

DISTRIBUTION OF MONEY ON CONNECTED GRAPHS WITH MULTIPLE BANKS

NICOLAS LANCHIER^{1,*} AND STEPHANIE REED²

Abstract. This paper studies an interacting particle system of interest in econophysics inspired from a model introduced in the physics literature. The original model consists of the customers of a single bank characterized by their capital, and the dynamics consists of monetary transactions in which a random individual x gives one coin to another random individual y , the transaction being canceled when x is in debt and there are no more coins in the bank. Using a combination of numerical simulations and heuristic arguments, physicists conjectured that the distribution of money (the random number of coins owned by a given individual) at equilibrium converges to an asymmetric Laplace distribution in the large population limit when the money temperature is large. We prove and extend this conjecture to a more general model including multiple banks and interactions among customers across banks. More importantly, we assume that customers are located on a general undirected connected graph (as opposed to the complete graph in the original model) where neighbors are interpreted as business partners, and transactions occur along the edges, thus modeling the flow of money across a social network. We first derive an exact expression for the distribution of money for all population sizes and money temperatures, then prove its convergence to an asymmetric Laplace distribution in the large population limit.

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1. INTRODUCTION

The main objective of this paper is to prove and extend the conjecture from [1] about an interacting particle system of interest in econophysics, the subfield of statistical mechanics that applies traditional concepts from physics to model and study economical systems. Many such stochastic models are reviewed in [2] and consist of large populations of individuals characterized by the amount of money they possess, which we identify with a *number of coins*. In all these models, two individuals, say x and y , are chosen uniformly at random and sequentially from the entire population at each time step to engage in a monetary transaction, and the models only differ in the exchange rules at each interaction. From the point of view of econophysics, the individuals can be thought of as particles, money as energy, and the mean number of coins per individual as the temperature, also referred to as the *money temperature*. The main objective in this field is to determine the so-called *distribution of money*, *i.e.*, the fraction of individuals that have a given number of coins at equilibrium, in the large

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¹ School of Mathematical and Statistical Sciences Arizona State University Tempe, AZ 85287, USA.

² Mathematics Department California State University Fullerton, CA 92834, USA.

* Corresponding author: nlanchie@asu.edu

population limit when the money temperature is large. The exchange rules reviewed in [2] along with the corresponding conjectures obtained by physicists based on numerical simulations and/or heuristic arguments are the following.

One-coin model [3]. Individual x gives one coin to individual y if she has at least one coin. In this case, it was conjectured that, in the large population limit, the distribution of money converges to the **exponential distribution** with mean the money temperature.

Uniform reshuffling model [3]. All the coins individuals x and y possess are uniformly redistributed between the two interacting individuals. In this case, it was conjectured that, in the large population limit, the distribution of money converges to the **exponential distribution** with mean the money temperature, just like in the one-coin model.

Immediate exchange model [4, 5]. Individuals x and y choose independently and uniformly a random number of their coins to give to the other individual. In this case, it was conjectured that the distribution of money converges to a **gamma distribution** with mean the money temperature and shape parameter two in the large population limit.

Saving propensity model [6, 7]. Individuals x and y independently save a random number of their coins, like in the immediate exchange model, and the remaining coins are uniformly redistributed between the two agents, like in the uniform reshuffling model. In this case, it was conjectured that the distribution of money converges to a **gamma distribution** with mean the money temperature and shape parameter two, just like in the immediate exchange model.

Individual debt model [3]. The rules at each interaction are the same as in the one-coin model except that individual x can now go into debt with an individual limit of L coins. In this case, it was conjectured that the distribution of money in the large population limit converges to a **shifted exponential distribution** shifted by $-L$ coins.

Single bank model [1]. The rules are again the same as in the one-coin model except that individual x can now borrow money from a central bank provided the bank is not empty. In this case, it was conjectured that, in the large population limit, the distribution of money at equilibrium converges to an **asymmetric Laplace distribution**.

Convergence of the one-coin model to the exponential distribution was proved in [8]. The authors also proved the conjectures about the uniform reshuffling, immediate exchange, and saving propensity models in [9], and the conjecture about the individual debt model in [10]. While they also studied some of the aspects of the single bank model in this last paper, they were unable to establish the convergence to the asymmetric Laplace distribution conjectured in [1]. In this paper, we prove this conjecture but also extend the result to a more realistic model that includes multiple banks and more importantly a general undirected connected graph/social network.

The one-coin model, the uniform reshuffling model, the immediate exchange model, and the single bank model were also studied rigorously in [11–15]. In these papers, the analysis relies on kinetic theory, and consists in taking first the large population limit to obtain a system of ordinary differential equations, and then taking the limit as time goes to infinity. One limitation of this approach is that it can only be implemented under the assumption of global interactions: all the pairs of individuals are equally likely to interact. In contrast, our analysis relies on probability theory and combinatorics, and consists in taking first the limit as time goes to infinity to obtain the stationary distribution of the stochastic process, and then taking the large population limit. Using reversibility in the same fashion as in [16], we obtain an expression of the stationary distribution that is not sensitive to the underlying network of interactions. In particular, in contrast with the models studied numerically in [1, 3–7] and analytically in [11–15] that assume global interactions, the individuals in our model are located on a general connected graph and can only interact (exchange money) if they are connected by an edge, which results in a more realistic model taking into account the presence of a social network.

2. MODEL DESCRIPTION AND MAIN RESULTS

To define the model with multiple banks, we start by letting $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a general finite connected graph. This graph represents a social network where each vertex $x \in \mathcal{V}$ is interpreted as an individual, while an edge $(x, y) \in \mathcal{E}$ indicates that x and y are business partners. The topology of the graph is incorporated in the dynamics of the economical system by assuming that the flow of money can only occur through the edges, meaning that only business partners can interact to exchange money. We also assume that the system includes K banks and that each individual is a customer at exactly one of the banks: let $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_K$ be a partition of the vertex set and assume that each $x \in \mathcal{V}_i$ is a customer at bank i only. Individuals/vertices are characterized by the number of coins they possess, which is negative for individuals in debt who borrowed money from their bank. More precisely, the state at time $t \in \mathbb{N}$ is a spatial configuration

$$\xi_t : \mathcal{V} \longrightarrow \mathbb{Z} \quad \text{where} \quad \xi_t(x) = \begin{cases} + \text{ number of coins } x \text{ possesses when } \xi_t(x) \geq 0, \\ - \text{ number of coins } x \text{ borrowed when } \xi_t(x) \leq 0. \end{cases}$$

Let M be the number of coins in the population, N_i be the number of customers in bank i , and R_i be the number of coins in bank i initially, and

$$N = N_1 + N_2 + \dots + N_K \quad \text{and} \quad R = R_1 + R_2 + \dots + R_K$$

be respectively the number of vertices (population size) and the total number of coins in all the banks initially. From now on, we will write

$$(N_i) = (N_1, N_2, \dots, N_K) \quad \text{and} \quad (R_i) = (R_1, R_2, \dots, R_K)$$

to avoid cumbersome notation. We also let

$$T = M/N = \text{money temperature} \quad \text{and} \quad \rho_i = R_i/M \quad \text{for} \quad i = 1, 2, \dots, K.$$

At each time step, we choose two neighbors, say x and y , sequentially, uniformly at random, or equivalently an oriented edge $\vec{x}y$ uniformly at random, and think of x as a potential buyer and y as a potential seller. Assuming that $x \in \mathcal{V}_i$ and $y \in \mathcal{V}_j$, we move one coin $x \rightarrow y$ if there is

- at least one coin at vertex x or
- less than one coin at vertex x but at least one coin in bank i that x can borrow.

Otherwise, the buyer x cannot pay or borrow money so the transaction is canceled and nothing happens. In case a transfer of money does happen, the seller y either

- increases her capital by one coin or
- decreases her debt by one coin by returning the coin to bank j .

To study the distribution of money, we first prove that the process (ξ_t) is ergodic, and use reversibility to prove that its unique stationary distribution is uniform on the set of admissible configurations, *i.e.*, the configurations such that the total fortune of all the individuals equals M and the total debt of all the customers at bank i does not exceed R_i . In particular, computing the probability that an individual, say x , possesses c coins at equilibrium reduces to counting the total number of configurations and the number of configurations with exactly c coins at vertex x . Using a combinatorial argument, we can prove the following general result which holds for all possible choices of the parameters M, N_i and R_i , not just in the large population limit.

