times C(\Omega) \to \mathbb{R}^+$ by

$$h_2(x, y) := \frac{a_1(x) - b_1(x)y}{f(x, y)},$$

by $(\mathbf{H_2})$, we have $h_2(x, y)$ is decreasing in $y$, $y \in C$, with $h_2(x, y_1) \le h_2(x, y_2)$ for all $x \in \bar{\Omega}$, if $y_1 \ge y_2$, and $y_1(x) > y_2(x)$ for some $x \in \bar{\Omega}$, and $y_i$, $i = 1, 2 \in C$. Clearly, the equation $h_2(x, y) = z$ has a unique positive solution for all $z > 0$, and $x \in \bar{\Omega}$, which we will refer this solution by $h_2^{-1}(x, z)$.

We let

$$\tilde{R}_0^* = \sup_{\phi\in H^1(\Omega),\phi\neq 0} \frac{\displaystyle\int_\Omega \mathrm{e}^{-b_4\tau} f\left(x, \frac{a_1}{b_1}\right)\frac{a_2}{b_3}\partial_2 g(x,0)\phi^2 \mathrm{d}x}{\displaystyle\int_\Omega (d_2|\nabla\phi|^2 + b_2\phi^2)\mathrm{d}x}, \tag{6.19}$$

and

$$\lambda_1^{**} := \lambda_1\left(d_2, f\left(x, \frac{a_1}{b_1}\right)\mathrm{e}^{-b_4\tau}\partial_2 g(x,0)\frac{a_2}{b_3} - b_2\right).$$

By similar reasoning to the proof of Lemma 2.5, we obtain for $d_1 \to 0$, $\lambda_1 \to \lambda_1^{**}$, and $R_0 \to \tilde{R}_0^*$. We let $\phi_1^{**}$ is the corresponding eigenfunction to the eigenvalue $\lambda_1^{**}$.

**Theorem 6.6.** *Assume that $\tilde{R}_0^* > 1$, and $(H_1) - (H_5)$ holds. Then, for any fixed $d_2 > 0$, there exists a sequence $\{d_{1,n}\}$ that satisfy $d_{1,n} \to 0$, as $n \to +\infty$, such that the corresponding PSS $(W_{1,n}, W_{2,n})$ of (6.1) satisfy $(W_{1,n}, W_{2,n}) \to (W_1^*, W_2^*)$ in $C$, with $W_1^*$ satisfy*

$$W_1^* = h_2^{-1}(x, g(x, \frac{a_2}{b_3}W_2^*)), \tag{6.20}$$

*and $W_2^*$ is the unique positive solution of the following nonlinear problem*

$$\begin{cases} d_2\Delta u + f\left(x, h_2^{-1}(x, g(x, \frac{a_2}{b_3}u))\right)g\left(x, \frac{a_2}{b_3}u\right) - b_2 u = 0, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{6.21}$$

*Proof.* Notice that $V$ is the solution of (2.28), and satisfy $V \to \frac{a_1}{b_1}$ as $d_1 \to 0$. We investigate the convergence of $(W_{1,n}, W_{2,n})$ as $n \to +\infty$. From the proof of Theorem 6.2, we have the uniform the boundedness of $W_1$ in $C$ and the uniform boundeness of $W_2$ in $W^{2,p}(\Omega)$ for all $d_1 > 0$ and $d_2 > 0$. Therefore, $W_{1,n} \to W_1^*$ weakly in $L^p(\Omega)$, and $W_{2,n} \to W_2^*$ weakly in $W^{2,p}(\Omega)$ and by the embedding theorem of $W^{2,p}(\Omega)$ in $C$, we have $W_{2,n} \to W_2^*$ strongly in $C$ as $n \to +\infty$, for some positive $W_1^*$ in $L^p(\Omega)$, and nonnegative $W_2^*$ in $W^{2,p}(\Omega)$ as $n \to \infty$, where $W_i, \quad i = 1, 2$ satisfy

$$a_1 - f(x, W_1^*)g\left(x, \frac{a_2}{b_3}W_2^*\right) - b_1 W_1^* = 0. \tag{6.22}$$

Hence, (6.22) can be rewritten as

$$W_1^* = h_2^{-1}(x, g(x, \frac{a_2}{b_3}W_2^*)). \tag{6.23}$$

We let $y_i, \quad i = 1, 2$ such that $y_1 \geq y_2$, and $y_1(x) > y_2(x)$ for some $x \in \bar{\Omega}$, and $y_i,\ i = 1, 2 \in C$. Hence, the monoticity of $g$ implies that $h_2^{-1}(x, g(x, y_1)) \leq h_2^{-1}(x, g(x, y_2))$, hence $h_2^{-1}(x, g(x, y))$ is decreasing in $y$. By (6.23), and the second equation of (6.1), we have $W_2^*$ is the nonnegative solution of (6.21). Now, we show that for $R_0 > 1$, the problem (6.21) has a unique positive solution $W_2^*$. As in the proof of Theorem 6.2, we construct an upper and lower solution for (6.21). Notice that as $d_1 \to 0$, we have $\lambda_1 \to \lambda_1^{**}$, and $R_0 \to \tilde{R}_0^*$, as $d_1 \to 0$.

