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NON-LOCAL HYPERBOLIC DYNAMICS OF CLUSTERS

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Abstract. The formation, movement and gluing of clusters can be described through a system of non-local conservation laws. Here, the well-posedness of this system is obtained, as well as various stability estimates. Remarkably, qualitative properties of the solutions are proved, providing information on stationary solutions and on the propagation speed. In some cases, fragmentation leads to clusters developing independently. Moreover, these equations may serve as an encryption/decryption tool. This poses new analytical problems and asks for improved numerical methods.

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1. Introduction

In various real-world phenomena, cluster dynamics results from the interaction among the *"individuals"* composing the system. In many cases, fragmentation may lead to different, separate *clusters*, or *patterns*, having different sizes according to the amount of individuals focusing into them. An obvious example is opinion dynamics, where the polarization into a single opinion, *i.e.* consensus, should not be considered as the standard type of evolution. (Here and in what follows we adopt the terminology in [1], Sect. II).

From a macroscopic point of view, we describe the *population* at hand through its density ρ, which is a function of both time t, varying in $\mathbb{R}$, and of some sort of *position* x, varying in a suitable $\mathbb{R}^n$. As long as individuals neither appear nor disappear, ρ solves a *continuity equation*, *i.e.*, a *conservation law* [2], Section 1.4 of the form

$$\partial_t \rho + \nabla \cdot (\rho\, v) = 0 \tag{1.1}$$

for a suitable velocity field v in $\mathbb{R}^n$.

In fluid dynamics, basic physical principles justify the use of further conservation (or balance) laws, such as that of linear momentum or that of energy. Social dynamics lacks these basic principles. At the same time, striving for simplicity induces to complete the model with an ansatz specifying how v depends on ρ, thereby closing (1.1). Hence, in general, the agent (or the opinion/individual) p, that at time t_o is at x_o, moves (or

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evolves) according to

$$\begin{cases} \dot p = v\,(t, p(t), \rho(t)) \\ p(t_o) = x_o \end{cases} \tag{1.2}$$

where the dependence of v on $\rho(t)$ is of a *functional* nature. Indeed, in many cases, we expect p to react to the values of $\rho(t,x)$ for all x in a neighborhood of p and not only to the value $\rho\,(t, p(t))$ computed exclusively at $(t, p(t))$.

In the dynamics of $p(t)$ the relevance of the value of $\rho(t,x)$ at some $x \in \mathbb{R}^n$ realistically depends on the distance $\|x - p(t)\|$ or, more generally, on the difference $x - p(t)$. This naturally leads to postulate that $v\,(t, p(t), \rho)$ depends on ρ through some sort of weighted average $\int_{\mathbb{R}^n} \eta\,(x - p(t))\; \rho(t,x)\, \mathrm{d}x$, where $\eta\,(x - p(t))$ is the (non necessarily positive) weight of $\rho(t,x)$ in the evolution of $p(t)$.

Whenever $\rho(t,x)$ is constant in x on all $\mathbb{R}^n$, it is reasonable to expect that v in (1.2) vanishes. In other words, the *variations* of $\int_{\mathbb{R}^n} \eta\,(x - p(t))\; \rho(t,x)\, \mathrm{d}x$ appear to be crucial for the dynamics of $p(t)$. Thus, we consider the model

$$\partial_t \rho + \nabla \cdot \left(\rho\; V\left(\nabla(\rho * \eta)\right)\right) = 0 \tag{1.3}$$

where $V \colon \mathbb{R}^n \to \mathbb{R}^n$ is a given function and, as usual, $(\rho * \eta)(x) = \int_{\mathbb{R}^n} \eta(\xi - x)\, \rho(t,\xi)\, \mathrm{d}\xi$. As long as ρ and η are sufficiently smooth, the elementary properties of the convolution product ensure that $\nabla(\rho * \eta) = (\nabla\rho) * \eta$, meaning that the gradient of the average $\rho * \eta$ is the average of the gradient of ρ.

A straightforward extension of (1.3) to the case of several interacting populations is then

$$\partial_t \rho_i + \nabla \cdot (\rho_i\; V_i(\nabla\rho * \eta)) = 0 \qquad \begin{array}{l} (t,x) \in \mathbb{R} \times \mathbb{R}^n \\ \rho_i = \rho_i(t,x) \in \mathbb{R} \\ i \in \{1, \ldots, m\}. \end{array} \tag{1.4}$$

Under two rather simple assumptions on V and η, see **(V)** and (η) below, we establish the well-posedness of (1.4), provide various stability estimates and prove qualitative properties of the solutions. In particular, we present conditions ensuring that symmetries in the initial datum persist in the solution, a large set of stationary solutions is exhibited, estimates on the propagation speed of the initial data are provided as well as conditions ensuring the fragmentation of the solution.

A peculiar property of (1.4) is its reversibility in time. In spite of its ability to describe fragmentation, polarization or consensus, (1.4) is well posed also going backward in time. As a consequence, for instance, it can be used to encrypt and decrypt scalar signals (where, say, $n = 1$) or images (where $n = 2$ and m can be thought as the number of colors). New problems arise from this possibility. At the analytic level, one may ask whether this encryption method is *"unbreakable"*, leading to the need of investigating the properties of the group generated by (1.4). At the numerical level, the need of a *reversible* algorithm arises. This lack is particularly relevant in 2 (or higher) spatial dimensions as it stems out from the comparison of the errors in (3.11) versus (3.15) and (3.16). Refer to Section 3.4 for further discussions.

Once the macroscopic evolution $t \mapsto \rho(t)$ is known, the dynamics of every single member p_i of the i-th population is given by (1.2), which in the present multi-population case reads

$$\begin{cases} \dot p_i = V_i\left(\nabla\left(\rho(t) * \eta\right)(p_i)\right) \\ p_i(t_o) = x_o. \end{cases} \tag{1.5}$$

The general well-posedness of (1.5), which we leave to a future work, seems reachable extending the results in [3, 4]. On the contrary, we note that the reverse connection, *i.e.*, a general rigorous derivation of (1.4) from a microscopic model, is apparently still an open problem if $n \geqslant 2$. For the case $n = 1$, this connection was obtained in [5, 6], a recent result devoted to the case $n = 2$ is [7].

A milestone in the modeling of flocks is the Cucker–Smale model [8]. Differently from the model presented therein, (1.4) is of a macroscopic nature and may describe, besides consensus, also polarization and fragmentation, already in the case of a single population. Moreover, with *ad hoc* choices of the function η, it can also describe interactions that are attractive in some regions and repelling in other, for instance.

Non-local equations similar to (1.4) have been considered in the literature in a variety of applications. Their use dates back at least to [9], devoted to opinion formation. Recall [10, 11], devoted to crowd dynamics, and [12], where a system of balance laws describes flocking phenomena. A controlled diffusion equivalent to that proved in Lemma 2.4 is shown in [13] in the case of a gradient flow resulting in a system of partial differential equations. A 1-dimensional case of (1.4) has been thoroughly studied in [14], where solutions are constructed as limits of solutions to interacting particle systems, also obtaining well-posedness. In [15], non-local terms represent the competition between species for resources. The issue of describing the effects of communications among individuals is investigated in [16] through a non-local hyperbolic system.

Alternative to the use, here pursued, of density functions typically in $\mathbf{L^1}$, is the use of measures and of the various types of Wasserstein distances. Paradigmatic in this direction is the recent work [17], where a general framework for population dynamics is based on Radon measures in Polish spaces.

The next section presents the analytical results. Section 3 is devoted to numerical integrations. Proofs are exposed in Section 4, a more technical one being deferred to the Appendix.

2. Well-posedness and qualitative properties

In the non-linear and non-local equation (1.4), $(\nabla\rho * \eta)$ is the $n \times m$ matrix

$$\nabla\rho * \eta = \begin{bmatrix} \nabla\rho_1 * \eta \\ \vdots \\ \nabla\rho_m * \eta \end{bmatrix}^{\intercal} \qquad \begin{array}{c} (\nabla\rho * \eta)_{ji} = \partial_{x_j}(\rho_i * \eta) \text{ for } j \in \{1,\ldots,n\} \text{ and } i \in \{1,\ldots,m\} \\ \text{where} \quad (\rho_i(t) * \eta)(x) = \displaystyle\int_{\mathbb{R}^n} \rho_i(t,\xi)\, \eta(x-\xi)\, \mathrm{d}\xi \,. \end{array} \tag{2.1}$$

The extension to different kernels η_{ij} acting on the different populations requires, from the analytical point of view, essentially only formal modifications, see [18, 19].

Throughout, we pose the following assumptions:

(η) $\eta \in (\mathbf{C^3} \cap \mathbf{W^{3,1}} \cap \mathbf{W^{3,\infty}})(\mathbb{R}^n;\mathbb{R})$.

(**V**) $V \in \mathbf{C^2}(\mathbb{R}^{n\times m};\mathbb{R}^{n\times m})$, $V(0) = 0$ and $\|DV\|_{\mathbf{W^{1,\infty}}(\mathbb{R}^{n\times m};\mathbb{R}^{(n\times m)^2})} \leqslant L_V$, for a $L_V \in \mathbb{R}_+$.

We set $V = [V_1 \cdots V_m]$, with $V_i \in \mathbb{R}^n$.

The next definition is intrinsic to (1.4) and independent of the way in which any particular solution may be constructed.

Definition 2.1. Fix a non trivial real interval I. A map $\rho \in \mathbf{C^0}\left(I;\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)\right)$ is a *solution* to (1.4) on I if setting, for $i \in \{1,\ldots,m\}$ and $(t,x) \in I \times \mathbb{R}^n$,

$$v_i(t,x) = \left[V_{1i}\left((\nabla\rho * \eta)(t,x)\right) \quad V_{2i}\left((\nabla\rho * \eta)(t,x)\right) \quad \cdots \quad V_{ni}\left((\nabla\rho * \eta)(t,x)\right)\right]^{\intercal}, \tag{2.2}$$

for $i \in \{1,\ldots,m\}$ the component ρ_i is a solution to

$$\partial_t\rho_i + \nabla\cdot(\rho_i\, v_i(t,x)) = 0 \qquad (t,x) \in I \times \mathbb{R}^n \tag{2.3}$$

on the interval I.

Refer to (4.2) and to the discussion therein for the precise meaning of *solution* to (2.3).

Here follows the main result of the present paper.

Theorem 2.2. *Let (η) and (**V**) hold. Then (1.4) generates a unique map*

$$\mathcal{G}\colon \mathbb{R} \times (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m) \to (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$$

such that

(1) *$\mathcal{G}$ is a group, in the sense that*

$$\mathcal{G}_0 = \mathbf{Id} \qquad \text{and} \qquad \mathcal{G}_{t_1} \circ \mathcal{G}_{t_2} = \mathcal{G}_{t_1+t_2}$$

for every $t_1, t_2 \in \mathbb{R}$.

(2) *For any $\rho_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$, the map $t \mapsto \mathcal{G}_t\rho_o$ is the unique global solution to (1.4) with initial datum ρ_o assigned at time $t = 0$, in the sense of Definition 2.1.*

(3) *For any $\rho_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$ and for every $t \in \mathbb{R}$,*

$$\|\mathcal{G}_t\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} = \|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}.$$

(4) *For any $\hat\rho_o, \check\rho_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$ and for every $t \in \mathbb{R}$,*

$$\|\mathcal{G}_t\hat\rho_o - \mathcal{G}_t\check\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \leqslant \|\check\rho_o - \hat\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\, \mathrm{e}^{C(|t|)}$$

where the function $C \in \mathbf{C^0}(\mathbb{R}_+;\mathbb{R}_+)$ is non decreasing, depends only on η, L_V and on the $\mathbf{L^1}$ and $\mathbf{L}^\infty$ norms of $\hat\rho_o$, $\check\rho_o$; moreover $\limsup_{t\to 0+} C(t)/t$ is bounded.

(5) *If $\hat{\mathcal{G}}$, respectively $\check{\mathcal{G}}$, is generated by (1.4) with speed $\hat V$, respectively $\check V$, satisfying (**V**) and such that L_V is an upper bound for the $\mathbf{W^{1,\infty}}$ norms of both $D\hat V$ and $D\check V$, then, for every $t \in \mathbb{R}$ and for any $\rho_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$*

$$\left\|\hat{\mathcal{G}}_t\rho_o - \check{\mathcal{G}}_t\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \leqslant C\,|t| \left(\left\|\hat V - \check V\right\|_{\mathbf{L}^\infty(B;\mathbb{R}^{n\times m})} + \left\|D\hat V - D\check V\right\|_{\mathbf{L^1}(B;\mathbb{R}^{(n\times m)^2})} \right) \mathrm{e}^{C\,|t|}$$

where $B = B_{\mathbb{R}^{n\times m}}(0, \|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)})$ and C depends on various norms of η and of the initial datum.

(6) *For $k = 0,1,2$, if $\rho_o \in \mathbf{C^k}(\mathbb{R}^n;\mathbb{R}^m)$, then also $\rho(t) \in \mathbf{C^k}(\mathbb{R}^n;\mathbb{R}^m)$ for all $t \in \mathbb{R}_+$.*

(7) *For any $i \in \{1,\ldots,m\}$, if $\rho_{o,i} \geqslant 0$ then for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, also $\rho_i(t,x) \geqslant 0$.*

The proof relies on a fixed point argument and is deferred to Section 4. Note that the group property stated at **(1)** implies that problem (1.4) is *reversible*.

Next, we provide conditions under which (1.4) is invariant with respect to symmetries in $O(n)$, *i.e.* orthogonal matrices.