Theorem 2.1. *Let e_j be the j th unit vector in \mathbb{Z}^K . For all connected graphs \mathcal{G} , all M, N_i and R_i , and all $x \in \mathcal{V}_j$, the probability that vertex x has c coins at equilibrium is given by*

$$\begin{aligned} \Lambda((N_i) - e_j, (R_i), M - c) / \Lambda((N_i), (R_i), M) & \quad \text{for all } 0 \leq c \leq M, \\ \Lambda((N_i) - e_j, (R_i) + ce_j, M - c) / \Lambda((N_i), (R_i), M) & \quad \text{for all } 0 \geq c \geq -R_j, \end{aligned}$$

where the function $\Lambda : \mathbb{N}^K \times \mathbb{N}^K \times \mathbb{N} \rightarrow \mathbb{N}$ is defined as

$$\begin{aligned} \Lambda((N_i), (R_i), M) = & \sum_{a_1=0}^{R_1} \cdots \sum_{a_K=0}^{R_K} \sum_{b_1=0}^{N_1} \cdots \sum_{b_K=0}^{N_K} \binom{N_1}{b_1} \cdots \binom{N_K}{b_K} \\ & \binom{a_1 - 1}{b_1 - 1} \cdots \binom{a_K - 1}{b_K - 1} \binom{M + a_1 + \cdots + a_K + N - b_1 - \cdots - b_K - 1}{N - b_1 - \cdots - b_K - 1}, \end{aligned}$$

with the convention $(-1 \text{ choose } k) = 1$, and $(n \text{ choose } k) = 0$ when $k < 0$ or $k > n$.

The quantity $\Lambda((N_i), (R_i), M)$ represents the total number of configurations, with the summands corresponding to the number of configurations with a_i coins borrowed from bank i , and b_i customers in debt from that bank. The last term in each summand is the number of ways one can distribute the coins in circulation to the individuals who are not in debt. When the initial number of coins and/or the number of customers in each bank are different, the probability that a given individual has c coins depends on the bank she goes to. In particular, the distribution of money consists of a convex combination of the probabilities given in the previous theorem: letting $x_1 \in \mathcal{V}_1, \dots, x_K \in \mathcal{V}_K$, the fraction of individuals with c coins at equilibrium is given by

$$\begin{aligned} \frac{1}{N} \sum_{x \in \mathcal{V}} \lim_{t \rightarrow \infty} P(\xi_t(x) = c) &= \sum_{j=1}^K \frac{N_j}{N} \lim_{t \rightarrow \infty} P(\xi_t(x_j) = c) \\ &= \sum_{j=1}^K \frac{N_j}{N} \frac{\Lambda((N_i) - e_j, (R_i), M - c)}{\Lambda((N_i), (R_i), M)} \mathbf{1}\{0 \leq c \leq M\} \\ &\quad + \sum_{j=1}^K \frac{N_j}{N} \frac{\Lambda((N_i) - e_j, (R_i) + ce_j, M - c)}{\Lambda((N_i), (R_i), M)} \mathbf{1}\{0 > c \geq -R_j\}. \end{aligned} \tag{2.1}$$

Looking now at the distribution of money in the large population limit, a natural approach to prove the conjecture in [1] is to study the right-hand side of equation (2.1) when both the population size N and the money temperature $T = M/N$ are large. However, we were not able to simplify this expression enough to deduce convergence to an asymmetric Laplace distribution, and ended up using a completely different approach. Because the convergence of the process (ξ_t) to the uniform distribution on the set of configurations holds *regardless* of the choice of the connected graph, we may focus on the complete graph, in which case the location of the individuals is unimportant and the dynamics can be simply described by the process (u_t) that keeps track of the fraction of individuals who have c coins and go to bank i . Assuming also some symmetry among the banks, we prove that, regardless of the population size and the money temperature, the stationary distribution of this process is an asymmetric two-sided geometric distribution. Computing the parameters of this distribution is quite challenging so we follow the physics literature and take the large population limit and the money temperature large in which case the asymmetric two-sided geometric distribution is well-approximated by an asymmetric Laplace distribution (in the same way a properly rescaled geometric distribution with a small success probability is well-approximated by an exponential distribution), and the calculations are greatly simplified. To find the parameters of this distribution, we use the conservation of the money temperature and prove that, in the symmetric case where all the banks have the same number of customers and start with the same number

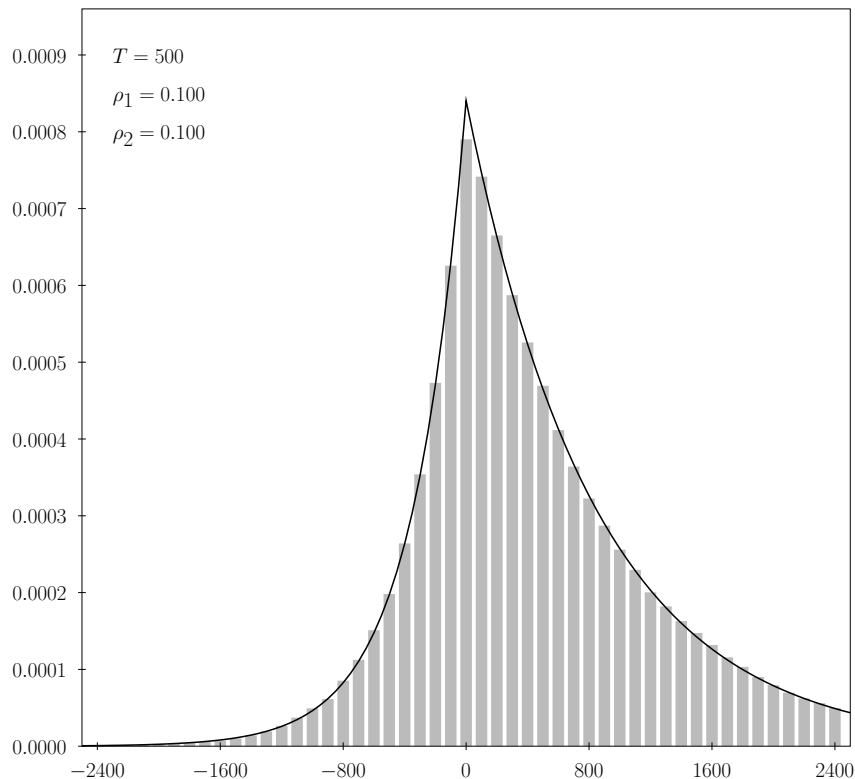


FIGURE 1. Distribution of money for the model with two identical banks (with the same number of customers and the same initial number of coins). The gray histogram is obtained from numerical simulation of the stochastic process on the complete graph with $N = 10K$ vertices, $M = 5M$ coins, and $R = 1M$ coins initially in the two banks. The solid curve represents the density function f found in Theorem 2.2.

of coins, the total number of coins in the banks in the large population limit can be neglected. In conclusion, we have the following theorem that extends the conjecture in [1] to multiple symmetric banks and general finite connected graphs/social networks.