By letting $\tilde{u} = \varepsilon\phi_1^{**}$ with $\varepsilon$ sufficiently small constant , we get that $\tilde{u}$ is a lower solution for (6.21), and we put $\hat{U} = A$, with $A$ is sufficiently large constant, we obtain that $\hat{U} = A$ is an upper solution for (6.21). Therefore, we guarantee the existence of solution for (6.21). By performing a similar reasoning in the last part of the proof

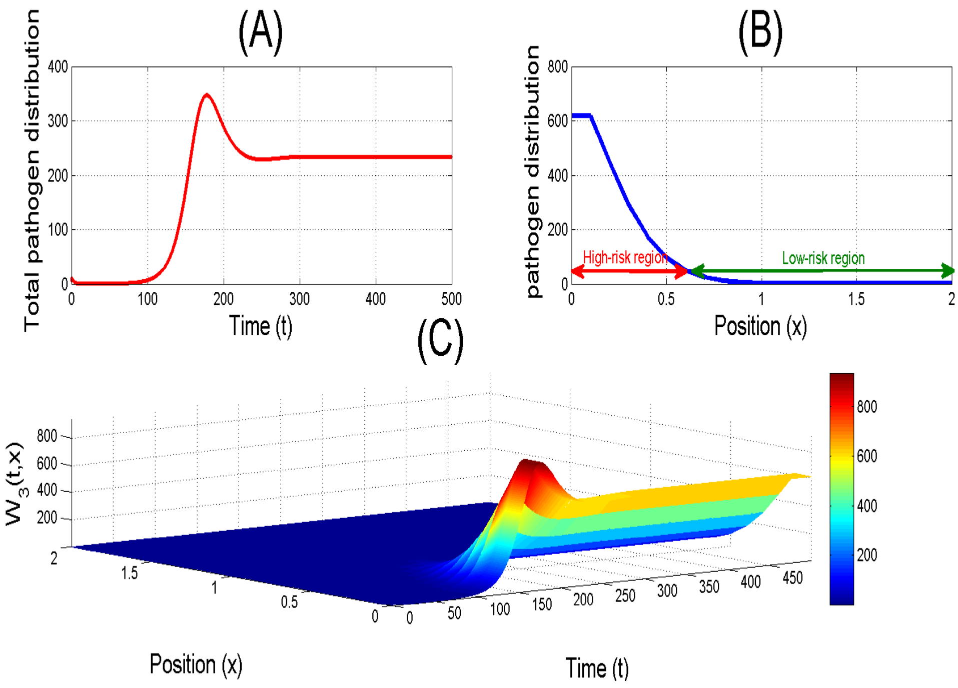


FIGURE 1. The dynamics of the solution for $d_2 = 0$, and the global stability of the positive steady state.

of Theorem 6.2, we deduce that the solution of (6.21) is unique. Therefore, for $\tilde{R}_0^* > 1$ the problem (6.21) has a unique positive solution. □

## 7. Numerical simulation

In this section, we utilize the numerical simulations to validate the theoretical findings and provide the proprieties of the basic reproduction number. Moreover, we determine the effect of the distinct diffusion coefficient on the temporal behavior of the solution. Motivated by [1], We consider that the spatial domain $\Omega$ is one-dimensional interval $\Omega = [0mm, 2mm]$, and we let $b_1(x) = 0.02 \ \ day^{-1}$, $a_1(x) = 10 \ \ day^{-1}mm^{-3}$, and we suppose that $b_2, b_3, a_2$ are space dependent and notice that these coefficients have the same unit as $b_1$, and expressed as $a_2(x) = 24(1 - 0.5x)$, $b_2(x) = 0.24(1 + x)$, and $b_3(x) = 2.4(1 + x)$. Moreover, we consider that the nonlinear incidence function takes the bilinear form, that is $G(x, W_1, W_3) = \beta(x)W_1W_3$, with $\beta$ is a $C^1(\bar{\Omega})$ function is the transmission rate. In this section, we vary $\beta$, and the dispersal rates $d_1$, $d_2$ to distinguish the effect of the transmission rate on the pathogen distribution and verify the theoretical results. From Lemma 2.5, $R_0$ is a decreasing function of $d_2$, where the maximum value can be reached for $d_2 = 0$, then, by the same lemma we have $R_0^+ = \lim_{t\to 0^+} R_0$. However, the lowest value can be reached when $d_2$ tends to $+\infty$, and by Lemma 2.5, we have $R_0^- = \lim_{d_2\to+\infty} R_0$. In the followed numerical simulations, we choose value of $d_1$ and $\beta$ in such a way $R_0$ is larger than one for $d_2 = 0$ and less than one as $d_2 \to +\infty$.

