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(3+1)-DIMENSIONAL GARDNER EQUATION DEFORMED FROM (1+1)-DIMENSIONAL GARDNER EQUATION AND ITS CONSERVATION LAWS

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Abstract. Through the application of the deformation algorithm, a novel (3+1)-dimensional Gardner equation and its associated Lax pair are derived from the (1+1)-dimensional Gardner equation and its conservation laws. As soon as the (3+1)-dimensional Gardner equation is set to be y or z independent, the Gardner equations in (2+1)-dimension are also obtained. To seek the exact solutions for these higher dimensional equations, the traveling wave method and the symmetry theory are introduced. Hence, the implicit expressions of traveling wave solutions to the (3+1)-dimensional and (2+1)-dimensional Gardner equations, the Lie point symmetry and the group invariant solutions to the (3+1)-dimensional Gardner equation are well investigated. In particular, after selecting some specific parameters, both the traveling wave solutions and the symmetry reduction solutions of hyperbolic function form are given.

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1. Introduction

Integrable systems have extensive applications in almost all branches of physics, such as condensed matter physics, plasma physics, optics, particle physics, *etc.* as well as in other branches of science such as chemistry, communications and biology. Some representative integrable systems include the Korteweg–de Vries (KdV) equation for investigating the shallow water waves in the ocean [1, 2], the nonlinear Schrödinger (NLS) equation for describing the transmission of optical solitons in optical fibers[3, 4], the sine-Gordon (sG) equation for modeling the propagation of fluxons in a junction between two superconductors [5], and so on. Usually, the classical nonlinear integrable equations are (1+1)-dimensional. However, when higher dimensional physical phenomena are considered, one has to extend (1+1)-dimensional nonlinear equations to higher dimensional ones. For example, while the beams traveling in higher dimensional space are taken into account, the (1+1)-dimensional NLS equations are then extended to (2+1)- or even (3+1)-dimensional ones [6, 7], the Nizhnik–Novikov–Veselov (NNV) equation and the Davey–Stewartson (DS) equation are well known the (2+1)-dimensional generalizations of the KdV equation and the NLS equation [8, 9]. Wang *et al.* have derived (2+1)-dimensional sG equation from the

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Ablowitz–Kaup–Newell–Segur (AKNS) system [10]. The propagation of gravity waves over the water surface, in particular the head-on collision of oblique waves have been simulated by (3+1)-dimensional Boussinesq equation [11].

Up to now, for the sake of constructing higher dimensional nonlinear equations, several effective methods have been proposed. For instance, using the Miura-type transformation, the (1+1)- and (2+1)-dimensional sG equations and the Mikhailov–Dodd–Bullough (MDB) equation were obtained from the (0+1)-dimensional Riccati equation [12]. Through the Painlevé analysis, some explicit (3+1)-dimensional integrable equations were obtained from the (2+1)-dimensional Kadomtsev–Petviashvili (KP) equation, the NLS equation, and the Schwarz KdV equation[13]. By complexifying the independent variables, the integrable generalizations in (4+2)- and (3+1)-dimension were presented from the (2+1)-dimensional KP and DS equations [14]. Very recently, to acquire more higher dimensional integrable systems, Lou *et al.* proposed another new powerful method, *i.e.* the deformation algorithm, which makes it possible for us to transform arbitrary lower dimensional integrable systems to higher dimensional ones [15–17]. And the correctness of the deformation algorithm has also been strictly proved [18].

In this article, starting from the (1+1)-dimensional Gardner equation, we try to derive its Lax integrable generalization in (3+1)-dimension *via* the deformation algorithm uniting its conservation laws, and then obtain the exact solutions, including the traveling wave solutions and the symmetry reduction solutions, for this (3+1)-dimensional Gardner equation. In Section 2, we employ the deformation algorithm to get the (3+1)-dimensional Gardner equation and its Lax pairs. Specifically, by setting (3+1)-dimensional Gardner equation to be y or z independent respectively, the nonlinear evolution forms for two (2+1)-dimensional Gardner equations are given at the same time. Section 3 is devoted to searching for the traveling wave solution of the (3+1)-dimensional Gardner equation, as well as those of (2+1)-dimensional equations in the cases of $y = 0$ or $z = 0$. Section 4 focuses on exploring the Lie point symmetry of the (3+1)-dimensional Gardner equation and obtaining the group invariant solutions through the symmetry reduction technique. The last section is allotted for some conclusions.

2. (3+1)-dimensional Gardner equation and its Lax pairs

In this study, we shall extend the (1+1)-dimensional Gardner equation of the following form [19]

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} = 0, \tag{2.1}$$

where $u = u(x,t)$ stands for the wave function, and x and t denote the spatial and temporal coordinate, respectively. In most applications, u represents the perturbation of an isopycnal surface of the relevant wave mode, the terms uu_x and u^2u_x symbol the nonlinear wave steepening and the third-order derivative term u_{xxx} characterizes the dispersive wave effects. The coefficients of the nonlinear terms α and β as well as the dispersive term γ are determined by the steady oceanic background density and flow stratification through the linear eigenmode of the internal waves [20–22]. In addition, the Gardner equation has also been widely used in other fields of physics, such as simulating the propagation of dust ion acoustic waves with isothermal electrons [23] and modeling various nonlinear wave phenomena in solids [24]. Due to the favorable integrability, many exact solutions of the Gardner equation have been well constructed by various mathematical physics methods. For example, the explicit analytic solutions related to the Jacobi elliptic functions have been given by bringing the nonlocal symmetry combining the symmetry reduction technique into the Gardner equation [25]. In Ref. [26], by the application of the modified double sub-equation method, authors derived analytical exact solutions for Gardner equation. Wang obtained traveling wave solutions of the Gardner equation by the exp-function method [27]. Using three effective integration techniques abundant solitary wave solutions of Gardner equation have been provided [28].

