

A PARAMETRIC OPTIMAL CONTROL PROBLEM APPLIED TO DAILY IRRIGATION

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Abstract. We conduct sensitivity analysis of an optimal control problem applied to agricultural irrigation. The aim is to minimize the square of the amount of irrigation water while ensuring the health grow of the crop. A crucial parameter of our model is the percentage of water loss due to deep percolation, β , a parameter hard to estimate and subject to perturbations. Thus, our problem is a parametric state constrained optimal control problem with a L^2 cost. Our goal is to study how perturbations of β affect the optimal solutions of our problem using sensitivity analysis of our problem using sensitivity analysis.

To solve numerically our optimal control problem we use the direct method, transcribing the problem into a non-linear programming problem. We show how sensitivity analysis applied to the non-linear programming problem provides information on the variation of optimal solutions of the original problem in terms of β . Valid approximations of optimal solution and cost, provided by sensitivity analysis, are computed for values of β within of a certain neighbourhood. Remarkably, we show that for all β in such neighbourhood, the irrigation period is kept constant. Only the flow rate of irrigation water changes.

Mathematics Subject Classification. 49K40, 49M25, 49M37, 90C30, 90C31.

Received May 13, 2024. Accepted November 9, 2024.

1. INTRODUCTION

While agriculture irrigation may be responsible for about 70% of freshwater withdrawals in the world, it is crucial to provide food for large populations: see, for example, [1]. Thus, the efficient and conscious use of water to produce healthy crop is nowadays of the foremost importance. Not surprisingly, one has witnessed a resurgence of interest to study control measures to minimize the water used to produce healthy crops (see, for example, [2–6] and references therein).

In this paper we focus on sensitivity analysis of a parametric optimal control problem based on a simple agrohydrological model establishing interactions between soil, water, crop and atmosphere, in the vein of [4, 7–9]. The model depends on the percentage of water loss due to run-off and deep infiltration; this percentage will be denoted by β . This is a parameter hard to estimate and subject to uncertainties. The aim of the optimal

Keywords and phrases: Irrigation problem, variation of water loss percentage, sensitivity analysis, non-linear programming methods, sensitivity differential.

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control problem we consider here is to determine irrigation policies on a fixed finite horizon, minimizing the square of the amount of water used, while ensuring that the water needs of the crop are satisfied throughout. Although the solution of such problem for a reference (or nominal) value for β can be numerically computed, it is essential to understand how solution and cost are affected by perturbations of the parameter β . To do so, here we conduct a sensitivity analysis to our problem following closely the approach proposed in [10]. The goal is to study approximate solutions and approximate cost values for values of the parameter β in a vicinity of the nominal value β_0 of the parameter.

Optimal control problems are commonly solved *via* the direct method which involves the transcription of the problem into a large scale non-linear programming (NLP) problem *via* discretization. This technique, although adequate for computation of optimal solutions for a nominal parameter, is computationally heavy and time consuming when different values of the parameter have to be considered. Sensitivity analysis is of help in this regard. For optimal control problems, a common approach is to carry out sensitivity analysis over a transcribed NLP problem and to translate back the results into the data of the original optimal control problem. An impressive panoply of papers on sensitivity analysis for different optimization problems can be found in [11–14]. Sensitivity analysis and computational methods are also studied in [15], where an optimal control problem involving impulsive systems with some coefficients subject to variations is studied. Sensitivity analysis for real-time optimization, although a relatively recent area of research, has fast found broad applications in different areas: see [16–22], for implementations of sensitivity analysis in aerospace, robotics, fluid dynamics and automotive applications.

Within the scope of optimization problems there are two different conceptions of sensitivity analysis: see [23]. In this paper, we are interested in parameter sensitivity analysis; this focuses on the sensitivity of the nominal solution and its cost to perturbations of the parameter. When the solutions are differentiable with respect to the parameter, [10] and [23] (and references therein) propose schemes to compute approximations of the solutions and the cost at neighbourhood points of the nominal parameter. These results have their roots on the work developed in [24–26], where the classical Implicit Function Theorem is considered to prove solution differentiability. In addition to [10] and [23], see also [27–30].

Focusing on parametric NLP problems, Büskens and Maurer in [23] analyse numerical issues associated with the computation of the Hessian with high accuracy and the second order sufficient conditions, essential to perform the sensitivity analysis. The main goal is the application of sensitivity analysis tools to optimize perturbed optimal solutions in real time. They propose computation of approximations of the perturbed solutions *via* first Taylor expansions with respect to the parameter, as well as explicit formulae for the parameter sensitivity derivatives of the optimal solution, the corresponding Lagrange multiplier and the optimal cost. The approach considered in [23] is then generalized in [10]. In [10] NLP methods to solve optimal control problems with state-control inequality constraints are studied: suitable discretizations of variables are used to transcribe the optimal control problem into a finite dimensional NLP problem. In this respect, it is imperative to keep the dimension of the resulting NLP problem as small as possible. This is because the evaluation of sensitivity formulae requires the exact computation of the Hessian of the Lagrangian. In [23] Büskens and Maurer suggest to conduct the discretization by a recursive approach where the state variables are written as functions of the control. In this case the only optimization variables of the NLP problem are the discretized control variables which decreases the dimension of the NLP problem, producing sensitivity analysis results with high accuracy. Once the NLP problem is solved numerically, a post-optimal calculation provides information on the adjoint variables of the original problem from the Lagrange multipliers associated with the NLP problem: see [10]. Noteworthy, the sensitivity analysis developed in [23] allows the computation of the sensitivity derivatives of the control variables. The parameter sensitivity derivatives with respect to state and adjoint variables may be recovered recursively from the control sensitivities. In this way a complete picture of sensitivity analysis is performed. It is worth mentioning that the approximate solutions provided by parameter sensitivity analysis are only valid around a nominal value of the parameter. Clearly, it is of foremost importance to know the size of the neighbourhood of the nominal value. In this respect help can be found in [22], where a method to determine the neighbourhood's size is proposed for some classes of problems.

In our optimal control problem of interest, the control variable is denoted by u , with $u(t)$ representing the flow rate of irrigation water at t , and the cost is $\int_0^{t_f} u^2(t)dt$, where t_f is the fixed final time. This L^2 cost may come

as a surprise, since the L^1 cost $\int_0^{t_f} u(t)dt$ considered in [9] comes naturally to mind. However, the literature on sensitivity analysis for the L^1 case, it does not apply effectively to our specific problem. In [31] Büskens *et al.* propose sensitivity analysis tools for parametric optimal control problems with mixed state-control constraints that hold when their optimal solutions have bang-bang and/or singular arcs. On the other hand, in [32], Maurer and Vossen study parametric optimal control problems with state constraints assuming that the optimal control have finitely many bang-bang and boundary arcs, discontinuous at junction times. Taking into account the profile of control solution for our irrigation problem with L^1 cost, one would expect to apply the results in [32]. However, the discontinuity assumption of the boundary control at junction times in [32] is a feature we can not guarantee for our optimal control problem with L^1 cost. Unable to overcome this difficulty, we leave the sensitivity analysis for the irrigation problem with L^1 cost (based on [9]) for future work.

In this paper we dwell on the irrigation optimal control with L^2 cost. Since $u(t) \in [0, \gamma]$ for all $t \geq 0$, with $\gamma > 1$, the choice of L^2 cost distinguishes between *large* (greater than 1) and *small* flow rates of irrigation water. In contrast to the L^1 cost, the L^2 cost penalizes situations when $u(t) > 1$ decreasing the penalization when $u(t) < 1$, in line with sustainable practices in agriculture where small quantities of irrigation water per unit of time are allowed, while large quantities are to be avoided (notice that in [9] the control constraint is $0 \leq u(t) \leq 1$ instead of $0 \leq u(t) \leq \gamma$ for all $t \geq 0$).

This paper is organized as it follows. In Section 2, we present the parametric optimal control problem applied to agricultural irrigation of interest. We also present the corresponding necessary and sufficient optimality conditions. In Section 3 we transcribe the parametric optimal control problem into a NLP problem through a recursive approach. For the NLP problem we also deduce the associated necessary and sufficient optimality conditions. Those are crucial to validate numerical solutions and to check differentiable properties with respect to the parameter in consideration. We end Section 3 by providing a post-optimal estimation of the adjoint variables. Sensitivity analysis associated with the optimal solution and the optimal value function is carried out in Section 4. We finish our paper illustrating our findings *via* numerical simulations presented in Section 5.

2. PARAMETRIC OPTIMAL CONTROL PROBLEM

In this section we introduce the parametric optimal control problem applied to irrigation. Next, we state the necessary and sufficient optimality conditions applying well known results. In particular, these conditions are important for notational purpose.

2.1. Formulation

We intend to optimize the irrigation of an agricultural field so that the healthy growth of crop is preserved, under a criterion which penalizes the use of large quantities of water in irrigation. The optimal control problem we consider relies on a model introduced in [9]. It is based on the quantity of water in the soil. A remarkable feature of the model in [9] is that the water's variations in the soil depend not only on the irrigation, but also on the evapotranspiration, precipitation and water loss. Mathematically, this perspective leads to the following dynamic system governing the variation of water in the soil:

$$\begin{cases} \dot{x}(t) = u(t) + g(t) - \beta x(t), & \text{for all } t \geq 0, \\ x(0) = a, \end{cases}$$

where

- x is the state variable with $x(t)$ expressing the amount of water in the soil, in mm (per m^2), at t ;
- u is the control with $u(t)$ denoting the flow rate of irrigation water, in mm/day (per m^2), at t ;
- $g(t)$ is defined as

$$g(t) = \text{rfall}(t) - \text{evtp}(t),$$

where $\text{rfall}(t)$ is the daily precipitation and $\text{evtp}(t)$ is the estimated evapotranspiration for the type of crop, in mm/day (per m^2), at t ;

- $\beta \in [0, \beta_M]$ represents the daily percentage of water loss due to the run-off and deep infiltration, being $\beta_M > 0$ the maximum value for β ;
- a represents the initial quantity of the water in the soil, in mm (per m^2).

To guarantee the healthy production of the crop, the amount of water in the soil $x(t)$ can never go below a certain level $x_{\min}$; this is known as *the hydrological need of the crop*. Such requirement is translated by the *state constraint*:

$$x(t) \geq x_{\min}, \quad \forall t \geq 0.$$

As for the control, it has to satisfy the following requirement

$$0 \leq u(t) \leq \gamma, \quad \forall t \geq 0, \quad (2.1)$$

for some fixed $\gamma > 0$. Here, γ is the maximum flow rate of irrigation water entering the system and it is assumed to be very large.

We want to determine the amount of water used for irrigation at each instant $t \in [0, t_f]$, where $[0, t_f]$ is the fixed period under study and $t_f > 0$, so as to minimize

$$\int_0^{t_f} u^2(t) dt. \quad (2.2)$$

As we have mentioned in the introduction, the L^2 cost (2.2) may come as a surprise, since the L^1 cost considered in [9] $\int_0^{t_f} u(t) dt$, comes naturally to mind. Taking into account (2.1) and the fact that γ is large (much greater than 1), the choice of (2.2) distinguishes between large and small flow rates of irrigation water used at each instant t : when compared to the L^1 cost, (2.2) penalizes situations when $u(t) > 1$, while decreasing the penalization of situations when $u(t) < 1$. Observe also that in [9] the control constraint is $0 \leq u(t) \leq 1$ instead of (2.1). The L^2 cost (2.2) is in line with a common practice in agriculture, where small quantities of irrigation water per unit of time are allowed while large quantities are avoided.

To simplify the analysis, we reformulate our problem in the Mayer form appealing to the usual state augmentation technique: we introduce a new function $w : [0, t_f] \mapsto \mathbb{R}$ defined by $w(t) = \int_0^t u^2(\tau) d\tau$. Moreover, to get an autonomous problem, we add an auxiliary state variable y to accommodate t . It will be seen in Section 2.2 that this detail is crucial to apply the results of [10]. For any $\beta \in [0, \beta_M]$ this leads to the following parametric optimal control problem:

$$\left\{ \begin{array}{ll} \min & w(t_f) \\ \text{s. t.} & \begin{array}{l} \dot{x}(t) = u(t) + g(t) - \beta x(t), \text{ a.e. } t \in [0, t_f], \\ \dot{y}(t) = 1, \text{ a.e. } t \in [0, t_f], \\ \dot{w}(t) = u^2(t), \text{ a.e. } t \in [0, t_f], \\ x(t) \geq x_{\min}, \quad \forall t \in [0, t_f], \\ u(t) \in [0, \gamma], \text{ a.e. } t \in [0, t_f], \\ x(0) = a, \\ y(0) = 0, \\ w(0) = 0. \end{array} \end{array} \right.$$

Setting $X = (x, y, w)$, we can rewrite the above problem in the form

$$\begin{cases} \min & G(X(0), X(t_f), \beta) \\ \text{s. t.} & \dot{X}(t) = f(X(t), u(t), \beta), \text{ a.e. } t \in [0, t_f], \\ & C(X(t), u(t), \beta) \leq 0, \forall t \in [0, t_f], \\ & \psi(X(0), X(t_f), \beta) = 0, \end{cases} \quad \mathbf{OCP}(\beta)$$

where the cost function is given by

$$G(X(0), X(t_f), \beta) = w(t_f), \quad (2.3)$$

the dynamic is defined by

$$f(X(t), u(t), \beta) = \begin{bmatrix} f_1(X(t), u(t), \beta) \\ f_2(X(t), u(t), \beta) \\ f_3(X(t), u(t), \beta) \end{bmatrix} = \begin{bmatrix} u(t) + g(t) - \beta x(t) \\ 1 \\ u^2(t) \end{bmatrix},$$

the inequality constraints are

$$C(X(t), u(t), \beta) = \begin{bmatrix} C_1(X(t), u(t), \beta) \\ C_2(X(t), u(t), \beta) \\ C_3(X(t), u(t), \beta) \end{bmatrix} = \begin{bmatrix} -x(t) + x_{\min} \\ -u(t) \\ u(t) - \gamma \end{bmatrix}, \quad (2.4)$$

and the initial state is fixed:

$$\psi(X(0), X(t_f), \beta) = \begin{bmatrix} \psi_1(X(0), X(t_f), \beta) \\ \psi_2(X(0), X(t_f), \beta) \\ \psi_3(X(0), X(t_f), \beta) \end{bmatrix} = \begin{bmatrix} x(0) - a \\ y(0) \\ w(0) \end{bmatrix}.$$

2.2. Necessary and sufficient optimality conditions

We now derive necessary and sufficient optimality conditions for $\mathbf{OCP}(\beta)$, through Proposition 2.1.