For a further understanding the asymptotic profile of the positive steady state as $d_2 \to 0$, we consider a small diffusion rate $d_2 = 0.0001\text{mm}^2\text{day}^{-1}$. In Figure 3, it is obtained that the positive steady state is stable (which is the result of the numerical results only), and the concentration of the pathogen particles is focused in a specific region which is referred to as a high-risk region. Hence, the intervention can be limited to this region only. Now, we fix $d_1 = 0.02\ \text{mm}^2\ \text{day}^{-1}$, and $\beta = 2.4 \times 10^{-4}\ \text{mm}^3\ \text{day}^{-1}$. Therefore, the highest value of the basic reproduction number can be obtained for $d_2 = 0\,\text{mm}^2\,\text{day}^{-1}$ (see Fig. 1) where in this case $R_0 = R_0^+ = 5 > 1$ ($R_0^+$

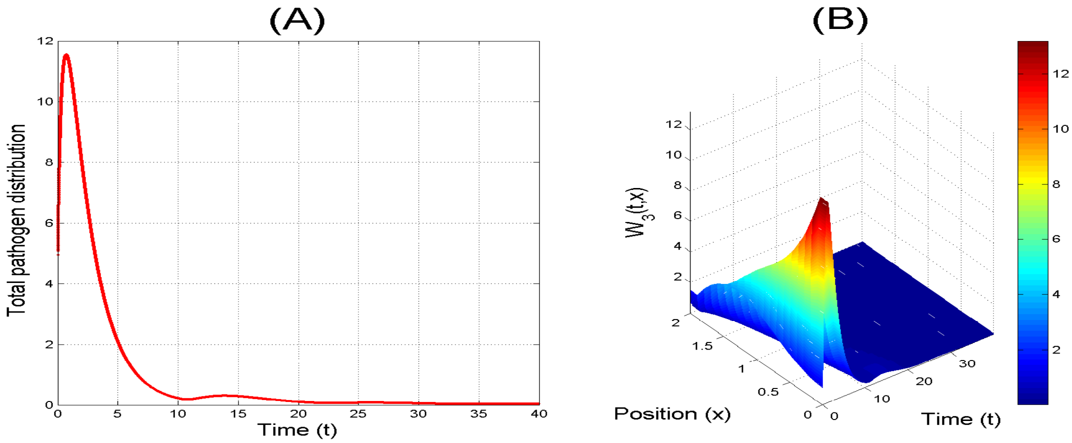


FIGURE 2. The dynamics of the solution for $d_2 = 0.4$, and the global stability of the pathogen free steady state.

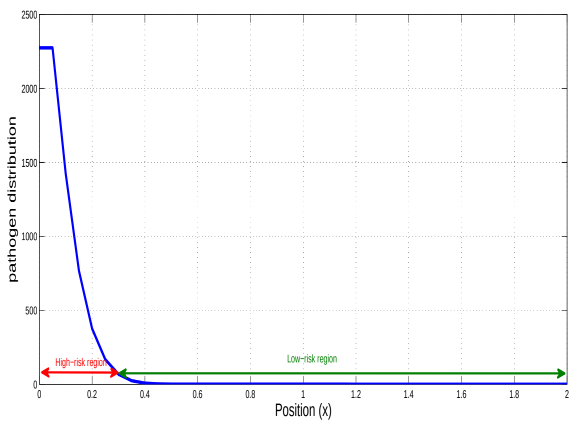


FIGURE 3. The asymptotic profile of the positive steady state as $d_2 \to 0$.

is defined in Lem. 2.5). The numerical findings show that the positive steady state is globally asymptotically stable, however, in our mathematical analysis we proved the attraction for some particular cases, namely, homogeneous case, and $d_1 = 0,\ d_2 > 0$. Notice that Figure 1B shows the profile positive steady state, where two principal regions can be distinguished, the first is the highly-risked region, which represents the favorites sites for pathogen reproduction, and the lowly-risked region, which represents the regions with a small concentration of pathogen particles. Therefore, the efforts must focus on the high-risk region to reduce the concentration of the pathogen particles instead of the overall domain. Moreover, as $d_2 \to +\infty$ we obtain that $R_0 \to R_0^- = 0.0166 < 1$, hence there exists $d_0^* > 0$ such that at $d_2 = d_2^* \approx 0.37$, $R_0 - 1$ switch signs. However, for $d_2 = 0.4 < d_2^*$, we obtain that $R_0 < 1$, and by Theorem 3.4 we deduce that the PFSS is globally asymptotically stable.