Proposition 2.3. *Let (**V**) and (η) hold. Fix $\rho_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$. Assume moreover that for a $R \in O(n)$:*

(1) *$V_i(w_1\,R,\ldots,w_m\,R) = R^{-1}\,V_i(w_1,\ldots,w_m)$ for $i \in \{1,\ldots,m\}$ and all $w_1,\ldots,w_m \in \mathbb{R}^{1\times n}$.*

(2) *$\eta(R\,x) = \eta(x)$ for all $x \in \mathbb{R}^n$.*

(3) *$\rho_o(R\,x) = \rho_o(x)$ for all $x \in \mathbb{R}^n$.*

Then, the solution $t \mapsto \mathcal{G}_t\rho_o$ exhibited in Theorem 2.2 is invariant with respect to R, i.e.,

$$\forall (t,x) \in \mathbb{R} \times \mathbb{R}^n \qquad (\mathcal{G}_t\rho_o)(R\,x) = (\mathcal{G}_t\rho_o)(x).$$

On the basis of **(V)** it is immediate to provide the following general bound on the growth of the support of the solutions to (1.4). This shows that solutions to (1.4) propagate with finite speed.

Lemma 2.4. *Let* (η) *and* **(V)** *hold. Fix* $\rho_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$, $x_o \in \mathbb{R}^n$ *and* $r > 0$. *If* $\operatorname{spt}\rho_o \subseteq B(x_o, r)$, *then for all* $t \in \mathbb{R}$

$$\operatorname{spt}\rho(t) \subseteq B\left(x_o, r + W\,|t|\right) \qquad \textit{where} \qquad W = L_V\,\|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}. \tag{2.4}$$

As a consequence, a compactly supported initial datum yields a solution which is compactly supported at any positive time.

The qualitative behavior of the solutions to (1.4) is governed by the direction of the vectors V_i. Indeed, when V_i and $\nabla(\rho_i * \eta)$ have the same direction, *i.e.* $V_i\left(\nabla(\rho * \eta)\right) \cdot \nabla(\rho_i * \eta) > 0$, then solutions to (1.4) do not propagate, in the sense of the following Proposition.

Proposition 2.5. *Let* **(V)**, (η) *hold and* $\rho_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$. *Assume moreover*

- **(1)** *For all* $w \in \mathbb{R}^{n\times m}$, $V(w) = v\left(\|w\|\right) w$ *with* v *such that* $v(s) > 0$ *for all* $s > 0$.
- **(2)** *For all* $x \in \mathbb{R}^n$, $\eta(x) = \tilde\eta\left(\|x\|\right)$ *with* $\tilde\eta' \leqslant 0$ *and* $\operatorname{spt}\tilde\eta = [-\ell, \ell]$.
- **(3)** *For all* $i \in \{1, \ldots, m\}$, $\rho_{o,i}(x) \geqslant 0$ *for a.e.* $x \in \mathbb{R}^n$.

Then,

- **(i)** *If* $C \subseteq \mathbb{R}^n$ *is closed and convex and* $i \in \{1, \ldots, m\}$,

$$\operatorname{spt}\rho_{o,i} \subset C \implies \forall\, t \in \mathbb{R}_+ \quad \operatorname{spt}(\mathcal{G}_t\rho)_i \subseteq C.$$

- **(ii)** *If* $\operatorname{spt}\rho_o$ *is bounded, then for all* $t \in \mathbb{R}_+$ *and* $i \in \{1, \ldots, m\}$,

$$\operatorname{spt}(\mathcal{G}_t\rho)_i \subseteq \overline{\operatorname{co}}\operatorname{spt}\rho_{o,i}.$$

Above, we adopt the usual definition of the support of a function $\rho \in \mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)$:

$$\operatorname{spt}\rho = \mathbb{R}^n \Big\backslash \bigcup_{A\in\mathcal{A}} A \qquad \text{where} \qquad \mathcal{A} = \left\{A \subseteq \mathbb{R}^n \colon A \text{ is open and } \rho = 0 \text{ a.e. in } A\right\}, \tag{2.5}$$

see [20], Proposition IV.17, and $\overline{\operatorname{co}}\,S$ stands for the closed convex hull of the set S.

Recall that ℓ is the radius of the support of η, *i.e.*, the maximal distance at which the non-local interaction may act. We now show that, under the key assumption **(1)** of Proposition 2.5, if the initial datum is supported over balls at a distance at least ℓ from each other, then the evolution of the solution on these balls proceeds independently on the different spheres, see Figure 1.

Corollary 2.6. *Under the assumptions* **(1)**, **(2)** *and* **(3)** *in Proposition 2.5, assume that, for a suitable* $i \in \{1, \ldots, m\}$, *there exist* $x_1, \ldots, x_k \in \mathbb{R}^n$ *and* $r_1, \ldots, r_k \in \left]0, +\infty\right[$, *such that*

$$\operatorname{spt}\rho_{o,i} \subseteq \bigcup_{h=1}^{k} B(x_h, r_h) \textit{ with } \|x_h - x_j\| > r_h + r_j + \ell \textit{ for all } h, j \in \{1, \ldots, k\}, h \neq j. \tag{2.6}$$

Then, the solution $\mathcal{G}\rho_o$ *to* (1.4) *satisfies for all* $t \in \mathbb{R}_+$

$$\operatorname{spt}(\mathcal{G}_t\rho_o)_i \subseteq \bigcup_{h=1}^{k} B(x_h, r_h) \qquad\qquad \textit{and} \tag{2.7}$$

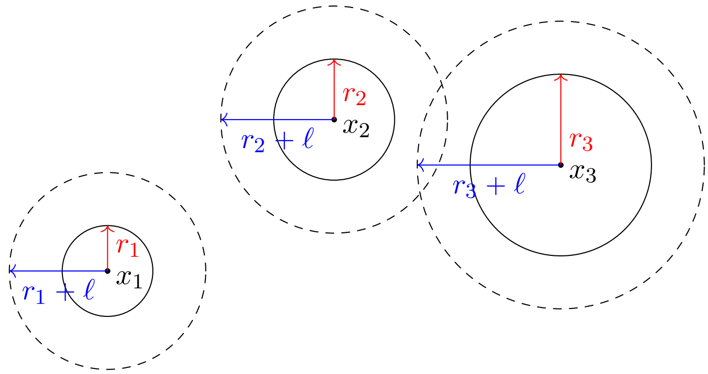


Figure 1. An example of a situation complying with (2.6) of Corollary 2.6 in the case $k = 3$. The dashed circumferences may not overlap with the solid ones.

$$\int_{B(x_h,r_h)} (\mathcal{G}_t\rho_o)_i(x)\,\mathrm{d}x = \int_{B(x_h,r_h)} \rho_{o,i}(x)\,\mathrm{d}x \qquad \text{for } h \in \{1,\ldots,k\}. \tag{2.8}$$

The next proposition describes a set of stationary solutions to (1.4): they are the most natural candidates of time asymptotic limits of solutions to (1.4).

Proposition 2.7. *Let* **(V)** *and* *(η)* *hold. Fix* $\rho_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$. *Assume moreover that for fixed* $r, \ell \in \mathbb{R}$ *with* $\ell > r > 0$ *and for* $x_1,\ldots,x_k \in \mathbb{R}^n$,

- **(1)** $\operatorname{spt}\nabla\eta \subseteq B(0,\ell)\backslash B(0,r)$.
- **(2)** $\operatorname{spt}\rho_o \subseteq \bigcup_{h=1}^k B(x_h, r/2)$ *with* $\|x_h - x_j\| > r + \ell$ *for all* $h, j \in \{1,\ldots,k\}$ *with* $h \neq j$.

Then, the solution to (1.4) *is stationary:* $\mathcal{G}_t\rho_o = \rho_o$ *for all* $t \geqslant 0$.

In the case $k = 1$, assumption **(2)** in Proposition 2.7 simplifies to $\operatorname{spt}\rho_o \subseteq B(x_1, r/2)$.

3. Examples

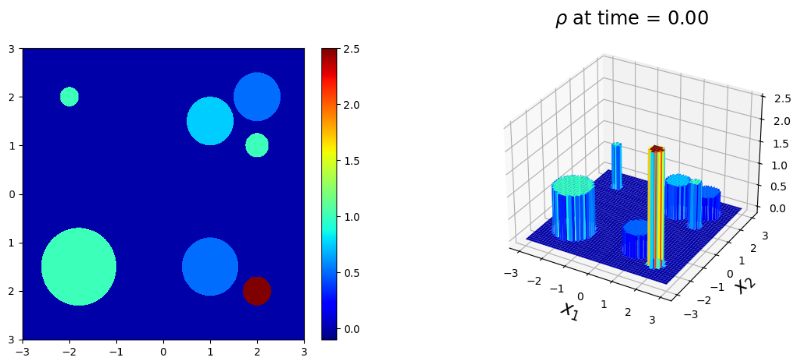
The paragraphs below present numerical integrations of (1.4). They are obtained by means of the upwind method [21], formula (4.35) in Section 4.8. The space mesh is uniform while the time step is chosen to meet the (adapted) CFL condition [21], Section 4.4 with coefficient 0.95. Whenever $n > 1$, we adopt the usual dimensional splitting [21], Section 19.5. The convolutions are approximated through integrals of functions that are piecewise constant on the spatial mesh. The numerical domains are indicated below and the values of the solution along the boundary are kept constantly equal to those of the initial datum.

Remark that also other methods can be exploited to numerically integrate (1.4). Here we recall the Lax–Friedrichs method [22]; particle methods, for instance along the lines of [23], and characteristic methods, for instance as in [24].

Below, initial data are often chosen to be linear combination of characteristic functions. This choice, though particular, is balanced by the $\mathbf{L^1}$ stability proved in Theorem 2.2.

3.1. Fragmentation, or clusters formation

Here, we exemplify the situation described in Corollary 2.6. To this aim, choose the initial datum and the parameters as in (3.1):



$$\begin{aligned} n &= 2\,, \\ m &= 1\,, \\ V(w) &= \frac{w}{\sqrt{1+\|w\|^2}}\,. \end{aligned} \tag{3.1}$$

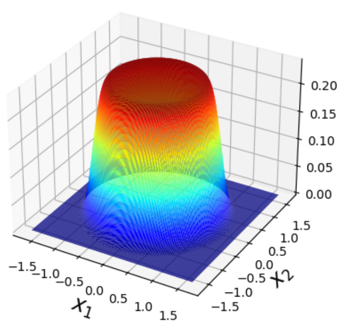
More precisely, the initial datum is a linear combination of characteristic functions of circles

$$\rho_o = \sum_{\nu=1}^{\hat\nu} h_\nu \, \chi_{B(c_\nu, r_\nu)} \tag{3.2}$$

where

$$\hat\nu = 7 \quad \begin{array}{cccccccc} \nu & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ h_\nu & 1.0 & 1.0 & 0.5 & 0.75 & 1 & 0.5 & 2.5 \\ c_\nu & (-2,2) & (-1.8,1.5) & (2,2) & (1,1.5) & (2,1) & (1,-1.5) & (2,-2) \\ r_\nu & 0.2 & 0.8 & 0.5 & 0.5 & 0.25 & 0.6 & 0.3 \end{array} \tag{3.3}$$

as shown in the two diagrams on the left in (3.1). As function η we use



$$\begin{aligned} a(\xi) &= \begin{cases} 0 & \xi < r \\ k\,(\xi - r)^3\,(\ell - \xi)^3 & \xi \in [r, \ell] \\ 0 & \xi > \ell \end{cases} \\ \eta(x) &= \int_{\|x\|}^{\ell} a(\xi)\,\mathrm{d}\xi \\ k &: \int_{\mathbb{R}^2} \eta(x)\,\mathrm{d}x = 1\,. \end{aligned} \tag{3.4}$$

with $r = 0.5$ and $\ell = 0.8$.

The behavior of the solution is consistent with Corollary 2.6. Indeed, as Figure 2 shows, the 4 clusters of characteristic functions in the 4 corners of the numerical domain evolve independently. In the top left corner we have a stationary cluster, consistent with Proposition 2.7. In the bottom left corner, the initial cluster is supported in a circle too wide to be stationary and evolves focusing in a smaller circle, its volume being conserved.

The 2 clusters on the right consist of different initial characteristic functions. Each of the 2 clusters focuses in a sort of barycenter of the initial mass.

Here, we as numerical domain the set $[-3,3]^2$ with a mesh consisting of 7500×7500 points.

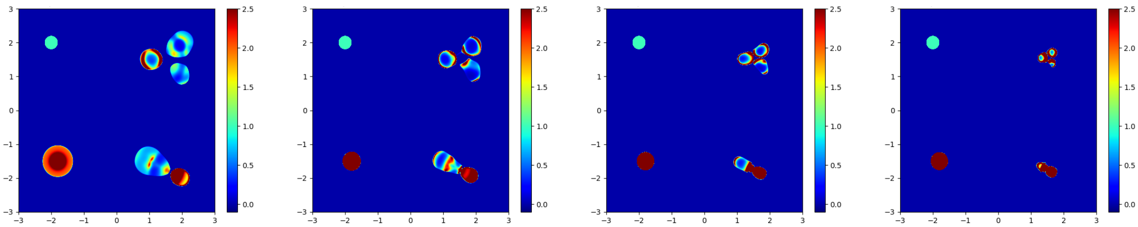


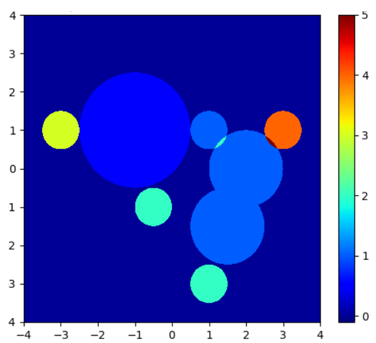
FIGURE 2. Solution to (1.4)–(3.1) with η as in (3.4), computed at times $t = 0.4,\ 0.8,\ 1.2,\ 1.6$. This evolution is consistent with Corollary 2.6, since the 4 parts of the solutions in the 4 corners evolve independently, as well as with Proposition 2.7, since the top left part is stationary.

