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ASYMPTOTIC DYNAMICS OF SIRS EPIDEMIC MODEL WITH DISPERSAL BUDGETS AND NONLINEAR RATES ABOUT HETEROGENOUS ENVIRONMENTS

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Abstract. This paper examines an SIRS epidemic model incorporating nonlinear incidence functions and nonlocal diffusion with scaled dispersal to enhance understanding of infectious disease spread in human populations. We establish the well-posedness of the model by proving both the existence and uniqueness of its solution. Additionally, we demonstrate the existence of a global compact attractor that describes the asymptotic behavior of all positive solutions. The basic reproduction number, \mathbb{R}_0 , is derived as the spectral radius of the linear and compact next-generation operator $\mathbf{R}(\cdot)$. When $\mathbb{R}_0 < 1$, the infection-free equilibrium (IFE) is globally asymptotically stable, leading to disease extinction, which has significant implications for public health policies. Conversely, when $\mathbb{R}_0 > 1$, persistence theory shows the system is strongly persistent, ensuring at least one positive endemic equilibrium state (PEES). The study investigates the system's asymptotic behavior under varying costs and scaling parameters of the dispersal kernel, revealing that when the dispersal kernel's support (σ) is sufficiently small and the cost parameter m < 2, the epidemic persists, posing public health risks. These results highlight the critical influence of scaling and cost parameters on disease dynamics.

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1. LITERARY WORKS, AND MODEL FORMULATION

Mathematical modeling plays a crucial role in analyzing the spread of infectious diseases and understanding their significant effects on society. These models are crucial for studying how infections move through populations, predicting their future course, and helping to develop strategies for controlling outbreaks. Classic models, such as SIS (Susceptible-Infected-Susceptible) [1–5], SIR (Susceptible-Infected-Recovered) [6–10], SEIR (Susceptible-Exposed-Infected-Recovered) [11], SVIR (Susceptible-Vaccinated-Infected-Recovered) [12], and SIRS (Susceptible-Infected-Recovered-Susceptible) [13–15], are commonly used to analyze the dynamics of diseases.

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Epidemics can spread quickly and have devastating effects, disrupting communities and economies. Take COVID-19, for example, which has dramatically altered our lives, causing millions of deaths worldwide. In light of such dangers, scientists have developed mathematical models to better understand how individuals interact and how diseases can be controlled. Epidemics like COVID-19, along with diseases such as measles, polio, Ebola, and SARS, have placed enormous pressure on global health systems and societies. This underscores the urgent need for effective modeling to understand these diseases and develop ways to mitigate their impact.

Infectious diseases profoundly affect human life, necessitating control of interactions between individuals and the spread of disease. However, the dynamics of these interactions, particularly between susceptible and infected individuals, remain complex and evolving. The nature of contact between these groups changes over time, complicating the modeling process and necessitating the assumption of nonlinear interactions. Mathematical modeling provides valuable insights into these complexities, enabling the study of infection dynamics and aiding in the development of effective intervention strategies. Numerous studies, such as those on SIS models [5] and SIR models [2, 16], have contributed significantly to this field, including variations like the SIRI model [17], which indicates that the recovered become directly infected, a factor in many diseases.

A particularly interesting aspect of disease modeling is the transmission dynamics, which involve understanding the interaction between susceptible and infected individuals. This requires simulating how these groups interact, particularly within spatially heterogeneous environments. In their pioneering work, [7], introduced the SIR model, which classified individuals into three categories: the susceptible S, infected I, and recovered R. In practice, nonlinear incidence is used in most SIR models (see for instance [1, 3, 5]) including $\frac{\beta SI}{S+I}$. This might not seem like the best way to describe the interaction, but it is more appropriate to assume that the denominator depends on the total population. The spread of infectious diseases is still challenging to understand. Are we dealing with a specific type of transmission, or does it come down to how strong the immune system is? To get closer to an answer, this study assumes that this interaction depends on the size of the population and follows a nonlinear pattern.

While these foundational models provide valuable insights, they often neglect the critical role of mobility and diffusion in disease spread. Mobility, representing the movement of individuals between regions, is an essential factor in realistic modeling. Incorporating diffusion into these models significantly enhances their ability to capture the spatial dynamics of epidemics. Recognizing this, many studies have explored how individual mobility can be effectively integrated into epidemic models, providing a more comprehensive understanding of disease transmission and its spatial effects.

This work builds upon previous advancements, highlighting the critical role of spatial interactions and mobility in heterogeneous environments to develop more accurate and practical models for analyzing infectious disease dynamics (see, for instance, [9, 13, 18–20] and references therein). While Laplacian operators are commonly used to model local and random mobility, they provide only a limited representation of movement, as they assume unrestricted diffusion in open areas. This assumption imposes significant limitations on the realistic depiction of mobility. To address this, introducing nonlocal diffusion offers a more natural and realistic approach, capturing the movement of individuals across nonlocal regions effectively. Moreover, to explain the effect of dispersal budget $\mathcal{L}_{\sigma} = \frac{1}{\sigma^n} L(\frac{z}{\sigma})$ (see [3, 21, 22]) in the area of mobility, we define the nonlocal diffusion as follows:

$$\frac{1}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma} \left(x - y \right) \left[\phi(t, y) - \phi(t, x) \right] \, \mathrm{d}y,$$

with $\Omega \subset \mathbb{R}^n$, $n \geq 1$, a bounded set with a smooth boundary. $\mathcal{L}_{\sigma}(x-y)$ represents the probability of jumping from y to x with \mathcal{L}_{σ} representing the dispersal kernel, and $\int_{\Omega} \mathcal{L}_{\sigma}(x-y) \phi(t,y) dy$ interprets as the function that represents the individuals collected at the area x. The term $-\phi(t, \cdot)$ shows the mobility of the individuals from x to any other area. Moreover, σ represents the scaling factor on the range of dispersal and m is considered as the cost parameter on the range of dispersal of the cost function f(z) which is proportional to $|y|^m$ with $f_0 = \frac{\mathbf{C}^0}{\int_{\mathbb{R}^n} L(z) |z|^m \, \mathrm{d}z}, \quad \frac{f_0 \mathcal{L}_{\sigma}(z)}{\sigma^m} \text{ . This nonlocal diffusion operator, driven by the general kernel } \mathcal{L}_{\sigma} \ge 0, \text{ differs}$ from the fractional Laplacian $(-\Delta)^s \phi(x) = C_{n,s} \int_{\Omega} \frac{\phi(x) - \phi(y)}{|x-y|^{n+2s}} \, \mathrm{d}y \text{ used in } [23] \text{ for a SARS-CoV-2 model, which}$ employs a singular kernel with polynomial decay to model anomalous diffusion, whereas our flexible kernel allows tailored dispersal patterns for the SIRS dynamics.

In the work proposed by $Hu \ et \ al.$ [3], they considered the following nonlocal dispersal SIS epidemic model with Neumann boundary conditions in a heterogeneous environment:

$$\begin{cases} \frac{\partial S}{\partial t} = \frac{d_1}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{S}(t,y) - \mathcal{S}(t,\cdot)] \, \mathrm{d}y - \frac{\beta(\cdot)\mathcal{S}\mathcal{I}}{\mathcal{S}+\mathcal{I}} + \alpha(\cdot)\mathcal{I}, & \text{on } \Omega \times \mathbb{R}^+, \\ \frac{\partial \mathcal{I}}{\partial t} = \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{I}(t,y) - \mathcal{I}(t,\cdot)] \, \mathrm{d}y + \frac{\beta(\cdot)\mathcal{S}\mathcal{I}}{\mathcal{S}+\mathcal{I}} - \alpha(\cdot)\mathcal{I}, & \text{on } \Omega \times \mathbb{R}^+, \\ \mathcal{S}(0,\cdot) = S_0(\cdot), \quad \mathcal{I}(0,\cdot) = I_0(\cdot), & \text{on } \Omega, \end{cases}$$
(1.1)

with

- S and \mathcal{I} represent the susceptible and infected individuals at time t > 0 and location $x \in \Omega$,
- d_1 and d_2 are the nonlocal diffusion coefficients for the susceptible and infected populations, respectively,
- $\mathcal{L}_{\sigma}(x-y)$ is the nonlocal dispersal kernel with a characteristic length scale σ ,
- $\beta(\cdot)$ is the infection transmission rate at location x,
- $\gamma(\cdot)$ is the recovery rate at location x,
- $S_0(\cdot)$ and $I_0(\cdot)$ are the initial conditions of the susceptible and infected individuals, respectively.