For a further understanding the asymptotic profile of the positive steady state as $d_2 \to 0$, we consider a small diffusion rate $d_2 = 0.0001\,\mathrm{mm}^2\mathrm{day}^{-1}$. In Figure 3, we obtained that the positive steady state is stable (which is the result of the numerical results only), and the concentration of the pathogen particles is focused in a specific region which is referred by a high-risk region, and hence the intervention can be limited in this region only.

## 8. Conclusion

In this paper, we investigated a global dynamics and profiles of a delayed reaction-diffusion host–pathogen system. In addition to the existence of the solution, we proved that the semiflow $\Psi(t)$ has a global attractor by applying [12], Theorem 2.4.6. The basic reproduction number $R_0$ is identified with its threshold role, if $R_0 \leq 1$ the PFSS is globally asymptotical stable, and for $R_0 > 1$ the semiflow $\Psi(t)$ is strongly uniformly persistent, and there exists at least one positive steady state, which it is confirmed using numerical simulations Figure 2.

The main purpose of this study is the investigate the asymptotic profile of the PSS as the dispersal rates tend to zero or infinity in the case of a generalized incidence function $G(s, x_1, x_2)$. Notice that this study is a generalization of the findings [11]. In this research, we investigated the large diffusion rates (means $d_1 \to +\infty$, or $d_2 \to +\infty$ or $d_1, d_2 \to +\infty$). However, due to the nonlinearity of the incidence function, we couldn't provide information on the asymptotic profile as $d_2 \to 0$. The asymptotic profile of the PSS is very important to show the feverous sites of reproduction of the pathogen particles, as is shown in the numerical simulation (Fig. 3). As in [19], we can define the high-risk, low-risk regions associated to the generalized system (1.3) as follows

$$\Omega_{high} = \{x \in \Omega | \frac{a_2}{b_3} \mathrm{e}^{-b_4 \tau} \partial_3 G(x, V(x), 0) - b_2(x) > 0\},$$

$$\Omega_{low} = \{x \in \Omega | \frac{a_2}{b_3} \mathrm{e}^{-b_4 \tau} \partial_3 G(x, V(x), 0) - b_2(x) < 0\},$$

respectively. Identifying the probable places where pathogens reproduce emphasizes how crucial it is to focus efforts in these directions in order to successfully stop the spread of infection among hosts, the probable places are mostly the highly risk regions $\Omega_{high}$. As such, choosing resources to these areas can greatly improve infection control strategies.

## References


[1] H. Shu, Z. Ma, X.S. Wang and L. Wang, Viral diffusion and cell-to-cell transmission: mathematical analysis and simulation study. *J. Math. Pures Appl.* **137** (2020) 290–313.

[2] J. Wang, J. Yang and T. Kuniya, Dynamics of a PDE viral infection model incorporating cell-to-cell transmission. *J. Math. Anal. Appl.* **444** (2016) 1542–1564.

[3] J. Wang, W. Wu and T. Kuniya, Global threshold analysis on a diffusive host–pathogen model with hyperinfectivity and nonlinear incidence functions. *Math. Comput. Simul.* **203** (2023) 767–802.

[4] J. Wang and X. Wu, Dynamics and profiles of a diffusive cholera model with bacterial hyperinfectivity and distinct dispersal rates. *J. Dynam. Diff. Equat.* (2021) 1–37.

[5] F.B. Wang, J. Shi and X. Zou, Dynamics of a host–pathogen system on a bounded spatial domain. *Commun. Pure Appl. Anal.* **14** (2015) 2535.

[6] S. Bentout, Analysis of global behavior in an age-structured epidemic model with nonlocal dispersal and distributed delay. *Math. Meth. Appl. Sci.* (2024) 1–24.

[7] F. Mahroug and S. Bentout, Dynamics of a diffusion dispersal viral epidemic modelwith age infection in a spatially heterogeneous environment with general nonlinear function. *Math. Meth. Appl. Sci.* **46** (2023) 14983–15010.