From the (1+1)-dimensional Gardner equation (2.1), one can find two conservation laws

$$\begin{aligned} u_t &= (-\frac{\alpha}{2}u^2 - \frac{\beta}{3}u^3 - \gamma u_{xx})_x, \\ (u^2)_t &= (-\frac{2\alpha}{3}u^3 - \frac{\beta}{2}u^4 - 2\gamma u u_{xx} + \gamma u_x^2)_x. \end{aligned} \tag{2.2}$$

The corresponding field derivative independent conservation densities are given as $\rho = u$ and $\rho = u^2$, and the conservation flows that depend on the field derivatives are

$$\begin{aligned} J_1 &= -\frac{\alpha}{2}u^2 - \frac{\beta}{3}u^3 - \gamma u_{xx}, \\ J_2 &= -\frac{2\alpha}{3}u^3 - \frac{\beta}{2}u^4 - 2\gamma u u_{xx} + \gamma u_x^2. \end{aligned} \tag{2.3}$$

By utilizing the deformation conjecture proposed in the literature [17], the (1+1)-dimensional Gardner equation (2.1) can be transformed to the (3+1)-dimensional one

$$\hat{T}u = -\alpha u \hat{L}u - \beta u^2 \hat{L}u - \gamma \hat{L}^3 u, \tag{2.4}$$

where the deformed operators $\hat{L}$ and $\hat{T}$ take the forms

$$\hat{L} = \partial_x + u\partial_y + u^2\partial_z, \quad \hat{T} = \partial_t + \bar{J}_1\partial_y + \bar{J}_2\partial_z \tag{2.5}$$

with the deformed flows

$$\begin{aligned} \bar{J}_1 &= -\frac{\alpha}{2}u^2 - \frac{\beta}{3}u^3 - \gamma \hat{L}^2 u, \\ \bar{J}_2 &= -\frac{2\alpha}{3}u^3 - \frac{\beta}{2}u^4 - 2\gamma u \hat{L}^2 u + r(\hat{L}u)^2. \end{aligned} \tag{2.6}$$

After expanding the deformed operators, the usual form of nonlinear evolution equation for equation (2.4) with (2.5) and (2.6) is

$$\begin{aligned} u_t =& -\gamma u_{xxx} - 3\gamma u u_{xxy} - 3\gamma u^2 u_{xxz} - 3\gamma u^2 u_{xyy} - 6\gamma u^3 u_{xyz} \\ & - 3\gamma u^4 u_{xzz} - \gamma u^3 u_{yyy} - 3\gamma u^4 u_{yyz} - 3\gamma u^5 u_{yzz} - \gamma u^6 u_{zzz} \\ & - 3\gamma(u_x + u u_y + u^2 u_z)[u_{xy} + 2u(u^2 u_{zz} + \frac{3}{2}u u_{yz} + \frac{1}{2}u_{yy} \\ & + u_{xz})] - 3\gamma u^4 u_z^3 - 6\gamma u^2 u_z^2 (u u_y + u_x) - \frac{1}{6}u_z(18\gamma u^2 u_y^2 \\ & + 3\beta u^4 + 36\gamma u u_x u_y + 2\alpha u^3 + 18\gamma u_x^2) - u_x(\beta u^2 + \alpha u) \\ & - \frac{2}{3}u^2 u_y(\beta u + \frac{3}{4}\alpha). \end{aligned} \tag{2.7}$$

Unsurprisingly, equation (2.7) will reduce to the ordinary (1+1)-dimensional Gardner equation (2.1) once u does not depend on variables y and z. And when u is y-independent, the (3+1)-dimensional Gardner equation (2.7) turns to the (2+1)-dimensional one

$$\begin{aligned} u_t =& -\gamma u_{xxx} - 3\gamma u^2 u_{xxz} - 3\gamma u^4 u_{xzz} - \gamma u^6 u_{zzz} - 6\gamma u u_{xz}(u_x + u^2 u_z) \\ & - 6\gamma u^3 u_{zz}(u_x + u^2 u_z) - 3\gamma u^4 u_z^3 - 6\gamma u^2 u_x u_z^2 - \frac{1}{6}u_z(3\beta u^4 + 2\alpha u^3 \\ & + 18\gamma u_x^2) - u u_x(\alpha + \beta u). \end{aligned} \tag{2.8}$$

Analogously, as soon as the dependent variable u does not depend on z, equation (2.7) degenerates into another (2+1)-dimensional Gardner equation

$$\begin{aligned} u_t = & -\gamma u_{xxx} - 3\gamma u u_{xxy} - 3\gamma u^2 u_{xyy} - \gamma u^3 u_{yyy} - 3\gamma u_{xy}(u_x + u u_y) \\ & - 3u[\gamma u_{yy}(u u_y + u_x) + \frac{1}{3}u_x(\beta u + \alpha) + \frac{2}{9}u u_y(\beta u + \frac{3}{4}\alpha)]. \end{aligned} \tag{2.9}$$

To check the Lax integrability of the (3+1)-dimensional Gardner equation, we try to bring in the similar deformation algorithm to the Lax pair of the original (1+1)-dimensional equation (2.1)

$$\begin{aligned} & (\partial_x^2 + \frac{2\beta u + \alpha}{m\gamma}\partial_x)\psi \equiv M\psi = 0, \\ & (\partial_t - \frac{\alpha}{m}\partial_x^2 + \frac{\beta u^2 + 2\alpha u + m\gamma u_x}{3}\partial_x)\psi \equiv N\psi = 0, \end{aligned} \tag{2.10}$$

where $m = \pm\sqrt{-\frac{6\beta}{\gamma}}$ and the compatibility condition $[M, N] = MN - NM = 0$ is just the Gardner equation (2.1), then it follows