Theorem 2.2. *Assume $N_i = N/K$ and $R_i = R/K$. For all $T = M/N$ large,*

$$\lim_{N \rightarrow \infty} \lim_{t \rightarrow \infty} P(\xi_t(x) = c) \approx f(c) = \begin{cases} \mu e^{-ac} & \text{for } c \geq 0, \\ \mu e^{+bc} & \text{for } c \leq 0, \end{cases} \quad (2.2)$$

where, letting $\rho = R/M = (R_1 + \dots + R_K)/M$,

$$\mu = \frac{1}{T} \left(\sqrt{1+\rho} - \sqrt{\rho} \right)^2, \quad a = \frac{1}{T} \left(1 - \sqrt{\frac{\rho}{1+\rho}} \right), \quad b = \frac{1}{T} \left(\sqrt{\frac{1+\rho}{\rho}} - 1 \right). \quad (2.3)$$

For a picture of the Laplace distribution (2.2) with parameters (2.3) along with the distribution of money obtained from numerical simulation of the stochastic process on a large complete graph in the presence of two identical banks, we refer the reader to Figure 1. We also used a computer program in Figure 2 to display the distribution of money obtained from Theorem 2.1 and equation (2.1) in the presence of two non-identical banks

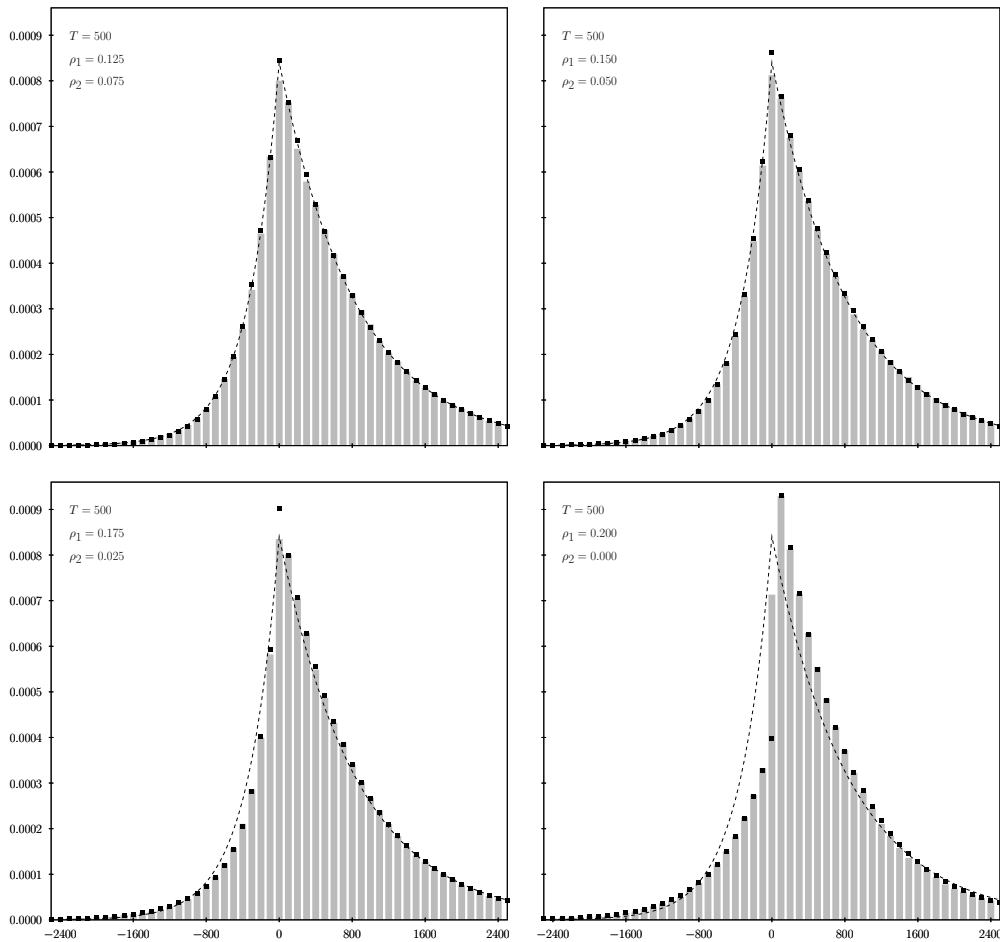


FIGURE 2. Distribution of money for the model with two banks with the same number of customers but different initial numbers of coins ($\rho_1 \neq \rho_2$). The gray histogram is again obtained from numerical simulation of the stochastic process on the complete graph with $N = 10K$ vertices, $M = 5M$ coins, and $R = 1M$ coins in the banks. The dashed curve is just a duplicate of the solid curve in Figure 1 displayed for comparison. The black squares are computed from combining Theorem 2.1 and (2.1) for $N = 100$, $M = 50K$ and $R = 10K$.

along with simulation of the process. The money temperature and the fraction $\rho_1 + \rho_2$ of coins initially in the two banks are the same as in Figure 1 and the banks again have the same number of customers but the banks now start with a different number of coins, meaning that $\rho_1 \neq \rho_2$. The pictures show that, as the initial number of coins in the first bank increases and the initial number of coins in the second bank decreases, the number of individuals in debt at equilibrium decreases. In addition, the value of c at which the maximum of the distribution of money is reached shifts to the right, suggesting that the convergence to an asymmetric Laplace distribution in Theorem 2.2 does not hold in the presence of non-identical banks.

3. CONVERGENCE TO THE UNIFORM DISTRIBUTION

This section is devoted to collecting some preliminary results that will be used in the next sections to prove the theorems. To begin with, we show that the process is irreducible (see Lem. 3.4) and aperiodic (see Lem. 3.5), and therefore ergodic: the process converges to a unique stationary distribution π that does not depend on the

initial configuration of coins. Note that this distribution is a probability measure on the set of configurations as opposed to the distribution of money which is defined as the distribution of the number of coins at a given vertex. Then, using reversibility, we deduce that the distribution π is uniform on the set of configurations (see Lem. 3.6). The proof of ergodicity and reversibility leading to an exact expression of the stationary distribution is similar for our process and a general class of particle systems introduced by Spitzer [16], known as zero-range processes. In these processes, particles are located on the vertices of a graph and jump in continuous time at a rate that depends on the number of particles at their location to one of their neighbors chosen uniformly at random. The dynamics implies that particles jump individually, and that the total number of particles is preserved by the dynamics. Thinking of particles as coins and vertices as individuals, the one-coin model can be viewed as a particular case of a zero-range process evolving in discrete time. In contrast, the uniform reshuffling, immediate exchange, and saving propensity models are not particular cases of zero-range processes because their dynamics involves the simultaneous jump of multiple particles. Similarly, the bank model is not a particular case of a zero-range process because, due to the presence of banks, the dynamics does not preserve the total number of coins in the population. In particular, ergodicity and reversibility for the bank model cannot be directly deduced from Spitzer's result [16], though the proofs have some similarities. To show ergodicity and reversibility, we first give an explicit expression of the transition probabilities of the process. This is done in the next lemma where τ_{xy} is the operator on the set of configurations that moves one coin from vertex x to vertex y , *i.e.*,

$$(\tau_{xy} \xi)(z) = (\xi(x) - 1) \mathbf{1}\{z = x\} + (\xi(y) + 1) \mathbf{1}\{z = y\} + \xi(z) \mathbf{1}\{z \neq x, y\},$$

and where S denotes the set of configurations, *i.e.*,

$$S = \{\xi \in \mathbb{Z}^{\mathcal{V}} : -R_i \leq \xi(x) \leq M + R \text{ for all } x \in \mathcal{V}_i, \\ \sum_{z \in \mathcal{V}} \xi(z) = M \text{ and } \sum_{z \in \mathcal{V}_i} \xi(z) \mathbf{1}\{\xi(z) < 0\} \geq -R_i \text{ for all } i\}.$$

Lemma 3.1. *For all $(x, y) \in \mathcal{V}_i \times \mathcal{V}$,*

$$p(\xi, \tau_{xy} \xi) = \frac{\mathbf{1}\{(x, y) \in \mathcal{E}\}}{2 \text{card}(\mathcal{E})} \mathbf{1}\left\{\xi(x) > 0 \text{ or } \sum_{z \in \mathcal{V}_i} \xi(z) \mathbf{1}\{\xi(z) < 0\} > -R_i\right\}.$$

Proof. Let $\xi \in S$ and $(x, y) \in \mathcal{V}_i \times \mathcal{V}$. Then, $p(\xi, \tau_{xy} \xi) > 0$ if and only if

1. the individual at x is not in debt or bank i is not empty, and
2. vertices x and y are connected by an edge.