They investigated the asymptotic behavior of the SIS epidemic model using a novel method that provides a useful framework for understanding the dynamics of complex systems such as the SIS epidemic model. The dynamics of the model in (1.1) were examined using this approach, which expanded on the ideas offered in [21]. Researchers from all across the world have been using nonlocal diffusion as a means of characterizing individuals' free movement in recent years (see [5, 24–27]). This method is thought to be more realistic and indicative of the real world. Moreover, there are many diseases that have the important characteristic of temporal immunity, which implies that the recovered individuals can lose immunity and become susceptible again (see, for instance, [17, 28, 29]) and the references therein. [29].

Motivated by all of these studies, but particularly by the work by [3], this study investigates a SIRS epidemic model with a nonlinear incidence function. It is assumed that the recovered individuals lose their immunity and become susceptible again, which is a fascinating way to explain many diseases that have been studied in the same case. It is also assumed that the kernel dispersal could be used to calculate the "budget dispersal", which represents a very precise movement of individuals, and that the nonlocal diffusion could be used to describe the mobility of individuals. Then, the model is defined as follows:

$$\begin{cases} \frac{\partial \mathcal{S}(t,\cdot)}{\partial t} = \frac{d_1}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{S}(t,y) - \mathcal{S}(t,\cdot)] dy - \frac{\beta(\cdot)\mathcal{S}\mathcal{I}}{\mathcal{S} + \mathcal{I} + \mathcal{R}} + \gamma(\cdot)\mathcal{R}(t,\cdot), \text{ on } \Omega \times \mathbb{R}^+, \\ \frac{\partial \mathcal{I}(t,\cdot)}{\partial t} = \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{I}(t,y) - \mathcal{I}(t,\cdot)] dy + \frac{\beta(\cdot)\mathcal{S}\mathcal{I}}{\mathcal{S} + \mathcal{I} + \mathcal{R}} - \alpha(\cdot)\mathcal{I}, \text{ on } \Omega \times \mathbb{R}^+, \\ \frac{\partial \mathcal{R}(t,\cdot)}{\partial t} = \frac{d_3}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)] dy + \alpha(\cdot)\mathcal{I} - \gamma(\cdot)\mathcal{R}, \text{ on } \Omega \times \mathbb{R}^+, \\ \mathcal{S}(0,\cdot) = S_0(\cdot), \ \mathcal{I}(0,\cdot) = I_0(\cdot), \ \mathcal{R}(0,\cdot) = R_0(\cdot), \text{ on } \Omega, \end{cases}$$
(1.2)

In this model, $\Omega \subset \mathbb{R}^n$, $n \ge 1$, is a bounded domain with a smooth boundary $\partial \Omega$, and (t, x) denote time and location, respectively. S(t, x), $\mathcal{I}(t, x)$, and $\mathcal{R}(t, x)$ represent the susceptible, infected, and recovered individuals,

respectively, with nonnegative initial conditions denoted as (S_0, I_0, R_0) on $\overline{\Omega}$. The dispersal coefficients for these groups are given by the parameters d_1 , d_2 , and d_3 , respectively, and $\beta(x)$ represents the transmission rate for susceptible individuals. This study makes the reasonable assumption that each group has distinct dispersal constants, reflecting their differing mobility patterns. Based on the aforementioned research, it is more reasonable to consider the dispersal of the three groups, using the nonlinearity of the incidence rates to model interactions between susceptible and infected individuals. The primary focus of this study is to investigate the asymptotic behavior of the SIRS model with nonlocal dispersal, which poses a challenge in analyzing, particularly, the asymptotic profile of the positive endemic equilibrium states (PEES). Recent studies have examined the asymptotic properties of the PEES with respect to the dispersal scaling parameter σ , providing further insights into this area, see for example [3, 21].

Disease outbreaks like COVID-19 show us we need better models to track how infections spread across space, especially when immunity doesn't last forever. In the present model, we consider that the densities are governed by nonlocal diffusion. This type of spatial movement is more appropriate for capturing realistic patterns of individual mobility, especially over larger distances, compared to classical diffusion based on the Laplace operator, which reflects only short-range displacements. We use a kernel $J_{\sigma} \in$ Schwartz space; which is smooth and drops off fast. This choice not only ensures mathematical tractability but also allows us to model individual mobility across varying spatial scales and allows us to shape dispersal to fit real scenarios. Furthermore, the interaction between susceptible and infected individuals is described by a nonlinear incidence term of the form $\frac{\beta(x) SI}{S + I + R}$, which reflects the complex and saturating nature of disease transmission dynamics, and it is a specific case of the Hattaf-Yousfi functional response [30, 31]. This formulation accounts for saturation effects and reflects the diminishing transmission potential as the population accumulates immunity—a critical aspect for accurately modeling SIRS-type diseases.

The main goal of this study is to investigate the dynamics of the SIRS epidemic model with nonlocal dispersal and nonlinear incidence rates, which represents a strategy of producing few offspring that are all spread out far from the source. The following is the paper's structure: This study builds the model and proves that its solutions exist and are unique in Section 2. It calculates and examines the basic reproduction number's characteristics in Section 3. For $\mathbb{R}_0 < 1$, the threshold behavior of the model (1.2) at the infection-free equilibrium is examined in Section 4, and global stability at $\mathbb{R}_0 = 1$ is analyzed using a Lyapunov function weighted by the principal eigenfunction. Additionally, this study examines the persistence conditions and the presence of positive endemic equilibrium states (PEES) in Section 5. The subsequent section examines the asymptotic profiles of PEES as $\sigma \to 0$, with variations in m. Lastly, the final section discusses the results and provides perspectives for future research.

2. Properties of solutions to (1.2)

As showed before, we assume that the individuals divide into three class: the susceptible individuals $(\mathcal{S}(t, \cdot))$, infected individuals $(\mathcal{I}(t, \cdot))$ and recovered individuals $(\mathcal{R}(t, \cdot))$ at time t and location $x \in \Omega$. Further, we assume that $\Omega \cup \mathbb{R}^n$ is assumed that bounded and smooth boundary. Moreover, we assume that the incidence function takes the form $\frac{\beta S \mathcal{I}}{S + \mathcal{I} + \mathcal{R}}$, which is a particular case of the Hattaf-Yousfi [30, 31] incidence function $\frac{\beta S \mathcal{I}}{\gamma_0 + \gamma_1 S + \gamma_2 \mathcal{I} + \gamma_3 S \mathcal{I}}$ with $\gamma_0 = 0$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0$, and the inclusion of \mathcal{R} to account for recovered individuals (see [17]) which represent the contact between the susceptible and infected individuals where β represent the transmission rate of disease. We define the term $\alpha(\cdot)\mathcal{I}(t, .)$ as the recovery rate from the infected individuals.

Furthermore, the term $\gamma(.)\mathcal{R}(t,.)$ represents the recovery process from disease. Moreover, we assume that the initial conditions S_0, I_0, R_0 satisfy the following assumptions:

1.
$$\int_{\Omega} I_0(\cdot), \int_{\Omega} R_0(\cdot) > 0, S_0 \ge 0, I_0 \ge 0, R_0 \ge 0,$$

2. Supposing that

$$\int_{\Omega} S_0(.) + I_0(.) + R_0(.) = \mathbf{N},$$
(2.1)

We suppose also that the model (1.2) have the following assumptions

(C0) $d_1 > 0, d_2 > 0$, and $d_3 > 0$. (C1) β , α and γ are strictly positive and Hölder continuous functions on $\overline{\Omega}$.

Next, we put that $\mathbb{X} = C(\overline{\Omega}, \mathbb{R}^3)$ represent the continuous functions on $\overline{\Omega}$, endowing by the norm

$$||d|| = \max\{\sup_{\text{on }\bar{\Omega}} |d_1(x)|, \sup_{\text{on }\bar{\Omega}} |d_2(x)|, \sup_{\text{on }\bar{\Omega}} |d_3(x)|\}, \forall d_1, d_2, d_3 \in C(\bar{\Omega}, \mathbb{R}^3).$$

we also define $\mathbb{X}^+ = C\left(\bar{\Omega}, \mathbb{R}^3_+\right)$ is the positive cone of \mathbb{X} .