$$\begin{aligned} & [(\partial_x + u\partial_y + u^2\partial_z)^2 + \frac{2\beta u + \alpha}{m\gamma}(\partial_x + u\partial_y + u^2\partial_z)]\psi \equiv \hat{M}\psi = 0, \\ & [\partial_t - (\gamma u_{xx} + 2\gamma u u_{xy} + 2\gamma u^2 u_{xz} + \gamma u^2 u_{yy} + 2\gamma u^3 u_{yz} + \gamma u^4 u_{zz} \\ & + \gamma u u_y^2 + 3\gamma u^2 u_y u_z - \gamma u_x u_y + 2\gamma u^3 u_z^2 + 2\gamma u u_x u_z + \frac{\beta}{3}u^3 + \frac{\alpha}{2}u^2)\partial_y \\ & - (2\gamma u u_{xx} + 4\gamma u^2 u_{xy} + 4\gamma u^3 u_{xz} + 2\gamma u^3 u_{yy} + 4\gamma u^4 u_{yz} + 2\gamma u^5 u_{zz} \\ & + 3\gamma u^4 u_z^2 + 4\gamma u^3 u_y u_z - 2\gamma u^2 u_x u_z + \gamma u^2 u_y^2 + \frac{\beta}{2}u^4 + \frac{2\alpha}{3}u^3 - \gamma u_x^2)\partial_z \\ & - \frac{\alpha}{m}(\partial_x + u\partial_y + u^2\partial_z)^2 + \frac{\beta u^2 + 2\alpha u}{3} + \frac{m\gamma}{3}(\partial_x + u\partial_y + u^2\partial_z)u]\psi \\ & \equiv \hat{N}\psi = 0. \end{aligned} \tag{2.11}$$

It is straightforward to prove that the integrable condition

$$[\hat{M}, \hat{N}] = 0 \tag{2.12}$$

is just the (3+1)-dimensional Gardner equation (2.7).

3. Traveling wave solutions

Due to the higher dimensionality and local integrability of the (3+1)-dimensional Gardner equation (2.7), it is hard to search for the explicit solutions of the equation in the frame of traditional mathematical physics methods, such as Darboux transformation, Hirota bilinear method, Bäcklund transformation, *etc.* Under the circumstances, we shall first consider seeking the exact solution to the (3+1)-dimensional Gardner equation by making use of the most commonly used traveling wave approach.

For the (3+1)-dimensional Gardner equation (2.7), the traveling wave solution holds the form

$$u = U(X), \quad X = kx + py + qz + \omega t \tag{3.1}$$

with k, p, q and ω being four real constants to be determined. By substituting solution (3.1) into the (3+1)-dimensional Gardner equation (2.7), the traveling wave reduction equation is

$$\begin{aligned}&6\gamma(qU^2+pU+k)^3U_{XXX}+U_X[18q\gamma(qU^2+pU+k)^2U_X^2\\&+18\gamma(2qU+p)(qU^2+pU+k)^2U_{XX}+3q\beta U^4+2(q\alpha+2p\beta)U^3\\&+3(p\alpha+2k\beta)U^2+6k\alpha U+6\omega]=0.\end{aligned}\tag{3.2}$$

With the help of Maple, it is found that the solution of equation (3.2) can be expressed by the elliptic integral

$$\int^U \frac{q^2\sqrt{6\gamma}(qf^2+pf+k)}{\sqrt{-\beta q^4 f^4-2\alpha q^4 f^3+\alpha_1 f^2+\alpha_2 f+\alpha_3}}\,\mathrm{d}f=\pm(X+X_0),\tag{3.3}$$

where

$$\begin{aligned}\alpha_1&=-6C_2\gamma q^4-\alpha pq^3-3\beta kq^3+\beta p^2q^2,\\\alpha_2&=12C_1\gamma q^4+8\alpha kq^3-2\alpha p^2q^2-8\beta kpq^2+2\beta p^3q,\\\alpha_3&=(6C_1\gamma p+6C_2\gamma k-6\omega)q^3+(5\alpha kp+3\beta k^2)q^2\\&\quad-(\alpha p^3+5\beta kp^2)q+\beta p^4,\end{aligned}$$

and C_1, C_2, X_0 are all arbitrary integral constants. Choosing suitable arbitrary parameters enables us to give the expressions for the particular solutions of the (3+1)-dimensional Gardner equation. For example, the selecting of parameters as

$$\begin{aligned}&b_1=\sqrt{6\gamma}q^3,\quad b_2=\sqrt{6\gamma}pq^2,\quad b_3=\frac{6\sqrt{6\gamma}q^2\omega}{2\alpha-\beta},\\&b_4=\frac{b_7(2\alpha-\beta)}{\beta},\quad b_5=-\frac{q^4\beta}{b_6b_7},\quad k=\frac{6\omega}{2\alpha-\beta},\\&C_1=\frac{(q\alpha-p\beta)(2\alpha p^2-\beta p^2-24q\omega)}{6\gamma q^3(2\alpha-\beta)},\\&C_2=\frac{2\alpha-\beta}{6\gamma}-\frac{2\alpha^2p-\alpha\beta p+18\beta\omega}{6\gamma q(2\alpha-\beta)}+\frac{p^2\beta}{6\gamma q^2}\end{aligned}$$

leads the traveling wave solution of equation (2.7)

$$\begin{aligned}\sqrt{\frac{b_4(U+1)}{b_7U+b_4}}=&\tanh[\frac{\pm b_7^2\sqrt{b_4b_5b_6}(X+X_0)}{b_4\alpha_4+2b_3b_7^2}+\frac{\alpha_4(b_4-b_7)E_\pi}{b_4\alpha_4+2b_3b_7^2}\\&+\frac{b_1b_7\sqrt{b_4(U+1)(b_7U+b_4)}}{b_4\alpha_4+2b_3b_7^2}]\end{aligned}\tag{3.4}$$

with $\alpha_4=b_1b_7+b_1b_4-2b_2b_7$, p, q, ω, X_0 still holding their arbitrariness and the incomplete elliptic integral $E_\pi\equiv E_\pi(\sqrt{\frac{b_4(U+1)}{b_7U+b_4}},\frac{b_7}{b_4},1)$.