These conditions can be expressed mathematically as

$$(\xi(x) > 0 \text{ or } \sum_{z \in \mathcal{V}_i} \xi(z) \mathbf{1}\{\xi(z) < 0\} > -R_i) \text{ and } (x, y) \in \mathcal{E}. \quad (3.1)$$

In addition, given that the conditions in (3.1) hold, the probability of transitioning from configuration ξ to configuration $\tau_{xy} \xi$ is the probability that the directed edge \vec{xy} is selected. Since each undirected edge results in two directed edges, this probability is equal to

$$\frac{1}{\text{card}\{x, y \in \mathcal{V} : (x, y) \in \mathcal{E}\}} = \frac{1}{2 \text{card}(\mathcal{E})}. \quad (3.2)$$

The result then follows from combining (3.1) and (3.2). \square

In order to prove irreducibility, we first use Lemma 3.1 to prove that if it is possible to move a coin through the directed edge $x\vec{y}$ in one time step then, right after this move, it is possible to move a coin from vertex y to any of its neighbors z , again in one time step.

Lemma 3.2. *Let $\xi \in S$ and $(x, y) \in \mathcal{E}$ such that $\xi' = \tau_{xy} \xi \in S$. Then,*

$$p(\xi, \xi') > 0 \quad \text{implies that} \quad \tau_{yz} \xi' \in S \quad \text{and} \quad p(\xi', \tau_{yz} \xi') > 0.$$

Proof. Letting $x \in \mathcal{V}_i$ and $y \in \mathcal{V}_j$, we prove the result by distinguishing two cases depending on whether the two vertices go to the same bank or not.

Case 1. Different banks $i \neq j$.

In this case, after the first move, there will be one more coin in bank j in case y was in debt therefore y can borrow a coin from this bank if needed. We now turn the intuition into rigorous equations. According to Lemma 3.1, it suffices to prove that

$$\xi'(y) > 0 \quad \text{or} \quad \sum_{z \in \mathcal{V}_j} \xi'(z) \mathbf{1}\{\xi'(z) < 0\} > -R_j. \quad (3.3)$$

Assuming that $\xi'(y) \leq 0$, we must have

$$\xi'(y) \mathbf{1}\{\xi'(y) < 0\} = \xi'(y) \mathbf{1}\{\xi'(y) \leq 0\} = \xi'(y) = \xi(y) + 1.$$

Using also that $\xi' \in S$, we deduce that

$$\begin{aligned} \sum_{z \in \mathcal{V}_j} \xi'(z) \mathbf{1}\{\xi'(z) < 0\} &= \sum_{z \in \mathcal{V}_j \setminus \{y\}} \xi(z) \mathbf{1}\{\xi(z) < 0\} + \xi(y) + 1 \\ &\geq \sum_{z \in \mathcal{V}_j} \xi(z) \mathbf{1}\{\xi(z) < 0\} + 1 \geq -R_j + 1 > -R_j, \end{aligned}$$

which shows that (3.3) holds in the first case.

Case 2. Same bank $i = j$.

In this case, either x is not in debt before the first move which brings one coin to either y or the common bank (in either case y can then use this coin) or x is in debt before the move but the fact that the move occurs indicates that the bank is not empty. To turn this into a rigorous proof, we again assume that $\xi'(y) \leq 0$, and observe that, according to Lemma 3.1,

$$\xi(x) > 0 \quad \text{or} \quad \sum_{z \in \mathcal{V}_i} \xi(z) \mathbf{1}\{\xi(z) < 0\} > -R_i \quad (3.4)$$

because $p(\xi, \tau_{xy} \xi) > 0$. This leads to two sub-cases:

Sub-case 2a. Assume that $\xi(x) > 0$.

The same reasoning as in the first case shows that the second inequality in (3.3) holds.

Sub-case 2b. Assume that $\xi(x) \leq 0$.

In this case, the second inequality in (3.4) must hold and

$$\begin{aligned} \xi'(x) \mathbf{1}\{\xi'(x) < 0\} &= \xi'(x) \mathbf{1}\{\xi(x) - 1 < 0\} = \xi'(x) = \xi(x) - 1, \\ \xi'(y) \mathbf{1}\{\xi'(y) < 0\} &= \xi'(y) \mathbf{1}\{\xi'(y) \leq 0\} = \xi'(y) = \xi(y) + 1, \end{aligned}$$

from which it follows that

$$\begin{aligned} \sum_{z \in \mathcal{V}_i} \xi'(z) \mathbf{1}\{\xi'(z) < 0\} &= \sum_{z \in \mathcal{V}_i \setminus \{x, y\}} \xi(z) \mathbf{1}\{\xi(z) < 0\} + (\xi(x) - 1) + (\xi(y) + 1) \\ &\geq \sum_{z \in \mathcal{V}_i} \xi(z) \mathbf{1}\{\xi(z) < 0\} > -R_i. \end{aligned}$$

This again shows that (3.3) holds. \square

We now use Lemma 3.2 to prove that if two configurations ξ and ξ' can be obtained from one another by moving one coin from x to y then the configurations communicate. In addition, the minimum number of time steps required is less than twice the number of vertices.

Lemma 3.3. *Let $\xi \in S$ and $x, y \in \mathcal{V}$ such that $\xi' = \tau_{xy} \xi \in S$. Then,*

$$p_t(\xi, \xi') > 0 \quad \text{for some } t < 2N.$$

Proof. Because the graph is connected, there exists a self-avoiding directed path

$$x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_t = y \quad \text{for some } t < N$$

going from vertex x to vertex y . According to Lemma 3.1, we can move a coin $x = x_0 \rightarrow x_1$ whenever vertex x or the bank vertex x goes to have at least one coin:

$$\xi(x) > 0 \quad \text{or} \quad \sum_{z \in \mathcal{V}_i} \xi(z) \mathbf{1}\{\xi(z) < 0\} > -R_i \tag{3.5}$$

in which case, after applying Lemma 3.2 repeatedly, we can move the coin along the path up to vertex y . In particular, to prove the lemma, it suffices to prove that (3.5) holds. It turns out, however, that this is the case for most but not all configurations $\xi \in S$ such that $\tau_{xy} \xi \in S$. The trouble appears when vertex x has no coin, vertex y is in debt, both vertices go to the same bank, and their bank is empty, in which case we will need to bring a coin from another vertex w . We now turn the argument into a rigorous proof. To begin with, assume that $x \in \mathcal{V}_i$ and $y \in \mathcal{V}_j$.

Case 1. Vertex x has at least one coin: $\xi(x) > 0$.

In this case, (3.5) holds so the result follows from the reasoning above using Lemmas 3.1 and 3.2.

Case 2. Vertex x does not have any coin: $\xi(x) \leq 0$.

This case is more complicated and we consider several sub-cases depending on whether y is in debt or not and on whether the two vertices go to the same bank or not.

Sub-case 2a. Vertex y is not in debt: $\xi(y) \geq 0$.

In this case, $\xi'(x) = \xi(x) - 1 < 0$ and $\xi'(y) = \xi(y) + 1 > 0$ therefore

$$\xi'(x) \mathbf{1}\{\xi'(x) < 0\} = \xi(x) - 1 \quad \text{and} \quad \xi'(y) \mathbf{1}\{\xi'(y) < 0\} = 0.$$

Using also that $\xi' = \tau_{xy} \xi \in S$, we deduce that

$$\begin{aligned} \sum_{z \in \mathcal{V}_i} \xi(z) \mathbf{1}\{\xi(z) < 0\} &= \sum_{z \in \mathcal{V}_i \setminus \{x\}} \xi(z) \mathbf{1}\{\xi(z) < 0\} + \xi(x) \mathbf{1}\{\xi(x) < 0\} \\ &> \sum_{z \in \mathcal{V}_i \setminus \{x\}} \xi(z) \mathbf{1}\{\xi(z) < 0\} + \xi(x) - 1 \\ &= \sum_{z \in \mathcal{V}_i} \xi'(z) \mathbf{1}\{\xi'(z) < 0\} \geq -R_i. \end{aligned} \tag{3.6}$$

In particular, (3.5) holds so we can move a coin $x \rightarrow y$ in less than N time steps.

Sub-case 2b. The vertices go to different banks: $i \neq j$.

In this case, we again have $\xi'(x) \mathbf{1}\{\xi'(x) < 0\} = \xi(x) - 1$. In addition, $y \notin \mathcal{V}_i$ therefore (3.6) and so (3.5) again hold, showing that we can move a coin $x \rightarrow y$.