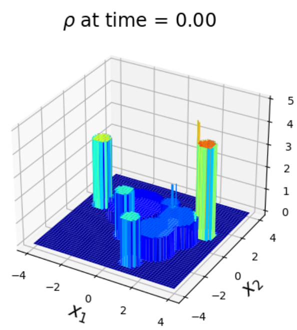
3.2. On the role of V

This paragraph exemplifies the role of V in (1.4). Consider the initial datum (3.2) with

$$\hat\nu = 4 \qquad \begin{array}{ccccccccc} \nu & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ h_\nu & 3.0 & 0.5 & 1.0 & 4.0 & 2.0 & 2.0 & 1.0 & 1.0 \\ c_\nu & (-3,1) & (-1,1) & (1,1) & (3,1) & (-0.5,-1) & (1,-3) & (1.5,-1.5) & (2,0) \\ r_\nu & 0.5 & 1.5 & 0.5 & 0.5 & 0.5 & 0.5 & 1.0 & 1.0 \end{array} \tag{3.5}$$

and with the parameters in (3.6)





$$\begin{aligned} n &= 2\,, \\ m &= 1\,, \\ r &= 0.6\,, \\ \ell &= 1.5 \end{aligned} \tag{3.6}$$

with η as in (3.4) and with the following 3 choices for V:

$$V^1(w) = \frac{w}{\sqrt{1+\|w\|^2}}, \qquad V^2(w) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{w}{\sqrt{1+\|w\|^2}}, \qquad V^3(w) = \frac{-w}{\sqrt{1+\|w\|^2}}. \tag{3.7}$$

Only the former choice V^1 complies with **(1)** in Proposition 2.5 and, indeed, in this case the support of the clusters concentrate. In the other 2 cases, some mass exits the numerical domain. Note that η as defined in (3.4) satisfies (η) and **(1)** in Proposition 2.7. All V^1, V^2 and V^3 comply with **(V)**. Moreover, the pattern formation resulting in all 3 cases apparently leads asymptotically to separate clusters, see Figure 3, corresponding to the stationary solutions exhibited in Proposition 2.7.

In the 3 integrations, the numerical domain is $[-4,4]^2$ with 4000×4000 mesh points.

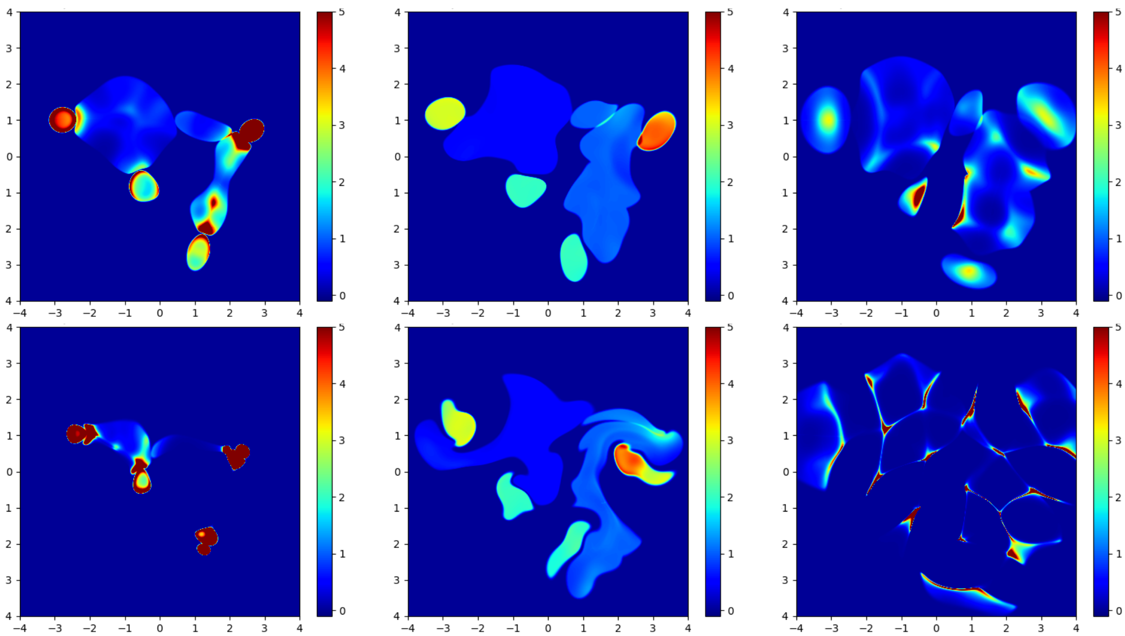


FIGURE 3. Evolution of the solutions to (1.4)–(3.6)–(3.4) at times $t = 1$, above, and $t = 3$, below. On the left, respectively middle and right column, the vector field is V^1, respectively V^2 and V^3 in (3.7). Only in the solution in the left column is the total mass conserved. In the middle and right cases, some mass exits the numerical domain.

3.3. A stationary solution

As an exemplification of the stationary solutions exhibited in Proposition 2.7, consider the following setting:

$$\begin{array}{cccc} n=2, & V(w) = w\,, & \rho_o(x) = \left(2+\sin(8\,x_2)\right)\chi_{[-1/4,1/4]^2}(x), & r=0.8, \\ m=1, & & & \ell=1.5, \end{array} \tag{3.8}$$

and as η we choose (3.4).

Note that η is constant for $\|x\| \leqslant r = 0.8$ and the support of the initial datum has diameter $1/\sqrt{2}$. Since $1/\sqrt{2} < 0.8$, the resulting numerical integration yields a stationary solution, coherently with Proposition 2.7, see Figure 4.

Here, the numerical domain is $[-0.5, 0.5]^2$ and the mesh consists of $10^3 \times 10^3$ points.

3.4. Encryption–decryption

Below, we write explicitly the dependence of the group $\mathcal{G}$ constructed in Theorem 2.2 on η and V as $\mathcal{G}^{\eta,V}$.

The well-posedness proved in Theorem 2.2, and in particular the reversibility shown therein, allows to consider the following procedure:

1. Interpret $\rho_o = \rho_o(x)$ as information to be encrypted.
2. Fix $T > 0$ and arbitrary V, η satisfying **(V)** and **(η)**. They are the encryption keys[1].
3. The encrypted signal is $\rho = \mathcal{G}^{\eta,V}_T \rho_o$.
4. To decrypt, integrate backwards since $\rho_o = \mathcal{G}^{\eta,V}_{-T}\rho$.

A result like the following one would play a key role in assessing the reliability of the above encryption procedure.

[1] Note however that rescalings of V allow to consider only $T = 1$ and $\int \eta = 1$, for instance.

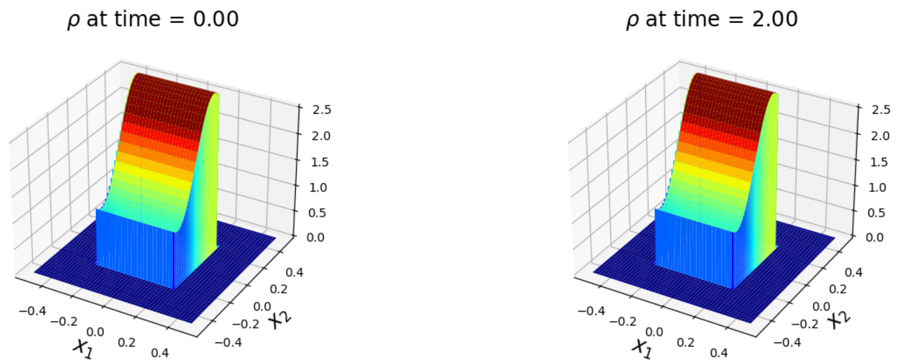


FIGURE 4. Numerical integration of equation (1.4) with parameters and initial datum as in (3.4)–(3.8). Coherently with Proposition 2.7, the resulting solution is stationary.

Problem 3.1. Call $\mathcal{K}_\eta$ and $\mathcal{K}_V$ the set of the keys η and of V that satisfy (η) and $(\mathbf{V})$. Find a sufficiently large class $\mathcal{R} \subset (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R}^n)$ such that for all $\rho_o \in \mathcal{R}$ the map $(\eta, V) \mapsto \mathcal{G}_1^{\eta,V} \rho_o$ is surjective onto $\mathcal{R}$.

Indeed, this surjectivity ensures that the above encryption procedure is essentially unbreakable in the class $\mathcal{R}$. On the basis of the properties of characteristic, it can be proved [25] that $\mathcal{R}$ can not be chosen as large as $(\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R}^n)$. The introduction of source terms [26] is likely to allow for a larger class $\mathcal{R}$.

At the numerical level, the above procedure needs an efficient numerical algorithm to compute approximate solutions to (1.4) that is *reversible*, thus respecting this key property of (1.4).

3.4.1. *The scalar 1-dimensional case*

Consider the encryption/decryption of, say, a signal. Indeed, consider (1.4) in the simplest case where, with reference to (1.4), we set $n = 1$, $m = 1$ and

$$\begin{matrix} n = 1 \\ m = 1 \end{matrix} \qquad V(w) = \frac{w}{\sqrt{1+w^2}} \qquad \eta(\xi) = \xi \, \exp\left(-\frac{1}{1-(\xi/\ell)^2}\right) \chi_{[-\ell,\ell]}(\xi) \qquad \ell = 1/4. \tag{3.9}$$

As initial datum, we choose

$$\begin{aligned} \rho_o(x) &= \vartheta(x;\, -0.8,\, -0.2) + \vartheta(x;\, -0.4,\, 0.4) + \vartheta(x;\, 0.2,\, 0.8) \\ \text{where} \quad \vartheta(x;\, a,\, b) &= \left(1 - \tfrac{x}{a}\right)^2 \, \left(1 - \tfrac{x}{b}\right)^4 \, \chi_{[a,b]}(x). \end{aligned} \tag{3.10}$$

Figure 5 shows the initial datum ρ_o, corresponding to the original signal, then the encrypted signal $\mathcal{G}_3 \rho_o$, *i.e.*, the solution to (1.4)–(3.9)–(3.10), and the image resulting from the decryption, *i.e.*, the numerical approximation $\tilde\rho$ of $\mathcal{G}_{-3}\, \mathcal{G}_3 \rho_o$. As described in the proof of Theorem 2.2, decryption is obtained through (1.4) where V is replaced by $-V$.

A measure of the reliability of the encryption – decryption process is given by the $\mathbf{L^1}$ difference between the original datum ρ_o in (3.10) and $\tilde\rho$, see the values in (3.11).

$$\begin{aligned} \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\approx 0.8781547619047426 \\ \|\mathcal{G}_3\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\approx 0.8781547619047422 \\ \|\tilde\rho\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\approx 0.8781547619047416 \end{aligned} \qquad \begin{aligned} \|\rho_o - \tilde\rho\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\approx 8.880075473908466 \times 10^{-4} \\ \frac{\|\rho_o - \tilde\rho\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}}{\|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}} &\approx 0,00101121987366 \end{aligned} \tag{3.11}$$

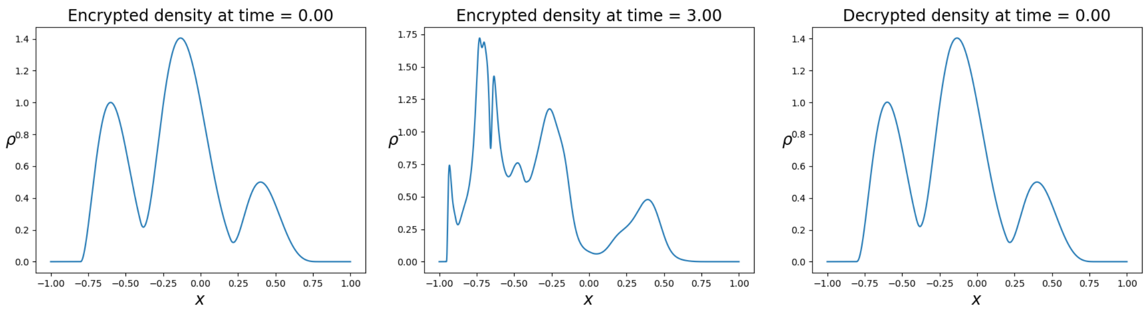


FIGURE 5. Left, the original data (3.10); center the encrypted data and, right, the decrypted data. Encryption is obtained through (1.4)–(3.9), decryption through an integration backward in time. Theorem 2.2 guarantees the feasibility of this procedure.

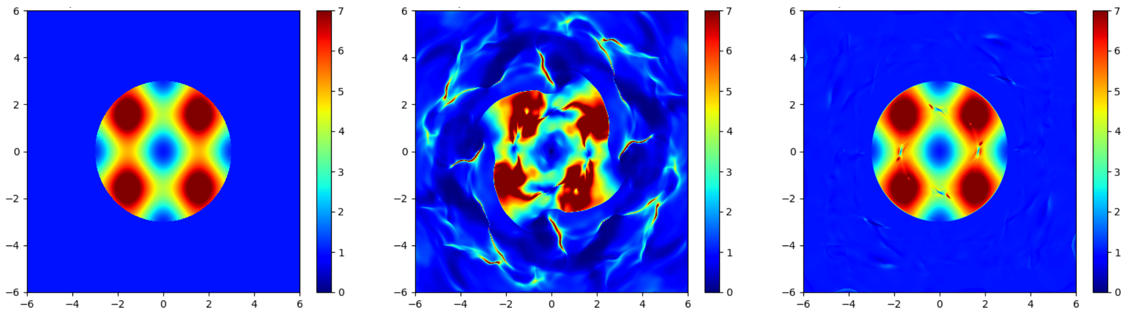


FIGURE 6. From left to right, the original ρ_o, its encryption and the decryption resulting from backward integration of (1.4)–(3.12)–(3.13). The numerical errors are in (3.15).

This integration was performed on $[-1, 1]$ with 10^5 mesh points.

As a side remark, note that the middle diagram in Figure 5 shows no clear formation of clusters, coherently with the fact that η changes sign, resulting in partly attractive and partly repulsive interactions.

3.4.2. 2-*Dimensional examples*

For completeness, we present two forward – backward integrations of (1.4) in the case $n = 2$, $m = 1$. We use dimensional splitting with a regular 24.000×24.000 mesh in the numerical domain $[-6, 6]^2$. The non reversibility of the numerical method imposes to consider a small time interval.