From the implicit expression (3.4), it is known that the shape of the kink is modulated by the small changes of the parameter ω. When $\omega=-\frac{1}{20}$, the kink of the Gardner equation varies slowly with the axis X, see Figure 1a, where other parameters are selected as $\alpha=-3$, $\gamma=-\beta=1$, $p=q=\frac{1}{5}$ and $X_0=0$. Figure 1b exhibits the kink expressed by (3.4) with the parameters $\omega=-\frac{1}{25}$, and other parameters are fixed as the same as those in Figure 1a. As ω decreases further, the kink deforms to a multi-valued kink, which is shown in Figure 1c.

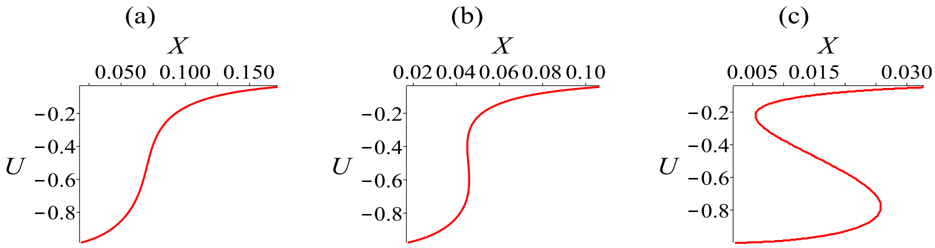

FIGURE 1. (a) A slowly varying kink profile of (3.4) with $\omega = -\frac{1}{20}$. (b) A kink solution of (3.4) with $\omega = -\frac{1}{25}$. (c) A folded kink under the small parameter selections $\omega = -\frac{1}{35}$. Other parameters are $\alpha = -3$, $\gamma = -\beta = 1$, $p = q = \frac{1}{5}$ and $X_0 = 0$.

Next, when we continue to apply the traveling wave method to (2+1)-dimensional Gardner equations (2.8) and (2.9), we have the traveling wave solutions

$$\int^U \frac{\sqrt{6}\gamma(pf^2+k)p}{\sqrt{-\gamma(\beta p^2 f^4 + 2\alpha p^2 f^3 + \beta_1 f^2 + \beta_2 f + \beta_3)}}\, \mathrm{d}f = \pm(X + X_0) \tag{3.5}$$

with

$$\begin{aligned} u &= U(X), \quad X = kx + qz + \omega t, \\ \beta_1 &= 12C_2\gamma p^3 + 3\beta kp, \\ \beta_2 &= -12C_1\gamma p^2 - 8\alpha kp, \\ \beta_3 &= -12kC_2\gamma p^2 - 3\beta k^2 + 6p\omega \end{aligned} \tag{3.6}$$

for equation (2.8) and

$$\int^U -\frac{\sqrt{6}\gamma(qf+k)q^2}{\sqrt{-\gamma(\beta q^4 f^4 + 2\alpha q^4 f^3 + \gamma_1 f^2 + \gamma_2 f + \gamma_3)}}\, \mathrm{d}f = \pm(X + X_0) \tag{3.7}$$

with

$$\begin{aligned} u &= U(X), \quad X = kx + qy + \omega t, \\ \gamma_1 &= 6C_2\gamma q^5 + 3\alpha kq^3 - 2\beta k^2 q^2, \\ \gamma_2 &= 12C_2\gamma kq^4 + 6\alpha k^2 q^2 - 4\beta k^3 q - 12\omega q^3, \\ \gamma_3 &= 6C_2\gamma k^2 q^3 - 6C_1\gamma q^3 + 3\alpha k^3 q - 2\beta k^4 - 6k\omega q^2 \end{aligned} \tag{3.8}$$

for equation (2.9). More specifically, when C_1, C_2, α, β and ω in solution (3.5) with (3.6) are selected as

$$\begin{aligned} C_1 &= \frac{p^2(c_3c_6 + c_4c_5)(b_1c_2 + 4b_3c_1)}{2b_1^3}, \\ C_2 &= -\frac{p(b_1c_1c_4c_6 + b_1c_2c_3c_5 - 3b_3c_1c_3c_5)}{2b_1^3}, \\ \alpha &= -\frac{3\gamma c_1 p^2(c_3c_6 + c_4c_5)}{b_1^2}, \\ \beta &= -\frac{6c_1c_3c_5\gamma p^2}{b_1^2}, \quad k = \frac{b_3 p}{b_1}, \\ \omega &= -\frac{\gamma p^3(b_1c_2c_4c_6 + b_3c_1c_4c_6 + b_3c_2c_3c_5)}{b_1^3}, \end{aligned}$$

the expression for the travelling solution of the (2+1)-dimensional Gardner equation (2.8) is

$$\begin{aligned} &c_5\sqrt{-c_2c_1}[m_1m_4(-c_1c_6^2E_\pi+(c_1c_6^2+c_2c_5^2)(E_e-E_f))U^2+c_6(c_5U+c_6)(c_1U^2+c_2)] \\ &+c_1^2c_6^3m_1m_4E_\pi U^2=\frac{[c_5^2c_6\sqrt{-c_1c_2^3c_3p^4U(c_5U+c_6)(c_1U^2+c_2)}(X+X_0)]}{2p^2b_3} \end{aligned} \tag{3.9}$$

with incomplete elliptic integrals $E_\pi \equiv E_\pi(m_1,m_2,m_3)$, $E_e \equiv E_e(m_1,m_3)$, $E_f \equiv E_f(m_1,m_3)$ and

$$\begin{aligned} m_1 &= \sqrt{\frac{\sqrt{-c_1c_2}(c_5U+c_6)}{(c_5\sqrt{-c_1c_2}-c_1c_6)\,U}}, \\ m_2 &= \frac{c_5\sqrt{-c_1c_2}-c_1c_6}{c_5\sqrt{-c_1c_2}}, \\ m_3 &= \sqrt{\frac{c_5\sqrt{-c_1c_2}-c_1c_6}{c_5\sqrt{-c_1c_2}+c_1c_6}}, \\ m_4 &= \frac{c_6(c_1U+\sqrt{-c_1c_2})}{(c_1c_6-\sqrt{-c_1c_2}c_5)U}. \end{aligned}$$