Furthermore, we define the following notations

$$\underline{m}(\cdot) = \min_{x \in \bar{\Omega}} m(\cdot), \quad \overline{m}(\cdot) = \max_{x \in \bar{\Omega}} m(x), \tag{2.2}$$

where $m \in \{\beta(\cdot), \alpha(\cdot), \gamma(\cdot)\}$. To study the well-posedness of the problem, we set

$$A_{S}^{\sigma} \mathcal{S} = \frac{d_{1}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{S}(t,y) - \mathcal{S}(t,\cdot)] dy + \gamma(\cdot)\mathcal{R},$$

$$A_{I}^{\sigma} \mathcal{I} = \frac{d_{1}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{I}(t,y) - \mathcal{I}(t,\cdot)] dy - \alpha(\cdot)\mathcal{I},$$

$$A_{R}^{\sigma} \mathcal{R} = \frac{d_{1}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)] dy + \alpha(\cdot)\mathcal{I}.$$
(2.3)

From the assumption (2), we can obtain the operators \mathbb{A}_{S}^{σ} , \mathbb{A}_{I}^{σ} , \mathbb{A}_{R}^{σ} are bounded and generates a semi-groups (see Thm. 1.2 in [32]) { \mathbb{T}_{S} }, { \mathbb{T}_{I} } and { \mathbb{T}_{R} }, respectively. Furthermore, we put that

$$\begin{split} \mathbb{F}_{S}^{\sigma}(\mathbb{U}) &= -\frac{\beta \mathcal{S}(t,.)\mathcal{I}(t,.)}{\mathcal{S} + \mathcal{I} + \mathcal{R}},\\ \mathbb{F}_{I}^{\sigma}(\mathbb{U}) &= \frac{\beta \mathcal{S}(t,.)\mathcal{I}(t,.)}{\mathcal{S} + \mathcal{I} + \mathcal{R}},\\ \mathbb{F}_{R}^{\sigma}(\mathbb{U}) &= -\gamma(.)\mathcal{R}(t,.), \end{split}$$

where $\mathbb{U}(t,.) = \begin{pmatrix} \mathcal{S}(t,.) \\ \mathcal{I}(t,.) \\ \mathcal{R}(t,.) \end{pmatrix}$. Thus, we can define the following Cauchy problem as follows

$$\frac{d\mathbb{U}}{dt} = \mathbb{A}^{\sigma}\mathbb{U} + \mathbb{F}^{\sigma}(\mathbb{U}), \quad U(0,x) = (S_0, I_0, R_0)^T$$

with $\mathbb{A} = \begin{pmatrix} \mathbb{A}_{S}^{\sigma}, \\ \mathbb{A}_{I}^{\sigma}, \\ \mathbb{A}_{R}^{\sigma} \end{pmatrix}$ and $\mathbb{F} = \begin{pmatrix} \mathbb{F}_{S}^{\sigma}, \\ \mathbb{F}_{I}^{\sigma}, \\ \mathbb{F}_{R}^{\sigma} \end{pmatrix}$.

The existence and uniqueness of solutions to model (1.2) are established in the following Theorem.

Theorem 2.1. Let $(S_0, I_0, R_0) \in \mathbb{X}^+$, then (1.2) admits a unique non-negative solution, that is globally defined. Proof. We know that the operators \mathbb{A}_S^{σ} , \mathbb{A}_I^{σ} , and \mathbb{A}_R^{σ} each generate a semigroups such that

$$\begin{cases} \mathcal{S}(t,\cdot) = \mathbb{T}_{\mathcal{S}}(t)S_{0}(\cdot) + \int_{0}^{t} \mathbb{T}_{\mathcal{S}}(t-\tau)(\mathbb{F}_{S}^{\sigma}(\mathbb{U}(\tau,\cdot))d\tau, \quad t > 0, \quad \text{on } \bar{\Omega}, \\ \mathcal{I}(t,\cdot) = \mathbb{T}_{\mathcal{I}}(t)I_{0}(\cdot) + \int_{0}^{t} \mathbb{T}_{\mathcal{I}}(t-\tau)(\mathbb{F}_{I}^{\sigma}(\mathbb{U}(\tau,\cdot))d\tau, \quad t > 0, \quad \text{on } \bar{\Omega}, \\ \mathcal{R}(t,\cdot) = \mathbb{T}_{\mathcal{S}}(t)R_{0}(\cdot) + \int_{0}^{t} \mathbb{T}_{\mathcal{S}}(t-\tau)(\mathbb{F}_{R}^{\sigma}(\mathbb{U}(\tau,\cdot))d\tau, \quad t > 0, \quad \text{on } \bar{\Omega}, \end{cases}$$
(2.4)

by using the first equation of model (1.2), we get

$$\begin{split} \mathcal{S}(t,\cdot) &= S_0(\cdot) \mathrm{e}^{-\int_0^t \frac{d_1}{\sigma^m} \int_\Omega \mathcal{L}_\sigma(x-y) [\mathcal{S}(\tau,y) - \mathcal{S}(\tau,\cdot)] \mathrm{d}y + \gamma(\cdot) \mathcal{R}(\tau,\cdot) \mathrm{d}\tau} \\ &- \int_0^t \mathrm{e}^{-\int_\tau^t \frac{d_1}{\sigma^m} \int_\Omega \mathcal{L}_\sigma(x-y) [\mathcal{S}(\tau,y) - \mathcal{S}(\tau,\cdot)] \mathrm{d}y + \gamma(\cdot) \mathcal{R}(\tau,\cdot)} \left(\frac{\beta(\cdot) \mathcal{S}(\tau,\cdot) \mathcal{I}(\tau,\cdot)}{\mathcal{S}(\tau,\cdot) + \mathcal{I}(\tau,\cdot) + \mathcal{R}(\tau,\cdot)} \right), \end{split}$$

since that $S_0(\cdot) \in \mathbb{X}^+, L(\cdot) > 0$, on Ω , and $(\mathcal{I}, \mathcal{R}) \in \mathbb{X}^+$, we can deduce that $\mathcal{S}(t, \cdot) \in \mathbb{X}^+$. Moreover, we need to show that $\mathcal{S}(t, \cdot) > 0$, on $[0, T^{max})$, we assume by contradiction that there exists T^{max} such that $\mathcal{S}(t_0, x^*) = 0$, $\frac{\partial \mathcal{S}}{\partial t}(t, x^*) \leq 0$ as $t = t_0, \mathcal{S}(t_0, .) > 0$, we consider

$$t_{I} = \inf \left\{ t \in [0, T^{\max}) \mid \mathcal{I}(t, \cdot) = 0 \right\}, t_{R} = \inf \left\{ t \in [0, T^{\max}) \mid \mathcal{R}(t, \cdot) = 0 \right\}, t_{0} = \min \left\{ t_{I}, t_{R} \right\},$$
(2.5)

from the first equation of model (1.2), we have

$$\frac{\partial}{\partial t}\mathcal{S}\left(t_{0},x^{*}\right) = \frac{d_{1}}{\sigma^{m}}\int_{\Omega}J\left(x^{*}-y\right)\mathcal{S}\left(t_{0},y\right)dy + \gamma(x^{*})\mathcal{R}(t_{0},x^{*}) > 0$$

this contradicts the assumption. Hence $\mathcal{S}(t, \cdot) \geq 0$, for $(t, \cdot) \in [0, T^{\max})$.

$$\begin{aligned} \mathcal{I} &= I_0(\cdot) \mathrm{e}^{-\int_0^t \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{I}(\tau,y) - \mathcal{I}(\tau,\cdot)] \mathrm{d}y - \alpha(\cdot) \mathcal{I}(\tau,\cdot) \mathrm{d}\tau} \\ &+ \int_0^t \mathrm{e}^{-\int_{\tau}^t \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{I}(\tau,y) - \mathcal{I}(\tau,\cdot)] \mathrm{d}y - \alpha(\cdot) \mathcal{I}(\tau,\cdot)} \left(\frac{\beta(\cdot) \mathcal{S}(\tau,x) \mathcal{I}(\tau,\cdot)}{\mathcal{S}(\tau,\cdot) + \mathcal{I}(\tau,\cdot) + \mathcal{R}(\tau,\cdot)} \right) \mathrm{d}\tau, t > 0, \quad \mathrm{on}\bar{\Omega}, \end{aligned}$$

for all $t \in [0, T^{\max})$, on Ω , we suppose that $I_0 \in \mathbb{X}^+$, $(S, \mathcal{I}, \mathcal{R}) \in \mathbb{X}^+$, then we conclude that $\mathcal{I}(t, \cdot) \in \mathbb{X}^+$ for all $t \in [0, T^{\max})$.