Proposition 2.1 (Necessary and Sufficient Optimality Conditions). *$(\hat{X}, \hat{u})$ is an optimal solution to $\mathbf{OCP}(\beta)$ if and only if there are a piecewise continuous multiplier function $\hat{\mu} : [0, t_f] \rightarrow (\mathbb{R}_0^+)^3$ and a piecewise continuous and piecewise continuously differentiable adjoint function $\hat{\lambda} : [0, t_f] \rightarrow \mathbb{R}^3$ such that:*

$$\begin{cases} \dot{\hat{\lambda}}_1(t) = \beta \hat{\lambda}_1(t) + \hat{\mu}_1(t) \\ \hat{\lambda}_1(t_f) = 0 \end{cases} \quad (2.5)$$

and

$$\begin{cases} \hat{\lambda}_2(t) = 0 \\ \hat{\lambda}_3(t) = 1. \end{cases} \quad (2.6)$$

Furthermore,

$$\hat{u}(t) = \frac{\hat{\mu}_2(t) - \hat{\mu}_3(t) - \hat{\lambda}_1(t)}{2}. \quad (2.7)$$

Proof. According to [10, 33, 34], if $(\hat{X}, \hat{u})$ is an optimal solution to **OCP**(β), then there is a piecewise continuous and piecewise continuously differentiable adjoint function $\hat{\lambda} : [0, t_f] \rightarrow \mathbb{R}^3$ and a piecewise continuous multiplier function $\hat{\mu} : [0, t_f] \rightarrow (\mathbb{R}_0^+)^3$ such that the following conditions are verified:

- the adjoint equation

$$\dot{\hat{\lambda}}(t) = - \left(\frac{\partial H}{\partial X} (\hat{X}(t), \hat{u}(t), \beta, \hat{\lambda}, \hat{\mu}) \right)^T; \quad (2.8)$$

- the transversality condition

$$\hat{\lambda}(t_f) = \left(\frac{\partial G}{\partial X(t_f)} (\hat{X}(0), \hat{X}(t_f), \beta) \right)^T; \quad (2.9)$$

- the optimality condition

$$\frac{\partial H}{\partial u} (\hat{X}(t), \hat{u}(t), \beta, \hat{\lambda}, \hat{\mu}) = 0, \quad (2.10)$$

where the function H , defined by

$$H(X, u, \beta, \lambda, \mu) = \lambda_1(u + g - \beta x) + \lambda_2 + \lambda_3 u^2 + \mu_1(x_{\min} - x) - \mu_2 u + \mu_3(u - \gamma),$$

is called the *augmented Hamiltonian*. From the adjoint equation (2.8) and the transversality condition (2.9), we get:

$$\begin{cases} \dot{\hat{\lambda}}_1(t) = \beta \hat{\lambda}_1(t) + \hat{\mu}_1(t) \\ \hat{\lambda}_1(t_f) = 0 \end{cases}, \quad \begin{cases} \dot{\hat{\lambda}}_2(t) = 0 \\ \hat{\lambda}_2(t_f) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{\hat{\lambda}}_3(t) = 0 \\ \hat{\lambda}_3(t_f) = 1 \end{cases}$$

which implies (2.5) and (2.6). Finally, from the optimality condition (2.10), we obtain

$$\hat{\lambda}_1(t) + 2\hat{\lambda}_3(t)\hat{u}(t) - \hat{\mu}_2(t) + \hat{\mu}_3(t) = 0$$

which is equivalent to (2.7), since $\hat{\lambda}_3(t) = 1$. Moreover, as **OCP**(β) is a convex optimal control problem, then the previous necessary optimality conditions are also sufficient. This concludes the proof. $\square$

Note that Proposition 2.1 is of great importance for Section 3.4, where a post-optimal estimation of the adjoint function is carried out.

3. NUMERICAL SOLUTION TO **OCP**(β) THROUGH A NLP METHOD

NLP methods to solve numerically optimal control problems with constraints are well established: see, for example, [35, 36]. These techniques involve the transcription of optimal control problems into NLP problems *via* a suitable discretization. In general, these methods can be divided into two types:

TABLE 1. Notation.

$\tau_i = \frac{i-1}{N-1}t_f$ for $i = 1, \dots, N$.
$v_i = v(\tau_i)$, where $v \in \{X, x, y, w, u, g\}$ and $i = 1, \dots, N$.
$X_i = (x_i, y_i, w_i)$ for $i = 1, \dots, N$.
$z = [u_1 \quad \dots \quad u_{N-1}]^T$
$\mu_{ji} = \mu_j(\tau_i)$ for $j = 1, 2, 3$ and $i = 1, \dots, N-1$.
$\mu_{\tau_i}^T = [\mu_{1i} \quad \mu_{2i} \quad \mu_{3i}]$ for $i = 1, \dots, N-1$.
$\mu_j^T = [\mu_{j1} \quad \mu_{j2} \quad \dots \quad \mu_{j(N-1)}]$ for $j = 1, 2, 3$.
$\mu^T = [\mu_1^T \quad \mu_2^T \quad \mu_3^T] = [\eta_1 \quad \eta_2 \quad \dots \quad \eta_{3N-3}] = \eta^T$
$\lambda_{ji} = \lambda_j(\tau_i)$ for $j = 1, 2, 3$ and $i = 1, \dots, N$.
$\lambda_{\tau_i}^T = [\lambda_{1i} \quad \lambda_{2i} \quad \lambda_{3i}]$ for $i = 1, \dots, N$.
$\mathcal{K}_1 = \{1, \dots, N-1\}$
$\mathcal{K}_2 = \{N, \dots, 2N-2\}$
$\mathcal{K}_3 = \{2N-1, \dots, 3N-3\}$

- (I) both the discretized state and control variables are considered as optimization variables: see, for example, [10];
- (II) only the discretized control variables are taken as optimization variables, while the state variables are computed as functions of the discretized control variables, using adequate numerical integration methods: see, for example, [10].

Type (I) gives rise to a high dimensional NLP problem with a sparse structure of the Jacobian and Hessian of the constraints and Lagrangian. Opting by type (II), we obtain a small, but dense NLP problem, where the dimension of the NLP problem does not depend on the dimension of the differential equation.

In Section 3.1, we show how **OCP**(β) is transcribed into a NLP problem *via* discretization. Then we state necessary and sufficient conditions for the NLP problem in Section 3.2, essential for the sensitivity analysis carried out in Section 4. Since we opt by the NLP method of type (II) in Section 3.1, we then compute adjoint variables by a post-optimal calculation, considering the Lagrange multipliers associated with the NLP problem, in Section 3.4. As from the current section, the notation is very extensive, we summarize it in Table 1.

3.1. Discretization of **OCP**(β) through a recursive approach

In the current section, we discretize **OCP**(β) problem, by transcribing it into a NLP problem. To do that we consider a NLP method of type (II).

Let N be a positive integer representing the mesh size. We use equidistant mesh points τ_i with

$$\tau_i = (i-1)\theta \quad \text{and} \quad \theta = \frac{t_f}{N-1}$$

for all $i = 1, \dots, N$. To simplify the writing, denote $v(\tau_i)$ simply by v_i , where $v \in \{X, x, y, w, u, g\}$. The discrete-time NLP problem is defined by

$$\left\{ \begin{array}{ll} \min & w_N \\ \text{s. t.} & x_{i+1} = x_i + \theta(u_i + g_i - \beta x_i), \forall i = 1, \dots, N-1, \\ & y_{i+1} = y_i + \theta, \forall i = 1, \dots, N-1, \\ & w_{i+1} = w_i + \theta(u_i)^2, \forall i = 1, \dots, N-1, \\ & -x_i + x_{\min} \leq 0, \forall i = 2, \dots, N, \\ & -u_i \leq 0, \forall i = 1, \dots, N-1, \\ & u_i - \gamma \leq 0, \forall i = 1, \dots, N-1, \\ & x_1 = a, \\ & y_1 = 0, \\ & w_1 = 0. \end{array} \right. \quad (3.1)$$

To simplify the writing, from now on let us consider that $\bar{\beta} = 1 - \theta\beta$. Taking into account (2.3) and (2.4), observe that (3.1) can be written as

$$\left\{ \begin{array}{ll} \min & \tilde{G}(z, \beta) := G(X_1(z, \beta), X_N(z, \beta), \beta) \\ \text{s. t.} & C_1(X_i(z, \beta), u_i, \beta) \leq 0, \forall i = 1, \dots, N-1, \\ & C_2(X_i(z, \beta), u_i, \beta) \leq 0, \forall i = 1, \dots, N-1, \\ & C_3(X_i(z, \beta), u_i, \beta) \leq 0, \forall i = 1, \dots, N-1, \end{array} \right. \quad \mathbf{NLP}(\beta)$$

where

- (a) $z = [u_1 \ \dots \ u_{N-1}]^T$ is the vector of discretized control variables;
- (b) $X_i(z, \beta) = (x_i(z, \beta), y_i(z, \beta), w_i(z, \beta))$ describes the discretized state variables written in terms of β and u_j for all $i = 1, \dots, N$ and $j = 1, \dots, N-1$ ¹;
- (c) $X_1(z, \beta) = X_1 = (a, 0, 0)$;
- (d) the cost function is defined by

$$G(X_1(z, \beta), X_N(z, \beta), \beta) = \theta \sum_{j=1}^{N-1} (u_j)^2; \quad (3.2)$$

- (e) the constraints $C_1(X_i(z, \beta), u_i, \beta) \leq 0$ describe the inequalities

$$-\bar{\beta}x_i(z, \beta) - \theta(u_i + g_i) + x_{\min} \leq 0 \Leftrightarrow -x_{i+1}(z, \beta) + x_{\min} \leq 0$$

for all $i = 1, \dots, N-1$, where $x_1(z, \beta) = a$ and $\bar{\beta} = 1 - \theta\beta$;

- (f) the constraints $C_2(X_i(z, \beta), u_i, \beta) \leq 0$ describe the inequalities $-u_i \leq 0$ for all $i = 1, \dots, N-1$;
- (g) the constraints $C_3(X_i(z, \beta), u_i, \beta) \leq 0$ describe the inequalities $u_i - \gamma \leq 0$ for all $i = 1, \dots, N-1$.

It is important to notice that in $\mathbf{NLP}(\beta)$

- we omit the constraints $\psi_2(X_1(z, \beta), X_N(z, \beta), \beta) = 0 \Leftrightarrow y_1 = 0$ and $y_{i+1} = y_i + \theta$ for all $i = 1, \dots, N-1$, because they do not depend on the control and other state variables. Knowing y_1 , we are able to obtain y_i for all $i = 1, \dots, N$, without solving any optimization problem.

¹Recall that we are using a NLP method of type (II).

- we omit the constraints $\psi_i(X_1(z, \beta), X_N(z, \beta), \beta) = 0$ for $i \in \{1, 3\}$, that is, the initial conditions $x_1 = a$ and $w_1 = 0$, since they are implicit in the constraint $C_1(X_1(z, \beta), u_1, \beta) \leq 0$ of **NLP**(β) (see the previous item (e)) and in the expression of cost function G (see the next item), respectively;
- the cost function (3.2) is derived from the constraints

$$\begin{cases} w_1 = 0, \\ w_{i+1} = w_i + \theta(u_i)^2 \text{ for all } i = 1, \dots, N-1. \end{cases} \quad (3.3)$$

Before going any further, some features of the constraints associated with the state variables x_i , $i = 1, \dots, N$, of (3.1) are discussed. Observe that

$$\begin{cases} x_1 = a, \\ x_{j+1} = x_j + \theta(u_j + g_j - \beta x_j), \quad \forall j = 1, \dots, N-1, \\ x_{\min} - x_i \leq 0, \quad \forall i = 2, \dots, N, \end{cases}$$

can be written as

$$\begin{cases} x_1 = a, \\ x_{\min} - (1 - \theta\beta)x_j - \theta(u_j + g_j) \leq 0, \quad \forall j = 1, \dots, N-1. \end{cases} \quad (3.4)$$

The next lemma allows us to rewrite (3.4) without the dependence on x_j , with $j = 1, \dots, N-1$.