Besides, once the arbitrary parameters C_1, α, β and ω in equation (3.7) with (3.8) are fixed as

$$\begin{aligned} C_1 = 0,\quad C_2 &= -\frac{c_2c_4c_6q}{b_3^2}, \\ \alpha &= -\frac{6c_1c_4c_6\gamma kq}{b_3^2}, \\ \beta &= -\frac{6c_1c_4c_6\gamma q^2}{b_3^2}, \\ \omega &= -\frac{c_1c_4c_6\gamma k^3}{b_3^2}, \end{aligned}$$

one can obtain the traveling wave solution for equation (2.9)

$$\pm\frac{n_2(\eta+\eta_0)(qU+k)}{q^2}=\ln\left(\frac{2c_2q^4c_5c_3U+q^4(c_3c_6+c_4c_5)c_2+2n_1b_3^2n_2}{2n_2b_3^2}\right) \tag{3.10}$$

with $n_1=\sqrt{c_2(c_5U+c_6)(c_3U+c_4)}\frac{q^2}{b_3}$ and $n_2=\sqrt{c_2c_3c_5}\frac{q^2}{b_3}$. Here $b_i(i=1...3)$ and $c_j(j=1...6)$ are all arbitrary constants.

4. Lie point symmetry and symmetry reduction solutions

Among numerous mathematical physics methods for solving nonlinear partial differential equations (NPDEs), the studying of symmetries and group invariant solutions for NPDEs is one of the most important and effective ways. By taking advantage of the symmetry theory, one can not only transform the known solutions of NPDEs to new ones through finite transformations, but also reduce the dimensions of NPDEs and proceed to construct group invariant solutions for these NPDEs. In this section, we would like to study the Lie point symmetry for the (3+1)-dimensional Gardner equation (2.7) and receive the group invariant solutions by means of the symmetry reduction technique.

It is known from the standard Lie point symmetry group approach that, to find the Lie point symmetry for equation (2.7), we can assume the symmetry owns the vector form

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u}, \tag{4.1}$$

where X, Y, Z, T, U are the functions of $\{x, y, z, t, u\}$, which implies that the (3+1)-dimensional Gardner equation is invariant under the transformations

$$\{x, y, z, t, u\} \to \{x + \epsilon X, y + \epsilon Y, z + \epsilon Z, t + \epsilon T, u + \epsilon U\} \tag{4.2}$$

with ϵ being an infinitesimal parameter. As the (3+1)-dimensional Gardner equation (2.7) is not explicitly space-time dependent, the symmetry in the vector form (4.1) can be rewritten as the corresponding function form

$$\sigma = U - Xu_x - Yu_y - Zu_z - Tu_t. \tag{4.3}$$

Here σ is the solution of the symmetry equation or the linearized equation of the (3+1)-dimensional Gardner equation

$$\begin{aligned}
&\sigma_t + 12\gamma\sigma u^3 u_z^3 + 9\gamma\sigma_z u^4 u_z^2 + 2\beta\sigma u^2 u_y + \alpha\sigma u u_y + 2\beta\sigma u^3 u_z + \alpha\sigma u^2 u_z \\
&+ 6\gamma\sigma_x u_x u_z + 3\gamma\sigma_z u^2 u_y^2 + 2\beta\sigma u u_x + 6\gamma\sigma_y u^3 u_z^2 + 6\gamma\sigma_x u^2 u_z^2 + 12\gamma\sigma u^3 u_{xzz} \\
&+ 3\gamma\sigma u^2 u_{yyy} + 12\gamma\sigma u^3 u_{yyz} + 15\gamma\sigma u^4 u_{yzz} + 6\gamma\sigma u^5 u_{zzz} + 3\gamma\sigma_z u^2 u_{xy} \\
&+ 3\sigma\gamma u_y u_{xy} + 3\gamma\sigma_y u u_{xy} + 3\gamma\sigma_{xy} u^2 u_z + 3\gamma\sigma_{xy} u u_y + 6\sigma\gamma u_x u_{xz} \\
&+ 6\gamma\sigma_z u^3 u_{xz} + 6\gamma\sigma_y u^2 u_{xz} + 6\gamma\sigma_x u u_{xz} + 6\gamma\sigma_{xz} u^3 u_z + 6\gamma\sigma_{xz} u^2 u_y \\
&+ 6\gamma\sigma_{xz} u u_x + 3\sigma\gamma u_x u_{yy} + 3\gamma\sigma_z u^3 u_{yy} + 3\gamma\sigma_y u^2 u_{yy} + 3\gamma\sigma_x u u_{yy} \\
&+ 3\gamma\sigma_{yy} u^3 u_z + 9\gamma\sigma_{yz} u^3 u_y + 9\gamma\sigma_{yz} u^2 u_x + 6\gamma\sigma_z u^5 u_{zz} + 6\gamma\sigma_y u^4 u_{zz} \\
&+ 6\gamma\sigma_x u^3 u_{zz} + 6\gamma\sigma_{zz} u^5 u_z + 6\gamma\sigma_{zz} u^4 u_y + 6\gamma\sigma_{zz} u^3 u_x + 3\gamma\sigma_{yy} u^2 u_y \\
&+ 3\gamma\sigma_{yy} u u_x + 9\gamma\sigma_z u^4 u_{yz} + 9\gamma\sigma_y u^3 u_{yz} + 9\gamma\sigma_x u^2 u_{yz} + 9\gamma\sigma_{yz} u^4 u_z \\
&+ 6\gamma\sigma u u_{xxz} + 6\gamma\sigma u u_{xyy} + 18\gamma\sigma u^2 u_{xyz} + \gamma\sigma_{xxx} + 6\gamma\sigma u_x u_y u_z + 6\gamma\sigma_x u u_y u_z \\
&+ 6\gamma\sigma_y u u_x u_z + 6\gamma\sigma_z u u_x u_y + 18\gamma\sigma u^2 u_y u_z^2 + 12\gamma\sigma u u_x u_z^2 + 12\gamma\sigma_z u^3 u_y u_z \\
&+ 12\gamma\sigma_z u^2 u_x u_z + 18\gamma\sigma u^2 u_z u_{xz} + 12\gamma\sigma u u_y u_{xz} + 9\gamma\sigma u^2 u_z u_{yy} + 6\gamma\sigma u u_y u_{yy} \\
&+ 36\gamma\sigma u^3 u_z u_{yz} + 27\gamma\sigma u^2 u_y u_{yz} + 18\gamma\sigma u u_x u_{yz} + 30\gamma\sigma u^4 u_z u_{zz} + 24\gamma\sigma u^3 u_y u_{zz} \\
&+ 18\gamma\sigma u^2 u_x u_{zz} + 6\gamma\sigma u u_z u_{xy} + 6\gamma\sigma u u_z u_y^2 + 6\gamma\sigma_y u^2 u_y u_z + 3\gamma\sigma_{yzz} u^5 \\
&+ 3\gamma\sigma_{yyz} u^4 + 3\gamma\sigma_x u_{xy} + 3\gamma\sigma_{xy} u_x + \frac{1}{2}\beta\sigma_z u^4 + \frac{1}{3}\alpha\sigma_z u^3 + 3\gamma\sigma_z u_x^2 + \alpha\sigma u_x \\
&+ \beta\sigma_x u^2 + \alpha\sigma_x u + \frac{2}{3}\beta\sigma_y u^3 + \frac{1}{2}\alpha\sigma_y u^2 + 3\gamma\sigma u_{xxy} + 3\gamma\sigma_{xxy} u \\
&+ \gamma\sigma_{zzz} u^6 + 3\gamma\sigma_{xzz} u^4 + \gamma\sigma_{yyy} u^3 + 3\gamma\sigma_{xxz} u^2 + 3\gamma\sigma_{xyy} u^2 + 6\gamma\sigma_{xyz} u^3 = 0.
\end{aligned} \tag{4.4}$$