Sub-case 2c. Everything else: $\xi(y) < 0$ and $i = j$.

In contrast with 2a and 2b, the common bank might be empty in this case and so (3.6) might not hold. To deal with this problem, the idea is to find another vertex w that has at least one coin, move this coin along a path $w \rightarrow y$, which will bring one coin in the bank that we can then move along a path $x \rightarrow y$. To begin with, observe that, because

$$\sum_{z \in \mathcal{V}} \xi(z) = M > 0, \quad \xi(x) \leq 0 \quad \text{and} \quad \xi(y) < 0,$$

there exists $w \in \mathcal{V}$, $w \neq x, y$, such that $\xi(w) > 0$. Using again that the graph is connected, there exist two self-avoiding directed paths

$$x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_s = w \quad \text{and} \quad w = y_0 \rightarrow y_1 \rightarrow \cdots \rightarrow y_t = y$$

for some $s, t < N$, going from vertex x to vertex w , and from vertex w to vertex y . Using as previously Lemmas 3.1 and 3.2, and the fact that $\xi(w) > 0$, we can move a coin along the path $w \rightarrow y$, after which the configuration is $\xi'' = \tau_{wy} \xi$. In this configuration, we have

$$\xi''(x) \mathbf{1}\{\xi''(x) < 0\} = \xi(x) \mathbf{1}\{\xi(x) < 0\} \quad \text{and} \quad \xi''(y) \mathbf{1}\{\xi''(y) < 0\} = \xi(y) + 1$$

from which it follows that

$$\sum_{z \in \mathcal{V}_i} \xi''(z) \mathbf{1}\{\xi''(z) < 0\} = \sum_{z \in \mathcal{V}_i} \xi(z) \mathbf{1}\{\xi(z) < 0\} + 1 > -R_i.$$

In particular, the second condition in (3.5) holds for configuration ξ'' so we can move a coin along the directed path $x \rightarrow w$, after which the configuration is

$$\tau_{xw} \xi'' = \tau_{xw} \tau_{wy} \xi = \tau_{xy} \xi = \xi'.$$

In conclusion, $p_{s+t}(\xi, \xi') > 0$ for some $s, t < N$, and the proof is complete. \square

Using Lemma 3.3, we can now deduce that the process is irreducible.

Lemma 3.4 (irreducibility). *The process (ξ_t) is irreducible.*

Proof. The idea is simple: given two configurations $\xi, \xi' \in S$, we can go from one configuration to the other one by moving coins around, which can be done in a finite number of time steps and with positive probability according to Lemma 3.3. To make this statement rigorous, define a distance D on the set of configurations by letting

$$D(\xi, \xi') = \sum_{z \in \mathcal{V}} |\xi(z) - \xi'(z)| \quad \text{for all } \xi, \xi' \in S.$$

As long as $\xi \neq \xi'$, because the number of coins is preserved by the dynamics and so the coordinates of each of the two configurations add up to M ,

$$\xi(x) > \xi'(x) \quad \text{and} \quad \xi(y) < \xi'(y) \quad \text{for some } x, y \in \mathcal{V}.$$

Then, using Lemma 3.3 to move one coin $x \rightarrow y$ reduces the distance:

$$\begin{aligned} D(\tau_{xy} \xi, \xi') &= \sum_{z \in \mathcal{V}} |\tau_{xy} \xi(z) - \xi'(z)| \\ &= \sum_{z \in \mathcal{V} \setminus \{x, y\}} |\xi(z) - \xi'(z)| + |\xi(x) - 1 - \xi'(x)| + |\xi(y) + 1 - \xi'(y)| \\ &= \sum_{z \in \mathcal{V}} |\xi(z) - \xi'(z)| - 2 = D(\xi, \xi') - 2. \end{aligned}$$

This shows that, starting from configuration ξ , we can move the coins around in such a way that, after a finite number of time steps and with positive probability, the new configuration is distance zero from ξ' and so equal to ξ' . In particular, the process is irreducible. \square

Aperiodicity directly follows from Lemma 3.1.

Lemma 3.5 (aperiodicity). *The process (ξ_t) is aperiodic.*

Proof. Because the process is irreducible and the period is a class property, it suffices to find one configuration with period one. Fix $x \in \mathcal{V}_i$ and let ξ be any configuration such that

$$\xi(x) \leq 0 \quad \text{and} \quad \sum_{z \in \mathcal{V}_i} \xi(z) \mathbf{1}\{\xi(z) < 0\} = -R_i.$$

Then, according to Lemma 3.1, $p(\xi, \tau_{xy} \xi) = 0$ for all $(x, y) \in \mathcal{E}$, so whenever the chosen directed edge starts from vertex x , the process stays in configuration ξ . This implies that

$$p(\xi, \xi) \geq \deg(x)/2 \text{card}(\mathcal{E}) \geq 1/2 \text{card}(\mathcal{E}) > 0,$$

which proves that ξ has period one. \square

Irreducibility and aperiodicity, together with the fact that the state space S is finite, imply the existence and uniqueness of a stationary distribution π to which the process converges regardless of the initial configuration. Using reversibility and doubly stochasticity, we now prove that π is the uniform distribution on the set of configurations.

Lemma 3.6 (reversibility). *The stationary distribution $\pi = \text{Uniform}(S)$.*

Proof. To prove reversibility and doubly stochasticity, it suffices to prove that

$$p(\xi, \xi') = p(\xi', \xi) \quad \text{for all } \xi, \xi' \in S. \quad (3.7)$$

The result is obvious when $\xi' = \xi$. When $\xi' \neq \tau_{xy} \xi$ for all $(x, y) \in \mathcal{E}$, it follows from Lemma 3.1 that the two transition probabilities in (3.7) are equal to zero so the result is again true in this case. To deal with the remaining cases, assume that

$$\xi' = \tau_{xy} \xi \quad \text{for some } (x, y) \in \mathcal{E}.$$

Then, applying Lemma 3.2, we get

$$p(\xi, \xi') = p(\xi, \tau_{xy} \xi) > 0 \quad \text{implies that} \quad p(\xi', \xi) = p(\tau_{xy} \xi, \tau_{yx} \tau_{xy} \xi) > 0. \quad (3.8)$$

Since in addition the transition probabilities are equal to either zero or $1/2 \text{card}(\mathcal{E})$ according to Lemma 3.1, the two probabilities in (3.8) must be equal, which proves (3.7). In particular, $\text{Uniform}(S)$ is a stationary distribution so the lemma follows from the uniqueness of π . \square

4. NUMBER OF CONFIGURATIONS OF COINS

Because the stationary distribution is $\text{Uniform}(S)$, meaning that all the configuration are equally likely, the distribution of money at equilibrium, *i.e.*, the probability that a given vertex has c coins under the distribution π , is equal to the number of configurations with exactly c coins at that vertex divided by the total number of configurations. This section is devoted to counting those configurations, from which Theorem 2.1 follows. As a warming-up, we first compute the number of configurations in the absence of banks, namely, in the special case $R = 0$.

Lemma 4.1. *Assume that $R = 0$. Then, the number of configurations is*

$$\Lambda((N_i), (R_i), M) = \Lambda((N_i), (0, 0, \dots, 0), M) = \binom{M + N - 1}{N - 1}.$$

Proof. By definition, the number of configurations $\Lambda((N_i), (R_i), M)$ is equal to the number of integer solutions $\xi(x)$, $x \in \mathcal{V}$, to the equation

$$\sum_{x \in \mathcal{V}} \xi(x) = M \quad \text{such that} \quad \sum_{y \in \mathcal{V}_i} \xi(y) \mathbf{1}\{\xi(y) < 0\} \geq -R_i \quad \text{for all } i = 1, 2, \dots, K.$$

Because $R = 0$, and so $R_i = 0$ for all i , this is the number of integer solutions to

$$\sum_{x \in \mathcal{V}} \xi(x) = M \quad \text{such that} \quad \xi(x) \geq 0 \quad \text{for all } x \in \mathcal{V},$$

which is also the number of sets of M items, each of which can be of N different types (combination with repetition). This quantity is known to be $(M + N - 1 \text{ choose } N - 1)$. \square

Generalizing Lemma 4.1, we now compute the number of configurations in the presence of banks.