We need to show the positivity of I, we assume by contradiction that $\mathcal{I}(t,.) > 0$, on $[0, T^{\max}) \times \overline{\Omega}$, $\mathcal{I}(t_0, x^*) = 0$, $\frac{\partial \mathcal{I}(t_0, x^*)}{\partial t} < 0$, if $t_I < t_R$, then $t_0 = t_I$, by the second equation of (1.2), we find that

$$\frac{\partial}{\partial t}\mathcal{I}\left(t_{0},x^{*}\right)=\frac{d_{2}}{\sigma^{m}}\int_{\Omega}\mathcal{L}_{\sigma}\left(x^{*}-y\right)\mathcal{I}\left(t_{0},y\right)dy>0,$$

which means a contradiction. Consequently, $\mathcal{I}(t, \cdot) > 0$, $\operatorname{on}\Omega, t \in [0, T^{\max})$.

From the third equation of (2.4), we obtain that

$$\mathcal{R} = R_0(\cdot) \mathrm{e}^{-\int_0^t \frac{d_3}{\sigma^m} \int_\Omega} \mathcal{L}_\sigma(x-y) [\mathcal{R}(\tau,y) - \mathcal{R}(\tau,\cdot)] \mathrm{d}y + \alpha(\cdot) \mathcal{I}(\tau,\cdot) \mathrm{d}\tau} \\ + \int_0^t \mathrm{e}^{-\int_\tau^t \frac{d_2}{\sigma^m} \int_\Omega} \mathcal{L}_\sigma(x-y) [\mathcal{R}(\tau,y) - \mathcal{R}(\tau,\cdot)] \mathrm{d}y + \alpha(\cdot) \mathcal{I}(\tau,\cdot)} (-\gamma(\cdot) \mathcal{R}(\tau,\cdot)) \, d\tau, t > 0, \quad \mathrm{on}\bar{\Omega},$$

from the fact that $R_0 \in \mathbb{X}^+$, $(S, \mathcal{I}, \mathcal{R}) \in \mathbb{X}^+$ we can get that $\mathcal{R}(t, \cdot) \in \mathbb{X}^+$, for all $t \in [0, T^{\max})$. To show the positivity of R, we assume by contradiction that $\mathcal{I}(t, .) > 0$, on $[0, T^{\max}) \times \overline{\Omega}$, $\mathcal{R}(t_0, x^*) = 0$, $\frac{\partial \mathcal{R}(t_0, x^*)}{\partial t} < 0$, $\mathcal{I}(t_0, .) > 0$ if $t_I \ge t_R$, then $t_0 = t_R$, by third equation of (1.2), we obtain that

$$\frac{\partial}{\partial t}\mathcal{R}\left(t_{0},x^{*}\right) = \frac{d_{3}}{\sigma^{m}}\int_{\Omega}\mathcal{L}_{\sigma}\left(x^{*}-y\right)R\left(t_{0},y\right)dy + \gamma(\cdot)\mathcal{I}(t_{0},x^{*}) > 0,$$

we get a contradiction, it's yields that $\mathcal{R}(t, \cdot) > 0$, on $[0, T^{\max}) \times \Omega$.

It is essential to show the difference between semi-flow and Semigroup.

- A semi-flow $\Phi(t) : \mathbb{X}^+ \to \mathbb{X}^+$, $t \ge 0$, is a nonlinear mapping on a metric space , satisfying identity, semigroup property, and joint continuity. Asymptotic smoothness ensures that orbits $\Phi(t)B$ approach a compact set.
- A semigroup $T(t) : \mathbb{X} \to \mathbb{X}, t \ge 0$, is a linear operator on a Banach space $(e.g., X = L^2(\Omega, \mathbb{R}^3))$, satisfying identity, semigroup property, and strong continuity. It may describe the linearized system around the disease-free equilibrium, where compactness of T(t) aids eigenvalue analysis for \mathbb{R}_0 (Sect. 3).

We now turn to showing the global existence of solutions to system (1.2).

Theorem 2.2. We consider $(S_0, I_0, R_0) \in \mathbb{X}^+$, then there exists an unique solution $\mathbb{U}(t, .; \mathbb{U}_0) = (\mathcal{S}(t, .), \mathcal{I}(t, .))$ of model (1.2) on $[0, \infty)$. Moreover, The semi-flow Φ_t generates by the solution \mathbb{U} is bounded dissipative.

Proof. Let $\mathcal{N}(t, \cdot) = \mathcal{S}(t, \cdot) + \mathcal{I}(t, \cdot) + \mathcal{R}(t, \cdot)$. By summing the equations in system (1.2), we obtain

$$\frac{\partial \mathcal{N}}{\partial t} \le \mathbb{C} - \mu(\cdot)\mathcal{N},\tag{2.6}$$

with $\mathbb{C} = \mathcal{N} \sup_{\Omega \times \Omega} \mathcal{L}_{\sigma} \left(\frac{d_1}{\sigma^m} + \frac{d_2}{\sigma^m} \right)$, and $\mu(\cdot) = \frac{d_1}{\sigma^m} + \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) dy$.

Applying constant variation method, we have

$$\mathcal{N}(t,x) \leq ||\mathcal{N}_0|| + \frac{\mathbb{C}}{\underline{\mu}} := \mathbb{M},$$

This implies that the semi-flow Φ_t , generated by the solutions of system (1.2), is bounded and dissipative. As a result, the global existence of solutions to model (1.2) follows immediately.

Furthermore, we need to establish the asymptotic smoothness of the semi-flow, which is important for establishing the validity of the constructed Lyapunov functional in proving global stability. A semiflow $\Phi(t)$: $\mathbb{X}^+ \to \mathbb{X}^+$, $t \ge 0$, is called asymptotically smooth if, for any closed, bounded, and positively invariant set $B \subset X_+$, there exists a compact set $K \subset \mathbb{X}^+$ such that

$$\lim_{t \to \infty} \sup_{x \in B} \operatorname{dist}(\Phi(t)x, K) = 0,$$

where $dist(x, K) = \inf_{y \in K} ||x - y||_{\mathbb{X}}$ (see [[33], Def. 2.25]).

Theorem 2.3. The semi-flow Φ_t is asymptotically smooth. Additionally, the semi-flow Φ_t has a global attractor compact.

Proof. For $(S_0, I_0, R_0) \in \mathbb{X}^+$, we have that $\Phi_t(t, (S_0, I_0, R_0)) = (\mathcal{S}(t, .), \mathcal{I}(t, .), \mathcal{R}(t, .)), \quad t \ge 0.$

We consider the operators $(\mathbb{A}_{S}^{\sigma}, \mathbb{A}_{I}^{\sigma}, \mathbb{A}_{R}^{\sigma})$ define as in (2.3), by using Lemma 2.1 in [25], we can find the operators $(\mathbb{A}_{S}^{\sigma}, \mathbb{A}_{I}^{\sigma}, \mathbb{A}_{R}^{\sigma})$ have a principal eigenvalues $\lambda_{i}, (i = S, \mathcal{I}, \mathcal{R})$, respectively, with $\lambda_{I} < -\omega$, we can rewrite model (1.2) as

$$\Phi_t(t) = \mathbb{T}(t)\phi_0 + \int_0^t \mathbb{T}(t-s)\mathbb{F}[\Phi_t](s)\,\mathrm{d}s, \quad \forall t \ge 0, \quad \phi_0 \in Y^+.$$

We decompose the semi-flow $\Phi(t)$ into two components:

$$\Phi_T(t) = \hat{u}(t) + \tilde{u}(t),$$

where

$$\hat{u}(t) := \mathbb{T}(t)\phi_0$$
, and $\tilde{u}(t) := \int_0^t T(t-s)\mathbb{F}[u](s) \,\mathrm{d}s$, $\forall t \ge 0$.

with $\mathbb{T}(t) = (\mathbb{T}_S, \mathbb{T}_I, \mathbb{T}_R)$, by applying the same approach of Theorem 5 in [34], we can conclude that \tilde{u} is compact, we need to show that $\hat{u}(t)$ is compact. Indeed, we have

$$\|\hat{u}(t)\| = \|T[\phi_0](t)\| \le e^{-\omega t} \|\phi_0\|,$$

thus,

$$\|\hat{u}(t)\| \le e^{-\omega t}, \quad \forall t > 0.$$

Consequently, for each t > 0 and $u \in B \subset Y^+$, we observe that

$$\rho(\Phi(t)(B)) \le \rho(\hat{u}[B](t)) + \rho(\tilde{u}[B](t)) \le \|\hat{u}(t)\|\rho(B) + 0 \le e^{-\omega t}\rho(B), \quad t > 0.$$

As a result, we investigate that the semi-flow Φ_t satisfies a ρ -contraction property. This implies that Φ_t is asymptotically smooth. Moreover, by invoking Theorem (2.2), we know that Φ_t is point-dissipative. Given that the semi-flow is also bounded and asymptotically smooth, we may apply Theorem 3.4.8 from [35] to conclude the existence of a compact global attractor in the positive cone \mathbb{X}^+ .