Lemma 3.1. *If $x_1 = a$ and $x_{j+1} = x_j + \theta(u_j + g_j - \beta x_j)$ for $j = 1, \dots, N-1$, then*

$$\bar{\beta}x_j + \theta(u_j + g_j) = \bar{\beta}^j a + \sum_{k=1}^j \bar{\beta}^{j-k} \theta(u_k + g_k), \quad \forall j = 1, \dots, N-1. \quad (3.5)$$

Proof. This is proved by Mathematical Induction. □

Observe that the left-side of (3.5) appears in (3.4) with a minus sign. From Lemma 3.1, we obtain that (3.4) is equivalent to

$$\begin{cases} x_1 = a, \\ x_{\min} - \bar{\beta}^j a - \sum_{k=1}^j \bar{\beta}^{j-k} \theta(u_k + g_k) \leq 0, \quad \forall j = 1, \dots, N-1. \end{cases}$$

Hence,

$$A \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} + B \leq 0_{\mathbb{R}^{N-1}},$$

where $A = [a_{ij}]$ is a $(N-1) \times (N-1)$ matrix defined by

$$A = -\theta \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \bar{\beta} & 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \bar{\beta}^2 & \bar{\beta} & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \bar{\beta}^3 & \bar{\beta}^2 & \bar{\beta} & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \bar{\beta}^4 & \bar{\beta}^3 & \bar{\beta}^2 & \bar{\beta} & \ddots & \ddots & & & \vdots & \vdots \\ \bar{\beta}^5 & \bar{\beta}^4 & \bar{\beta}^3 & \bar{\beta}^2 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \bar{\beta}^6 & \bar{\beta}^5 & \bar{\beta}^4 & \bar{\beta}^3 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \bar{\beta}^{N-2} & \bar{\beta}^{N-3} & \cdots & \cdots & \cdots & \cdots & \bar{\beta}^3 & \bar{\beta}^2 & \bar{\beta} & 1 \end{bmatrix},$$

$$B = [b_1 \quad \cdots \quad b_{N-1}]^T \text{ and } b_j = x_{\min} - \bar{\beta}^j a - \sum_{k=1}^j \bar{\beta}^{j-k} \theta g_k \text{ for all } j = 1, \dots, N-1.$$

Now, we turn to the inequality constraints of **NLP**(β), by setting

$$\begin{aligned} \bullet \quad Az + B &= \begin{bmatrix} C_1(X_1(z, \beta), u_1, \beta) \\ \vdots \\ C_1(X_{N-1}(z, \beta), u_{N-1}, \beta) \end{bmatrix}; \\ \bullet \quad -I_{N-1}z &= \begin{bmatrix} C_2(X_1(z, \beta), u_1, \beta) \\ \vdots \\ C_2(X_{N-1}(z, \beta), u_{N-1}, \beta) \end{bmatrix}; \\ \bullet \quad I_{N-1}z - \tilde{\gamma} &= \begin{bmatrix} C_3(X_1(z, \beta), u_1, \beta) \\ \vdots \\ C_3(X_{N-1}(z, \beta), u_{N-1}, \beta) \end{bmatrix}; \end{aligned}$$

where $\tilde{\gamma} = [\gamma \quad \cdots \quad \gamma]^T$. The inequality constraints of **NLP**(β) are then

$$\begin{bmatrix} A \\ -I_{N-1} \\ I_{N-1} \end{bmatrix} z + \begin{bmatrix} B \\ 0_{\mathbb{R}^{N-1}} \\ -\tilde{\gamma} \end{bmatrix} \leq 0_{\mathbb{R}^{3N-3}}. \quad (3.6)$$

Observe that, due to (3.2) and (3.6), only the discretized control variables $u_1, \dots, u_{N-1}$ are taken as optimization variables for **NLP**(β) that can be simply rewritten as it follows:

$$\begin{cases} \min & \theta \sum_{j=1}^{N-1} (u_j)^2 \\ \text{s. t.} & \begin{bmatrix} A \\ -I_{N-1} \\ I_{N-1} \end{bmatrix} z + \begin{bmatrix} B \\ 0_{\mathbb{R}^{N-1}} \\ -\tilde{\gamma} \end{bmatrix} \leq 0_{\mathbb{R}^{3N-3}}, \end{cases} \quad (3.7)$$

where $z = [u_1 \quad \cdots \quad u_{N-1}]^T$.

In what follows we provide a way to recover the state variables x_i with $i = 1, \dots, N$, when the controls are computed and β is fixed. Recall that when we consider $x_1 = a$ and $x_{j+1}(z, \beta) = x_j(z, \beta) + \theta(u_j + g_j - \beta x_j(z, \beta))$ for all $j = 1, \dots, N-1$, the constraints

$$C_1(X_i(z, \beta), u_i, \beta) \leq 0, \text{ with } i = 1, \dots, N-1$$

describe the inequalities $x_{\min} - x_j(z, \beta) \leq 0$ for all $j = 2, \dots, N$. So we can write

$$\begin{bmatrix} x_1(z, \beta) \\ \vdots \\ x_N(z, \beta) \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ & -A & \end{bmatrix} z + \begin{bmatrix} a - x_{\min} \\ -B \end{bmatrix} + \begin{bmatrix} x_{\min} \\ \vdots \\ x_{\min} \end{bmatrix} \quad (3.8)$$

which in turn is equivalent to

$$\begin{cases} x_1(z, \beta) &= a, \\ x_{i+1}(z, \beta) &= x_{\min} - b_i + \sum_{k=1}^i \bar{\beta}^{i-k} \theta u_k, \text{ for all } i = 1, \dots, N-1. \end{cases} \quad (3.9)$$

3.2. Necessary and sufficient optimality conditions for the $\text{NLP}(\beta)$

Before to derive necessary and sufficient optimality conditions for $\text{NLP}(\beta)$, let us define the set of indexes associated with the active constraints of a certain solution of $\text{NLP}(\beta)$.

Definition 3.2 (Indexes of the Active Constraints). Suppose that $z^* = [u_1^* \cdots u_{N-1}^*]^T$ is a solution to $\text{NLP}(\beta)$ and $\eta = [\eta_1 \cdots \eta_{3N-3}]^T = \mu$ is the corresponding multiplier. With respect to solution z^* , the set of the indexes of the multiplier η associated with the active state constraints is defined by

$$\mathcal{I}_1(z^*, \beta) = \{k \in \mathcal{K}_1 : C_1(X_k(z^*, \beta), u_k^*, \beta) = 0\},$$

where $\mathcal{K}_1 = \{1, \dots, N-1\}$. Moreover, the sets of the indexes of η related with the active control constraints are defined by

$$\mathcal{I}_2(z^*, \beta) = \{k \in \mathcal{K}_2 : C_2(X_{k-N+1}(z^*, \beta), u_{k-N+1}^*, \beta) = 0\}$$

and

$$\mathcal{I}_3(z^*, \beta) = \{k \in \mathcal{K}_3 : C_3(X_{k-2N+2}(z^*, \beta), u_{k-2N+2}^*, \beta) = 0\},$$

where $\mathcal{K}_2 = \{N, \dots, 2N-2\}$ and $\mathcal{K}_3 = \{2N-1, \dots, 3N-3\}$. Thus, for z^* , the set of all active indexes associated with multiplier η is the set

$$\mathcal{I}(z^*, \beta) = \mathcal{I}_1(z^*, \beta) \cup \mathcal{I}_2(z^*, \beta) \cup \mathcal{I}_3(z^*, \beta).$$

Next, we define the set of the left-side of all active constraints of a certain solution of $\text{NLP}(\beta)$.

Definition 3.3 (Active Constraints). Suppose that $z^* = [u_1^* \cdots u_{N-1}^*]^T$ is a solution to $\text{NLP}(\beta)$ problem and $\eta = [\eta_1 \cdots \eta_{3N-3}]^T = \mu$ is the corresponding multiplier. With respect to solution z^* , the set composed

by the left-side of active state constraints is defined by

$$C_{1a}(z^*, \beta) = \{-x_{k+1}(z^*, \beta) + x_{\min} : k \in \mathcal{I}_1(z^*, \beta)\}.$$

The sets composed by the left-side of active control constraints are defined by

$$C_{2a}(z^*, \beta) = \{-u_{k-N+1}^* : k \in \mathcal{I}_2(z^*, \beta)\}$$

and

$$C_{3a}(z^*, \beta) = \{u_{k-2N+2}^* - \gamma : k \in \mathcal{I}_3(z^*, \beta)\}.$$

The set of the left-side of all active constraints for z^* is then

$$C_a(z^*, \beta) = C_{1a}(z^*, \beta) \cup C_{2a}(z^*, \beta) \cup C_{3a}(z^*, \beta).$$

We denote the multiplier corresponding to active constraints for z^* by $\eta_a \in \mathbb{R}^{N_a}$, where $N_a = \#\mathcal{I}(z^*, \beta)$.

We are now in the position to derive necessary and sufficient optimality conditions for **NLP**(β).

Proposition 3.4 (Necessary and Sufficient Optimality Conditions). *Let $\beta_0 \in [0, \beta_M]$ be a fixed, or nominal, value for the parameter β . $\hat{z} = [\hat{u}_1 \ \cdots \ \hat{u}_{N-1}]^T$ is an optimal solution of **NLP**(β_0) such that*

$$\text{rank} \left(\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0) \right) = \#\mathcal{I}(\hat{z}, \beta_0)$$

if and only if there is a unique multiplier $\hat{\eta} = [\hat{\mu}_1^T \ \hat{\mu}_2^T \ \hat{\mu}_3^T]^T$ satisfying the following conditions:

$$\begin{cases} \theta \begin{bmatrix} 2\hat{u}_1 & \cdots & 2\hat{u}_{N-1} \end{bmatrix}^T + A^T \hat{\mu}_1 - \hat{\mu}_2 + \hat{\mu}_3 = 0_{\mathbb{R}^{N-1}}; \\ \hat{\eta} \in (\mathbb{R}_0^+)^{3N-3}; \\ \hat{\eta}_i = 0, \ \forall i \notin \mathcal{I}(\hat{z}, \beta_0). \end{cases} \quad (3.10)$$

Proof. To derive necessary and sufficient optimality conditions for non-linear programming problems, one can see, for example, [37]. Observe that $\beta_0 \in [0, \beta_M]$ is a nominal value and $\hat{z} = [\hat{u}_1 \ \cdots \ \hat{u}_{N-1}]^T$ is an optimal solution of **NLP**(β_0) for which the Jacobian $\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0)$ has full rank $\#\mathcal{I}(\hat{z}, \beta_0)$. The multiplier $\hat{\eta} = [\hat{\mu}_1^T \ \hat{\mu}_2^T \ \hat{\mu}_3^T]^T$ given by the Karush–Kuhn–Tucker (KKT) conditions:

$$\begin{cases} \frac{\partial L}{\partial z}(\hat{z}, \hat{\eta}, \beta_0) = 0; \\ \hat{\eta} \in (\mathbb{R}_0^+)^{3N-3}; \\ \hat{\eta}_i = 0, \ \forall i \notin \mathcal{I}(\hat{z}, \beta_0), \end{cases} \quad (3.11)$$

is unique, because $\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0)$ is of full rank (see [23], Thm. 1), where $L : \mathbb{R}^{N-1} \times \mathbb{R}^{3N-3} \times [0, \beta_M] \rightarrow \mathbb{R}$, defined by

$$L(z, \mu, \beta) = G(X_1(z, \beta), X_N(z, \beta), \beta) + \sum_{i=1}^{N-1} \left(\sum_{j=1}^3 \mu_{ji} C_j(X_i(z, \beta), u_i, \beta) \right)$$

with

$$\mu^T = [\mu_1^T \quad \mu_2^T \quad \mu_3^T], \quad \mu_j^T = [\mu_{j1} \quad \mu_{j2} \quad \cdots \quad \mu_{j(N-1)}] \quad \text{and} \quad \mu_{ji} = \mu_j(\tau_i)$$

for $j = 1, 2, 3$ and $i = 1, \dots, N-1$, is the Lagrangian associated with $\mathbf{NLP}(\beta)$. Note that it can be rewritten as it follows:

$$\begin{aligned} L(z, \mu, \beta) &= w_N(z, \beta) + \sum_{i=1}^{N-1} [\mu_{1i} C_1(X_i(z, \beta), u_i, \beta) + \mu_{2i} C_2(X_i(z, \beta), u_i, \beta) \\ &\quad + \mu_{3i} C_3(X_i(z, \beta), u_i, \beta)] \\ &= w_N(z, \beta) + \sum_{i=1}^{N-1} [\mu_{1i} (x_{\min} - x_{i+1}(z, \beta)) - \mu_{2i} u_i + \mu_{3i} (u_i - \gamma)]. \end{aligned}$$

Writing the state variables as functions of the controls (see (3.2) and (3.8)), we get

$$\begin{aligned} L(z, \mu, \beta) &= \theta \sum_{i=1}^{N-1} (u_i)^2 \\ &\quad + \mu_{11} \left(\sum_{j=1}^{N-1} a_{1j} u_j + b_1 \right) + \cdots + \mu_{1(N-1)} \left(\sum_{j=1}^{N-1} a_{(N-1)j} u_j + b_{N-1} \right) \\ &\quad + \sum_{i=1}^{N-1} [-\mu_{2i} u_i + \mu_{3i} (u_i - \gamma)] \\ &= \theta \sum_{i=1}^{N-1} (u_i)^2 + \sum_{i=1}^{N-1} \left\{ \mu_{1i} \left(\sum_{j=1}^{N-1} a_{ij} u_j + b_i \right) - \mu_{2i} u_i + \mu_{3i} (u_i - \gamma) \right\} \end{aligned} \quad (3.12)$$

$$= \theta \sum_{i=1}^{N-1} (u_i)^2 + (\mu_1^T A - \mu_2^T I_{N-1} + \mu_3^T I_{N-1}) z + \mu_1^T B - \mu_3^T \tilde{\gamma}. \quad (3.13)$$

Thus, from (3.13), (3.11) is equivalent to (3.10). Note that the KKT conditions (3.11) are also sufficient, since $\mathbf{NLP}(\beta_0)$ is a convex problem. This concludes the proof. $\square$

The Proposition 3.4 is crucial in the next section, where second order sufficient optimality conditions are established for the $\mathbf{NLP}(\beta)$ problem.

3.3. Second order sufficient optimality conditions for the $\mathbf{NLP}(\beta)$

The verification of second order sufficient optimality conditions are crucial for the Section 4, where a sensitivity analysis is carried out. They allow us to prove that the optimal state and control to $\mathbf{NLP}(\beta)$ are differentiable with respect to β (see [23]), an essential property for the study of sensitivity analysis associated with $\mathbf{NLP}(\beta)$ (see Sect. 4.1).