In (1.4), we set

$$\begin{array}{l} n=2 \\ m=1 \\ T=0.75 \end{array} \qquad V(w) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{w}{\sqrt{1+\|w\|^2}} \qquad \eta(\xi) = \cos^3\left(\frac{\pi}{4}\,\|\xi\|^2\right)\,\chi_{\{\|\xi\|<2\}}(\xi). \tag{3.12}$$

and choose the initial data

$$\rho_o(x) = 1 + \left(4\sin^2 x_1 + 3\sin^2 x_2\right)\,\chi_{\{\|x\|<3\}}(x); \tag{3.13}$$

$$\rho_o(x) = 1 + 4\chi_{\{x_1>0;x_2>0;\|x\|<3\}}(x) + \chi_{[-3,0]\times[0,3]}(x) + 2\chi_{\{x_1<0;x_2<0;\|x\|<3\}}(x) + 3\chi_{[0,3]\times[-3,0]}(x). \tag{3.14}$$

The numerical solutions are in Figure 6 and in Figure 7.

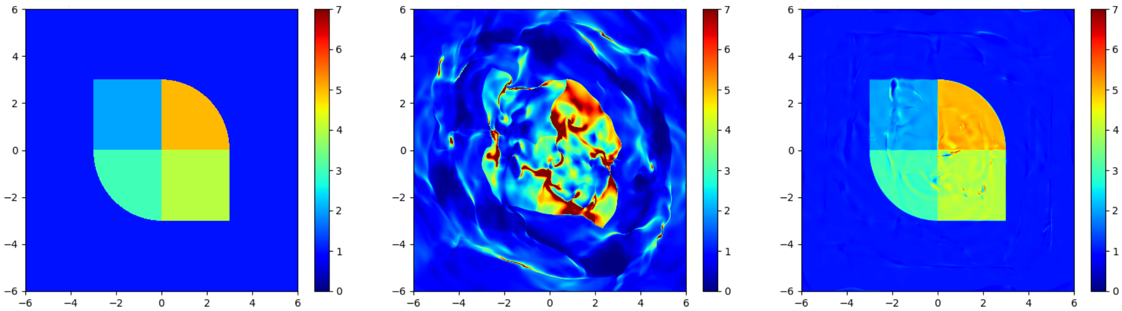


FIGURE 7. From left to right, the original ρ_o, its encryption and the decryption resulting from backward integration of (1.4)–(3.12)–(3.14). The numerical errors are in (3.16).

For completeness, we provide below estimates on the errors in the numerical integrations of the two examples.

$$\begin{array}{rl} \text{Example in} & \text{Figure 6} \\ \|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})} \approx & 252.0991 \\ \|\mathcal{G}_{0.75}\rho_o\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})} \approx & 252.0306 \\ \|\tilde\rho\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})} \approx & 252.1635 \end{array} \qquad \begin{array}{l} \|\rho_o - \tilde\rho\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})} \approx 4.8442 \\ \dfrac{\|\rho_o - \tilde\rho\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})}}{\|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})}} \approx 0.01922 \end{array} \tag{3.15}$$

$$\begin{array}{rl} \text{Example in} & \text{Figure 7} \\ \|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})} \approx & 222.4204 \\ \|\mathcal{G}_{0.75}\rho_o\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})} \approx & 221.8976 \\ \|\tilde\rho\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})} \approx & 222.2594 \end{array} \qquad \begin{array}{l} \|\rho_o - \tilde\rho\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})} \approx 6.1975 \\ \dfrac{\|\rho_o - \tilde\rho\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})}}{\|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^2;\mathbb{R})}} \approx 0.02786 \end{array} \tag{3.16}$$

We note that the errors in (3.15) and (3.16) are significantly higher than that in (3.11). This is due to the numerical algorithm, in particular to the use of the dimensional splitting method, which results in the 2 dimensional algorithm being *less reversible* than that used in dimension 1.

As a side remark, observe that a possibly better encryption can be obtained extending (1.4) adding suitable source terms, see [26].

4. PROOFS

We set $\mathbb{R}_+ = [0, +\infty[$. For $n \in \mathbb{N}\backslash\{0\}$, $\mathbb{R}^n$ is equipped with the Euclidean norm: this choice applies also to matrix spaces such as $\mathbb{R}^{n\times m}$. For $x \in \mathbb{R}^n$ and $r > 0$, $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at x with radius r. Throughout, ∇ is the usual differential operator with respect to the spatial coordinates, while DV is the total derivative of the map V with respect to all its variables.

$\mathcal{L}$ is the Lebesgue measure in $\mathbb{R}^n$. Concerning function spaces and norms, we convene that for a measurable function $\rho\colon \mathbb{R}^n \to \mathbb{R}^m$ and for any $p \in \{1, +\infty\}$

$$\begin{array}{rl} \|\rho\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} = & \int_{\mathbb{R}^n} \|\rho(x)\|\,\mathrm{d}x \\ \|\rho\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R}^m)} = & \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|\rho(x)\| \end{array} \quad \text{and} \quad \begin{array}{rl} \|\rho\|_{\mathbf{W^{1,p}}(\mathbb{R}^n;\mathbb{R}^m)} = & \|\rho\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^m)} + \|D\rho\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^{n\times m})} \\ \|\rho\|_{\mathbf{C^0}(\mathbb{R}_+;\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^m))} = & \sup_{t\in\mathbb{R}_+} \|\rho(t)\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^m)}. \end{array}$$

By $\mathbf{BV}(\mathbb{R}^n;\mathbb{R})$ we mean the set of functions in $\mathbf{L^1_{loc}}(\mathbb{R}^n;\mathbb{R})$ such that their distributional derivative is a (vector) Radon measure with finite total variation. In particular, a function in $\mathbf{BV}(\mathbb{R}^n;\mathbb{R})$ needs not be also in $\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})$, as in the case of the function $\rho \equiv 1$ on $\mathbb{R}^n$.

In connection with (2.3), for $i \in \{1, \ldots, m\}$, introduce the characteristics $X_i \colon \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ where

$$t \mapsto X_i(t; t_o, x_o) \quad \text{solves} \quad \begin{cases} \dot{x} = v_i(t, x) \\ x(t_o) = x_o. \end{cases} \tag{4.1}$$

It is immediate to check that when v_i is as in (2.2), for any $\rho \in \mathbf{C^0}\left(\mathbb{R}; \mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^m)\right)$, assumptions (η) and $(\mathbf{V})$ ensure the global existence and regularity of X_i.

For any $\rho_o \in \mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^m)$, recall that the solution to (2.3) with initial datum ρ_o assigned at time $t_o = 0$ is given, for $i \in \{1, \ldots, m\}$, through the Lagrangian representation

$$\rho_i(t, x) = \rho_{o,i}\left(X_i(0; t, x)\right) \exp\left(-\int_0^t (\nabla \cdot v_i)\left(\tau; X_i(\tau; t, x)\right) \mathrm{d}\tau\right) \tag{4.2}$$

which is a distributional solution and also a Kružkov solution in the sense of [27], Definition 1. Indeed, several results in the literature, see for instance [28], Lemma 5 or [29], Corollary II.1, ensure that in the case of the Cauchy problem for (2.3), the concepts of *weak* and *entropy* (or *Kružkov*) solutions coincide. Under assumptions (η) and $(\mathbf{V})$, when $\rho_{o,i}$ is sufficiently regular, (4.2) also yields the *strong* solution to (2.3) with v as in (2.2).

For $T > 0$, introduce the set

$$\mathcal{V}_T = \left\{ v \in (\mathbf{C^0} \cap \mathbf{L^\infty})\left([0, T] \times \mathbb{R}^n; \mathbb{R}^{n \times m}\right) : \begin{array}{r} v(t) \in \mathbf{C^2}(\mathbb{R}^n; \mathbb{R}^{n \times m}) \quad \forall\, t \in [0, T] \\ \nabla v \in \mathbf{L^\infty}([0, T] \times \mathbb{R}^n; \mathbb{R}^{n^2 \times m}) \\ \nabla(\nabla \cdot v) \in \mathbf{L^1}([0, T] \times \mathbb{R}^n; \mathbb{R}^{n \times m}) \end{array} \right\} \tag{4.3}$$

and, for a fixed $\rho_o \in \mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^m)$, the map

$$\begin{array}{rcl} \Sigma : \mathcal{V}_T & \to & \mathbf{C^0}\left([0, T]; \mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^m)\right) \\ v & \mapsto & \rho \end{array} \quad \text{where} \quad \begin{cases} \partial_t \rho_i + \nabla \cdot (\rho_i\, v_i) = 0 & i \in \{1, \ldots, m\} \\ \rho(0) = \rho_o. \end{cases} \tag{4.4}$$

The Cauchy problem in (4.4) consists of m linear, scalar, independent conservation laws. Thus, the proof of its well-posedness is very similar to that of various results in the literature. Here we refer to [28] and provide precise references.

Lemma 4.1. *Fix $T > 0$ and let $\rho_o \in \mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^m)$. Then, the map Σ in (4.4) is well defined and enjoys the following properties:*

($\mathbf{\Sigma 1}$) For all $t \in [0, T]$:

$$\|(\Sigma v)(t)\|_{\mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^m)} = \|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^m)}. \tag{4.5}$$

($\mathbf{\Sigma 2}$) If $\rho_o \in \mathbf{L^\infty}(\mathbb{R}^n; \mathbb{R}^m)$, for all $t \in [0, T]$:

$$\|(\Sigma v)(t)\|_{\mathbf{L^\infty}(\mathbb{R}^n; \mathbb{R}^m)} \leqslant \|\rho_o\|_{\mathbf{L^\infty}(\mathbb{R}^n; \mathbb{R}^m)} \exp\left(\|\nabla \cdot v\|_{\mathbf{L^1}([0,t]; \mathbf{L^\infty}(\mathbb{R}^n; \mathbb{R}^m))}\right). \tag{4.6}$$

($\mathbf{\Sigma 3}$) If $\rho_o \in \mathbf{BV}(\mathbb{R}^n; \mathbb{R}^m)$, for all $t \in [0, T]$:

$$\begin{aligned} \mathrm{TV}\left((\Sigma v)(t)\right) \leqslant{} & \left(1 + \|\nabla(\nabla \cdot v)\|_{\mathbf{L^1}([0,t]; \mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^{n \times m}))}\right) \\ & \times \exp \|\nabla v\|_{\mathbf{L^1}([0,t]; \mathbf{L^\infty}(\mathbb{R}^n; \mathbb{R}^{n^2 \times m}))} \left(\|\rho_o\|_{\mathbf{L^\infty}(\mathbb{R}^n; \mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right). \end{aligned} \tag{4.7}$$

(Σ4) If $\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$*,* $v_1, v_2 \in \mathcal{V}_T$ *and* $\nabla \cdot (v_2 - v_1) \in \mathbf{L^1}([0,t] \times \mathbb{R}^n;\mathbb{R}^m)$ *for a* $t \in [0,T]$*, then*

$$\begin{aligned}
&\left\|(\Sigma v_2 - \Sigma v_1)(t)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \\
\leqslant\ & \exp \int_0^t \max\left\{\left\|\nabla v_1(\tau)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}, \left\|\nabla v_2(\tau)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}\right\} \mathrm{d}\tau \\
&\times \left(1 + \int_0^t \max\left\{\left\|\nabla\left(\nabla\cdot v_1(\tau)\right)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}, \left\|\nabla\left(\nabla\cdot v_2(\tau)\right)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}\right\} \mathrm{d}\tau\right) \\
&\times \left(\left\|\rho_o\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right) \left\|v_2 - v_1\right\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m}))} \\
&+ \left\|\rho_o\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} \left\|\nabla\cdot v_2 - \nabla\cdot v_1\right\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^m)}.
\end{aligned}$$

(Σ5) If $\rho_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$*, for all* $t_1, t_2 \in [0,T]$*,*

$$\begin{aligned}
\left\|(\Sigma v)(t_2) - (\Sigma v)(t_1)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \leqslant\ & \left(1 + \left\|\nabla(\nabla\cdot v)\right\|_{\mathbf{L^1}([0,t_1\vee t_2];\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m}))}\right) \\
&\times \exp\left\|\nabla v\right\|_{\mathbf{L^1}([0,t_1\vee t_2];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m}))} \\
&\times \left(\left\|\rho_o\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right) |t_2 - t_1|.
\end{aligned}$$

(Σ6) If $\rho_{o,i} \geqslant 0$ *for an* $i \in \{1,\dots,n\}$*, then for all* $t \in [0,T]$*,* $((\Sigma v)(t))_i \geqslant 0$*.*

(Σ7) Let $\hat\rho_o, \check\rho_o \in \mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)$ *and call* $\hat\Sigma, \check\Sigma$ *the corresponding maps defined as in* (4.4) *with* $\hat\rho_o, \check\rho_o$ *as initial datum. Then, for all* $t \in [0,T]$*,*

$$\left\|(\hat\Sigma v)(t) - (\check\Sigma v)(t)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} = \left\|\hat\rho_o - \check\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}$$

and if moreover $\hat\rho_o, \check\rho_o \in \mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)$*, for all* $t \in [0,T]$*,*

$$\left\|(\hat\Sigma v)(t) - (\check\Sigma v)(t)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} \leqslant \left\|\hat\rho_o - \check\rho_o\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} \exp\left(\left\|\nabla\cdot v\right\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m))}\right).$$