3. Basic reproduction number

In this section, we aim to characterize the basic reproduction number \mathbb{R}_0 by performing a linearization of model (1.2) around the disease-free equilibrium E_0 . This leads to the following linearized system:

$$\frac{\partial \mathcal{S}}{\partial t} = \frac{d_1}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{S}(t,y) - \mathcal{S}(t,\cdot)) dy - \gamma(\cdot)\mathcal{R} - \beta(\cdot)\mathcal{I}, \text{ on } \Omega \times \mathbb{R}^+,$$
$$\frac{\partial \mathcal{I}}{\partial t} = \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{I}(t,y) - \mathcal{I}(t,\cdot)) dy + \beta(\cdot)\mathcal{I} - \alpha(\cdot)\mathcal{I}, \text{ on } \Omega \times \mathbb{R}^+,$$
$$\frac{\partial \mathcal{R}}{\partial t} = \frac{d_3}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)) dy + \alpha(\cdot)\mathcal{I}(t,\cdot) - \gamma(\cdot)\mathcal{R}, \text{ on } \Omega \times \mathbb{R}^+,$$

then, we consider the following model as follows,

$$\frac{\partial \mathcal{I}}{\partial t} = \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{I}(t,y) - \mathcal{I}(t,\cdot)) dy + \beta(\cdot)\mathcal{I} - \alpha(\cdot)\mathcal{I}, \text{ on } \Omega \times \mathbb{R}^+,
\frac{\partial \mathcal{R}}{\partial t} = \frac{d_3}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)) dy + \alpha(\cdot)\mathcal{I} - \gamma(\cdot)\mathcal{R}, \text{ on } \Omega \times \mathbb{R}^+,$$
(3.1)

By applying the method of variation of constants to equation (3.1), we obtain the following expressions:

$$\mathcal{I}(t,\cdot) = \mathrm{e}^{\frac{d_2}{\sigma^m}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{I}(t,y) - \mathcal{I}(t,\cdot)) \mathrm{d}yt I_0(\cdot) + I_0(\cdot) \int_0^t \mathrm{e}^{A_I(t-s)} \left(\beta(\cdot)\mathcal{I}(\tau,\cdot) + \alpha(\cdot)\mathcal{I}(\tau,\cdot)\right) \mathrm{d}\tau, \quad (3.2)$$

and

$$\begin{aligned} \mathcal{R}(t,\cdot) &= \mathrm{e}^{\frac{d_{2}}{\sigma^{m}}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)) \mathrm{d}yt \\ &+ \int_{0}^{t} \mathrm{e}^{\frac{d_{3}}{\sigma^{m}}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)) \mathrm{d}yt \\ &+ \int_{0}^{t} \mathrm{e}^{\frac{d_{3}}{\sigma^{m}}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)) \mathrm{d}yt \\ &+ \int_{0}^{t} \mathrm{e}^{\frac{d_{3}}{\sigma^{m}}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)) \mathrm{d}yt \\ &+ \int_{0}^{t} \mathrm{e}^{\frac{d_{3}}{\sigma^{m}}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)) \mathrm{d}yt \\ &+ \int_{0}^{t} \mathrm{e}^{\frac{d_{3}}{\sigma^{m}}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)) \mathrm{d}yt \\ &+ \int_{0}^{t} \mathrm{e}^{\frac{d_{3}}{\sigma^{m}}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)) \mathrm{d}yt \\ &+ \int_{0}^{t} \mathrm{e}^{\frac{d_{3}}{\sigma^{m}}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)) \mathrm{d}yt \\ &+ \int_{0}^{t} \mathrm{e}^{\frac{d_{3}}{\sigma^{m}}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) (\mathcal{R}(t,y) - \mathcal{R}(t,\cdot)) \mathrm{d}yt \\ &+ \int_{0}^{t} \mathrm{e}^{\frac{d_{3}}{\sigma^{m}}} \mathrm{d}yt \\ &+ \int_{0}^{t} \mathrm{d}yt \\ &+$$

we need to construct the next generation operator $\mathbb{R}:\mathbb{X}\to\mathbb{X},$ such as

$$\mathbb{R}\psi(\cdot) = \mathbb{F}(\cdot) \left(\mathbb{A}_{I}^{\sigma}\right)^{-1} \phi(\cdot), \quad \text{on}\Omega,$$
(3.3)

with \mathbb{A}_{I}^{σ} is already defined and $\mathbb{F}(\psi(\cdot)) = (\beta(\cdot) - \gamma(\cdot))\psi(\cdot)$, by applying Theorem 3.2 in [36], we have

$$\left(\mathbb{A}_{I}^{\sigma}\right)^{-1}\phi(\cdot) = \int_{0}^{\infty} \mathbb{T}_{I}^{\sigma}\phi(\cdot)\mathrm{d}\sigma, \quad \phi \in \mathbb{X},$$
(3.4)

we replace (3.4) in (3.3), we obtain

$$\mathbb{R}\psi(\cdot) = \mathbb{F}(\cdot) \int_0^\infty \mathbb{T}_I^\sigma \phi(\cdot) \mathrm{d}\sigma, \quad \mathrm{on}\Omega,$$
(3.5)

Hence, we define \mathbb{R}_0 as the spectral radius of \mathbb{R} , expressed as

$$\mathbb{R}_0 = r(\mathbb{R}),$$

with r represents the spectral radius.

In the next, we define the problem for the first equation of (3.1), we put $\mathcal{I}(t, \cdot) = e^{\lambda t} \omega(\cdot)$, we get that

$$\lambda\omega(\cdot) = \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \left[\omega(y) - \omega(\cdot)\right] \, \mathrm{d}y + \beta(\cdot)\omega(\cdot) - \alpha(\cdot)\omega(\cdot), \quad \text{on}\Omega, \tag{3.6}$$

To validate the existence of the principal eigenvalue λ^* , we need to assume the following assumptions.

(H1) $\beta \neq \alpha$ and by using Theorem 2.1 in [37]() we can conclude that the problem (3.6) has a principal eigenvalue λ^* associated with a positive strictly ψ^* .

In the following proposition, we establish the relationship between \mathbb{R}_0 and λ^* .

Proposition 3.1. We consider $s(\mathbb{R}) = \sup \{Re(\lambda) : \lambda \in \xi(\mathbb{R})\}$, with ξ represents the spectrum of \mathbb{R} , then $\mathbb{R}_0 - 1$ has the same sign of $\lambda^* = s(\mathbb{A}_I^{\sigma} + \mathbb{F})$.

Proof. By applying Theorem 3.1 in [36], we have that

$$(\lambda I - \mathbb{A}_I^{\sigma})^{-1}\psi_2 = \int_0^\infty e^{-\lambda t} \mathbb{T}_{\mathcal{I}}(t)\psi_2 \, dt, \quad \phi_2 \in \mathbb{X}.$$

and we choose $\lambda = 0$ which gives

$$(-\mathbb{A}_{I}^{\sigma})^{-1}\varphi_{2}(\cdot) = \int_{0}^{\infty} \mathbb{T}_{\mathcal{I}}(t)\varphi_{2}(\cdot) dt,$$

Furthermore, we observe that $\lambda^* = s \left(\mathbb{A}_I^{\sigma} + \mathbb{F}(\cdot) \right)$. According to Proposition 2.4 in [38], the spectral bound $s \left(\mathbb{A}_I^{\sigma} \right)$ is strictly negative. Recalling the definition of the basic reproduction number, we have $\mathbb{R}_0 = r \left(\left(\mathbb{A}_I^{\sigma} \right)^{-1} \mathbb{F} \right)$. Consequently, by applying Theorem 3.5 from [36], it follows that the sign of λ^* is determined by the expression $r \left(\left(\mathbb{A}_I^{\sigma} \right)^{-1} \mathbb{F} \right) - 1$.