Proposition 3.5 (Second Order Sufficient Optimality Conditions). *For a given nominal value β_0 , let $\hat{z}$ and $\hat{\eta}$ be, respectively, an admissible point to $\mathbf{NLP}(\beta_0)$ and the corresponding multiplier which satisfy (3.10). Moreover, assume that H is a $(N-1) \times (N-1-N_a)$ matrix with full column rank, whose columns span the kernel of $\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0)$. If*

- $\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0) = \#\mathcal{I}(\hat{z}, \beta_0)$;
- the multiplier corresponding to active constraints $\eta_a \in (\mathbb{R}^+)^{N_a}$;
- the matrix $2\theta H^T H$ has only positive eigenvalues;

then $\hat{z}$ is a strong local minimum of $\mathbf{NLP}(\beta_0)$.

Proof. Being β_0 a nominal value, let us start by considering that $\hat{z}$ and $\hat{\eta}$ are, respectively, an admissible point to $\mathbf{NLP}(\beta_0)$ and the corresponding multiplier which satisfy (3.10). Additionally, suppose that H is a $(N-1) \times (N-1-N_a)$ matrix with full column rank, whose columns span the kernel of $\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0)$. From [23], Theorem 2, if $\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0)$ has full rank $\#\mathcal{I}(\hat{z}, \beta_0)$, $\eta_a \in (\mathbb{R}^+)^{N_a}$ and the matrix

$$H^T \frac{\partial^2 L}{\partial z^2}(\hat{z}, \hat{\eta}, \beta_0) H = H^T (2\theta I_{N-1}) H = 2\theta H^T H$$

has only positive eigenvalues, then $\hat{z}$ is a strong local minimum of $\mathbf{NLP}(\beta_0)$. The eigenvalues of $2\theta I_{N-1}$ are always positive, independently of solution $\hat{z}$. Then, we conclude that $\hat{z}$ is indeed a strong local minimum of $\mathbf{NLP}(\beta_0)$. This concludes the proof. $\square$

3.4. Post-optimal estimation of the adjoint function

In this section we get a post-optimal estimation of the adjoint function introduced in the Section 2.2. For that, let us first introduce the following notation:

$$\lambda_{\tau_i}^T = [\lambda_{1i} \quad \lambda_{2i} \quad \lambda_{3i}] \quad \text{and} \quad \mu_{\tau_j}^T = [\mu_{1j} \quad \mu_{2j} \quad \mu_{3j}]$$

for $i = 1, \dots, N$ and $j = 1, \dots, N-1$, where $\lambda_{si} = \lambda_s(\tau_i)$ for $s = 1, 2, 3$ and λ_{τ_i} is an approximation of $\lambda(\tau_i)$ associated with $\mathbf{OCP}(\beta)$.

Proposition 3.6 (Post-optimal Estimation of the Adjoint Function). *For a given nominal value β_0 , let $\hat{z}$ and $\hat{\mu}$ be, respectively, an optimal solution of $\mathbf{NLP}(\beta_0)$ and the corresponding multiplier. Assuming that $\theta \hat{\mu}_{1j} \approx \hat{\mu}_{1j}$ for all $j = 1, \dots, N-1$, a post-optimal estimation for the adjoint function values is given by:*

$$\begin{cases} \hat{\lambda}_{1i} = (1 - \theta\beta) \hat{\lambda}_{1(i+1)} - \hat{\mu}_{1i}, \quad \forall i = 1, \dots, N-1, \\ \hat{\lambda}_{1N} = 0, \end{cases} \quad (3.14)$$

and

$$\begin{cases} \hat{\lambda}_{2i} = 0, \quad \forall i = 1, \dots, N, \\ \hat{\lambda}_{3i} = 1, \quad \forall i = 1, \dots, N. \end{cases} \quad (3.15)$$

Proof. The value of $\hat{\lambda}_{\tau_i}$ can be computed after solving $\mathbf{NLP}(\beta_0)$. Once the $\mathbf{NLP}(\beta_0)$ problem solved, we obtain the multiplier $\hat{\mu}_{1j}$ for $j = 1, \dots, N-1$, related to the inequality constraints

$$-x_i + x_{\min} \leq 0, \quad \forall i = 2, \dots, N$$

of $\mathbf{NLP}(\beta_0)$, that is needed to compute (3.14) and (3.15).

From the second equation of (2.5), we get $\hat{\lambda}_1(t_f) = 0$. Recalling that $t_f = (N-1)\theta = \tau_N$, we obtain

$$\hat{\lambda}_{1N} = \hat{\lambda}_1(\tau_N) = 0$$

that is the second equation of (3.14). From the first equation of (2.5), we get

$$\dot{\hat{\lambda}}_1(t) = \beta \hat{\lambda}_1(t) + \hat{\mu}_1(t).$$

Approximating $\dot{\hat{\lambda}}_1(t)$ through the finite backward difference method, we obtain:

$$\begin{aligned} \hat{\lambda}_1(\tau_{i+1}) - \hat{\lambda}_1(\tau_i) &= \theta \beta \hat{\lambda}_1(\tau_{i+1}) + \hat{\mu}_1(\tau_i) \Leftrightarrow \hat{\lambda}_{1(i+1)} - \hat{\lambda}_{1i} = \theta \beta \hat{\lambda}_{1(i+1)} + \hat{\mu}_{1i} \\ &\Leftrightarrow \hat{\lambda}_{1i} = \hat{\lambda}_{1(i+1)} - \theta \beta \hat{\lambda}_{1(i+1)} - \hat{\mu}_{1i} \\ &\Leftrightarrow \hat{\lambda}_{1i} = (1 - \theta \beta) \hat{\lambda}_{1(i+1)} - \hat{\mu}_{1i} \end{aligned}$$

for all $i = 1, \dots, N-1$, under the approximation $\theta \hat{\mu}_{1j} \approx \hat{\mu}_{1j}$ for all $j = 1, \dots, N-1$. Thus, we get the first equation of (3.14).

Finally, from (2.6), we have that $\hat{\lambda}_2(t) = 0$ and $\hat{\lambda}_3(t) = 1$. So, the equations of (3.15) are obvious. This concludes the proof. $\square$

In conclusion, we solve $\mathbf{NLP}(\beta_0)$ and, consequently, we get the values of multiplier μ_{1i} which appear in (3.14). Using the second equation of (3.14) as the starting point, we determine estimations for $\lambda_1(\tau_i)$ for all $i = 1, \dots, N-1$, by the recursive rule in (3.14). Additionally, from the $\mathbf{NLP}(\beta_0)$ problem solving, we can also recover the discretized state variables x_i and w_i , for all $i = 1, \dots, N$, using the computed discretized controls in (3.9) and (3.3), respectively.

4. SENSITIVITY ANALYSIS

In this section we provide a sensitivity analysis with respect to $\mathbf{OCP}(\beta)$, following the methodology of [10, 23]. Being β_0 a fixed value of $\beta \in [0, \beta_M]$, we consider $\mathbf{OCP}(\beta_0)$ as the *unperturbed* optimal control problem, also called *nominal* optimal control problem. Since we solve numerically $\mathbf{OCP}(\beta_0)$ using NLP techniques and the discretization described in Section 3.1, the $\mathbf{NLP}(\beta_0)$ is the *unperturbed* NLP problem associated with $\mathbf{OCP}(\beta_0)$.

4.1. First order sensitivity analysis of the optimal solution

In the current section we proceed with a sensitivity analysis for an optimal solution of $\mathbf{NLP}(\beta)$. Assume that

$$[\hat{X}^T \quad \hat{z}^T] = [\hat{X}_1 \quad \dots \quad \hat{X}_N \quad \hat{u}_1 \quad \dots \quad \hat{u}_{N-1}] \quad \text{and} \quad \hat{\eta} = [\hat{\eta}_1 \quad \dots \quad \hat{\eta}_{3N-3}]^T = [\hat{\mu}_1^T \quad \hat{\mu}_2^T \quad \hat{\mu}_3^T]^T$$

are an admissible point and a multiplier for the unperturbed problem $\mathbf{NLP}(\beta_0)$, satisfying the conditions of Propositions 3.4 and 3.5. Then, [23], Theorem 3 asserts that there is a neighbourhood $B_0 \subset [0, \beta_M]$ of β_0 and that there are continuously differentiable functions $z : B_0 \rightarrow \mathbb{R}^{N-1}$ and $\eta : B_0 \rightarrow \mathbb{R}^{3N-3}$ with the following properties:

- (A) $z(\beta_0) = \hat{z}$ and $\eta(\beta_0) = \hat{\eta}$;
- (B) the active sets are constant in B_0 , that is, $\mathcal{I}(z(\beta), \beta) \equiv \mathcal{I}(\hat{z}, \beta_0)$ for all $\beta \in B_0$;
- (C) $\text{rank} \left(\frac{\partial C_a}{\partial z} \right) (z(\beta), \beta) = N_a$ for all $\beta \in B_0$;
- (D) for all $\beta \in B_0$, the pair $(z(\beta), \eta(\beta))$ satisfies the second order sufficient conditions of Proposition 3.5 for the perturbed problem $\mathbf{NLP}(\beta)$. In particular, $(z(\beta), \eta(\beta))$ is a strong local minimum of $\mathbf{NLP}(\beta)$.

Next, we provide approximate solutions to $\mathbf{NLP}(\beta)$, determined by the first order Taylor expansion around the nominal solution to the unperturbed problem $\mathbf{NLP}(\beta_0)$ for a fixed β_0 . However, let us first recall that for all $t \in [0, t_f]$ and $\beta \in [0, \beta_M]$, $(X(t, \beta), u(t, \beta)) = (x(t, \beta), y(t, \beta), w(t, \beta), u(t, \beta))$ and $\lambda(t, \beta)$ are family of solutions

and adjoint functions associated with **OCP**(β), and that $(\hat{X}(t), \hat{u}(t)) = (\hat{x}(t), \hat{y}(t), \hat{w}(t), \hat{u}(t))$ and $\hat{\lambda}(t)$ are a solution and the associated adjoint function to **OCP**(β_0).

Proposition 4.1. *For a given nominal value β_0 , let $[\hat{X}^T \ \hat{z}^T]$ be an optimal solution of **NLP**(β_0). Assuming that the state functions $x(t, \beta)$ and $w(t, \beta)$ are of class $\mathcal{C}^1$, with respect to both variables t and β , and that the control $u(t, \beta)$ and adjoint function $\lambda(t, \beta)$ are, at least, piecewise of class $\mathcal{C}^1$, then we get the following approximations:*

$$\left\{ \begin{array}{l} x(\tau_i, \beta) \approx \hat{x}_i - [a_{(i-1)1} \ a_{(i-1)2} \ \cdots \ a_{(i-1)(N-1)}] \frac{dz}{d\beta}(\beta_0)(\beta - \beta_0) \\ \quad - \left(\theta(i-1)\bar{\beta}^{i-2} + \theta^2 \sum_{k=1}^{i-2} (i-1-k)\bar{\beta}^{i-2-k}(g_k + u_k) \right) (\beta - \beta_0), \ \forall i = 2, \dots, N; \\ w(\tau_i, \beta) \approx \hat{w}_i + 2\theta u_{i-1} e_{i-1} \frac{dz}{d\beta}(\beta_0)(\beta - \beta_0), \ \forall i = 2, \dots, N; \\ u(\tau_i, \beta) \approx \hat{u}_i + \frac{du_i}{d\beta}(\beta_0)(\beta - \beta_0), \ \forall i = 1, \dots, N-1; \\ \lambda_1(\tau_i, \beta) \approx [\mathbf{0}_{1 \times (N-1)} \ -\mathbf{1}_{\mathcal{I}_1(\hat{z}, \beta_0)}(1) \ \cdots \ -\mathbf{1}_{\mathcal{I}_1(\hat{z}, \beta_0)}(N-1) \ \mathbf{0}_{1 \times (2N-2)}] \begin{bmatrix} \frac{dz}{d\beta}(\beta_0) \\ \frac{d\eta_a}{d\beta}(\beta_0) \end{bmatrix} (\beta - \beta_0) \\ \quad + \hat{\lambda}_{1i} - \theta \hat{\lambda}_{1(i+1)}(\beta - \beta_0), \ \forall i = 1, \dots, N-1; \end{array} \right. \quad (4.1)$$

where $e_j \in \mathbb{R}^{N-1}$ is the canonical vector of coordinate $j \in \{1, \dots, N-1\}$, $\mathbf{1}_D$ is the indicator function defined by

$$\mathbf{1}_D(p) = \begin{cases} 1, & \text{if } p \in D \\ 0, & \text{if } p \notin D \end{cases},$$

$$\begin{bmatrix} \frac{dz}{d\beta}(\beta_0) \\ \frac{d\eta_a}{d\beta}(\beta_0) \end{bmatrix} = - \begin{bmatrix} 2\theta I_{N-1} & \left(\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0) \right)^T \\ \frac{\partial C_a}{\partial z}(\hat{z}, \beta_0) & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial u_1} \sum_{i \in \mathcal{I}_1(\hat{z}, \beta_0)} \eta_i \frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N-1} a_{ij} u_j + b_i \right) \Big|_{(\hat{z}, \hat{\eta}_a, \beta_0)} \\ \vdots \\ \frac{\partial}{\partial u_{N-1}} \sum_{i \in \mathcal{I}_1(\hat{z}, \beta_0)} \eta_i \frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N-1} a_{ij} u_j + b_i \right) \Big|_{(\hat{z}, \hat{\eta}_a, \beta_0)} \\ \frac{\partial C_a}{\partial \beta}(\hat{z}, \beta_0) \end{bmatrix}$$

and $\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0)$ and $\frac{\partial C_a}{\partial \beta}(\hat{z}, \beta_0)$ are matrices, with dimensions $N_a \times (N-1)$ and $N_a \times 1$, obtained from the matrices

$$\frac{\partial C}{\partial z}(\hat{z}, \beta_0) = \begin{bmatrix} A|_{\beta=\beta_0} \\ -I_{N-1} \\ I_{N-1} \end{bmatrix} \quad \text{and} \quad \frac{\partial C}{\partial \beta}(\hat{z}, \beta_0) = \begin{bmatrix} \frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N-1} a_{1j} u_j + b_1 \right) \Big|_{(\hat{z}, \beta_0)} \\ \vdots \\ \frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N-1} a_{(N-1)j} u_j + b_{N-1} \right) \Big|_{(\hat{z}, \beta_0)} \\ 0_{\mathbb{R}^{2N-2}} \end{bmatrix},$$

respectively, by deleting all rows i such that $i \notin \mathcal{I}(\hat{z}, \beta_0)$.