Proof of Lemma 4.1. The equality in **(Σ1)** directly follows from (4.2), also using the change of coordinates $\xi = X_i(0;t,x)$ whose Jacobian determinant $J_i = J_i(t,x)$ satisfies $J_i(t,x) = \exp\int_0^t \nabla\cdot v_i\left(\tau, X_i(\tau;0,x)\right) \mathrm{d}\tau$, see [28], (H3) in Proposition 3 for the details. The bound in **(Σ2)** follows from a direct estimate based on (4.2), see [28], (H4) in Proposition 3. The bound **(Σ3)** is proved through an approximation argument, see [28], (H6) in Proposition 3 for the details. The estimate in **(Σ4)** follows from [28], (H5) in Proposition 3. The continuity in time **(Σ5)** is [28], (H7) in Proposition 3. The proof of **(Σ6)** is immediate, thanks to (4.2). Finally, **(Σ7)** directly follows from **(Σ1)**, thanks to the linearity of the differential equation in (4.4). □

For a V as in **(V)**, define the maps

$$\begin{array}{rcl}\tilde\Pi : \mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m) & \to & \mathbf{C^2}(\mathbb{R}^n;\mathbb{R}^{n\times m}) \\ \rho & \mapsto & V\left(\nabla(\rho * \eta)\right)\end{array} \quad \text{where } \left((\nabla(\rho*\eta))(x)\right)_{ji} = \int_{\mathbb{R}^n} \rho_i(x-\xi)\, \partial_{x_j}\eta(\xi)\, \mathrm{d}\xi \tag{4.8}$$

and for a $T > 0$

$$\begin{array}{rcccl}\Pi & : & \mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)\right) & \to & \mathbf{L}^\infty\left([0,T];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)\right) \\ & & \rho & \mapsto & \left[t \mapsto \tilde\Pi\left(\rho(t)\right)\right].\end{array} \tag{4.9}$$

Lemma 4.2. *Fix $p \in [1,+\infty]$, V satisfying* (**V**) *and let*

$$\eta \in (\mathbf{C^3} \cap \mathbf{W^{3,p}})(\mathbb{R}^n;\mathbb{R}) \quad \text{and} \quad \nabla\eta \in \mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n) \quad \text{and} \quad \Delta\eta \in \mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}). \tag{4.10}$$

Then, $\tilde\Pi$ in (4.8) is well defined and for all $\rho \in \mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)$

$$\tilde\Pi\rho \in \mathbf{C^2}(\mathbb{R}^n;\mathbb{R}^{n\times m}) \tag{4.11}$$

$$\left\|\tilde\Pi\rho\right\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^{n\times m})} \leqslant L_V \, \|\nabla\eta\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^n)} \, \|\rho\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \tag{4.12}$$

$$\left\|\nabla\cdot\tilde\Pi\rho\right\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^m)} \leqslant L_V \, \|\Delta\eta\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R})} \, \|\rho\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \tag{4.13}$$

$$\left\|\nabla\tilde\Pi\rho\right\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^{n^2\times m})} \leqslant L_V \, \left\|D^2\eta\right\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^{n\times n})} \, \|\rho\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \tag{4.14}$$

$$\begin{aligned}\left\|\nabla(\nabla\cdot\tilde\Pi\rho)\right\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^{n\times m})} \leqslant{}& L_V \, \left\|D^2\eta\right\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^{n\times n})} \, \|\Delta\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \, \|\rho\|^2_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \\ &+ L_V \, \left\|D^3\eta\right\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^{n\times n\times n})} \, \|\rho\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}.\end{aligned} \tag{4.15}$$

Moreover, for all $\rho_1, \rho_2 \in \mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)$, setting

$$C_\Pi = L_V \left(1 + \min\left\{\|\rho_1\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}, \|\rho_2\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right\} \|\nabla\eta\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^n)}\right)$$

we have

$$\left\|\tilde\Pi(\rho_2) - \tilde\Pi(\rho_1)\right\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^{n\times m})} \leqslant L_V \, \|\nabla\eta\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^n)} \, \|\rho_2 - \rho_1\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \tag{4.16}$$

$$\left\|\nabla\cdot\tilde\Pi(\rho_2) - \nabla\cdot\tilde\Pi(\rho_1)\right\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^m)} \leqslant C_\Pi \, \|\Delta\eta\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R})} \, \|\rho_2 - \rho_1\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \tag{4.17}$$

$$\left\|\nabla\tilde\Pi(\rho_2) - \nabla\tilde\Pi(\rho_1)\right\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^{n^2\times m})} \leqslant C_\Pi \, \left\|D^2\eta\right\|_{\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^{n\times n})} \, \|\rho_2 - \rho_1\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}. \tag{4.18}$$

Observe that Lemma 4.2 admits a straightforward extension to Π as defined in (4.9), where $\|\cdot\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}$ is replaced by $\|\cdot\|_{\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m))}$, see [18].

Proof of Lemma 4.2. The regularity (4.11) and all subsequent estimates are immediate consequences of (**V**) and of the classical properties of the convolution product, see for instance [20], Section IV.4. □

Proof of Theorem 2.2. Fix a positive T and a $\rho_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)$. Introduce the set

$$\mathcal{R} = \left\{\rho \in \mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)\right) : \begin{array}{l}\rho(0) = \rho_o \text{ and} \\ \forall\, t \in [0,T] \quad \|\rho(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} = \|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\end{array}\right\} \tag{4.19}$$

and define the composition

$$\begin{array}{rccc}\mathcal{T} & : & \mathcal{R} & \to & \mathcal{R} \\ & & \rho & \mapsto & (\Sigma\circ\Pi)(\rho)\end{array} \tag{4.20}$$

with Σ defined in (4.4) and Π in (4.9). We now prove that $\mathcal{T}$ is well defined and a contraction. Indeed, by Definition 2.1 a map ρ is a solution to (1.4) if and only if it is a fixed point of $\mathcal{T}$ on the time interval where it is defined.

Claim 1: $\mathcal{T}$ is well defined. Prove first that if $\rho \in \mathcal{R}$ then $\Pi\rho \in \mathcal{V}_T$, with $\mathcal{V}_T$ as in (4.3). For any $\rho \in \mathcal{R}$, $\Pi\rho \in \mathbf{L}^\infty([0,T] \times \mathbb{R}^n; \mathbb{R}^{n\times m})$ by (4.12). To prove the continuity of $\Pi\rho$, fix $(\bar t, \bar x) \in [0,T] \times \mathbb{R}^n$ and $\varepsilon > 0$. Then, since $\rho \in \mathcal{R}$ and (4.11), there exists $\delta > 0$ such that

$$\|\rho(t) - \rho(\bar t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} < \frac{\varepsilon}{2L_V \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}}$$

for every $t \in [0,T]$, $|t - \bar t| < \delta$ and

$$\left\|\tilde\Pi\left(\rho(\bar t)\right)(x) - \tilde\Pi\left(\rho(\bar t)\right)(\bar x)\right\| < \frac{\varepsilon}{2}$$

for every $\|x - \bar x\| < \delta$. Hence, using (4.16), we deduce that

$$\begin{aligned}
\|\Pi\rho(t)(x) - \Pi\rho(\bar t)(\bar x)\| &\leqslant \|\Pi\rho(t)(x) - \Pi\rho(\bar t)(x)\| + \|\Pi\rho(\bar t)(x) - \Pi\rho(\bar t)(\bar x)\| \\
&\leqslant \left\|\tilde\Pi\left(\rho(t)\right)(x) - \tilde\Pi\left(\rho(\bar t)\right)(x)\right\| + \left\|\tilde\Pi\left(\rho(\bar t)\right)(x) - \tilde\Pi\left(\rho(\bar t)\right)(\bar x)\right\| \\
&\leqslant \left\|\tilde\Pi\left(\rho(t)\right) - \tilde\Pi\left(\rho(\bar t)\right)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m})} + \frac{\varepsilon}{2} \\
&\leqslant L_V \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} \|\rho(t) - \rho(\bar t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} + \frac{\varepsilon}{2} < \varepsilon,
\end{aligned}$$

proving that $\Pi\rho \in \mathbf{C^0}\left([0,T] \times \mathbb{R}^n; \mathbb{R}^{m\times n}\right)$. The requirement $\nabla\Pi\rho \in \mathbf{L}^\infty([0,T] \times \mathbb{R}^n; \mathbb{R}^{m\times n\times n})$ follows from (4.14) and (4.19), while $\nabla(\nabla\cdot\Pi\rho) \in \mathbf{L^1}([0,T] \times \mathbb{R}^n; \mathbb{R}^{n\times m})$ by (4.15).

Prove now that if $v \in \mathcal{V}_T$, then $\Sigma v \in \mathcal{R}$, with $\mathcal{R}$ as in (4.19). Simply apply (4.5) in **(Σ1)** and **(Σ5)** using $\rho_o \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R}^m)$.

Claim 1 is proved.

Claim 2: For T small, $\mathcal{T}$ is a contraction. Fix ρ_1, ρ_2 in $\mathcal{R}$ and call $v_i = \Pi\rho_i$ for $i \in \{1,2\}$. Then, using (4.17), for $t \in [0,T]$,

$$\begin{aligned}
&\|\nabla\cdot v_2(t) - \nabla\cdot v_1(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \\
\leqslant\ & L_V \left(1 + \|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \|\nabla\eta\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^n)}\right) \|\Delta\eta\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \|\rho_2(t) - \rho_1(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}
\end{aligned}$$

which is finite, so that we can apply **(Σ4)** in Lemma 4.2 and obtain:

$$\begin{aligned}
&\|(\mathcal{T}\rho_2)(t) - (\mathcal{T}\rho_1)(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \\
=\ & \|(\Sigma v_2)(t) - (\Sigma v_1)(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \\
\leqslant\ & \exp\int_0^t \max\left\{\|\nabla v_1(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}, \|\nabla v_2(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}\right\} \mathrm{d}\tau \\
&\times\left(1 + \int_0^t \max\left\{\|\nabla\left(\nabla\cdot v_1(\tau)\right)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}, \|\nabla\left(\nabla\cdot v_2(\tau)\right)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}\right\} \mathrm{d}\tau\right) \\
&\times\left(\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right) \|v_2 - v_1\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m}))} \\
&+\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} \|\nabla\cdot v_2 - \nabla\cdot v_1\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^m)} \\
\leqslant\ & \exp\int_0^t \max\left\{\|\nabla\Pi\rho_1(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}, \|\nabla\Pi\rho_2(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}\right\} \mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&\times \left(1 + \int_0^t \max\left\{\left\|\nabla\left(\nabla\cdot\Pi\rho_1(\tau)\right)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}, \left\|\nabla\left(\nabla\cdot\Pi\rho_2(\tau)\right)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}\right\} \mathrm{d}\tau\right)\\
&\times \left(\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right) \ \|\Pi\rho_2 - \Pi\rho_1\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m})}\\
&+\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} \ \|\nabla\cdot\Pi\rho_2 - \nabla\cdot\Pi\rho_1\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^m)}\\
\leqslant{}& \exp\left(L_V \left\|D^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \int_0^t \max\left\{\|\rho_1(\tau)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}, \|\rho_2(\tau)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right\}\mathrm{d}\tau\right)\\
&\times\left(1 + L_V\left(\left\|D^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\Delta\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \left\|D^3\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n\times n})}\right)\right.\\
&\left.\times\int_0^t \max\left\{\begin{array}{c}\left(1+\|\rho_2(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right)\|\rho_2(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)},\\ \left(1+\|\rho_1(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right)\|\rho_1(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\end{array}\right\}\mathrm{d}\tau\right)\\
&\times\left(\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right) \ L_V \ \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} \ \|\rho_2-\rho_1\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^m)}\\
&+\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} \ L_V\left(1+\min\left\{\|\rho_1\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}, \|\rho_2\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right\}\|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\right)\\
&\times\|\Delta\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \ \|\rho_2-\rho_1\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^m)}\\
\leqslant{}& \exp\left(L_V \ \left\|D^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \ \|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \ t\right)\\
&\times\Big[1 + L_V\left(\left\|D^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\Delta\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \left\|D^3\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n\times n})}\right)\\
&\qquad\times\left(1+\|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right)\|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} t\Big]\\
&\times\left(\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right) \ L_V \ \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} \ \|\rho_2-\rho_1\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^m)}\\
&+\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} \ L_V\left(1+\|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)}\right)\|\Delta\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\\
&\quad\times\|\rho_2-\rho_1\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^m)}\\
\leqslant{}& C \ \left(\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right)\mathrm{e}^{C\,t}\,\|\rho_2-\rho_1\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^m)}
\end{aligned}
$$

where C is a positive constant depending on η, L_V and $\|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}$. In particular, C is independent of $\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)}$, $\mathrm{TV}(\rho_o)$, ρ_1, ρ_2, t and T.

We thus obtain

$$
\begin{aligned}
&\|\mathcal{T}(\rho_2)-\mathcal{T}(\rho_1)\|_{\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m))}\\
\leqslant{}& C\ \left(\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right)\mathrm{e}^{C\,T}\,\|\rho_2-\rho_1\|_{\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m))}\,T
\end{aligned}
$$

proving that for $T\,\mathrm{e}^{C\,T} < \frac{1}{C\left(\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)}+\mathrm{TV}(\rho_o)\right)}$, $\mathcal{T}$ is a contraction and admits a unique fixed point ρ^*. By Definition 2.1, ρ^* solves on $[0,T]$ the Cauchy problem for (1.4) with initial datum ρ_o assigned at time $t_o = 0$. Moreover, ρ^* satisfies $\|\rho^*(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} = \|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}$ for $t\in[0,T]$.

Claim 3: Extension of ρ^* for all times. A repeated application of the previous step allows to iteratively construct maps $\rho^\nu \in \mathbf{C^0}\left([T_{\nu-1},T_\nu];\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)\right)$ that solve

$$
\left\{\begin{array}{lll}
\partial_t\rho_i^\nu + \nabla\cdot\left(\rho_i^\nu\ V_i(\nabla\rho^\nu * \eta)\right) = 0 & (t,x)\in[T_{\nu-1},T_\nu]\times\mathbb{R}^n & i\in\{1,\ldots,m\}\\
\rho^\nu(T_{\nu-1},x) = \rho^{\nu-1}(T_{\nu-1},x) & x\in\mathbb{R}^n. &
\end{array}\right.
$$

The juxtaposition ρ^* of ρ_ν for $\nu \in \mathbb{N}\backslash\{0\}$ solves (1.4) with initial datum ρ_o assigned at time $t_o = 0$ on the time interval $[0, \sup_\nu T_\nu[$. At the same time, ρ^* is a fixed point for $\mathcal{T}$ extended to the time interval $[0, \sup_\nu T_\nu[$.