In the next step, we introduce the following eigenvalue problem as follows

$$\frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \left[\psi(y) - \psi(\cdot) \right] dy - \alpha(\cdot)\psi(\cdot) = -\lambda\beta(\cdot)\psi(\cdot), \quad x \in \Omega,$$
(3.7)

and by applying the same arguments as [5], we can define \mathbb{R}_0 as follows

$$\mathbb{R}_{0} = \inf_{\psi \in L^{2}(\Omega)\psi \neq 0} \frac{\int_{\Omega} \beta(\cdot)\psi(\cdot)^{2} \mathrm{d}x}{\frac{d_{2}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \left[\psi(y) - \psi(\cdot)\right]^{2} \mathrm{d}y + \int_{\Omega} \alpha(\cdot)\psi^{2}(\cdot)\mathrm{d}x}$$
(3.8)

Now, we are ready to give the following Theorem.

Theorem 3.2. Let $\lambda^*(m, \sigma, d_2)$ represent the principal eigenvalue of problem (3.6). The following statements are equivalent.

- 1. $\sigma \to \infty$ and $0 < m \leq 2$ than $\lambda^* \to \min_{\Omega} \frac{\alpha(\cdot)}{\beta(\cdot)}$
- 2. $\sigma \to 0$ and $0 < m \le 2$ than $\lambda^* \to \min_{\Omega} \frac{\alpha(\cdot)}{\beta(\cdot)}$
- 3. For m = 2, and we assume that $\alpha, \beta \in C^{0,\nu}(\Omega)$ where $\nu > 0$ and we assume that $J \in C(\mathbb{R}^n)$ is positive, symmetric and $|z|^2 L(z) \in L^1(\mathbb{R}^n)$, then we have that

$$\lim_{\sigma \to 0} \lambda^* (m, d_2) \to \lambda^* (\frac{d_2 \mathbb{D}_2(J)}{2\mathbf{N}} \Delta),$$

with

$$\lambda^*(\frac{d_2\mathbb{D}_2}{2\mathbf{N}}\Delta) := \inf_{\psi \in H_0^1(\Omega), \psi \neq 0} \frac{\int_\Omega \left(\frac{\mathbb{D}^2(J)}{2\mathbf{N}} |\nabla\varphi|^2(\cdot) \,\mathrm{d}x\right)}{\int_\Omega \beta(\cdot)\psi^2(\cdot) \,\mathrm{d}x} + \frac{\int_\Omega \alpha(\cdot)\psi^2(\cdot) \,\mathrm{d}x}{\int_\Omega \beta(\cdot)\psi^2(\cdot) \,\mathrm{d}x}$$

with $\lambda_*(\frac{d_2\mathbb{D}_2}{2\mathbf{N}}\Delta)$ is the principal eigenvalue verifies:

$$\begin{cases} \frac{d_{\mathcal{S}}\mathbb{D}^{2}(J)}{2\mathbf{N}}\Delta\mathcal{S}(\cdot) - \frac{\beta(\cdot)\mathcal{S}(\cdot)\mathcal{I}(\cdot)}{\mathcal{S}(\cdot)+\mathcal{I}(\cdot)+\mathcal{R}(\cdot)} + \gamma(\cdot)\mathcal{R}(\cdot) = 0, \quad on\Omega, \\\\ \frac{d_{I}\mathbb{D}^{2}(J)}{2\mathbf{N}}\Delta\mathcal{I}(\cdot) + \frac{\beta(\cdot)\mathcal{S}(\cdot)\mathcal{I}(\cdot)}{\mathcal{S}(\cdot)+\mathcal{I}(\cdot)+\mathcal{R}(\cdot)} - \alpha(\cdot)\mathcal{I}(\cdot) = 0, \quad on\Omega, \\\\ \frac{d_{I}\mathbb{D}^{2}(J)}{2\mathbf{N}}\Delta\mathcal{R}(\cdot) + -\gamma(\cdot)\mathcal{R}(\cdot) + \alpha(\cdot)\mathcal{I}(\cdot) = 0, \quad on\Omega, \\\\ \frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial R}{\partial n} = 0, \qquad x \in \partial\Omega, \end{cases}$$

where
$$\mathbb{D}^2(J) = \int_{\Omega} \mathcal{J}(z) |z|^2 \mathrm{d}z.$$

Proof. We begin the first case when $0 < m \leq 2$ and $\sigma \to 0$, by applying Lemma 4.1 in [21], we have that

$$\lambda^*(d_2, m) \le \sigma^{2-m} \mathbb{J}(\psi) + \frac{\int_{\Omega} \alpha(\cdot)\psi^2(\cdot) \mathrm{d}x}{\int_{\Omega} \beta(\cdot)\psi^2(\cdot) \mathrm{d}x},$$
(3.9)

with

$$\mathbb{J}(\psi) := \mathbb{D}^2(J) \int_{\Omega} |\nabla \psi(\cdot)|^2 \, \mathrm{d}x, \quad \psi \in L^2(\Omega)$$

and $\psi \in H_0^1(\Omega)$. We suppose that there exists a sequence $\psi_n \in H_0^1(\Omega)$ such that $\operatorname{supp}(\psi_n) \subset B_r(x_n)$ with B_r represents the open ball centered, by (3.9), we have

$$\limsup_{\sigma \to 0} \lambda^*(d_2, m) \le \frac{\int_{\Omega} \alpha(\cdot) \psi^2(\cdot) \mathrm{d}x}{\int_{\Omega} \beta(\cdot) \psi^2(\cdot) \mathrm{d}x} = -\int_{B_r} \psi^2(\cdot) \,\mathrm{d}x, \quad \psi \in L^2(\Omega), \quad \le \min_{B_r} \frac{\alpha(\cdot)}{\beta(\cdot)}$$

then, by converging $\sigma \to 0$, we obtain that $\limsup_{\sigma \to 0} \lambda^*(d_2, m) \leq \min_{\Omega} \frac{\alpha(x_n)}{\beta(x_n)}$, therefore

$$\limsup_{\sigma \to 0} \lambda^*(d_2, m, \sigma) \le \min_{\Omega} \frac{\alpha(\cdot)}{\beta(\cdot)} + \frac{1}{n},$$

for $n \to \infty$, we have that $\limsup_{\sigma \to 0} \lambda^*(d_2, m) \le \min_{\Omega} \frac{\alpha(\cdot)}{\beta(\cdot)}$, Now, we use the following test functions $(\lambda, \psi) = (\min_{\Omega} \frac{\alpha(\cdot)}{\beta(\cdot)}, 1)$, we can obtain that

$$\lambda^*(d_2, m) \ge \min_{\Omega} \frac{\alpha(\cdot)}{\beta(\cdot)},$$

Then, we get

$$\min_{\Omega} \frac{\alpha(\cdot)}{\beta(\cdot)} \le \liminf_{\sigma \to 0} \lambda^*(d_2, m) \le \limsup_{\sigma \to 0} \lambda^*(d_2, m) \le \min_{\Omega} \frac{\alpha(\cdot)}{\beta(\cdot)}.$$

Next, we focus to show that $\sigma \to \infty$, by using the same arguments as The proposition 2.3 in [21], we cann find easily the result.

Now, we focus to show the last result, we consider the following function

$$\zeta_{\sigma}(z) := \frac{1}{\sigma^2 \mathbb{D}^2(\mathcal{L}_{\sigma})} \mathcal{L}_{\sigma}(z) |z|^2, \quad \text{then for } \psi \in H^1_0(\Omega),$$

with the following proprieties of ζ_{σ}

$$\begin{split} \zeta_{\sigma} &\geq 0 \text{ in } \mathbb{R}^{N}, \\ \int_{\mathbb{R}^{N}} \zeta_{\sigma}(z) \, \mathrm{d}z = 1, \quad \forall \sigma > 0, \\ \lim_{\sigma \to 0} \int_{\{|z| \geq \delta\}} \zeta_{\sigma}(z) \, \mathrm{d}z = 0, \quad \forall \delta > 0, \end{split}$$

and by using the characterization of Sobolev space (see [39]), we obtain when $\sigma \to 0$,

$$\int_{\Omega \times \Omega} \frac{\zeta_{\sigma}(x-y)(\psi(\cdot) - \psi(y))^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y = \mathbb{K}_{2,N} \|\nabla \psi\|_{L^2(\Omega)}^2, \tag{3.10}$$

for any $\psi \in H_0^1(\Omega)$, with Let $\mathbb{K}_{2,N}$ be defined as:

$$K_{2,N} := \frac{1}{|S^{N-1}|} \int_{S^{N-1}} (s \cdot e_1)^2 \, \mathrm{d}s = \frac{1}{N},$$