Proof. Assuming that the state functions $x(t, \beta)$ and $w(t, \beta)$ are of class $\mathcal{C}^1$, with respect to both variables t and β , and, moreover, that the control $u(t, \beta)$ and adjoint function $\lambda(t, \beta)$ are, at least, piecewise of class $\mathcal{C}^1$, then we can construct an approximation of the perturbed solution $\zeta(t, \beta)$ considering the following first order Taylor expansion

$$\zeta(t, \beta) \approx \hat{\zeta}(t) + \frac{\partial \zeta}{\partial \beta}(t, \beta_0)(\beta - \beta_0), \quad (4.2)$$

for any $\zeta \in \{x, w, u, \lambda\}$ and $\hat{\zeta} \in \{\hat{x}, \hat{w}, \hat{u}, \hat{\lambda}\}$. Note that in (4.2):

- ζ can represent the control solution u , or any state solution x , or w of **OCP**(β), or the corresponding adjoint function λ , with $\beta \in [0, \beta_M]$;
- $\hat{\zeta}$ can represent the nominal control solution $\hat{u}$, or the nominal state solution $\hat{x}$, or $\hat{w}$ of **OCP**(β_0), or the corresponding nominal adjoint function $\hat{\lambda}$, with β_0 being a fixed value for parameter β .

The derivative $\frac{\partial \zeta}{\partial \beta}(t, \beta_0)$ is called the *sensitivity differential* for $\zeta \in \{x, w, u, \lambda\}$ with respect to β . Observe that (4.2) holds for all $t \in [0, t_f]$. In particular, it holds for the mesh points

$$\tau_i = (i-1) \frac{t_f}{N-1} \in [0, t_f], \quad i = 1, \dots, N.$$

So, we have

$$\begin{cases} \zeta(\tau_i, \beta) \approx \hat{\zeta}_i + \frac{\partial \zeta}{\partial \beta}(\tau_i, \beta_0)(\beta - \beta_0), \quad \forall i = 2, \dots, N \quad \text{and} \quad \forall \zeta \in \{x, w\}; \\ u(\tau_i, \beta) \approx \hat{u}_i + \frac{\partial u}{\partial \beta}(\tau_i, \beta_0)(\beta - \beta_0), \quad \forall i = 1, \dots, N-1; \\ \lambda(\tau_i, \beta) \approx \hat{\lambda}_i + \frac{\partial \lambda}{\partial \beta}(\tau_i, \beta_0)(\beta - \beta_0), \quad \forall i = 1, \dots, N. \end{cases} \quad (4.3)$$

The *sensitivity differentials* at each mesh point can be obtained by the Chain Rule, as it follows:

$$\begin{cases} \frac{\partial \zeta}{\partial \beta}(\tau_i, \beta_0) = \frac{\partial \zeta_i}{\partial z}(\hat{z}, \beta_0) \frac{dz}{d\beta}(\beta_0) + \frac{\partial \zeta_i}{\partial \beta}(\hat{z}, \beta_0), \quad \forall i = 2, \dots, N, \text{ where } \zeta \in \{x, w\}, \\ \frac{\partial u}{\partial \beta}(\tau_i, \beta_0) = \frac{du_i}{d\beta}(\beta_0), \quad \forall i = 1, \dots, N-1, \\ \frac{\partial \lambda}{\partial \beta}(\tau_i, \beta_0) = \begin{bmatrix} \frac{\partial \lambda_{\tau_i}}{\partial z}(\hat{z}, \hat{\eta}, \beta_0) & \frac{\partial \lambda_{\tau_i}}{\partial \eta_a}(\hat{z}, \hat{\eta}, \beta_0) \end{bmatrix} \begin{bmatrix} \frac{dz}{d\beta}(\beta_0) \\ \frac{d\eta_a}{d\beta}(\beta_0) \end{bmatrix} + \frac{\partial \lambda_{\tau_i}}{\partial \beta}(\hat{z}, \hat{\eta}, \beta_0), \quad \forall i = 1, \dots, N. \end{cases} \quad (4.4)$$

To compute the previous sensitivity differentials, we need to know $\frac{dz}{d\beta}(\beta_0)$ and $\frac{d\eta_a}{d\beta}(\beta_0)$ which can be obtained from

$$\begin{bmatrix} \frac{dz}{d\beta}(\beta_0) \\ \frac{d\eta_a}{d\beta}(\beta_0) \end{bmatrix} = \begin{bmatrix} \frac{du_1}{d\beta}(\beta_0) \\ \vdots \\ \frac{du_{N-1}}{d\beta}(\beta_0) \\ \frac{d\eta_a}{d\beta}(\beta_0) \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 L}{\partial z^2}(\hat{z}, \hat{\eta}_a, \beta_0) & \left(\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0) \right)^T \\ \frac{\partial C_a}{\partial z}(\hat{z}, \beta_0) & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^2 L}{\partial z \partial \beta}(\hat{z}, \hat{\eta}_a, \beta_0) \\ \frac{\partial C_a}{\partial \beta}(\hat{z}, \beta_0) \end{bmatrix}.$$

With respect to $\frac{\partial^2 L}{\partial z^2}(\hat{z}, \hat{\eta}_a, \beta_0)$ and to $\frac{\partial^2 L}{\partial z \partial \beta}(\hat{z}, \hat{\eta}_a, \beta_0)$, we have $\frac{\partial^2 L}{\partial z^2}(\hat{z}, \hat{\eta}_a, \beta_0) = 2\theta I_{N-1}$, from (3.13), and

$$\frac{\partial L}{\partial \beta}(\hat{z}, \hat{\eta}_a, \beta_0) = \sum_{i \in \mathcal{I}_1(\hat{z}, \beta_0)} \eta_i \frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N-1} a_{ij} u_j + b_i \right) \Big|_{(\hat{z}, \hat{\eta}_a, \beta_0)}, \quad (4.5)$$

from (3.12), which implies that

$$\frac{\partial^2 L}{\partial z \partial \beta}(\hat{z}, \hat{\eta}_a, \beta_0) = \begin{bmatrix} \frac{\partial}{\partial u_1} \sum_{i \in \mathcal{I}_1(\hat{z}, \beta_0)} \eta_i \frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N-1} a_{ij} u_j + b_i \right) \Big|_{(\hat{z}, \hat{\eta}_a, \beta_0)} \\ \vdots \\ \frac{\partial}{\partial u_{N-1}} \sum_{i \in \mathcal{I}_1(\hat{z}, \beta_0)} \eta_i \frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N-1} a_{ij} u_j + b_i \right) \Big|_{(\hat{z}, \hat{\eta}_a, \beta_0)} \end{bmatrix}_{(N-1) \times 1}.$$

Moreover, we have

$$\frac{\partial C}{\partial z}(\hat{z}, \beta_0) = \begin{bmatrix} A|_{\beta=\beta_0} \\ -I_{N-1} \\ I_{N-1} \end{bmatrix} \quad \text{and} \quad \frac{\partial C}{\partial \beta}(\hat{z}, \beta_0) = \begin{bmatrix} \frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N-1} a_{1j} u_j + b_1 \right) \Big|_{(\hat{z}, \beta_0)} \\ \vdots \\ \frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N-1} a_{(N-1)j} u_j + b_{N-1} \right) \Big|_{(\hat{z}, \beta_0)} \\ 0_{\mathbb{R}^{2N-2}} \end{bmatrix}.$$

Consequently, $\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0)$ and $\frac{\partial C_a}{\partial \beta}(\hat{z}, \beta_0)$ are matrices, with dimensions $N_a \times (N-1)$ and $N_a \times 1$, respectively, which are obtained from the matrices $\frac{\partial C}{\partial z}(\hat{z}, \beta_0)$ and $\frac{\partial C}{\partial \beta}(\hat{z}, \beta_0)$, respectively, by deleting all rows i such that $i \notin \mathcal{I}(\hat{z}, \beta_0)$.

Joining the first equations of (4.3) and (4.4), we get:

$$\begin{aligned} x(\tau_i, \beta) &\approx \hat{x}_i + \frac{\partial x}{\partial \beta}(\tau_i, \beta_0)(\beta - \beta_0) \\ &= \hat{x}_i + \left(\frac{\partial x_i}{\partial z}(\hat{z}, \beta_0) \frac{dz}{d\beta}(\beta_0) + \frac{\partial x_i}{\partial \beta}(\hat{z}, \beta_0) \right) (\beta - \beta_0) \\ &= \hat{x}_i - [a_{(i-1)1} \quad a_{(i-1)2} \quad \cdots \quad a_{(i-1)(N-1)}] \frac{dz}{d\beta}(\beta_0)(\beta - \beta_0) \\ &\quad - \left(\theta(i-1)\bar{\beta}^{i-2} + \theta^2 \sum_{k=1}^{i-2} (i-1-k)\bar{\beta}^{i-2-k}(g_k + u_k) \right) (\beta - \beta_0) \end{aligned}$$

and

$$\begin{aligned} w(\tau_i, \beta) &\approx \hat{w}_i + \frac{\partial w}{\partial \beta}(\tau_i, \beta_0)(\beta - \beta_0) \\ &= \hat{w}_i + \left(\frac{\partial w_i}{\partial z}(\hat{z}, \beta_0) \frac{dz}{d\beta}(\beta_0) + \frac{\partial w_i}{\partial \beta}(\hat{z}, \beta_0) \right) (\beta - \beta_0) \\ &= \hat{w}_i + 2\theta u_{i-1} e_{i-1} \frac{dz}{d\beta}(\beta_0)(\beta - \beta_0) \end{aligned}$$

for all $i \in \{2, \dots, N\}$. From the second equations of (4.3) and (4.4), we obtain for the control

$$u(\tau_i, \beta) \approx \hat{u}_i + \frac{\partial u}{\partial \beta}(\tau_i, \beta_0)(\beta - \beta_0) = \hat{u}_i + \frac{du_i}{d\beta}(\beta_0)(\beta - \beta_0)$$

for all $i \in \{1, \dots, N-1\}$. Finally, from the third equations of (4.3) and (4.4), for the adjoint function, we have

$$\begin{aligned} \lambda_1(\tau_i, \beta) &\approx \hat{\lambda}_{1i} + \frac{\partial \lambda_1}{\partial \beta}(\tau_i, \beta_0)(\beta - \beta_0) \\ &= \hat{\lambda}_{1i} + \left(\begin{bmatrix} \frac{\partial \lambda_{1i}}{\partial z}(\hat{z}, \hat{\eta}, \beta_0) & \frac{\partial \lambda_{1i}}{\partial \eta_a}(\hat{z}, \hat{\eta}, \beta_0) \end{bmatrix} \begin{bmatrix} \frac{dz}{d\beta}(\beta_0) \\ \frac{d\eta_a}{d\beta}(\beta_0) \end{bmatrix} + \frac{\partial \lambda_{1i}}{\partial \beta}(\hat{z}, \hat{\eta}, \beta_0) \right) (\beta - \beta_0) \\ &= \hat{\lambda}_{1i} + [\mathbf{0}_{1 \times (N-1)} \quad -\mathbf{1}_{\mathcal{I}(\hat{z}, \beta_0)}(1) \quad \cdots \quad -\mathbf{1}_{\mathcal{I}(\hat{z}, \beta_0)}(N-1) \quad \mathbf{0}_{1 \times (2N-2)}] \begin{bmatrix} \frac{dz}{d\beta}(\beta_0) \\ \frac{d\eta_a}{d\beta}(\beta_0) \end{bmatrix} (\beta - \beta_0) \\ &\quad - \theta \hat{\lambda}_{1(i+1)}(\beta - \beta_0) \end{aligned}$$

for all $i \in \{1, \dots, N-1\}$. □

4.2. First order sensitivity analysis of the optimal value function

We are now in the position to obtain a sensitivity analysis associated with the optimal cost, which is of foremost importance in this work, since it allows us to know how the optimal cost is affected by the perturbations of β in a neighbourhood B_0 . In the next proposition, we provide an approximate optimal cost associated with $\mathbf{NLP}(\beta)$ through the first order Taylor expansion around the optimal cost of the unperturbed problem $\mathbf{NLP}(\beta_0)$ for $\beta \in B_0$.