[8] R.D. Nussbaum, Eigenvectors of nonlinear positive operators and the linear Krein–Rutman theorem. Fixed Point Theory. Springer, Berlin, Heidelberg (1981) 309–330.

[9] K. Deimling, Nonlinear Functional Analysis. Springer, Berlin (1988).

[10] F.B. Wang, Y. Huang and X. Zou, Global dynamics of a PDE in-host viral model. *Applic. Anal.* **93** (2014) 2312–2329.

[11] Y. Wu and X. Zou, Dynamics and profiles of a diffusive host–pathogen system with distinct dispersal rates. *J. Diff. Equat.* **264** (2018) 4989–5024.

[12] J.K. Hale, Asymptotic behavior of dissipative systems. Mathematical Surveys and Monographsvol, Vol. 25. American Mathematical Society, Providence, RI (1988).

[13] X.Q. Zhao, Dynamical Systems in Population Biology, Vol. 16. Springer, New York (2017).

[14] S. Djilali, Threshold asymptotic dynamics for a spatial age-dependent cell-to-cell transmission model with nonlocal disperse. *Discrete Continuous Dyn. Syst. Ser. B* **28** (2023).

[15] N. Mouhcine, Y. Sabbar and Z. Anwar, Stability characterization of a fractional-order viral system with the non-cytolytic immune assumption. *Math. Model. Numer. Simul. Applic.* **2** (2022) 164–176.

[16] M. Naim, Z. Yaagoub, A. Zeb, M. Sadki and A. Karam, Global analysis of a fractional-order viral model with lytic and non-lytic adaptive immunity. *Model. Earth Syst. Environ.* **10** (2023) 1749–1769.

[17] S. Djilali, S. Bentout and A. Zeb, Dynamics of a diffusive delayed viral infection model in a heterogeneous environment. *Math. Methods Appl. Sci.* **46** (2023) 16596–16624.

[18] H. Yang and J. Wei, Dynamics of spatially heterogeneous viral model with time delay. *Commun. Pure Appl. Anal.* **19** (2020) 85.

[19] L.J.S. Allen, B.M. Bolker, Y. Lou and A.L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, Discrete Contin. *Dyn. Syst.* **21** (2008) 1–20.

[20] Y. Wu and X. Zou, Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism. *J. Diff. Equat.* **261** (2016) 4424–4447.

[21] G.F. Webb, Theory of Nonlinear Age-dependent Population Dynamics. CRC Press (1985).

[22] H.L. Smith, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems. American Mathematical Society. Providence (1995).

[23] W. Wang and X. Zhao, Basic reproduction numbers for reaction–diffusion epidemic models. *SIAM J. Appl. Dyn. Syst.* **11** (2012) 1652–1673.

[24] N.S. Trudinger, Elliptic Partial Differential Equations of Second Order. (1983).

[25] H.R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity. *SIAM J. Appl. Math.* **70** (2009) 188–211.

[26] W. Kerscher and R. Nagel, Asymptotic behavior of one-parameter semigroups of positive operators. *Acta Appl. Math.* **2** (1984) 297–309.

[27] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag, New York, Berlin, Heidelberg, Tokyo (1983).

[28] W. Desch and W. Schappacher, Linearized stability for nonlinear semigroups, in Differential Equations in Banach Spaces, Lecture Notes in Math. 1223, edited by A. Favini and E. Obrecht. Springer-Verlag, Berlin, Heidelberg (1986) 61–67.

[29] R. Cui, K.Y. Lam and Y. Lou, Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments. *J. Diff. Equat.* **263** (2017) 2343–2373.

[30] P. Magal and X.-Q. Zhao, Global attractors and steady states for uniformly persistent dynamical systems. *SIAM J. Math. Anal.* **37** (2005) 251–275.

[31] H. Smith and X.Q. Zhao, Robust persistence for semidynamical systems. *Nonlin. Anal. Theo. Meth. Appl.* **47** (2001) 6169–6179.

[32] L. Liu, R. Xu and Z. Jin, Global dynamics of a spatial heterogeneous viral infection model with intracellular delay and nonlocal diffusion. *Appl. Math. Model.* **82** (2020) 150–167.

[33] A. Korobeinikov, Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-linear transmission. *Bull. Math. Biol.* **68** (2006) 615–626.

[34] B. Soufiane and T.M. Touaoula, Global analysis of an infection age model with a class of nonlinear incidence rates. *J. Math. Anal. Appl.* **434** (2016) 1211–1239.

[35] D. Henry, Geometric theory of Semilinear Parabolic Equations, Vol. 840. Springer (2006).