The inserting of symmetry (4.3) into equation (4.4) and vanishing u_t according to the (3+1)-dimensional Gardner equation, one will get a set of equations for the functions X, Y, Z, T and U. By solving these equations yields the general solutions

$$
\begin{aligned}
X &= \frac{c_1}{3}x - \frac{c_1\alpha^2}{6\beta}t + c_3,\\
Y &= -\frac{c_1\alpha}{6\beta}x + c_4,\\
Z &= -\frac{c_1}{3}z - \frac{c_1\alpha}{3\beta}y + c_5,\\
T &= c_1 t + c_2,\\
U &= -\frac{c_1(2\beta u + \alpha)}{6\beta}
\end{aligned}
\tag{4.5}
$$

with $c_i(i = 1 \cdots 5)$ being arbitrary constants. Thus the Lie point symmetry of the (3+1)-dimensional Gardner equation (2.7) is

$$
\begin{aligned}
\sigma = \frac{1}{6\beta}\{&[c_1\alpha^2 t - \beta(2c_1 x + 6c_3)]u_x + [2c_1\alpha y\\
&+ \beta(2c_1 z - 6c_5)]u_z - 6\beta(c_1 t + c_2)u_t\\
&+ (c_1\alpha x - 6c_4\beta)u_y - 2c_1(\beta u + \frac{\alpha}{2})\}.
\end{aligned}
\tag{4.6}
$$

After determining the infinitesimal form (4.6) for (3+1)-dimensional Gardner equation (2.7), the group reduction solutions of this system can be acquired by making $\sigma = 0$, which is equivalent to solving the characteristic equations

$$
\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}y}{Y} = \frac{\mathrm{d}z}{Z} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{d}u}{U}. \tag{4.7}
$$

While solving equations (4.7), one has to distinguish between the case in which c_1 is identical to zero and the case where it is not, which will lead to different relations between the similarity variables and the original variables. Consequently, in the next paragraphs, the following two cases will be considered in detail: (i) $c_1 \neq 0$ and (ii) $c_1 = 0$.

Case (i) $c_1 \neq 0$. In this case, solving equations (4.7) leads to the following similarity variables

$$
\begin{aligned}
\xi &= \frac{4c_1\beta x + c_1\alpha^2 t + 3c_2\alpha^2 + 12c_3\beta}{4c_1\beta(c_1 t + c_2)^{\frac{1}{3}}},\\
\eta &= \frac{1}{12c_1\beta^2}[6c_1\alpha\beta x + 12c_1\beta^2 y + c_1\alpha^3 t + 4c_2\alpha^3 + 18c_3\alpha\beta\\
&\quad - (c_2\alpha^3 + 6c_3\alpha\beta + 12c_4\beta^2)\ln(c_1 t + c_2)],\\
\theta &= \frac{1}{32c_1\beta^3}[(c_1 t + c_2)^{\frac{1}{3}}(8c_1\alpha^2\beta x + 32c_1\alpha\beta^2 y + 32c_1\beta^3 z\\
&\quad + c_1\alpha^4 t - 3c_2\alpha^4 - 24c_3\alpha^2\beta - 96c_4\alpha\beta^2 - 96c_5\beta^3)]
\end{aligned}
\tag{4.8}
$$

and the similarity solution

$$
u = \frac{U(\xi, \eta, \theta)}{(c_1 t + c_2)^{\frac{1}{3}}} - \frac{\alpha}{2\beta}. \tag{4.9}
$$