Proposition 4.2. *For a given nominal value β_0 , let $\tilde{G}(\hat{z}, \beta_0)$ be the optimal cost of $\mathbf{NLP}(\beta_0)$. An approximation of the perturbed optimal cost $\tilde{G}(z(\beta), \beta)$ for $\beta \in B_0$, is given by*

$$\tilde{G}(z(\beta), \beta) \approx \tilde{G}(\hat{z}, \beta_0) + \sum_{i \in \mathcal{I}_1(\hat{z}, \beta_0)} \eta_i \frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N-1} a_{ij} u_j + b_i \right) \Big|_{(\hat{z}, \hat{\eta}_a, \beta_0)} (\beta - \beta_0). \quad (4.6)$$

Proof. Following [23], the first order Taylor expansion

$$\tilde{G}(z(\beta), \beta) \approx \tilde{G}(\hat{z}, \beta_0) + \frac{d\tilde{G}}{d\beta}(\hat{z}, \beta_0)(\beta - \beta_0)$$

yields an approximation of the perturbed optimal cost $\tilde{G}(z(\beta), \beta)$ for $\beta \in B_0$, where

$$\frac{d\tilde{G}}{d\beta}(\hat{z}, \beta_0) = \frac{\partial L}{\partial \beta}(\hat{z}, \hat{\eta}_a, \beta_0)$$

is given by the right-side of (4.5). □

5. A NUMERICAL EXAMPLE

Next, we provide some numerical simulations associated with $\mathbf{NLP}(\beta)$ in order to illustrate the results of Section 4. For such computations, we consider the set of values of Table 2.

We recall that $g(t)$ is defined as $g(t) = \text{rfall}(t) - \text{evtp}(t)$, where $\text{rfall}(t)$ is the daily precipitation and $\text{evtp}(t)$ is the estimated evapotranspiration for the type of crop, at t . The precipitation data is collected from a weather station that takes place in *Carrazeda de Ansiães* – Portugal and the evapotranspiration is obtained from the product between the crop’s coefficient and the referenced evapotranspiration which is calculated according with [38]. Note that the jump in the values of the vector $[g_1 \ \cdots \ g_{10}]$, corresponding to the passage from the seventh to the eighth day, is due to sudden rainfall.

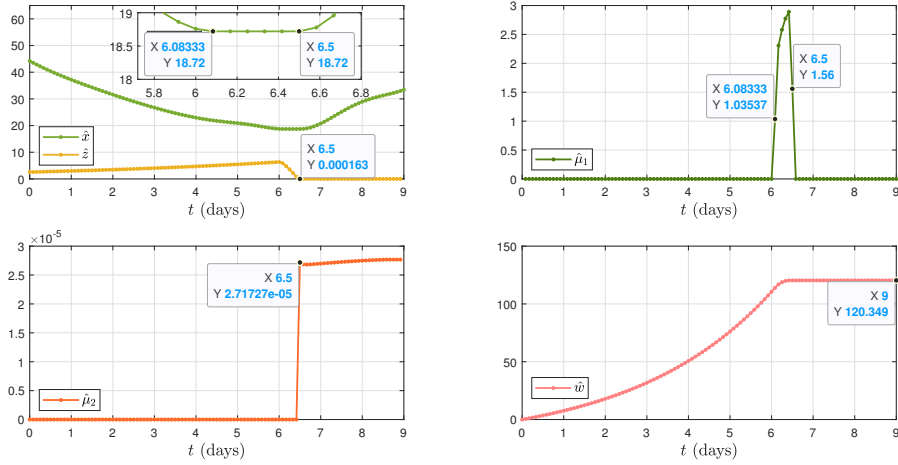
Note that we only have daily information with respect to g . As we intend to work numerically with a mesh-grid with more than 10 points, we do an interpolation of g values of Table 2, using MATLAB function `interp1`, considering the method `spline` and the vector $xq = [0 \ \theta \ 2\theta \ \cdots \ t_f]$ for the coordinates of the query points.

5.1. Numerical results of the nominal problem

The discretized problem (3.7) for $\beta = \beta_0$ is formulated with the help of the Applied Modeling Programming Language – AMPL (see [39, 40]) interfaced with the Interior-Point optimization solver – IPOPT – which is developed by Wächter and Biegler [41] to solve non-linear programs, within the NEOS Server. As one can observe by accessing the website <https://neos-server.org/neos/solvers/nco:Ipopt/AMPL.html>, “IPOPT is an open-source software package for large-scale nonlinear optimization. It implements a primal-dual interior point algorithm, which uses a filter line-search method to ensure global convergence”. For the integration method, we implement the Euler’s method with the values of Table 2. The tolerance and the maximum number of iterations for IPOPT are set to 10^{-8} and 5×10^3 .

TABLE 2. Values for parameters.

Parameter	Value	SI Unity	Parameter	Value	SI Unity
N	109	—	g_1	-4.3415	mm/day (per m ²)
θ	1/12	Day	g_2	-3.3820	mm/day (per m ²)
β_0	0.15	—	g_3	-3.9805	mm/day (per m ²)
β_M	0.25	—	g_4	-4.3890	mm/day (per m ²)
t_f	9	Day	g_5	-4.1230	mm/day (per m ²)
a	44.2362	mm	g_6	-4.0565	mm/day (per m ²)
$x_{\min}$	18.7200	mm	g_7	-4.0090	mm/day (per m ²)
γ	1000	mm/day (per m ²)	g_8	11.0315	mm/day (per m ²)
			g_9	10.1385	mm/day (per m ²)
			g_{10}	11.5635	mm/day (per m ²)

FIGURE 1. (Colour only in online version) Nominal solutions $\hat{x}$, $\hat{z}$ and $\hat{w}$, as well as the nominal multipliers $\hat{\mu}_1$ and $\hat{\mu}_2$ of $\mathbf{NLP}(\beta_0)$, under the parameter values of Table 2.

Nominal solutions $\hat{x}$, $\hat{z}$ and $\hat{w}$, as well as the nominal multipliers $\hat{\mu}_1$ and $\hat{\mu}_2$ are drawn in Figure 1. Notice that we do not present $\hat{\mu}_3$, since it is a null vector. Moreover, we get $\mathcal{I}(\hat{z}, \beta_0) = \{74, 75, 76, 77, 78, 79\} \cup \{187, 188, 189, \dots, 216\}$, $N_a = 36$ and the optimal cost $\tilde{G}(\hat{z}, \beta_0) \simeq 120.34867$.

5.2. Verification of the optimality conditions

In what follows, we verify that indeed $\text{rank}(\frac{\partial C_a}{\partial \hat{z}}(\hat{z}, \beta_0)) = N_a$ and $\hat{z}$ together with $\hat{\eta}^T = [\hat{\mu}_1^T \ \hat{\mu}_2^T \ \hat{\mu}_3^T]$ satisfy the KKT conditions (3.10) of Proposition 3.4.

From our numerical calculations, we obtain $\text{rank}(\frac{\partial C_a}{\partial \hat{z}}(\hat{z}, \beta_0)) = 36 = N_a$ and $\frac{\partial L}{\partial \hat{z}}(\hat{z}, \hat{\eta}, \beta_0)$ is approximately the null vector in $\mathbb{R}^{108}$. Thus, we deduce that the first condition of (3.10) is verified. Observe in Figure 1 that $\hat{\mu}_i, \forall i \in \{1, 2\}$, are not negative and recall that $\hat{\mu}_3$ is a null vector. Therefore, the second condition of (3.10) is satisfied. Confronting the graphs of $\hat{x}$ and $\hat{\mu}_1$ on the top of Figure 1, we conclude that $\hat{\mu}_1$ is equal to zero when the trajectory $\hat{x}$ is not active. On the other hand, the two left side graphs in Figure 1 show that $\hat{\mu}_2$ is equal to zero when the trajectory $\hat{z}$ does not touch the lower bound. So, we also conclude that the third condition of (3.10) is checked. It remains to verify the second order sufficient conditions of Proposition 3.5. We have that $\hat{\eta}_a$ is a positive vector in $\mathbb{R}^{36}$ and, as we expected, the second order sufficient conditions of Proposition 3.5 are

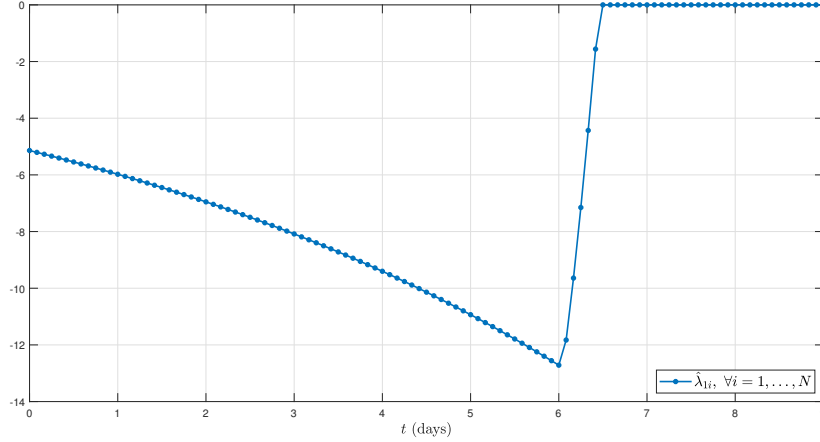


FIGURE 2. (Colour only in online version) Adjoint variables $\hat{\lambda}_{1i}$ for all $i = 1, \dots, 109$.

checked, since the matrix

$$H^T \frac{\partial^2 L}{\partial z^2}(\hat{z}, \hat{\eta}, \beta_0) H = 2\theta I_{72}$$

has obviously only positive eigenvalues, where H is a $(N - 1) \times (N - 1 - N_a)$ matrix with full column rank, whose columns span the kernel of $\frac{\partial C_a}{\partial z}(\hat{z}, \beta_0)$. Thus $\hat{z}$ is a strong local minimum of $\mathbf{NLP}(\beta_0)$ and we can also carry out a sensitivity analysis. Next we estimate the adjoint variables.

5.3. Post-optimal estimation of the adjoint functions

A post-optimal estimation of the adjoint function $\hat{\lambda}_1(\tau_i)$, associated with the state variable x , is denoted by $\hat{\lambda}_{1i}$ for all $i = 1, \dots, 109$, that is, $\hat{\lambda}_1(\tau_i) \simeq \hat{\lambda}_{1i}$, $\forall i = 1, \dots, 109$. To get such estimation, we have to consider the multiplier $\hat{\mu}_1$ drawn in Figure 1 and (3.14) with $\beta = \beta_0$. As result, we get the adjoint function drawn in Figure 2. From (3.15), the post-optimal estimation of adjoint functions $\hat{\lambda}_2(\tau_i)$ and $\hat{\lambda}_3(\tau_i)$ is obvious for all $i = 1, \dots, 109$.

Moreover, having a post-optimal estimation of the adjoint function $\hat{\lambda}_1(\tau_i)$, for all $i = 1, \dots, 109$, we can verify that the characterization of the optimal control, given by (2.7), is fulfilled by the numerical solution of the nominal problem: see Figure 3. Recall that $\lambda_3(\tau_i) = 1$ for all $i = 1, \dots, 109$.

5.4. Sensitivity results

In this section, we provide the results of the sensitivity analysis. As we have seen before, by solving the nominal problem $\mathbf{NLP}(\beta_0)$, we get

$$\hat{x} = [\hat{x}_1 \quad \cdots \quad \hat{x}_{109}]^T \quad \text{and} \quad \hat{z} = [\hat{u}_1 \quad \cdots \quad \hat{u}_{108}]^T$$

(see Fig. 1). Moreover, the adjoint functions are estimated a posteriori: see Section 5.3. Consequently, we determine a numerical and approximate solution to $\hat{\lambda}_1$ which is presented in Figure 2.

Following the steps outlined in Section 4, namely in Propositions 4.1 and 4.2, we now provide approximate optimal solutions and an approximate optimal cost to $\mathbf{NLP}(\beta)$, for values of $\beta \in B_0$, around the optimal solution and optimal cost to $\mathbf{NLP}(\beta_0)$.

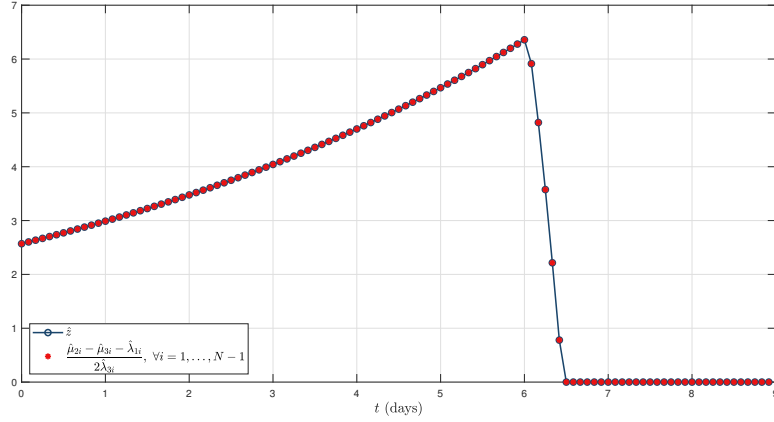


FIGURE 3. (Colour only in online version) Nominal solution $\hat{z}$ and numerical information provided from (2.7), (3.14) and (3.15).