Lemma 4.2. *The total number of configurations is*

$$\begin{aligned} \Lambda((N_i), (R_i), M) &= \sum_{a_1=0}^{R_1} \cdots \sum_{a_K=0}^{R_K} \sum_{b_1=0}^{N_1} \cdots \sum_{b_K=0}^{N_K} \binom{N_1}{b_1} \cdots \binom{N_K}{b_K} \\ &\quad \binom{a_1 - 1}{b_1 - 1} \cdots \binom{a_K - 1}{b_K - 1} \binom{M + a_1 + \cdots + a_K + N - b_1 - \cdots - b_K - 1}{N - b_1 - \cdots - b_K - 1}. \end{aligned}$$

Proof. To compute $\text{card}(S)$, we denote by

$$\phi((a_i), (b_i), M) = \phi((a_1, \dots, a_K), (b_1, \dots, b_K), M)$$

the number of configurations with a_i coins borrowed from bank i , and b_i individuals in debt from that bank. Because b_i cannot exceed the number N_i of customers at bank i or the total number of coins a_i borrowed from that bank, summing over all the possible choices, we get

$$\text{card}(S) = \sum_{a_1=0}^{R_1} \cdots \sum_{a_K=0}^{R_K} \sum_{b_1=0}^{a_1 \wedge N_1} \cdots \sum_{b_K=0}^{a_K \wedge N_K} \phi((a_i), (b_i), M). \quad (4.1)$$

In addition, for each bank i ,

$$\# \text{ ways to choose the individuals in debt} = (N_i \text{ choose } b_i). \quad (4.2)$$

Letting x_1, x_2, \dots, x_{b_i} be these individuals, $A_k = -\xi(x_k)$ and $B_k = -\xi(x_k) - 1$, and assuming that $a_i, b_i > 0$, the number of ways to allocate the debt among these individuals is

$$\begin{aligned} & \# \text{ integer solutions to } A_1 + \dots + A_{b_i} = a_i \text{ with } A_k > 0 \\ & = \# \text{ integer solutions to } B_1 + \dots + B_{b_i} = a_i - b_i \text{ with } B_k \geq 0 \\ & = \binom{a_i - b_i + b_i - 1}{b_i - 1} = \binom{a_i - 1}{b_i - 1} \end{aligned} \quad (4.3)$$

as in the proof of Lemma 4.1. Similarly, the number of ways to distribute the remaining $M + a_1 + \dots + a_K$ coins among the other $N_1 + \dots + N_K - b_1 - \dots - b_K$ individuals is

$$\begin{aligned} & \binom{(M + a_1 + \dots + a_K) + (N_1 + \dots + N_K - b_1 - \dots - b_K) - 1}{(N_1 + \dots + N_K - b_1 - \dots - b_K) - 1} \\ & = \binom{M + a_1 + \dots + a_K + N - b_1 - \dots - b_K - 1}{N - b_1 - \dots - b_K - 1}. \end{aligned} \quad (4.4)$$

Combining (4.2)–(4.4), we deduce that

$$\begin{aligned} \phi((a_i), (b_i), M) & = \binom{N_1}{b_1} \dots \binom{N_K}{b_K} \binom{a_1 - 1}{b_1 - 1} \dots \binom{a_K - 1}{b_K - 1} \\ & \quad \times \binom{M + a_1 + \dots + a_K + N - b_1 - \dots - b_K - 1}{N - b_1 - \dots - b_K - 1}. \end{aligned} \quad (4.5)$$

Finally, combining (4.1) and (4.5), we get

$$\begin{aligned} \Lambda((N_i), (R_i), M) & = \sum_{a_1=0}^{R_1} \dots \sum_{a_K=0}^{R_K} \sum_{b_1=0}^{a_1 \wedge N_1} \dots \sum_{b_K=0}^{a_K \wedge N_K} \binom{N_1}{b_1} \dots \binom{N_K}{b_K} \\ & \quad \binom{a_1 - 1}{b_1 - 1} \dots \binom{a_K - 1}{b_K - 1} \binom{M + a_1 + \dots + a_K + N - b_1 - \dots - b_K - 1}{N - b_1 - \dots - b_K - 1}, \end{aligned}$$

which, using the convention $(-1 \text{ choose } k) = 1$ and $(n \text{ choose } k) = 0$ when $k < 0$ or $k > n$, is equal to the expression given in the statement of the lemma. \square

To deduce Theorem 2.1, assume that $x \in \mathcal{V}_j$ and observe that, given that there are c coins at vertex x , there is a total of $M - c$ coins scattered across the rest of the connected graph. In case x is not in debt, the customers of bank j other than x can altogether borrow R_j coins therefore the number of configurations such that $\xi(x) = c \geq 0$ is given by

$$\Lambda((N_1, \dots, N_{j-1}, N_j - 1, N_{j+1}, \dots, N_K), (R_i), M - c) = \Lambda((N_i) - e_j, (R_i), M - c). \quad (4.6)$$

In case x is in debt, the customers of bank j other than x can altogether borrow $R_j + c$ coins therefore the number of configurations such that $\xi(x) = c < 0$ is given by

$$\begin{aligned} & \Lambda((N_i) - e_j, (R_1, \dots, R_{j-1}, R_j + c, R_{j+1}, \dots, R_K), M - c) \\ & = \Lambda((N_i) - e_j, (R_i) + ce_j, M - c). \end{aligned} \quad (4.7)$$

The theorem follows from combining (4.6)–(4.7) and Lemma 4.1, and using that the stationary distribution π is uniform on the set of configurations according to Lemma 3.6.

5. CONVERGENCE TO THE LAPLACE DISTRIBUTION

This section is devoted to the proof of Theorem 2.2 which gives the distribution of money in the large population limit when the money temperature is large and

$$N_i = \text{card}(\mathcal{V}_i) = N/K \quad \text{and} \quad R_i = R/K \quad \text{for all } i = 1, 2, \dots, K. \quad (5.1)$$

The first ingredient to prove the theorem is to observe that, according to Lemma 3.6, the stationary distribution of the process (ξ_t) is the uniform distribution on the set of configurations *regardless of the topology of the graph*. In particular, the limiting distribution of money on any connected graph is the same as the limiting distribution of money on the complete graph, in which case the evolution is simply described by the Markov chain (u_t) that keeps track of the fraction of individuals who are customers at bank i and have c coins, but ignores their location. Using symmetry under the assumption (5.1), we can prove that the unique stationary distribution of this Markov chain is an asymmetric two-sided geometric distribution regardless of the population size and regardless of the money temperature (see Lem. 5.1). In the large population limit and when the money temperature is large, this two-sided geometric distribution approaches an asymmetric Laplace distribution characterized by three parameters. The second ingredient is to use that the total number of coins across the population is preserved by the dynamics and that the number of coins in each bank can be neglected (see Lem. 5.4) to identify these three parameters (see Lem. 5.6).

From now on, we assume that \mathcal{G} is the complete graph on N vertices. Because in this case the location of the individuals is unimportant, the process that only keeps track of the fraction of individuals with a given number of coins along with the bank they go to:

$$u_t = \{u_t(i, c) = \sum_{x \in \mathcal{V}_i} \mathbf{1}\{\xi_t(x) = c\}/N : 1 \leq i \leq K, -R/K \leq c \leq M + R\}$$

is a well-defined Markov chain. The ergodicity of the process (ξ_t) implies that the process (u_t) is ergodic as well, and thus converges to its unique stationary distribution, say $\bar{\pi}$, regardless of its initial state. To describe the dynamics and find $\bar{\pi}$, let

$$u_t^-(i) = \sum_{c < 0} u_t(i, c), \quad u_t^+(i) = \sum_{c > 0} u_t(i, c), \quad B_t(i) = R/K + \sum_{c < 0} cN u_t(i, c)$$

be the fraction of individuals who go to bank i and are in debt, the fraction of individuals who go to bank i and have at least one coin, and the number of coins in bank i . Assuming in addition the conditions (5.1), it follows from symmetry that the limits

$$\lim_{t \rightarrow \infty} u_t(i, c) = u(c)/K, \quad \lim_{t \rightarrow \infty} u_t^\pm(i) = u^\pm/K, \quad \lim_{t \rightarrow \infty} P(B_t(i) > 0) = p$$

do not depend on the choice of the bank i . We are now ready to prove that the stationary distribution $\bar{\pi}$ is an asymmetric two-sided geometric distribution.