If $\sup_\nu T_\nu = +\infty$, then ρ^* is globally defined on $\mathbb{R}_+$. Otherwise, note that applying **(Σ5)**, (4.14), (4.15), we have for $t_1, t_2 \in [0, \sup T_\nu[$ with $t_1 < t_2$,

$$
\begin{aligned}
& \left\|\rho^*(t_2) - \rho^*(t_1)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \\
= {} & \left\|\left(\Sigma(\Pi\rho^*)\right)(t_2) - \left(\Sigma(\Pi\rho^*)\right)(t_1)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \\
\leqslant {} & \left(1 + \left\|\nabla(\nabla\cdot\Pi\rho^*)\right\|_{\mathbf{L^1}([0,t_2]\times\mathbb{R}^n;\mathbb{R}^{n\times m})}\right) \\
& \times \exp\left\|\nabla\Pi\rho^*\right\|_{\mathbf{L^1}([0,t_2];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m}))} \\
& \times \left(\left\|\rho_o\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right)\left|t_2 - t_1\right| \\
\leqslant {} & \left(1 + L_V\left(\left\|D^2\eta\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times n})}\left\|\Delta\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \left\|D^3\eta\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times n\times n})}\right)\right. \\
& \qquad \left.\times\left(1 + \left\|\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right)\left\|\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} t_2\right) \\
& \times \exp\left(L_V\left\|D^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\left\|\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} t_2\right) \\
& \times \left(\left\|\rho_o\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right)\left|t_2 - t_1\right| \\
\leqslant {} & \left(1 + L_V\left(\left\|D^2\eta\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times n})}\left\|\Delta\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \left\|D^3\eta\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times n\times n})}\right)\right. \\
& \qquad \left.\times\left(1 + \left\|\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right)\left\|\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \sup_\nu T_\nu\right) \\
& \times \exp\left(L_V\left\|D^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\left\|\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \sup_\nu T_\nu\right) \\
& \times \left(\left\|\rho_o\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right)\left|t_2 - t_1\right|
\end{aligned}
$$

which shows that ρ^* is uniformly Lipschitz continuous on $[0, \sup_\nu T_\nu[$. Thus, $\rho^*(\sup_\nu T_\nu)$ is well defined and can be used as initial datum for (1.4) to further extend ρ^*. This proves that ρ^* can be extended to all $\mathbb{R}_+$ and satisfies $\left\|\rho^*(t)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} = \left\|\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}$ for $t \in \mathbb{R}_+$.

Claim 4: Uniqueness of the Solution to the Cauchy problem. Let $\tilde\rho$ be a solution to (1.4) on the time interval $[0, \tilde T]$ in the sense of Definition 2.1 with initial datum ρ_o at time $t_o = 0$. Then, for a $\bar\nu \in \mathbb{N}$, $\tilde T \in [T_{\bar\nu}, T_{\bar\nu+1}[$, where we convene that $T_0 = 0$. By the uniqueness of the fixed point of a contraction, $\tilde\rho_{|[T_\nu, T_{\nu+1}[}$ coincides with $\rho^*_{|[T_\nu, T_{\nu+1}[}$ for all $\nu = 0, \ldots, \bar\nu - 1$. The same argument also applies to the time interval $[T_{\bar\nu}, \tilde T]$.

Claim 5: Definition of the group $\mathcal{G}$. By the arbitrariness of ρ_o, we can define a map

$$
\mathcal{G}^+ : \mathbb{R}_+ \times (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m) \to (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R}^m)
$$

such that $t \mapsto \mathcal{G}^+_t \rho_o$ is the solution to (1.4) with datum ρ_o assigned at time 0 and satisfies $\left\|\mathcal{G}^+_t\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n,\mathbb{R}^m)} = \left\|\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}$ for every $t \in \mathbb{R}_+$.

Note that the map $-V$ satisfies **(V)**, hence the above claims all apply to the equation

$$
\partial_t \tilde\rho_i - \nabla\cdot\left(\tilde\rho_i\, V_i(\nabla\tilde\rho * \eta)\right) = 0 \qquad i \in \{1, \ldots, m\} \tag{4.21}
$$

thus generating a map

$$\mathcal{G}^- : \mathbb{R}_+ \times (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R}^m) \to (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^n; \mathbb{R}^m)$$

satisfying $\left\|\mathcal{G}_t^- \rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n,\mathbb{R}^m)} = \|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}$ for all $t \in \mathbb{R}_+$. Since (1.4) and (4.21) are autonomous and thanks to the representation (4.2), the orbits of $\mathcal{G}^-$ are those of $\mathcal{G}^+$ but reversed in time, so that $\mathcal{G}_t^- (\mathcal{G}_t^+ \rho_o) = \rho_o$, refer to Appendix A for more details. We thus define

$$\mathcal{G}(t, \rho_o) = \begin{cases} \mathcal{G}^-(-t, \rho_o) & t < 0 \\ \mathcal{G}^+(t, \rho_o) & t \geqslant 0. \end{cases}$$

proving **(1)** and **(2)**. Note that $\mathcal{G}$ satisfies $\|\mathcal{G}(t, \rho_o)\|_{\mathbf{L^1}(\mathbb{R}^n,\mathbb{R}^m)} = \|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}$ for all $t \in \mathbb{R}$, so that **(3)** is proved.

Claim 6: Estimate (4). Fix $\hat{\rho}_o, \check{\rho}_o \in \left(\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV}\right)(\mathbb{R}^n; \mathbb{R}^m)$ and introduce the maps

$$\hat{v}(t, x) = V\left(\left((\mathcal{G}_t \hat{\rho}_o) * \eta\right)(x)\right) \qquad \text{and} \qquad \check{v}(t, x) = V\left(\left((\mathcal{G}_t \check{\rho}_o) * \eta\right)(x)\right)$$

as well as the solution $\bar{\rho}$ to the linear decoupled system

$$\begin{cases} \partial_t \bar{\rho}_i + \nabla \cdot (\bar{\rho}_i \; \check{v}_i(t, x)) = 0 & i \in \{1, \dots, m\} \\ \bar{\rho}(0) = \hat{\rho}_o. \end{cases}$$

Then,

$$\|\mathcal{G}_t \hat{\rho}_o - \mathcal{G}_t \check{\rho}_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \leqslant \|\mathcal{G}_t \hat{\rho}_o - \bar{\rho}(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} + \|\bar{\rho}(t) - \mathcal{G}_t \check{\rho}_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \tag{4.22}$$

and we compute the two terms separately. First, by **($\boldsymbol{\Sigma}$4)**, the first term in the right hand side of (4.22) can be bounded as follows:

$$\begin{aligned}
& \|\mathcal{G}_t \hat{\rho}_o - \bar{\rho}(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \\
\leqslant & \exp \int_0^t \max\left\{\|\nabla \check{v}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}, \|\nabla \hat{v}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}\right\} \mathrm{d}\tau && [\text{By } (\boldsymbol{\Sigma}4)] \\
& \times \left(1 + \int_0^t \max\left\{\|\nabla\left(\nabla \cdot \check{v}(\tau)\right)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}, \|\nabla\left(\nabla \cdot \hat{v}(\tau)\right)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}\right\} \mathrm{d}\tau\right) \\
& \times \left(\|\hat{\rho}_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\hat{\rho}_o)\right) \; \|\hat{v} - \check{v}\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m})} \\
& + \|\hat{\rho}_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} \; \|\nabla \cdot \hat{v} - \nabla \cdot \check{v}\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^m)} \\
\leqslant & \exp\left(L_V \left\|D^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \int_0^t \max\left\{\|\mathcal{G}_\tau \check{\rho}_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}, \|\mathcal{G}_\tau \hat{\rho}_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right\} \mathrm{d}\tau\right) && [\text{By } (4.14)] \\
& \times \left(1 + L_V \left(\left\|D^2\eta\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times n})} \|\Delta\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \left\|D^3\eta\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times n\times n})}\right)\right. \\
& \quad \times \int_0^t \max\left\{\left(1 + \|\mathcal{G}_\tau \check{\rho}_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right) \|\mathcal{G}_\tau \check{\rho}_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)},\right. \\
& \qquad \left.\left.\left(1 + \|\mathcal{G}_\tau \hat{\rho}_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right) \|\mathcal{G}_\tau \hat{\rho}_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\right\} \mathrm{d}\tau\right) && [\text{By } (4.15)] \\
& \times \left(\|\hat{\rho}_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\hat{\rho}_o)\right) L_V \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} \|\mathcal{G}\check{\rho}_o - \mathcal{G}\hat{\rho}_o\|_{\mathbf{L^1}([0,t];\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m))} && [\text{By } (4.16)]
\end{aligned}$$

$$
\begin{aligned}
&+ \|\widehat\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} \\
&\quad\times L_V \left(1 + \min\left\{\|\mathcal{G}\widehat\rho_o\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^m)}, \|\mathcal{G}\check\rho_o\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^n;\mathbb{R}^m)}\right\} \|\nabla\eta\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\right) \\
&\quad\times \|\Delta\eta\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|\mathcal{G}\widehat\rho_o - \mathcal{G}\check\rho_o\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m))} && \text{[By (4.17)]} \\
&\leqslant \exp\left(L_V \|D^2\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})} \max\left\{\|\check\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)}, \|\widehat\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)}\right\} t\right) && \text{[By (4.5)]} \\
&\quad\times \Bigg(1 + L_V \left(\|D^2\eta\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^{n\times n})} \|\Delta\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} + \|D^3\eta\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^{n\times n\times n})}\right) \\
&\qquad\quad \times \max\Big\{\left(1 + \|\check\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)}\right) \|\check\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)}, \\
&\qquad\qquad\qquad \left(1 + \|\widehat\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)}\right) \|\widehat\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)}\Big\}\, t\Bigg) && \text{[By (4.5)]} \\
&\quad\times \left(\|\widehat\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\widehat\rho_o)\right) L_V \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} \|\mathcal{G}\check\rho_o - \mathcal{G}\widehat\rho_o\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m))} \\
&+ \|\widehat\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} L_V \left(1 + \min\left\{\|\widehat\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)}, \|\check\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)}\right\} t\, \|\nabla\eta\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^n)}\right) && \text{[By (4.5)]} \\
&\quad\times \|\Delta\eta\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})} \|\mathcal{G}\widehat\rho_o - \mathcal{G}\check\rho_o\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m))}.
\end{aligned}
$$

Introducing a positive, continuous and non decreasing map $C = C(t)$ dependent on η and on the initial data $\check\rho_o$, $\widehat\rho_o$, we obtain a bound of the form, for all $t \in \mathbb{R}_+$,

$$
\|\mathcal{G}_t\widehat\rho_o - \bar\rho(t)\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)} \leqslant C(t)\, \|\mathcal{G}\widehat\rho_o - \mathcal{G}\check\rho_o\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m))}. \tag{4.23}
$$

The second term in the right hand side in (4.22) is estimated by means of **(Σ7)**:

$$
\|\bar\rho(t) - \mathcal{G}_t\check\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)} \leqslant \|\check\rho_o - \widehat\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)}. \tag{4.24}
$$

Grouping (4.23) and (4.24) we get

$$
\|\mathcal{G}_t\check\rho_o - \mathcal{G}_t\widehat\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)} \leqslant \|\check\rho_o - \widehat\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)} + C(t)\, \|\mathcal{G}\widehat\rho_o - \mathcal{G}\check\rho_o\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m))}
$$

so that, after renaming the constant C, an application of Gronwall Lemma yields the desired estimate in **(4)**.

Claim 7: Stability Estimate (5). Assume for simplicity $t \geqslant 0$, the other case being analogous. Set

$$
\widehat v(t,x) = \widehat V\left(\nabla(\widehat{\mathcal{G}}_t\rho_o * \eta)(x)\right) \quad \text{and} \quad \check v(t,x) = \check V\left(\nabla(\check{\mathcal{G}}_t\rho_o * \eta)(x)\right).
$$