5.4.1. Neighbourhood's size

The necessary and sufficient optimality conditions allow us to assert the existence of a neighbourhood B_0 of β_0 where the approximations (4.1) and (4.6) are valid. This means that there are continuously differentiable functions $z : B_0 \rightarrow \mathbb{R}^{N-1}$ and $\eta : B_0 \rightarrow \mathbb{R}^{3N-3}$ with the properties (A)–(D) of Section 4.1, for all $\beta \in B_0$. So, functions z and η are locally Lipschitz continuous within this neighbourhood B_0 of β_0 . Due to this, the boundary of B_0 can be found, *via* properties of Lipschitz continuity, as it is done in [22], where a method to determine the size of a neighbourhood is proposed. Note that, for all $j \in \{1, 2, 3\}$, the function C_j , associated with inequality constraints, depends directly on the solution $z(\beta)$ and on the parameter β , since the discretized state variables are written in function of the discretized control variables, *i.e.*,

$$C_j(X_i(z(\beta), \beta), u_i, \beta) =: \tilde{C}_{ji}(z(\beta), \beta)$$

for all $j \in \{1, 2, 3\}$ and for all $i \in \{1, \dots, N-1\}$. Furthermore, we have

$$\tilde{C}(z(\beta), \beta) = \begin{bmatrix} \tilde{C}_{11}(z(\beta), \beta) \\ \vdots \\ \tilde{C}_{1(N-1)}(z(\beta), \beta) \\ \tilde{C}_{21}(z(\beta), \beta) \\ \vdots \\ \tilde{C}_{2(N-1)}(z(\beta), \beta) \\ \tilde{C}_{31}(z(\beta), \beta) \\ \vdots \\ \tilde{C}_{3(N-1)}(z(\beta), \beta) \end{bmatrix}.$$

Thus, we can state the relationship of the constraint space to the solution and parameter spaces as it follows:

$$\|\tilde{C}(z(\beta), \beta) - \tilde{C}(\hat{z}, \beta_0)\|_q \leq L_C \|z(\beta) - \hat{z}\|_q + L_\beta \|\beta - \beta_0\|_q, \quad (5.1)$$

where $\|\cdot\|_q$ denotes any q -norm with $q \in \mathbb{N} \cup \{\infty\}$ and L_C and L_β are the Lipschitz constants corresponding to the mapping $z \mapsto \tilde{C}(z, \beta)$ and $\beta \mapsto \tilde{C}(z, \beta)$, respectively. Considering the properties of Lipschitz continuity, one can extend (5.1) to consider the relationship of solution space to parameter space:

$$\|\tilde{C}(z(\beta), \beta) - \tilde{C}(\hat{z}, \beta_0)\|_q \leq (L_C L_z + L_\beta) \|\beta - \beta_0\|_q, \quad (5.2)$$

where L_z is the Lipschitz constant corresponding to the mapping $\beta \mapsto z(\beta)$. From Section 4.1, the active sets are constant in B_0 , that is, $\mathcal{I}(z(\beta), \beta) \equiv \mathcal{I}(\hat{z}, \beta_0)$ for all $\beta \in B_0$. Therefore, inactive constraints remain inactive for all $\beta \in B_0$. So, in [22] the authors exploit this fact and derive a conservative bound on $\|\beta - \beta_0\|_q$ such that the inactive constraints remain inactive. To this goal, they propose to merely investigate inactive constraints and (5.2) implies

$$C_j(X_i(z(\beta), \beta), u_i, \beta) \leq C_j(X_i(\hat{z}, \beta_0), \hat{u}_i, \beta_0) + (L_C L_z + L_\beta) \|\beta - \beta_0\|_q \stackrel{!}{\leq} 0$$

for all $i \notin \mathcal{I}(\hat{z}, \beta_0)$ and for all $j \in \{1, 2, 3\}$. Thus, we get a conservative size of a ball, by which we can define the neighbourhood of β :

$$|\beta - \beta_0| \leq \frac{1}{L_C L_z + L_\beta} \times \min_{i \notin \mathcal{I}(\hat{z}, \beta_0), j \in \{1, 2, 3\}} \{-C_j(X_i(\hat{z}, \beta_0), \hat{u}_i, \beta_0)\}$$

which is an upper bound for the neighbourhood B_0 , where L_C , L_z and L_β are, respectively, the maximum value of $\frac{\partial C}{\partial z}(\hat{z}, \beta_0)$, $\frac{dz}{d\beta}(\beta_0)$ and $\frac{\partial C}{\partial \beta}(\hat{z}, \beta_0)$.

From our numerical simulations, we get $L_C = 1$, $L_z \simeq 51.36449$ and $L_\beta \simeq 129.5804$. Furthermore, we also get:

$$\min_{i \notin \mathcal{I}(\hat{z}, \beta_0), j \in \{1, 2, 3\}} \{-C_j(X_i(\hat{z}, \beta_0), \hat{u}_i, \beta_0)\} \simeq 3.878 \times 10^{-2}.$$

With these values we now deduce that $|\beta - \beta_0| \simeq 2.1 \times 10^{-4}$. So, from [22], we obtain $B_0 = [0.14979, 0.15021]$. Since this interval around β_0 is rather small, we conclude that the method proposed in [22] is not the best one for our case. Actually, the smaller the value of θ , the smaller the amplitude of B_0 , because $L_C = 1$ is independently of the value of θ , L_z and L_β are not very sensible to variations of θ and

$$\min_{i \notin \mathcal{I}(\hat{z}, \beta_0), j \in \{1, 2, 3\}} \{-C_j(X_i(\hat{z}, \beta_0), \hat{u}_i, \beta_0)\}$$

decreases as θ decreases. Note that for all $i \notin \mathcal{I}_3(\hat{z}, \beta_0)$

$$-C_3(X_i(z, \beta), u_i, \beta) = \gamma - u_i \gg 0,$$

since γ is much bigger than u_i . On the other hand, for all $i = 1, \dots, N-1$ we get

$$-C_2(X_i(z, \beta), u_i, \beta) = u_i$$

and u_i could be, or not, close to 0. However, for all $i \notin \mathcal{I}_1(\hat{z}, \beta_0)$, we get

$$-C_1(X_i(z, \beta), u_i, \beta) = x_{i+1}(z, \beta) - x_{\min}.$$

TABLE 3. Best fitted values for $\beta_{\min}$ and $\beta_{\max}$ and corresponding quadratic errors.

$\beta_{\min}$	$\beta_{\max}$	$\varepsilon_{x\beta_{\min}}$	$\varepsilon_{x\beta_{\max}}$	$\varepsilon_{u\beta_{\min}}$	$\varepsilon_{u\beta_{\max}}$
0.132	0.168	0.0182	0.0110	0.0237	0.0324

Furthermore, we have that

$$\lim_{\theta \rightarrow 0^+} \min\{x_{i+1}(z, \beta) - x_{\min}\} = 0^+$$

and this is the reason why the smaller the value of θ , the smaller the amplitude of B_0 .

So, let us suggest

$$B_0 = [\beta_{\min}, \beta_{\max}] = [0.132, 0.168]$$

which gives raise to acceptable quadratic errors between the solutions provided by sensitivity analysis and the ones obtained by solving numerically the transcribed **NLP**(β). Let $\varepsilon_{x\beta}$ and $\varepsilon_{u\beta}$ be, respectively, the quadratic errors between the state and control solutions provided by sensitivity analysis and the ones obtained by solving numerically the transcribed **NLP**(β) problem for $\beta \in \{\beta_{\min}, \beta_{\max}\}$ and $\theta = \frac{1}{12}$: see Table 3. The comparison between the solutions provided by sensitivity analysis and the solutions obtained by solving numerically the transcribed **NLP**(β) problem for $\beta = \beta_{\min}$ and $\beta = \beta_{\max}$ can be found in Figure 4 for the state x , control u and adjoint variable λ_1 . The nominal solutions, obtained from $\beta = \beta_0$, are drawn in Figure 4 as red curves.

5.4.2. Optimal solution

The light green curve of top left plot of Figure 1 ($\hat{x}$) appears in Figures 4a and 5a as red curves. The others curves of Figure 5a are the computed approximations of the optimal state trajectory x to **NLP**(β) for different values of $\beta \in B_0$. Those are functions of t and they are computed through the first relation of (4.1).

The yellow curve of top left plot of Figure 1 ($\hat{z}$) appears in Figures 4b and 5b as red dashed curves. Similarly, approximations of the optimal control for **NLP**(β) for different values of $\beta \in B_0$ are also shown in Figure 5b. For each β , these values depend on t and are computed through the third relation of (4.1).

Post-estimated values presented in Figure 2, through the blue curve, appear in Figures 4c and 5c as red dashed curves. The others curves of Figure 5c are the computed approximations of the adjoint function λ_1 , associated with the state trajectory x , to **NLP**(β) for different values of $\beta \in B_0$. Those are functions of t and they are computed through the fourth relation of (4.1).

Summarizing, the plots of the nominal solutions (for β_0) and the approximations for different values of $\beta \in B_0$ are shown in Figure 5.

5.4.3. Optimal value function

With respect to (4.6), we get $\tilde{G}(\hat{z}, \beta_0) \simeq 120.34867$ and $\frac{\partial L}{\partial \beta}(\hat{z}, \hat{\eta}_a, \beta_0) \simeq 1406.78095$. Using these values in (4.6), we compute the approximate values for the optimal cost corresponding to different values of $\beta \in B_0$. They are in Figure 6. We can now repeat all the procedure of Section 5 for several nominal values β_0 , so as to cover an interval of admissible values for β .

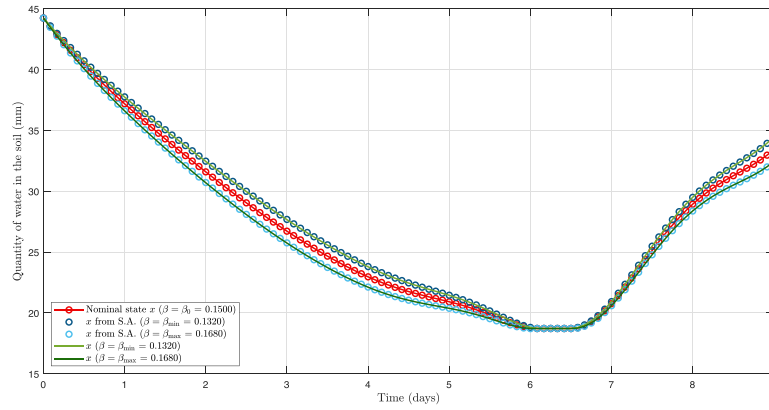
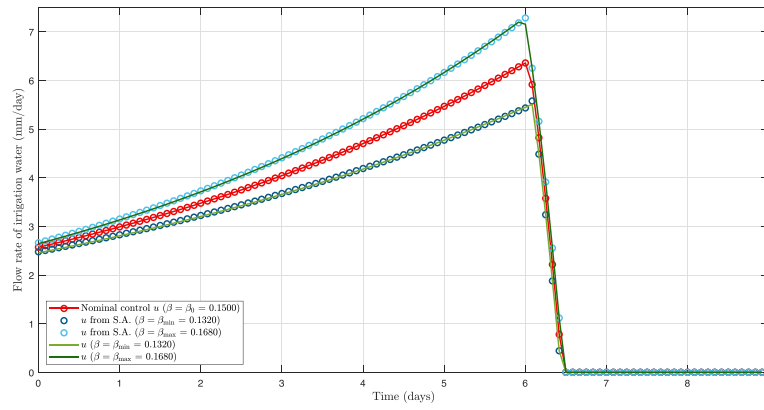
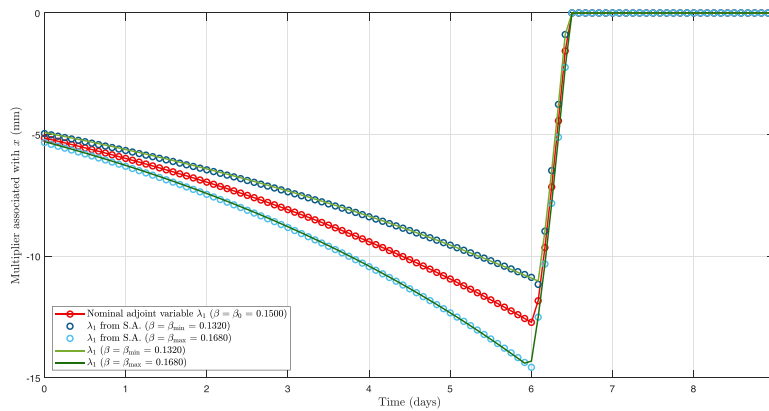
(A) State trajectory x .(B) Control u .(C) Adjoint variable λ_1 .

FIGURE 4. (Colour only in online version) Nominal solutions obtained from $\beta = \beta_0$ and the comparison between the solutions provided by sensitivity analysis and the ones obtained by solving numerically the $\text{NLP}(\beta)$ problem for $\beta \in \{\beta_{\min}, \beta_{\max}\}$.

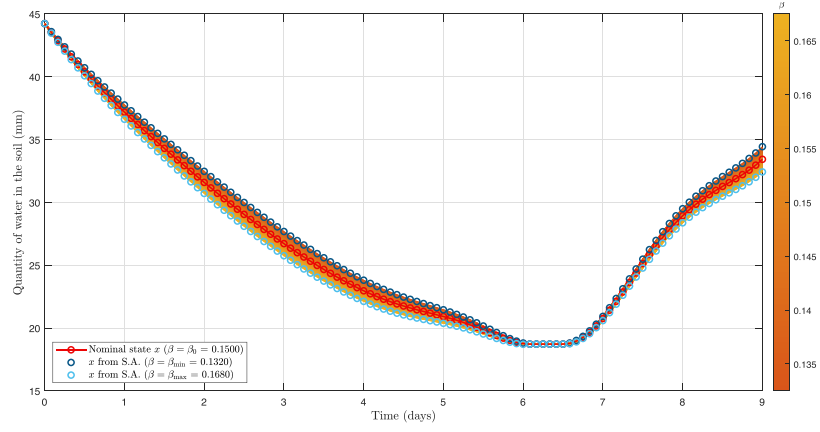
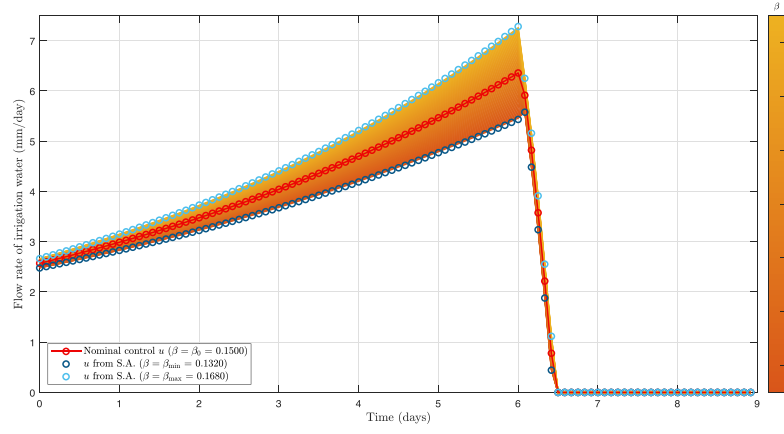
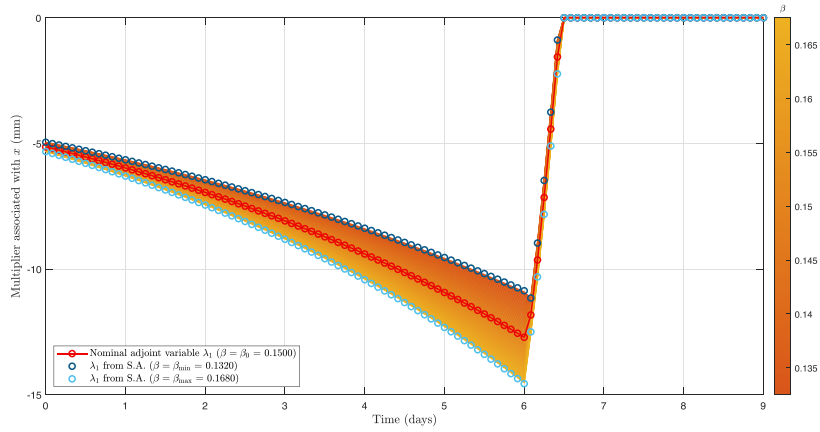
(A) State trajectory x .(B) Control u .(C) Adjoint variable λ_1 .