Hence, the substituting of solution (4.9) into the (3+1)-dimensional Gardner equation (2.7) yields the related similarity reduction equation

$$
\begin{aligned}
&12\beta^2\gamma U_{\theta\theta\theta}U^6 + 36\beta^2\gamma(2U_\theta U_{\theta\theta} + U_{\eta\theta\theta})U^5 + 6\beta^2(6\gamma U_\theta^3 + 18\gamma U_\theta U_{\eta\theta}\\
&+ 12\gamma U_\eta U_{\theta\theta} + 6\gamma U_{\theta\theta\xi} + 6\gamma U_{\eta\eta\theta} + \beta U_\theta)U^4 + 4\beta^2(18\gamma U_\eta U_\theta^2\\
&+ 27\gamma U_\eta U_{\eta\theta} + 18\gamma U_\xi U_{\theta\theta} + 18\gamma U_\theta U_{\theta\xi} + 9\gamma U_\theta U_{\eta\eta} + 18\gamma U_{\eta\theta\xi}\\
&+ 3\gamma U_{\eta\eta\eta} + 2\beta U_\eta)U^3 + 12\beta^2(6\gamma U_\xi U_\theta^2 + 3\gamma U_\theta U_\eta^2 + 9\gamma U_\xi U_{\eta\theta}\\
&+ 6\gamma U_\eta U_{\theta\xi} + 3\gamma U_\eta U_{\eta\eta} + 3\gamma U_\theta U_{\eta\xi} + \beta U_\xi + 3\gamma U_{\eta\eta\xi} + 3\gamma U_{\theta\xi\xi})U^2\\
&+ 4\beta^2(18\gamma U_\xi U_\theta U_\eta + 18\gamma U_\xi U_{\theta\xi} + 9\gamma U_\xi U_{\eta\eta} + 9\gamma U_\eta U_{\eta\xi}\\
&+ 9\gamma U_{\eta\xi\xi} - c_1)U + 36\beta^2\gamma U_\theta U_\xi^2 + 36\gamma\beta^2 U_\xi U_{\eta\xi} - 4c_1\beta^2\xi U_\xi\\
&+ 4c_1\beta^2\theta U_\theta - c_2\alpha^3 U_\eta + 12\beta^2\gamma U_{\xi\xi\xi} - 6c_3\alpha\beta U_\eta - 12c_4\beta^2 U_\eta = 0.
\end{aligned}
\tag{4.10}
$$

Case (ii) When $c_1 = 0$, the group invariant solution obtained from the invariant condition $\sigma = 0$ has the form

$$u = U(\xi, \eta, \theta) \tag{4.11}$$

with the similarity variables

$$\xi = \frac{c_3 y - c_4 x}{c_3},\ \eta = \frac{c_3 z - c_5 x}{c_3},\ \theta = \frac{c_2 t - c_2 x}{c_3}. \tag{4.12}$$

After inserting the similarity reduction solution (4.11) into the (3+1)-dimensional Gardner equation (2.7), we have the corresponding similarity reduction equation

$$
\begin{aligned}
&6c_3^3\gamma U_{\eta\eta\eta}U^6 + 18c_3^3\gamma(2U_\eta U_{\eta\eta} + U_{\eta\eta\xi})U^5 + [18c_3^2\gamma(c_3U_\eta^3 + 3c_3U_\eta U_{\eta\xi}\\
&+ 2c_3U_\xi U_{\eta\eta} - c_5U_{\eta\eta\eta} + c_3U_{\eta\xi\xi} - c_4U_{\eta\eta\xi} - c_2U_{\eta\eta\theta}) + 3c_3^3\beta U_\eta]U^4\\
&+ [6c_3^2\gamma(6c_3U_\xi U_\eta^2 - 6c_2U_\eta U_{\eta\theta} - 6c_4U_\eta U_{\eta\xi} - 12c_5U_\eta U_{\eta\eta} + 3c_3U_\eta U_{\xi\xi}\\
&- 6c_2U_\theta U_{\eta\eta} + 9c_3U_\xi U_{\eta\xi} - 6c_4U_\xi U_{\eta\eta} - 6c_4U_{\eta\xi\xi} - 6c_5U_{\eta\eta\xi} + c_3U_{\xi\xi\xi}\\
&- 6c_2U_{\eta\theta\xi}) + 2c_3^3(\alpha U_\eta + 2\beta U_\xi)]U^3 - [18c_3\gamma(2c_3c_5U_\eta^3 + 2c_2c_3U_\theta U_\eta^2\\
&+ 2c_3c_4U_\xi U_\eta^2 - c_3^2U_\eta U_\xi^2 + 4c_3c_5U_\eta U_{\eta\xi} + c_3c_4U_\eta U_{\xi\xi} + c_2c_3U_\eta U_{\theta\xi}\\
&+ 2c_2c_3U_\xi U_{\eta\theta} + 3c_2c_3U_\theta U_{\eta\xi} + 5c_3c_4U_\xi U_{\eta\xi} + 2c_3c_5U_\xi U_{\eta\eta} - c_3^2U_\xi U_{\xi\xi}\\
&- c_5^2U_{\eta\eta\eta} + c_3c_5U_{\eta\xi\xi} - c_4^2U_{\eta\xi\xi} + c_2c_3U_{\theta\xi\xi} - 2c_4c_5U_{\eta\eta\xi} - 2c_2c_5U_{\eta\eta\theta}\\
&- c_2^2U_{\eta\theta\theta} + c_3c_4U_{\xi\xi\xi} - 2c_2c_4U_{\eta\theta\xi} + 3c_3^2(2c_5\beta U_\eta + 2c_2\beta U_\theta - c_3\alpha U_\xi\\
&+ 2c_4\beta U_\xi)]U^2 - [18c_3\gamma(2c_3c_5U_\xi U_\eta^2 + 2c_2c_3U_\eta U_\theta U_\xi + 2c_3c_4U_\eta U_\xi^2\\
&- 2c_2c_5U_\eta U_{\eta\theta} - 2c_4c_5U_\eta U_{\xi\eta} - 2c_5^2U_\eta U_{\eta\eta} + c_3c_5U_\eta U_{\xi\xi} - 2c_2^2U_\theta U_{\eta\theta}\\
&- 2c_2c_4U_\xi U_{\eta\theta} - 2c_2c_4U_\theta U_{\xi\eta} - 2c_2c_5U_\theta U_{\eta\eta} + c_2c_3U_\theta U_{\xi\xi} + c_3c_5U_\xi U_{\eta\xi}\\
&- 2c_4^2U_\xi U_{\xi\eta} - 2c_4c_5U_\xi U_{\eta\eta} + 2c_3c_4U_\xi U_{\xi\xi} + c_2c_3U_\xi U_{\theta\xi} - 2c_4c_5U_{\eta\xi\xi}\\
&- 2c_2c_4U_{\theta\xi\xi} - c_5^2U_{\eta\eta\xi} - c_2^2U_{\theta\theta\xi} - c_4^2U_{\xi\xi\xi} - 2c_2c_5U_{\eta\theta\xi}) + 6c_3^3\alpha(c_5U_\eta\\
&+ c_2U_\theta + c_4U_\xi)]U + \gamma[18c_3(c_2U_\theta + c_4U_\xi + c_5U_\eta)(c_2U_\eta U_\theta + c_2U_{\theta\xi}\\
&+ c_4U_{\xi\xi} + c_4U_\eta U_\xi + c_5U_\eta^2 + c_5U_{\eta\xi}) - 6(c_5^3U_{\eta\eta\eta} + 3c_4^2c_5U_{\eta\xi\xi} + 3c_2c_4^2U_{\theta\xi\xi}\\
&+ 3c_4c_5^2U_{\eta\eta\xi} + 3c_2^2c_4U_{\theta\theta\xi} + 3c_2c_5^2U_{\eta\eta\theta} + 3c_2^2c_5U_{\eta\theta\theta} + c_4^3U_{\xi\xi\xi}\\
&+ 6c_2c_4c_5U_{\eta\theta\xi} + c_2^3U_{\theta\theta\theta})] + 6c_3^3U_\theta = 0.
\end{aligned}
\tag{4.13}
$$