In view of the application of **(Σ4)**, call $B = B_{\mathbb{R}^{n\times m}}(0, \|\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)} \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)})$, note that $\left\|\nabla\left(\check{\mathcal{G}}_t\rho_o * \eta\right)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m})} \leqslant \|\rho_o\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)} \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}$ for every $t \in \mathbb{R}$. Compute for $t \in \mathbb{R}$,

$$
\begin{aligned}
&\|\widehat v(t) - \check v(t)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m})} \\
\leqslant\ & \left\|\widehat v(t) - \widehat V\left(\nabla(\check{\mathcal{G}}_t\rho_o * \eta)\right)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m})} + \left\|\widehat V\left(\nabla(\check{\mathcal{G}}_t\rho_o * \eta)\right) - \check v(t)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m})} \\
\leqslant\ & L_V \left\|\nabla(\widehat{\mathcal{G}}_t\rho_o * \eta) - \nabla(\check{\mathcal{G}}_t\rho_o * \eta)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m})} + \left\|\widehat V - \check V\right\|_{\mathbf{L}^\infty(B;\mathbb{R}^{n\times m})} \\
\leqslant\ & L_V \left\|\widehat{\mathcal{G}}_t\rho_o - \check{\mathcal{G}}_t\rho_o\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^{n\times m})} \|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)} + \left\|\widehat V - \check V\right\|_{\mathbf{L}^\infty(B;\mathbb{R}^{n\times m})}
\end{aligned}
$$

so that

$$\begin{aligned}
&\|\hat{v}-\check{v}\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m}))}\\
&\leqslant L_V \left\|\hat{\mathcal{G}}\rho_o-\check{\mathcal{G}}\rho_o\right\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^{n\times m})}\|\nabla\eta\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}+\left\|\hat{V}-\check{V}\right\|_{\mathbf{L}^\infty(B;\mathbb{R}^{n\times m})}t. \qquad (4.25)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|\nabla\cdot\hat{v}(t)-\nabla\cdot\check{v}(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\\
&=\|\nabla\cdot(\hat{v}(t)-\check{v}(t))\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\\
&\leqslant \left\|D\left(\hat{V}\left(\nabla(\hat{\mathcal{G}}_t\rho_o*\eta)\right)-\check{V}\left(\nabla(\check{\mathcal{G}}_t\rho_o*\eta)\right)\right)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{(n\times m)^2})}\\
&\quad\times\left\|D^2\left((\hat{\mathcal{G}}_t\rho_o*\eta)-(\check{\mathcal{G}}_t\rho_o*\eta)\right)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}\\
&\leqslant\left[\left\|D\left(\hat{V}\left(\nabla(\hat{\mathcal{G}}_t\rho_o*\eta)\right)-\hat{V}\left(\nabla(\check{\mathcal{G}}_t\rho_o*\eta)\right)\right)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{(n\times m)^2})}\right.\\
&\qquad\left.+\left\|D\left(\hat{V}\left(\nabla(\check{\mathcal{G}}_t\rho_o*\eta)\right)-\check{V}\left(\nabla(\check{\mathcal{G}}_t\rho_o*\eta)\right)\right)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{(n\times m)^2})}\right]\\
&\quad\times\left\|D^2\left((\hat{\mathcal{G}}_t\rho_o-\check{\mathcal{G}}_t\rho_o)*\eta\right)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}\\
&\leqslant\left[L_V\left\|\nabla(\hat{\mathcal{G}}_t\rho_o*\eta)-\nabla(\check{\mathcal{G}}_t\rho_o*\eta)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}+\left\|D\hat{V}-D\check{V}\right\|_{\mathbf{L^1}(B;\mathbb{R}^{(n\times m)^2})}\right]\\
&\quad\times\left\|\hat{\mathcal{G}}_t\rho_o-\check{\mathcal{G}}_t\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}\left\|D^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\\
&\leqslant\left[L_V\left\|\hat{\mathcal{G}}_t\rho_o-\check{\mathcal{G}}_t\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}\|\nabla\eta\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^n)}+\left\|D\hat{V}-D\check{V}\right\|_{\mathbf{L^1}(B;\mathbb{R}^{(n\times m)^2})}\right]\\
&\quad\times\left\|\hat{\mathcal{G}}_t\rho_o-\check{\mathcal{G}}_t\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}\left\|D^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}
\end{aligned}$$

so that

$$\begin{aligned}
&\|\nabla\cdot\hat{v}-\nabla\cdot\check{v}\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^m)}\\
&\leqslant\left[L_V\left\|\hat{\mathcal{G}}\rho_o-\check{\mathcal{G}}\rho_o\right\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^{n\times m})}\|\nabla\eta\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^n)}+\left\|D\hat{V}-D\check{V}\right\|_{\mathbf{L^1}(B;\mathbb{R}^{(n\times m)^2})}t\right]\\
&\quad\times 2\|\rho_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\left\|D^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}. \qquad (4.26)
\end{aligned}$$

Therefore we deduce, for every $t\in\mathbb{R}$,

$$\begin{aligned}
&\left\|\hat{\mathcal{G}}_t\rho_o-\check{\mathcal{G}}_t\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)}\\
&\leqslant\exp\int_0^t\max\left\{\|\nabla\check{v}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})},\|\nabla\hat{v}(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}\right\}\mathrm{d}\tau \qquad \text{[By } (\mathbf{\Sigma 4})]\\
&\quad\times\left(1+\int_0^t\max\left\{\|\nabla(\nabla\cdot\check{v}(\tau))\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})},\|\nabla(\nabla\cdot\hat{v}(\tau))\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}\right\}\mathrm{d}\tau\right)\\
&\quad\times\left(\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)}+\mathrm{TV}(\rho_o)\right)\|\hat{v}-\check{v}\|_{\mathbf{L^1}([0,t];\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m}))}
\end{aligned}$$

$$
\begin{aligned}
&+\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} \left\|\nabla\cdot\hat v - \nabla\cdot\check v\right\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^m)}\\
\leqslant{}& \exp\int_0^t \max\left\{\|\nabla\check v(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}, \|\nabla\hat v(\tau)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n^2\times m})}\right\} \mathrm{d}\tau \qquad \text{[By (4.25), (4.26)]}\\
&\times\left(1+\int_0^t \max\left\{\|\nabla\left(\nabla\cdot\check v(\tau)\right)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}, \|\nabla\left(\nabla\cdot\hat v(\tau)\right)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^{n\times m})}\right\}\mathrm{d}\tau\right)\\
&\times\left(\|\rho_o\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)} + \mathrm{TV}(\rho_o)\right)\\
&\times\left[L_V\left\|\hat{\mathcal{G}}\rho_o - \check{\mathcal{G}}\rho_o\right\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^{n\times m})}\|\nabla\eta\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^n)} + \left\|\hat V - \check V\right\|_{\mathbf{L}^\infty(B;\mathbb{R}^{n\times m})} t\right]\\
&+2\|\rho_o\|^2_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^m)}\left\|D^2\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\\
&\times\left[L_V\left\|\hat{\mathcal{G}}\rho_o - \check{\mathcal{G}}\rho_o\right\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^{n\times m})}\|\nabla\eta\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^n)} + \left\|D\hat V - D\check V\right\|_{\mathbf{L^1}(B;\mathbb{R}^{(n\times m)^2})} t\right]
\end{aligned}
$$

so that

$$
\begin{aligned}
\left\|\hat{\mathcal{G}}_t\rho_o - \check{\mathcal{G}}_t\rho_o\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)} \leqslant{}& C\,\mathrm{e}^{C\,t}\,t\left(\left\|\hat V - \check V\right\|_{\mathbf{L}^\infty(B;\mathbb{R}^{n\times m})} + \left\|D\hat V - D\check V\right\|_{\mathbf{L^1}(B;\mathbb{R}^{(n\times m)^2})}\right)\\
&+C\left\|\hat{\mathcal{G}}\rho_o - \check{\mathcal{G}}\rho_o\right\|_{\mathbf{L^1}([0,t]\times\mathbb{R}^n;\mathbb{R}^{n\times m})}
\end{aligned}
$$

An application of Gronwall Lemma completes the proof of the claim.

Regularity and Positivity: The statements **(6)** and **(7)** follow from formula (4.2) with v_i as in (2.2).

The proof of Theorem 2.2 is completed. □

Proof of Proposition 2.3. Denote $\rho(t,x) = (\mathcal{G}_t\rho_o)(x)$ and define $r_o(x) = \rho_o(R\,x)$, $r(t,x) = (\mathcal{G}_t r_o)(x)$.

For later use, observe that if $\psi \in \mathbf{C_c^1}(\mathbb{R}^2;\mathbb{R})$, setting $\varphi(t,x) = \psi(t,R^{-1}x)$, we then have

$$
\partial_t\varphi(t,x) = \partial_t\psi(t,R^{-1}x) \text{ and } \nabla\varphi(t,x)\;R = \nabla\psi(t,R^{-1}x). \tag{4.27}
$$

Moreover, for $i \in \{1,\ldots,m\}$,

$$
\begin{aligned}
\left(\nabla r_i(t) * \eta\right)(x) ={}& \nabla\int_{\mathbb{R}^n} r_i(t,\xi)\,\eta(x-\xi)\,\mathrm{d}\xi\\
={}& \int_{\mathbb{R}^n} r_i(t,\xi)\,\nabla\eta(x-\xi)\,\mathrm{d}\xi\\
={}& \int_{\mathbb{R}^n} \rho_i(t,R\,\xi)\,\nabla\eta(x-\xi)\,\mathrm{d}\xi\\
={}& \int_{\mathbb{R}^n} \rho_i(t,z)\,\nabla\eta(x-R^{-1}z)\,\mathrm{d}z\\
={}& \int_{\mathbb{R}^n} \rho_i(t,z)\,\nabla\eta(R\,x-z)\,\mathrm{d}z\;R \qquad \text{[Since } \nabla\eta(x) = \nabla\eta(R\,x)\,R \text{ by } \mathbf{(2)}]\\
={}& \left(\nabla\rho_i(t) * \eta\right)(R\,x)\,R.
\end{aligned} \tag{4.28}
$$

Using **(1)**, **(2)**, **(3)**, (4.27) and (4.28), we now verify that r solves (1.4) with initial datum r_o.

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} r_i(t,x)\,\partial_t\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_{\mathbb{R}_+}\int_{\mathbb{R}} r_i(t,x)\,\nabla\psi(t,x)\,V_i\left(\left(\nabla r_1(t)*\eta\right)(x),\ldots,\left(\nabla r_m(t)*\eta\right)(x)\right)\mathrm{d}x\,\mathrm{d}t \\
&-\int_{\mathbb{R}} r_{o,i}(t,x)\,\psi(0,x)\,\mathrm{d}x \\
=&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \rho_i^*(t,R\,x)\,\partial_t\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_{\mathbb{R}_+}\int_{\mathbb{R}} \rho_i^*(t,R\,x)\,\nabla\psi(t,x)\,V_i\left(\left(\nabla\rho_1(t)*\eta\right)(R\,x)\,R,\ldots,\left(\nabla\rho_m(t)*\eta\right)(R\,x)\,R\right)\mathrm{d}x\,\mathrm{d}t \\
&-\int_{\mathbb{R}} \rho_{o,i}^*(t,R\,x)\,\psi(0,x)\,\mathrm{d}x \\
=&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \rho_i^*(t,y)\,\partial_t\psi(t,R^{-1}y)\,\mathrm{d}y\,\mathrm{d}t \\
&+\int_{\mathbb{R}_+}\int_{\mathbb{R}} \rho_i^*(t,y)\,\nabla\psi(t,R^{-1}y)\,V_i\left(\left(\nabla\rho_1(t)*\eta\right)(y)\,R,\ldots,\left(\nabla\rho_m(t)*\eta\right)(y)\,R\right)\mathrm{d}y\,\mathrm{d}t \\
&-\int_{\mathbb{R}} \rho_{o,i}^*(t,y)\,\psi(0,R^{-1}y)\,\mathrm{d}y \\
=&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \rho_i^*(t,y)\,\partial_t\varphi(t,y)\,\mathrm{d}y\,\mathrm{d}t \\
&+\int_{\mathbb{R}_+}\int_{\mathbb{R}} \rho_i^*(t,y)\,\nabla\varphi(t,y)\,R\,R^{-1}\,V_i\left(\left(\nabla\rho_1(t)*\eta\right)(y),\ldots,\left(\nabla\rho_m(t)*\eta\right)(y)\right)\mathrm{d}y\,\mathrm{d}t \\
&-\int_{\mathbb{R}} \rho_{o,i}^*(t,y)\,\varphi(0,y)\,\mathrm{d}y = 0.
\end{aligned}
$$

The uniqueness proved in Theorem 2.2 thus implies that $\rho^* = r$, completing the proof. □

Observe for later use that the formula (4.2) implies that, for every $i \in \{1,\ldots,m\}$,

$$\operatorname{spt}\rho_i(t) = X_i\left(t;0,\operatorname{spt}\rho_{o,i}\right). \tag{4.29}$$

Proof of Lemma 2.4. Using (4.1) and (2.2), we obtain, for $i \in \{1,\ldots,m\}$, the estimate

$$
\begin{aligned}
\left\|\dot X_i(t;0,x_o)\right\| =& \left\|V_i\left(\nabla\left(\rho(t)*\eta\right)(x)\right)\right\| && \text{[By (2.2)–(4.1)]}\\
\leqslant& L_V\left\|\nabla\left(\rho(t)*\eta\right)\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^{n\times m})} && \text{[By (V)]}\\
\leqslant& L_V\left\|\nabla\eta\right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R}^n)}\left\|\rho_o\right\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R}^m)}, && \text{[By (4.12) and (3) in Theorem 2.2]}
\end{aligned}
$$

the latter quantity being W as in (2.4). Hence, by (4.29), (4.2) and the assumption on $\operatorname{spt}\rho_o$,

$$\operatorname{spt}\rho(t) \subseteq \bigcup_{i=1}^m X_i\left(t;0,\rho_{o,i}\right) \subseteq \bigcup_{i=1}^m X_i\left(t;0,B(x_o,r)\right) \subseteq B\left(x_o, r+W\,t\right),$$

completing the proof. □

Proof of Proposition 2.5. Consider first **(i)**. By contradiction, referring to (4.29), assume there exist $\bar t \in]0,+\infty[$ and $x_o \in \operatorname{spt}\rho_{o,i}$ such that $X_i(\bar t;0,x_o) \in \partial C$, $X_i\left([0,\bar t[;0,\operatorname{spt}\rho_{o,i}\right) \subseteq C$ and for a suitable $\varepsilon > 0$, $X_i(]\bar t,\bar t+\varepsilon[,0,x_o) \subset \mathbb{R}^n \backslash C$. Call $\bar x = X_i(\bar t;0,x_o)$. Since C is non empty and closed, there exists a vector ν in the (outer) normal cone [30], Section 1.2, formula (6) to C at $\bar x$ such that

$$V_i\left(\nabla\left(\rho(\bar t) * \eta\right)(\bar x)\right) \cdot \nu \geqslant 0. \tag{4.30}$$