where S^{N-1} is the (N-1)-dimensional unit sphere, e_1 is the first standard basis vector, and \cdot denotes the dot product, and by (3.10), we obtain

$$\begin{split} \lim_{\sigma \to 0} \lambda^*(d_2, m) &\leq \lim_{\sigma \to 0} \frac{1}{\|\psi\|_{L^2(\Omega)}^2} \left(\frac{\frac{d_2}{2\sigma^2} \int_{\Omega \times \Omega} \mathcal{L}_\sigma(x - y)(\psi(\cdot) - \psi(y))^2 \, \mathrm{d}x \, \mathrm{d}y}{\int_\Omega \beta(\cdot)\psi^2(\cdot) \mathrm{d}x} + \frac{\int_\Omega \alpha(\cdot)\psi^2(\cdot) \mathrm{d}x}{\int_\Omega \beta(\cdot)\psi^2(\cdot) \mathrm{d}x} \right), \\ &= \lim_{\sigma \to 0} \frac{1}{\|\psi\|_{L^2(\Omega)}^2} \left(\frac{D^2(J)}{2} \frac{\int_{\Omega \times \Omega} \zeta_\sigma(x - y)(\psi(\cdot) - \psi(y))^2 \, \mathrm{d}x \, \mathrm{d}y}{|x - y|^2 \int_\Omega \beta(\cdot)\psi^2(\cdot) \mathrm{d}x} + \frac{\int_\Omega \alpha(\cdot)\psi^2(\cdot) \mathrm{d}x}{\int_\Omega \beta(\cdot)\psi^2(\cdot) \mathrm{d}x} \right) \\ &= \lim_{\sigma \to 0} \frac{1}{\|\psi\|_{L^2(\Omega)}^2} \left(\frac{\mathbb{D}^2(J)}{2\mathbf{N}} \frac{\int_\Omega \|\nabla\psi\|_{L^2(\Omega)}^2 , \mathrm{d}x}{\int_\Omega \beta(\cdot)\psi^2(\cdot) \mathrm{d}x} + \frac{\int_\Omega \alpha(\cdot)\psi^2(\cdot) \mathrm{d}x}{\int_\Omega \beta(\cdot)\psi^2(\cdot) \mathrm{d}x} \right) \end{split}$$

From the last equation, we obtain that

$$\lim_{\sigma \to 0} \lambda^* \left(d_2, m \right) \le \lambda_* \left(\frac{d_2 \mathbb{D}_2(J)}{2\mathbf{N}} \Delta \right),$$

Now, we focus to show that

$$\lambda_*\left(\frac{d_2\mathbb{D}_2(J)}{2\mathbf{N}}\Delta\right) \le \liminf_{\sigma \to 0} \lambda^*(d_2, m)$$

we first claim there exists a test function ψ_* such that

$$\left[\lambda^*\left(d_2,m\right)\beta(\cdot)+\delta_*\right]\psi_*(\cdot)+\frac{d_2}{\sigma^2}\int_{\Omega}\mathcal{L}_{\sigma}(x-y)\left[\psi_*(y)-\psi_*(\cdot)\right]\,\mathrm{d}y-\alpha(\cdot)\psi_*(\cdot)\geq 0,\quad\text{on}\quad\Omega,\tag{3.11}$$

we know that there exists ψ_* that satisfies (3.11). Additionally, we define a function ν_{σ} which represents the smooth mollifier of unit mass with $\operatorname{supp}(nu_{\sigma}) \subset \mathbb{B}_1$ with \mathbb{B}_1 is the unit ball with $\nu_{\sigma} := \frac{1}{\chi^N} \nu_{\sigma} \left(\frac{s}{\chi}\right)$, for all $\chi > 0$. We define the convolution product as follows $\underline{\psi} = \nu_{\sigma} * \psi_*$, we have that

$$\nu_{\sigma} * \left(\left[\lambda^* \left(d_2, m \right) \beta(\cdot) + \delta_* \right] \psi_*(\cdot) + \frac{d_2}{\sigma^2} \int_{\Omega} \mathcal{L}_{\sigma}(x - y) \left[\psi_*(y) - \psi_*(\cdot) \right] \, \mathrm{d}y - \alpha(\cdot) \psi_*(\cdot) \right) \ge 0, \quad \text{on}\Omega, \tag{3.12}$$

we calculate the following term,

$$\frac{d_2}{\sigma^2} \int_{\Omega} \nu_{\sigma} \tau(x-s) \int_{\Omega} \mathcal{L}_{\sigma}(s-y) \psi_*(s) \,\mathrm{d}y \,\mathrm{d}s = \frac{d_2}{\sigma^2} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \underline{\psi}(y) \,\mathrm{d}y, \tag{3.13}$$

and

$$\begin{aligned} \frac{d_2}{\sigma^2} \int_{\Omega} \nu_{\sigma}(x-s) \int_{\Omega} \mathcal{L}_{\sigma}(s-y) \psi_*(s) \, \mathrm{d}y \, \mathrm{d}s &= \frac{d_2}{\sigma^2} \int_{\Omega} \nu_{\sigma}(x-s) \psi_*(s) \int_{\Omega} \mathcal{L}_{\sigma}(s-y) \mathrm{d}y \, \mathrm{d}s, \\ &= \frac{d_2}{\sigma^2} \int_{\Omega} \nu_{\sigma}(x-s) \psi_{\sigma}(s) \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \mathrm{d}y \, \mathrm{d}s \\ &+ \frac{d_2}{\sigma^2} \int_{\Omega} \nu_{\sigma}(x-s) \psi_*(s) [\int_{\Omega} \mathcal{L}_{\sigma}(s-y) \mathrm{d}y - \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \mathrm{d}y] \, \mathrm{d}s. \end{aligned}$$

we set $A_{\sigma}(\cdot) = \int_{\Omega} \mathcal{L}_{\sigma}(x-y) dy$, then we get

$$\frac{d_2}{\sigma^2} \int_{\Omega} \nu_{\sigma}(x-s) \int_{\Omega} \mathcal{L}_{\sigma}(s-y) \psi_*(s) \, \mathrm{d}y \, \mathrm{d}s = \frac{d_2}{\sigma^2} \int_{\Omega} \nu_{\sigma}(x-s) \psi_*(s) A_{\sigma}(\cdot) \, \mathrm{d}s + \frac{d_2}{\sigma^2} \int_{\Omega} \nu_{\sigma}(x-s) \psi_*(s) [A_{\sigma}(s) - A_{\sigma}(\cdot)] \, \mathrm{d}s.$$

$$(3.14)$$

By adding and subtracting the term $\int_{\Omega} \nu_{\sigma}(x-s)(-\alpha(\cdot))\psi_{*}(s) \mathrm{d}s$, we obtain

$$\int_{\Omega} \nu_{\sigma}(x-s)(-\alpha(\cdot))\psi_{*}(s)\mathrm{d}s = \int_{\Omega} \nu_{\sigma}(x-s)(-\alpha(\cdot))\psi_{*}(s)\mathrm{d}s + \int_{\Omega} \nu_{\sigma}(x-s)\psi_{*}(s)\left[\alpha(\cdot)-\alpha(s)\right], \quad (3.15)$$

by replacing (3.13)-(3.14)-(3.15) in (3.12), we obtain that

$$[\lambda^* (d_2, m) \beta(\cdot) + \delta_* - \alpha(\cdot)] \underline{\psi}(\cdot) + \frac{d_2}{\sigma^2} \int_{\Omega} \mathcal{L}_{\sigma} (x - y) \left[\underline{\psi}(y) - \underline{\psi}(\cdot) \right] dy + \frac{d_2}{\sigma^2} \int_{\Omega} \zeta_{\sigma} (x - s) \psi_*(s) \left[A_{\sigma}(s) - A_{\sigma}(\cdot) \right] ds$$

$$+ \int_{\Omega} \nu_{\sigma} (x - s) \psi_*(s) \left[\alpha(\cdot) - \alpha(s) \right] ds \ge 0.$$

$$(3.16)$$

We know that α is Hölder continuous, which implies that,

$$\left| \int_{\Omega} \zeta_{\sigma}(x-y)\psi_{*}(y)(\alpha(\cdot)-\alpha(s)) \,\mathrm{d}s \right| \leq \int_{\Omega} \nu_{\sigma}(x-y)\psi_{*}(y) \left| \frac{\alpha(\cdot)-\alpha(s)}{|x-s|^{\alpha}} \right| |x-s|^{\alpha} \,\mathrm{d}y,$$