FIGURE 5. (Colour only in online version) Numerical results provided by sensitivity analysis.

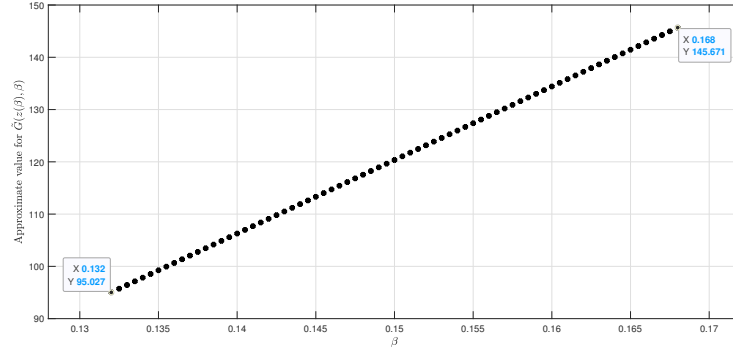
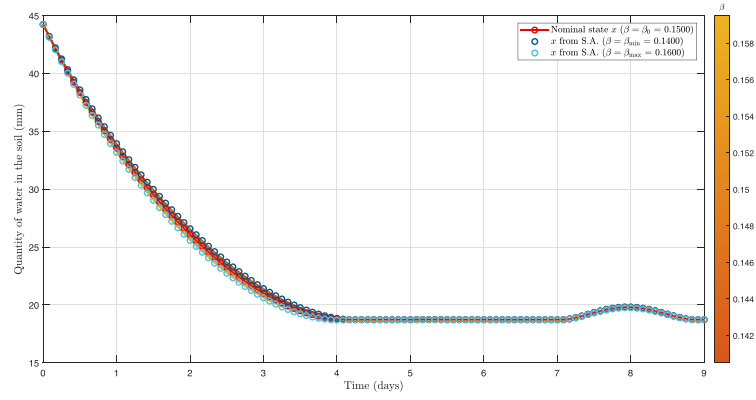
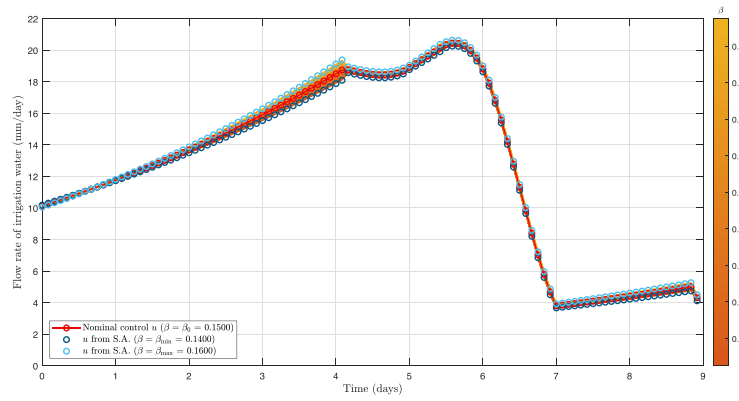


FIGURE 6. (Colour only in online version) Optimal cost in function of $\beta \in B_0$.

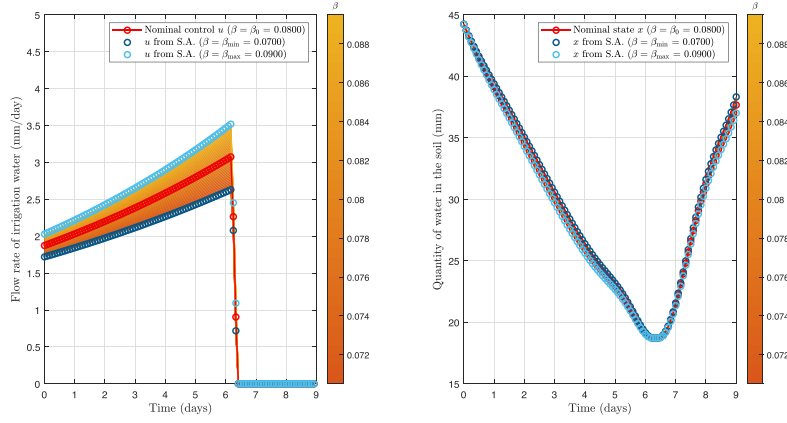
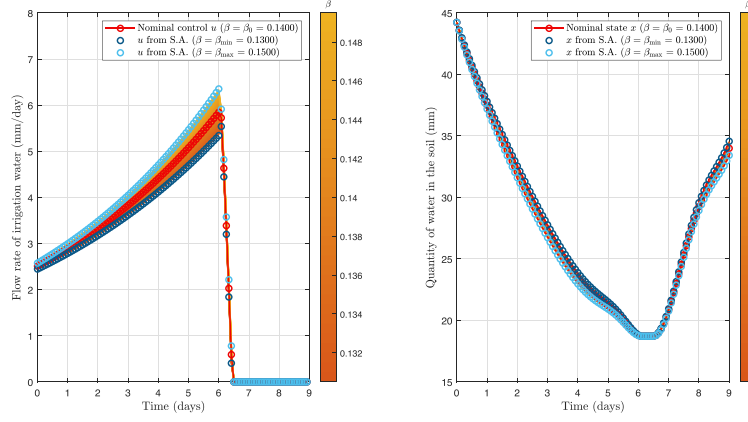
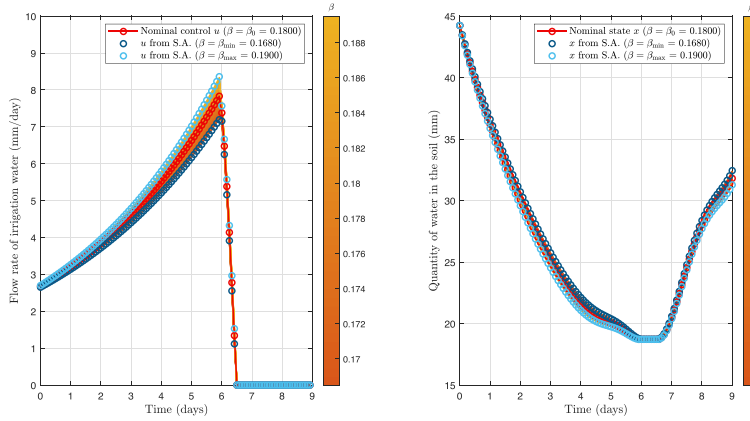


(A) State trajectory x .



(B) Control u .

FIGURE 7. (Colour only in online version) Numerical results provided by sensitivity analysis.

(A) Parameter sensitivity analysis for $\beta_0 = 0.08$.(B) Parameter sensitivity analysis for $\beta_0 = 0.14$.(C) Parameter sensitivity analysis for $\beta_0 = 0.18$.FIGURE 8. (Colour only in online version) Numerical results provided by sensitivity analysis for $\beta_0 \in \{0.08, 0.14, 0.18\}$.

6. CONCLUSIONS

In this paper we mathematically study an optimal control problem applied to agricultural irrigation. An important feature of this problem is related with parameter β that is very hard to estimate and also subject to perturbations. Recall that β represents the water loss percentage due to deep percolation. In order to answer to this challenge, we go further by studying a parametric optimal control problem, denoted by $\text{OCP}(\beta)$ for all $\beta \in [0, \beta_M]$, which allows us to study the desired mathematical application through an efficient process. Our $\text{OCP}(\beta)$ problem differs from the other optimal control problems for agricultural irrigation in the literature, because it allows perturbations of β . In this paper, $\text{OCP}(\beta)$ problem was analysed through its transcribed NLP problem, denoted by $\text{NLP}(\beta)$, for which we designed a parameter sensitivity analysis. To do that, we had to solve $\text{NLP}(\beta)$ for $\beta = \beta_0$, that is, we had to solve the nominal problem $\text{NLP}(\beta_0)$. The obtained nominal solution had to verify necessary optimality conditions, as well sufficient optimality conditions of first and second order, essential to design the corresponding parameter sensitivity analysis. As only the discretized control variables are the optimization variables of $\text{NLP}(\beta)$, after obtaining the nominal solution of $\text{NLP}(\beta_0)$, we estimated the state trajectory and the adjunct function associated with the proposed parametric optimal control problem. The estimation of the adjunct function was carried out with the help of necessary and sufficient optimality conditions of $\text{OCP}(\beta)$ for all $\beta \in [0, \beta_M]$.

With the purpose to illustrate the theoretical results summarized previously, we provided a parameter sensitivity analysis with respect to $\text{OCP}(\beta)$, assuming the set of parameter values of Table 2. Under this assumption, we concluded that the computed optimal trajectories have the same behaviour for different β in a neighbourhood B_0 of the nominal value $\beta_0 = 0.15$. Consequently, the irrigation periods remain the same for different β . We only have to adjust the flow rate of irrigation water in such periods, since as β increases the flow rate of irrigation water also increases (see the Fig. 5b). Although the optimal solution and optimal cost are sensible to variations of β , small perturbations of β in B_0 do not affect the state of preservation of the crop, because the minimum level $x_{\min}$ is always guaranteed for $\beta \in B_0$ (see the Fig. 5a). It is important to emphasize that all these conclusions were obtained for the set of parameter values of Table 2. We remark that the optimal solution is determined for a time window of ten days, using weather prevision data. So, for each ten days and according with the predictions, we determine the soil moisture from the model and the corresponding control for each day of time window, in function of the parameter β .

We highlight that non-parametric optimal control problems are commonly solved through their transcription into large scale NLP problems *via* discretization. However, the computation of their optimal solutions for different values of β is heavy and time consuming. This is the reason why we considered a sensitivity analysis to study the variations of the optimal solution and optimal cost caused by the perturbations of $\beta \in B_0$. We remark that we can identify different nominal values for β from a set of admissible ones such that the corresponding neighbourhoods do not overlap and cover all the set. Carrying out other numerical simulations for $\beta_0 \neq 0.15$ and assuming the remaining values of Table 2, we observed that the behaviour of the control is very similar, changing only the quantity of water used to irrigate. In practice, these changes in the control values are essentially important for the farmer who has to implement the irrigation. As the nominal value β_0 increases, the control values increase. As β_0 decreases, the control values decrease. However, the control action and, consequently, the behaviour of the state are similar to those presented in Figures 5a and 5b for $\beta_0 = 0.15$. Please, see, for example, Figures 8a, 8b and 8c which illustrate what is mentioned before, when $\beta_0 \in \{0.08, 0.14, 0.18\}$. We remark that as β means the percentage of water loss, it does not make sense to consider $\beta > 0.20$.

We emphasize that for each nominal value β_0 , we can get different and important evolutions for the state and control. One is when irrigation is not needed, that is, when the rain is enough to guarantee the minimum level $x_{\min}$ for the state. This case is not important, because there is nothing to control. Another behaviour is obtained, when we have some days with rain and some days without. As the control is quadratic, having some days without rain, even being few, the control is activated since the beginning, even when $x(0) > x_{\min}$. The behaviour of this case is similar to that presented here: see Figure 4. An extreme scenario can be obtained, when we have to irrigation almost all the time, that is, when function g is always negative within the time interval under study. A scenario of this type and its parameter sensitivity analysis is illustrated in Figure 7.

With respect to this parameter sensitivity analysis, we remark that the active indexes remain the same within a certain neighbourhood of β_0 .

The sensitivity analysis for the irrigation problem with L^1 cost is a challenge we hope to solve in a future work. The answer to this and the comparison with the findings of the current paper is part of our future plans of research.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the financial support provided by the Portuguese Foundation for Science and Technology (FCT – Fundação para a Ciência e a Tecnologia). Ana P. Lemos-Paião is supported by The Center for Research and Development in Mathematics and Applications (CIDMA) through the Portuguese Foundation for Science and Technology (FCT – Fundação para a Ciência e a Tecnologia), reference UIDB/04106/2020 (<https://doi.org/10.54499/UIDB/04106/2020>). Sofia O. Lopes is supported by The Physics Center of Minho and Porto Universities (CF-UM-UP) through the Portuguese Foundation for Science and Technology (FCT – Fundação para a Ciência e a Tecnologia), reference UIDB/04650/2020. M. d. R. de Pinho is supported by The Research Center for Systems and Technologies (SYSTEC) through the Portuguese Foundation for Science and Technology (FCT – Fundação para a Ciência e a Tecnologia), reference UIDB/00147/2020.

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