On the other hand, using **(1)**, **(2)**, **(3)** and the convexity of C,

$$V_i\left(\nabla\left(\rho(\bar t) * \eta\right)(\bar x)\right) \cdot \nu = v\left(\left\|\nabla\left(\rho(\bar t) * \eta\right)(\bar x)\right\|\right)\ \nabla\left(\rho_i(\bar t) * \eta\right)(\bar x) \cdot \nu\,; \tag{4.31}$$

$$\begin{aligned}\left(\nabla\left(\rho_i(\bar t) * \eta\right)(\bar x)\right) \cdot \nu &= \left(\left(\rho_i(\bar t) * \nabla\eta\right)(\bar x)\right) \cdot \nu \\ &= \int_{\mathbb{R}^n} \rho_i(\bar t,\xi)\ \nabla\eta(\bar x - \xi) \cdot \nu \,\mathrm{d}\xi \\ &= \int_{\operatorname{spt}\rho_i(\bar t) \cap B(\bar x,\ell)} \rho_i(\bar t,\xi)\ \nabla\eta(\bar x - \xi) \cdot \nu \,\mathrm{d}\xi \end{aligned}\tag{4.32}$$

$$= \int_{\operatorname{spt}\rho_i(\bar t) \cap B(\bar x,\ell)} \underbrace{\rho_i(\bar t,\xi)}_{>0 \text{ a.e.}}\ \underbrace{\tilde\eta'\left(\|\bar x - \xi\|\right)}_{<0 \text{ a.e.}}\ \underbrace{\frac{\bar x - \xi}{\|\bar x - \xi\|} \cdot \nu}_{>0 \text{ a.e.}} \,\mathrm{d}\xi \tag{4.33}$$

hence

$$V_i\left(\nabla\left(q(\bar t) * \eta\right)(\bar x)\right) \cdot \nu \quad \begin{cases} < 0 & \text{if } \mathcal{L}\left(\operatorname{spt}\rho_i(\bar t) \cap B(\bar x,\ell)\right) > 0 \\ = 0 & \text{if } \mathcal{L}\left(\operatorname{spt}\rho_i(\bar t) \cap B(\bar x,\ell)\right) = 0 \end{cases}$$

contradicting (4.30) in case $\mathcal{L}\left(\operatorname{spt}\rho_i(\bar t) \cap B(\bar x,\ell)\right) > 0$. On the other hand, by (2.5),

$$\mathcal{L}\left(\operatorname{spt}\rho_i(\bar t) \cap B(\bar x,\ell)\right) = 0 \implies \rho_i(\bar t) = 0 \text{ a.e. on } B(\bar x,\ell) \implies B(\bar x,\ell) \subseteq \left(\mathbb{R}^n \backslash \operatorname{spt}\rho_i(t)\right)$$

where we used the fact that $B(\bar x,\ell)$ is an open set. The latter statement contradicts the choice of $\bar x$. Indeed, by (4.29),

$$\bar x = X_i(\bar t;0,x_o) \in X_i\left(\bar t;0,\operatorname{spt}\rho_{o,i}\right) = \operatorname{spt}\rho_i(t),$$

completing the proof of **(i)**. Now, **(ii)** immediately follows. ▫

Proof of Corollary 2.6. First, we prove that if $x_o \in B(x_h,r_h)$, the curve $t \mapsto X_i(t;0,x_o)$ defined in (4.1) remains in the same ball $B(x_h,r_h)$ for all times $t \geqslant 0$. By contradiction, assume there exists a first time $\bar t \in]0,+\infty[$ when the characteristic $t \mapsto X_i(t;0,x_o)$ exits the ball $B(x_h,r_h)$. Then, $X_i(\bar t;0,x_o) \in \partial B(x_h,r_h)$, $X_i\left([0,\bar t[;0,x_o\right) \subseteq B(x_h,r_h)$ and for a suitable $\varepsilon > 0$, $X_i(]\bar t,\bar t+\varepsilon[;0,x_o) \notin B(x_h,r_h)$.

Call $\bar x = X_i(\bar t;0,x_o)$. Fix an exterior normal ν to $B(x_h,r_h)$ at $\bar x$, use equality (4.31) and repeat computations analogous to those between (4.32) and (4.33) to obtain:

$$\left(\nabla\left(\rho_i(\bar t) * \eta\right)(\bar x)\right) \cdot \nu = \int_{B(x_h,r_h) \cap B(\bar x,\ell)} \underbrace{\rho_i(\bar t,\xi)}_{>0 \text{ a.e.}}\ \underbrace{\tilde\eta'\left(\|\bar x - \xi\|\right)}_{<0 \text{ a.e.}}\ \underbrace{\frac{\bar x - \xi}{\|\bar x - \xi\|} \cdot \nu}_{>0 \text{ a.e.}} \,\mathrm{d}\xi < 0$$

since by construction $\mathcal{L}\left(B(x_{\bar h},r_{\bar h}) \cap B(\bar x,\ell)\right) > 0$, getting a contradiction.

From expression (4.2) the inclusion (2.7) then follows. Equality (2.8) is a consequence of the conservative form of (1.4). ▫

Proof of Proposition 2.7. We verify that the map ρ given by $\rho(t,x) = \rho_o(x)$ solves (1.4) in the sense of Definition 2.1. To this aim, fix $x \in \operatorname{spt}\rho_o$ so that $x \in B(x_{\bar h}, r/2)$ for a suitable $\bar h \in \{1,\dots,k\}$. Note that for $i \in \{1,\dots,m\}$,

$$\begin{aligned}
(\nabla\rho_i(t) * \eta)(x) &= \int_{\mathbb{R}^n} (\rho_o)_i(y)\, \nabla\eta(x-y)\,\mathrm{d}y && [\text{Since } \rho(t) = \rho_o]\\
&= \int_{\bigcup_{h=1}^k B(x_h, r/2)} (\rho_o)_i(y)\, \nabla\eta(x-y)\,\mathrm{d}y && [\text{By } (2)]\\
&= \sum_{h=1}^k \int_{B(x_h, r/2)} (\rho_o)_i(y)\, \nabla\eta(x-y)\,\mathrm{d}y\,. && [\text{By } (2)]
\end{aligned}$$

Each term in the latter sum vanishes by **(1)**. Indeed, when $h = \bar h$

$$\begin{array}{lll} x \in B(x_{\bar h}, r/2) & \Longrightarrow & \|x - x_{\bar h}\| \leqslant r/2 \\ y \in B(x_{\bar h}, r/2) & \Longrightarrow & \|y - x_{\bar h}\| \leqslant r/2 \end{array} \Longrightarrow \|x-y\| \leqslant r \Longrightarrow \nabla\eta(x-y) = 0.$$

On the other hand, when $h \neq \bar h$ using **(2)**

$$\begin{array}{lll} x \in B(x_{\bar h}, r/2) & \Longrightarrow & \|x - x_{\bar h}\| \leqslant r/2 \\ y \in B(x_{h}, r/2) & \Longrightarrow & \|y - x_{h}\| \leqslant r/2 \end{array} \Longrightarrow \|x-y\| \geqslant \ell \Longrightarrow \nabla\eta(x-y) = 0.$$

Thus, the velocity v_i in (2.2) vanishes for all $t \geqslant 0$ and $x \in \operatorname{spt}\rho_o$. Hence, $\rho_i(t,x)\, v_i(t,x) = 0$ for all $t \geqslant 0$ and $x \in \mathbb{R}^n$, completing the proof. □

5. Conclusion

This paper presents an equation, namely (1.4), establishes its well-posedness and proves some of its qualitative properties. The resulting picture is that of a model prone to describe *collective behaviors* in the widest sense. In contrast, much of the existing literature takes the opposite approach, beginning with quantitative experiments or qualitative/heuristic insights into specific real-world phenomena, and then developing a model. Many of the specific real features captured by these models appear to be within the range of the qualitative properties of the solutions to (1.4).

The absence of a parabolic diffusion allows a (non standard) analytic treatment essentially based on hyperbolic techniques. Moreover, qualitative properties of the solutions to (1.4) are rigorously proved, then exemplified through numerical integrations.

This new model suggests various related open analytical problems, such as its connections with microscopic models, the characterization and the stability of its asymptotic states and the development of *ad hoc* efficient numerical algorithms, for instance.

Moreover, the use of this equation as an encryption – decryption tool rises new analytical and numerical questions. The search for a class where the map $(\eta, V) \mapsto \mathcal{G}_T^{\eta,V}\rho_o$ turns out to be surjective is likely to require the development of tools for the fine analysis of non-local conservation laws. Present general numerical methods for the integration of conservation laws hardly respect the reversibility of (1.4), in particular in more than 1 space dimension.

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Data availability statement

The research data associated with this article are included in the article.

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Appendix A. Time reversibility

In this appendix, for readability purposes, we insert the proof of reversibility in time, which makes the map $\mathcal{G}$ in Theorem 2.2 a group. More precisely, using the notation in **Claim 5** in the proof of Theorem 2.2, we prove that, for $\rho_o \in \left(\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV}\right)(\mathbb{R}^n;\mathbb{R}^m)$, the relation $\mathcal{G}^-_t(\mathcal{G}^+_t\rho_o) = \rho_o$ holds for every $t \in \mathbb{R}_+$.

Fix $\rho_o \in \left(\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV}\right)(\mathbb{R}^n;\mathbb{R}^m)$, $t > 0$ and define, for $\tau \in [0,t]$, $r(\tau) = \mathcal{G}^+_{t-\tau}\rho_o$. We prove that r solves (4.21) with initial condition $\mathcal{G}^+_t\rho_o$, so that $r(\tau) = \mathcal{G}^-_\tau(\mathcal{G}^+_t\rho_o)$. To this aim, fix $i \in \{1,\ldots,m\}$ and $\psi \in \mathbf{C_c^\infty}([0,t]\times\mathbb{R}^n;\mathbb{R})$ and use the change of variables $s = t - \tau$:

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} \left[r_i(\tau,x)\partial_t\psi(\tau,x) - r_i(\tau,x)V_i\left(\left(\nabla r(\tau) * \eta\right)(x)\right)\cdot\nabla\psi(\tau,x)\right]\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_{\mathbb{R}^n} r(0,x)\psi(0,x)\,\mathrm{d}x - \int_{\mathbb{R}^n} r(t,x)\psi(t,x)\,\mathrm{d}x \\
={}& \int_0^t \int_{\mathbb{R}^n} \left[(\mathcal{G}^+_{t-\tau}\rho_o)_i(x)\partial_t\psi(\tau,x) - (\mathcal{G}^+_{t-\tau}\rho_o)_i(x)V_i\left(\left(\nabla r(\tau) * \eta\right)(x)\right)\cdot\nabla\psi(\tau,x)\right]\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_{\mathbb{R}^n} \mathcal{G}^+_t\rho_o(x)\psi(0,x)\,\mathrm{d}x - \int_{\mathbb{R}^n} \rho_o(x)\psi(t,x)\,\mathrm{d}x \\
={}& \int_0^t \int_{\mathbb{R}^n} \left[(\mathcal{G}^+_s\rho_o)_i(x)\partial_t\psi(t-s,x) - (\mathcal{G}^+_s\rho_o)_i(x)V_i\left(\left(\nabla r(t-s) * \eta\right)(x)\right)\cdot\nabla\psi(t-s,x)\right]\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_{\mathbb{R}^n} \mathcal{G}^+_t\rho_o(x)\psi(0,x)\,\mathrm{d}x - \int_{\mathbb{R}^n} \rho_o(x)\psi(t,x)\,\mathrm{d}x\,.
\end{aligned}
$$

Define, for every $s \in [0,t]$ and $x \in \mathbb{R}^n$, the function $\tilde{\psi}(s,x) = \psi(t-s,x)$, so that $\tilde{\psi} \in \mathbf{C_c^\infty}([0,t] \times \mathbb{R}^n;\mathbb{R})$, $\partial_s \tilde{\psi}(s,x) = -\partial_t \psi(t-s,x)$, and $\nabla \tilde{\psi}(s,x) = \nabla \psi(t-s,x)$. Thus

$$
\begin{aligned}
& \int_0^t \int_{\mathbb{R}^n} \left[r_i(\tau,x)\partial_t \psi(\tau,x) - r_i(\tau,x) V_i\left(\left(\nabla r(\tau) * \eta\right)(x)\right) \cdot \nabla \psi(\tau,x)\right] \mathrm{d}x \, \mathrm{d}\tau \\
& \quad + \int_{\mathbb{R}^n} r(0,x)\psi(0,x)\,\mathrm{d}x - \int_{\mathbb{R}^n} r(t,x)\psi(t,x)\,\mathrm{d}x \\
= & - \int_0^t \int_{\mathbb{R}^n} \left[(\mathcal{G}_s^+ \rho_o)_i(x) \partial_s \tilde{\psi}(s,x) + (\mathcal{G}_s^+ \rho_o)_i(x) V_i\left(\left(\nabla r(t-s) * \eta\right)(x)\right) \cdot \nabla \tilde{\psi}(s,x)\right] \mathrm{d}x \, \mathrm{d}s \\
& \quad + \int_{\mathbb{R}^n} \mathcal{G}_t^+ \rho_o(x) \tilde{\psi}(t,x)\,\mathrm{d}x - \int_{\mathbb{R}^n} \rho_o(x)\tilde{\psi}(0,x)\,\mathrm{d}x \\
= & - \int_0^t \int_{\mathbb{R}^n} \left[(\mathcal{G}_s^+ \rho_o)_i(x) \partial_s \tilde{\psi}(s,x) + (\mathcal{G}_s^+ \rho_o)_i(x) V_i\left(\left(\nabla (\mathcal{G}_s^+ \rho_o) * \eta\right)(x)\right) \cdot \nabla \tilde{\psi}(s,x)\right] \mathrm{d}x \, \mathrm{d}s \\
& \quad + \int_{\mathbb{R}^n} \mathcal{G}_t^+ \rho_o(x) \tilde{\psi}(t,x)\,\mathrm{d}x - \int_{\mathbb{R}^n} \rho_o(x)\tilde{\psi}(0,x)\,\mathrm{d}x = 0
\end{aligned}
$$

since $\tau \mapsto \mathcal{G}_\tau^+ \rho_o$ solves the forward problem. The uniqueness proved in Theorem 2.2 thus implies that $r(\tau) = \mathcal{G}_\tau^-(\mathcal{G}_t^+ \rho_o)$, and so $\mathcal{G}_t^-(\mathcal{G}_t^+ \rho_o) = \rho_o$, completing the proof.