It is known that given any one solution $U(\xi, \eta, \theta)$ of the symmetry reduction equation (4.10) or (4.13), the group invariant solutions of the (3+1)-dimensional Gardner equation (2.7) can be constructed by solution (4.9) or (4.11). For instance, after performing the traveling wave transformation

$$U(\xi, \eta, \theta) = U(k_1\xi + l_1\eta + w_1\theta) \tag{4.14}$$

on the reduction equation (4.13), we can obtain a special exact solution for the original (3+1)-dimensional Gardner equation (2.7)

$$u = \frac{1}{2a_2^{\frac{3}{2}}}[(a_0a_2 + a_2 - \frac{a_1^2}{4})\sinh(\sqrt{2}\zeta) - (a_0a_2 - a_2 - \frac{a_1^2}{4})\cosh(\sqrt{2}\zeta) + \sqrt{2}a_1], \tag{4.15}$$

where

$$\begin{aligned}
\zeta &= -\frac{k_1c_4 + c_2w_1}{c_3}x + k_1y + w_1t,\\
a_1 &= -\frac{k_1c_6\alpha + \beta}{3k_1^3c_6\gamma}, \quad a_2 = -\frac{\beta}{6k_1^2\gamma}\\
c_3 &= c_6(k_1c_4 + c_2w_1),\\
c_6 &= \frac{6^{\frac{2}{3}}k_1w_1\alpha + 6^{\frac{1}{3}}k_1^{\frac{2}{3}}w_1^{\frac{4}{3}}(6\beta + \sqrt{\frac{36w_1\beta^2 - 6k_1\alpha^3}{w_1}})^{\frac{2}{3}}}{6k_1^{\frac{4}{3}}w_1^{\frac{5}{3}}(6\beta + \sqrt{\frac{36w_1\beta^2 - 6k_1\alpha^3}{w_1}})^{\frac{1}{3}}},
\end{aligned}$$

and c_2, c_4, c_5, k_1, w_1 and a_0 are all random constants.

5. Conclusion

To sum up, the deformation algorithm makes it possible for us to derive more different mathematical and physical models in higher dimensions and investigate their integrable properties as well. However, due to the local integrability and higher dimensionality of these higher dimensional systems deformed by this method, it is not easy to seek the explicit solutions for them. In this paper, we have successfully derived a novel (3+1)-dimensional Gardner equation from the (1+1)-dimensional Gardner equation by utilizing the deformation algorithm and its conservation laws. And the integrability of this equation has been rigorously verified through its (3+1)-dimensional Lax pairs. For the sake of obtaining the explicit solutions for the (3+1)-dimensional Gardner equation, two approaches including the traveling wave method and the symmetry reduction method have been adopted. In the frame of the traveling wave technique, the general traveling wave solution in the elliptic integral form has been obtained to the (3+1)-dimensional Gardner equation. Specially, the implicit expression of traveling wave solution of tanh function form has been given after fixing some arbitrary constants. And in terms of the symmetry reduction approach, we have discovered the Lie point symmetry and two different types of group invariant solutions for the (3+1)-dimensional Gardner equation. In addition, by setting the (3+1)-dimensional Gardner equation (2.7) to be y or z independent, two (2+1)-dimensional Gardner equations (2.8) and (2.9) were also derived, and the traveling wave solutions of incomplete elliptic integral function and rational function form were constructed.

Up to now, the deformation algorithm is still limited to the (1+1)-dimensional evolution equation, how to apply this method to extend (2+1)-dimensional equations to higher dimensional ones looks worthy of further investigation. Besides, how to get more abundant exact solutions for these higher dimensional systems also needs further discussion.

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