Lemma 5.1. *Let $u_0 = \lim_{t \rightarrow \infty} P(\xi_t(x) = 0)$. Then,*

$$\lim_{t \rightarrow \infty} P(\xi_t(x) = c) = \bar{\pi}(c) = \begin{cases} u_0 \bar{u}^c & \text{when } c > 0, \\ u_0 (\bar{u}/p)^c & \text{when } c < 0, \end{cases}$$

where $\bar{u} = u^+ + p(u_0 + u^-)$.

Proof. To check that the distribution $\bar{\pi}$ given in the statement is indeed the distribution of money at equilibrium, it suffices to prove that, under this distribution,

$$P_{\bar{\pi}}(u_{t+1}(c) = u_t(c) + 1/N | u_t) = P_{\bar{\pi}}(u_{t+1}(c) = u_t(c) - 1/N | u_t) \quad \text{for all } c. \quad (5.2)$$

To compute these two transition probabilities, note that an individual can give a coin either because she has at least one coin or because her bank is not empty. In particular, the fraction of individuals who can give a coin at equilibrium is

$$\lim_{t \rightarrow \infty} \sum_i (u_t^+(i) + (u_t(i, 0) + u_t^-(i))P(B_t(i) > 0)) = u^+ + p(u_0 + u^-) = \bar{u}.$$

Using also that the number of individuals with c coins increases by one when

- a vertex with $c - 1$ coins gets a coin from a vertex not having c coins or
- a vertex with $c + 1$ coins gives a coin to a vertex not having c coins,

we deduce that, depending on the value of c ,

$$\begin{aligned} c < 0: \quad & P_{\bar{\pi}}(u_{t+1}(c) = u_t(c) + 1/N | u_t) \\ &= (\bar{u} - p\bar{\pi}(c)) \bar{\pi}(c-1) + p\bar{\pi}(c+1) (1 - \bar{\pi}(c)) \\ &= (\bar{u} - pu_0(\bar{u}/p)^c) u_0(\bar{u}/p)^{c-1} + pu_0(\bar{u}/p)^{c+1} (1 - u_0(\bar{u}/p)^c), \\ c = 0: \quad & P_{\bar{\pi}}(u_{t+1}(0) = u_t(0) + 1/N | u_t) \\ &= (\bar{u} - p\bar{\pi}(0)) \bar{\pi}(-1) + \bar{\pi}(1) (1 - \bar{\pi}(0)) \\ &= (\bar{u} - pu_0) u_0(p/\bar{u}) + u_0\bar{u}(1 - u_0), \\ c > 0: \quad & P_{\bar{\pi}}(u_{t+1}(c) = u_t(c) + 1/N | u_t) \\ &= (\bar{u} - \bar{\pi}(c)) \bar{\pi}(c-1) + \bar{\pi}(c+1) (1 - \bar{\pi}(c)) \\ &= (\bar{u} - u_0\bar{u}^c) u_0\bar{u}^{c-1} + u_0\bar{u}^{c+1} (1 - u_0\bar{u}^c). \end{aligned} \quad (5.3)$$

Similarly, the number of individuals with c coins decreases by one when

- a vertex with c coins gets a coin from a vertex not having $c + 1$ coins or
- a vertex with c coins gives a coin to a vertex not having $c - 1$ coins

therefore, depending on the value of c ,

$$\begin{aligned} c < 0: \quad & P_{\bar{\pi}}(u_{t+1}(c) = u_t(c) - 1/N | X_t) \\ &= (\bar{u} - p\bar{\pi}(c+1)) \bar{\pi}(c) + p\bar{\pi}(c) (1 - \bar{\pi}(c-1)) \\ &= (\bar{u} - pu_0(\bar{u}/p)^{c+1}) u_0(\bar{u}/p)^c + pu_0(\bar{u}/p)^c (1 - u_0(\bar{u}/p)^{c-1}), \\ c = 0: \quad & P_{\bar{\pi}}(u_{t+1}(0) = u_t(0) - 1/N | X_t) \\ &= (\bar{u} - \bar{\pi}(1)) \bar{\pi}(0) + p\bar{\pi}(0) (1 - \bar{\pi}(-1)) \\ &= (\bar{u} - u_0\bar{u}) u_0 + pu_0 (1 - u_0(p/\bar{u})), \\ c > 0: \quad & P_{\bar{\pi}}(u_{t+1}(c) = u_t(c) - 1/N | u_t) \\ &= (\bar{u} - \bar{\pi}(c+1)) \bar{\pi}(c) + \bar{\pi}(c) (1 - \bar{\pi}(c-1)) \\ &= (\bar{u} - u_0\bar{u}^{c+1}) u_0\bar{u}^c + u_0\bar{u}^c (1 - u_0\bar{u}^{c-1}). \end{aligned} \quad (5.4)$$

One easily checks that the three expressions for the transition probabilities in (5.3) are equal to their counterparts in (5.4), which shows (5.2) and completes the proof of the lemma. \square

Even though the number of coins each individual has is an integer, and the distribution of money is appropriately described by a discrete random variable, to follow the physics literature (and also simplify the calculations of some key parameters in the next lemmas), we take the large population limit and the money temperature large, in which case the asymmetric two-sided geometric distribution $\bar{\pi}$ in the previous lemma is well-approximated by its continuous analog: an asymmetric Laplace distribution with probability density function f of the form

$$f(c) = \begin{cases} \mu e^{-ac} & \text{when } c \in \mathbb{R}_+, \\ \mu e^{+bc} & \text{when } c \in \mathbb{R}_-, \end{cases} \quad \text{for some } \mu, a, b > 0.$$

To complete the proof of the theorem, the last step is to find explicit expressions of the three parameters μ, a, b by deriving a system of three equations involving these three parameters. The first two equations are obtained in the next lemma by using that f is a density function and that the money temperature is preserved by the dynamics.

Lemma 5.2. *In the limit as $N \rightarrow \infty$, we have $\mu/a + \mu/b = 1$ and $\mu/a^2 - \mu/b^2 = T$.*

Proof. To begin with, observe that

$$\int_{-\infty}^{\infty} f(c) dc = \int_0^{\infty} \mu e^{-ac} dc + \int_0^{\infty} \mu e^{-bc} dc = \mu/a + \mu/b.$$

Because the function f is a density function, the integral above must be equal to one, which gives the first equation in the lemma. To derive the second equation, observe that

$$\begin{aligned} \int_{-\infty}^{\infty} cf(c) dc &= \int_0^{\infty} \mu ce^{-ac} dc - \int_0^{\infty} \mu ce^{-bc} dc \\ &= \int_0^{\infty} \frac{\mu e^{-ac}}{a} dc - \int_0^{\infty} \frac{\mu e^{-bc}}{b} dc = \mu/a^2 - \mu/b^2. \end{aligned}$$

Because the system is conservative and the money temperature is preserved by the dynamics, the integral above, which represents the mean number of coins per individual at equilibrium, must be equal to the money temperature T . This proves the second equation. \square

The third equation involving μ, a, b expresses the fact that, at equilibrium, the total number of coins shared by the individuals not in debt is of the order of $M + R$, the total number of coins in the system, which holds because the number of coins in each of the banks in the large population limit becomes much smaller than R . The number of coins in the banks has a negative drift that scales like the density $u_0(t)$ so, to show that the number of coins in the banks indeed becomes negligible, the next step is to prove that, at equilibrium, the fraction of individuals with exactly zero coin is not too small, of the order of the reciprocal of the money temperature.

Lemma 5.3. *In the limit as $N \rightarrow \infty$, there exists $C_0 > 0$ such that $\mu \geq C_0/T$.*

Proof. Let $\delta = b/a > 1$. Using the second then the first equation in Lemma 5.2, we get

$$\begin{aligned} T &= \mu/a^2 - \mu/b^2 = \mu(1/a + 1/b)(1/a - 1/b) \\ &= 1/a - 1/b = 1/a - 1/\delta a = (1 - 1/\delta)(1/a). \end{aligned}$$

In particular, using again the first equation, we get $a = (1 - 1/\delta)/T$ and

$$\mu = \frac{1}{1/a + 1/b} = \frac{ab}{a + b} = \frac{(1 - 1/\delta)(\delta - 1)}{(1 - 1/\delta) + (\delta - 1)} \frac{1}{T} = \frac{(\delta - 1)^2}{\delta^2 - 1} \frac{1}{T} = \left(\frac{\delta - 1}{\delta + 1} \right) \frac{1}{T}$$

which proves the lemma. \square

Using Lemma 5.3, we can now prove that, at equilibrium, the number of coins in each of the banks is much smaller than R , the initial number of coins in the banks.

Lemma 5.4. *In the limit as $N \rightarrow \infty$, $\lim_{t \rightarrow \infty} B_t(i)/R \rightarrow 0$.*

Proof. The number of coins in bank i increases by one each time an individual with at least one coin and/or not in bank i gives a coin to a customer of bank i who is in debt therefore

$$P_{\bar{\pi}}(B_{t+1}(i) = B_t(i) + 1 | u_t) = (1 - u^-/K) u^-/K. \quad (5.5)$$

Similarly, the number of coins in bank i decreases by one each time a customer of bank i with no coins or in debt gives a coin to an individual not in bank i or not in debt therefore

$$P_{\bar{\pi}}(B_{t+1}(i) = B_t(i) - 1 | u_t, B_t(i) > 0) = (1 - u^-/K)(u_0 + u^-)/K. \quad (5.6)$$

The transition probabilities (5.5) and (5.6) show that, once there is at least one individual in debt, the process $(B_t(i))$ is dominated by a one-dimensional symmetric random walk with reflecting boundary at zero so the process reaches zero after a time of the order of at most R^2 . We now look at the time it takes for bank i to return away from state zero. Because $\mu \geq C_0/T$ by Lemma 5.3, near the stationary distribution, $u_0 \geq \mu/2 = C_0/2T$ and

$$\frac{(1 - u^-/K)(u_0 + u^-)/K}{(1 - u^-/K) u^-/K} = \frac{u_0 + u^-}{u^-} \geq 1 + u_0 \geq 1 + C_0/2T.$$

In particular, letting $\tau = \inf\{t : B_t(i) = 0 \text{ or } B_t(i) > \sqrt{R_i T}\}$, recalling (5.5) and (5.6), and applying the optional stopping theorem, we deduce that

$$\begin{aligned} P(B_{\tau}(i) > \sqrt{R_i T} | B_0(i) = 1) &\leq \frac{1 - (1 + C_0/2T)}{1 - (1 + C_0/2T)\sqrt{R_i T}} \leq \left(\frac{1}{1 + C_0/2T} \right)^{\sqrt{R_i T}-1} \\ &\leq \left(1 - \frac{C_0}{4T} \right)^{\sqrt{R_i T}-1} \approx \exp\left(-C_0 \sqrt{\frac{R_i}{4T}} \right) = \exp(-C_0 \sqrt{\rho_i N}/2). \end{aligned}$$

This, together with the strong Markov property, implies that the expected amount of time for the number of coins in bank i to return above the threshold $\sqrt{R_i T}$ is larger than

$$E(\text{Geometric}(\exp(-C_0 \sqrt{\rho_i N}/2))) = \exp(C_0 \sqrt{\rho_i N}/2).$$

Observing also that $\sqrt{R_i T}/R \rightarrow 0$ in the limit as $N \rightarrow \infty$, the result follows. \square

Using the previous lemma, we can now derive a third equation involving the parameters of the asymmetric Laplace distribution.

Lemma 5.5. *In the limit as $N \rightarrow \infty$, we have $\mu/a^2 = (1 + \rho)T$.*

Proof. According to Lemma 5.4, in the large population limit and at equilibrium, the total number of coins in all the banks is much smaller than R , so the number of coins shared by all the individuals not in debt converges to $M + R$. In particular, in the limit as $N \rightarrow \infty$,

$$\int_0^\infty cf(c) dc = \frac{M + R}{N} = (1 + \rho)T.$$

Because the left-hand side is also equal to

$$\int_0^\infty cf(c) dc = \int_0^\infty \mu ce^{-ac} dc = \int_0^\infty \frac{\mu e^{-ac}}{a} dc = \mu/a^2,$$

the lemma follows. \square

The final step to prove the theorem is to use the three equations in Lemmas 5.2 and 5.5 to express the three parameters μ, a, b of the asymmetric Laplace distribution as a function of the money temperature T and the fraction of coins from the banks ρ .

Lemma 5.6. *In the limit as $N \rightarrow \infty$,*

$$\mu = \frac{1}{T} \left(\sqrt{1 + \rho} - \sqrt{\rho} \right)^2, \quad a = \frac{1}{T} \left(1 - \sqrt{\frac{\rho}{1 + \rho}} \right), \quad b = \frac{1}{T} \left(\sqrt{\frac{1 + \rho}{\rho}} - 1 \right).$$

Proof. Recall from Lemmas 5.2 and 5.5 that

$$\mu/a + \mu/b = 1, \quad \mu/a^2 - \mu/b^2 = T, \quad \mu/a^2 = (1 + \rho)T. \quad (5.7)$$

Subtracting the second equation from the third equation in (5.7), then taking the square root of the ratio with the third equation, we get

$$\frac{\mu}{b^2} = \frac{\mu}{a^2} - \left(\frac{\mu}{a^2} - \frac{\mu}{b^2} \right) = \rho T \quad \text{and} \quad \frac{a}{b} = \sqrt{\frac{\mu}{b^2}} / \sqrt{\frac{\mu}{a^2}} = \sqrt{\frac{\rho}{1 + \rho}}. \quad (5.8)$$

Using the first two equations in (5.7), we also have

$$1/a - 1/b = (\mu/a + \mu/b)(1/a - 1/b) = \mu/a^2 - \mu/b^2 = T. \quad (5.9)$$

From (5.9) and the last equation in (5.8), we deduce that

$$\begin{aligned} \frac{1}{a} - \frac{1}{b} &= \frac{1}{a} \left(1 - \sqrt{\frac{\rho}{1 + \rho}} \right) = T \quad \text{so} \quad a = \frac{1}{T} \left(1 - \sqrt{\frac{\rho}{1 + \rho}} \right), \\ \frac{1}{a} - \frac{1}{b} &= \frac{1}{b} \left(\sqrt{\frac{1 + \rho}{\rho}} - 1 \right) = T \quad \text{so} \quad b = \frac{1}{T} \left(\sqrt{\frac{1 + \rho}{\rho}} - 1 \right). \end{aligned}$$

Finally, using the last equation in (5.7) and the expression of a ,

$$\mu = \frac{1+\rho}{T} \left(1 - \sqrt{\frac{\rho}{1+\rho}}\right)^2 = \frac{1}{T} \left(\sqrt{1+\rho} - \sqrt{\rho}\right)^2.$$

This completes the proof. \square

From the previous lemma, we can also deduce that, at equilibrium, the fraction of individuals with at least one coin and the fraction of individuals in debt are given by

$$u^+ = \int_0^\infty \mu e^{-ac} dc = \frac{\mu}{a} = \frac{\sqrt{1+\rho}(\sqrt{1+\rho} - \sqrt{\rho})^2}{\sqrt{1+\rho} - \sqrt{\rho}} = \sqrt{1+\rho} (\sqrt{1+\rho} - \sqrt{\rho}),$$

$$u^- = \int_0^\infty \mu e^{-bc} dc = \frac{\mu}{b} = \frac{\sqrt{\rho}(\sqrt{1+\rho} - \sqrt{\rho})^2}{\sqrt{1+\rho} - \sqrt{\rho}} = \sqrt{\rho} (\sqrt{1+\rho} - \sqrt{\rho}).$$

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