$$\leq \kappa_{\alpha}\tau^{\kappa}\underline{\psi}(\cdot),$$
(3.17)

and

$$\left| \int_{\Omega} \zeta_{\sigma}(x-s)\psi_{*}(s)[A_{\sigma}(s) - A_{\sigma}(\cdot)] \,\mathrm{d}s \right| \leq \int_{\Omega} \nu_{\sigma}(x-y)\psi_{\sigma}(y) \left| \frac{A_{\sigma}(s) - A_{\sigma}(\cdot)}{|s-x|^{\alpha}} \right| |x-s|^{\alpha} \,\mathrm{d}y,$$

$$\leq \kappa_{A}\tau^{\kappa}\underline{\psi}(\cdot),$$
(3.18)

with κ_{α} and κ_{A} represent the Hölder semi-norms of $\alpha(\cdot)$ and A, respectively. Additionally, we assume that τ satisfies the condition $\tau \leq \inf \left\{ \tau_{0}, \left(\frac{\delta *}{2\kappa\alpha}\right)^{\frac{1}{\kappa}} \right\}$ and $\inf \left\{ \tau_{0}, \left(\frac{\delta *}{2\kappa\alpha}\right)^{\frac{1}{\kappa}} \right\}$. By substituting (3.17)-(3.18) into (3.16), we derive the following result:

$$\left[\lambda^*\left(d_2,m\right)\beta(\cdot) + 3\delta_* - \alpha(\cdot)\right]\underline{\psi}(\cdot) + \frac{d_2}{\sigma^2}\int_{\Omega}\mathcal{L}_{\sigma}\left(x-y\right)\left[\underline{\psi}(y) - \underline{\psi}(\cdot)\right]\mathrm{d}y \ge 0, \quad \forall x \in \Omega.$$
(3.19)

Next, we need to show the $\lambda^*(d_2, m) \to \lambda_*\left(\frac{d_s\mathbb{D}_2}{2\mathbf{N}}\right)$, we set $\psi^* := \mu \underline{\psi}$, then we obtain that

$$\left[\lambda^*\left(d_2,m\right)\beta(\cdot) + 3\delta_* - \alpha(\cdot)\right]\psi^*(\cdot) + \frac{d_2}{\sigma^2}\int_{\Omega}\mathcal{L}_{\sigma}\left(x-y\right)\left[\psi^*(y) - \psi^*(\cdot)\right]\mathrm{d}y \ge 0, \quad \forall x \in \Omega.$$
(3.20)

with $\mu = \frac{\int_{\Omega} \underline{\psi} dx}{\int_{\Omega} \psi^* dx}$ then, $\frac{\int_{\Omega} \psi_*(\cdot)^2 dx}{\int_{\Omega} (\psi(\cdot)^*)^2} = 1$, we multiply the first term of (3.20) by ψ_* we obtain that

$$-\int_{\Omega} \int_{\Omega} \frac{1}{\sigma^2} \mathcal{L}_{\sigma}(x-y) \left(\psi^*(y) - \psi^*(\cdot)\right) \psi^*(\cdot) \, \mathrm{d}y \, \mathrm{d}x = \frac{d_2}{2\sigma^2} \int_{\Omega} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \left(\psi^*(y) - \psi^*(\cdot)\right)^2 \, \mathrm{d}x \, \mathrm{d}y,$$

$$= \frac{\mathbb{D}^2(J)}{2\mathbf{N}} \int_{\Omega} \int_{\Omega} \mathcal{\nu}_{\sigma}(s) \frac{\left(\psi^*(s+\cdot) - \psi^*(\cdot)\right)^2}{|s|^2} \, \mathrm{d}s \, \mathrm{d}x,$$
(3.21)

By substituting (3.21) into (3.20), we obtain the following result

$$\frac{\mathbb{D}^2(J)}{2\mathbf{N}} \int_{\Omega} \int_{\Omega} \nu_{\sigma}(s) \frac{\left(\psi^*(x+s) - \psi^*(\cdot)\right)^2}{|s|^2} \,\mathrm{d}s \,\mathrm{d}x + \int_{\Omega} \alpha(\cdot)(\psi^*(\cdot))^2 \,\mathrm{d}x \le \left(\lambda^*(d_2, m) + 3\delta\right) \int_{\Omega} (\psi^*(\cdot))^2 \,\mathrm{d}x,$$

since $\varphi_{\sigma} \in C^{\infty}(\Omega)$, by Taylor's expansion, we have the following estimate:

$$\begin{aligned} |\psi^{*}(x+s) - \psi^{*}(\cdot) - s \cdot \nabla \psi^{*}(\cdot)| &\leq \sum_{1 \leq i,j \leq N} |x_{i} - y_{i}| |x_{j} - y_{j}| \int_{0}^{1} r\left(\int_{0}^{1} |\partial_{ij}\psi^{*}(x+r\sigma s)| \,\mathrm{d}\sigma\right) \mathrm{d}r, \\ &\leq |s \cdot \nabla \psi_{*}(\cdot)| \leq \sum_{1 \leq i,j \leq N} |s_{i}s_{i}| \int_{0}^{1} r\left(\int_{0}^{1} |\partial_{ij}\psi^{*}(x+r\sigma s)| \,\mathrm{d}s\right) \mathrm{d}r + |\psi^{*}(x+s) - \psi^{*}(\cdot)|. \end{aligned}$$

$$(3.22)$$

Next, we apply the Hölder inequality for every $\xi > 0$, we obtain that

$$\begin{aligned} |s \cdot \nabla \psi^*(\cdot)|^2 &\leq \mathcal{C}_{\xi} \left[\sum_{i,j} |s_i s_j| \int_0^1 r \left(\int_0^1 |\partial_{ij} \varphi^{\sigma}(x + r\sigma s)| \,\mathrm{d}\sigma \right) \mathrm{d}r \right]^2 + (1 + \xi) |\psi^*(x + s) - \psi^*(\cdot)|^2, \\ &\leq \mathcal{C}_{\xi} \sum_{i,j} |s_i s_j|^2 \int_0^1 \int_0^1 r^2 |\partial_{ij} \psi^*(x + r\sigma s)|^2 \,\mathrm{d}\sigma \,\mathrm{d}r + (1 + \xi) |\psi^*(x + s) - \psi^*(\cdot)|^2. \end{aligned}$$

Integrating this formula in s and x over $\Omega \times \Omega$, we obtain that and for σ small, $\operatorname{supp}(\rho_{\sigma}) \subset B_1(0)$, and we have for all $x \in \Omega$,

$$\int_{\Omega} \zeta_{\sigma}(|s|) \frac{|s \cdot \nabla \psi^*(\cdot)|^2}{|s|^2} \,\mathrm{d}s \,\mathrm{d}x = \mathbb{K}_{2,N} |\nabla \psi^*(\cdot)|^2 = \frac{1}{\mathbf{N}} |\nabla \psi^*(\cdot)|^2, \tag{3.23}$$

and the following inequality holds

$$\frac{\mathbb{D}^{2}(J)}{2\mathbf{N}} \int_{\Omega} |\nabla\psi^{*}(\cdot)|^{2} \mathrm{d}x \leq \mathcal{C}_{\xi} \int_{\Omega} \zeta_{\sigma}(|s|) \sum_{i,j} \frac{|s_{i}s_{j}|^{2}}{|s|^{2}} \left(\int_{0}^{1} \int_{0}^{1} r^{2} |\partial_{ij}\psi^{*}(x+r\sigma s)|^{2} \, \mathrm{d}\sigma \, \mathrm{d}r \right) \mathrm{d}s \, \mathrm{d}x + (1+\xi) \int_{\omega} \zeta_{\sigma}(|s|) \frac{|\psi^{*}(x+s) - \psi^{*}(\cdot)|^{2}}{|s|^{2}} \, \mathrm{d}s \, \mathrm{d}x.$$
(3.24)

we divide the both terms of (3.24) by $\int_{\Omega} \beta(\cdot)(\psi^*)^2(\cdot) dx$ and adding and subtracting the term $\int_{\Omega} \alpha(\cdot)(\psi^*(\cdot))^2 dx$, we obtain that

$$\frac{\frac{\mathbb{D}^{2}(J)}{2\mathbf{N}}\int_{\Omega}|\nabla\psi^{*}(\cdot)|^{2}\mathrm{d}x+\int_{\Omega}\alpha(\cdot)\psi^{*}(\cdot)\mathrm{d}x}{\int_{\Omega}\beta(\cdot)(\psi^{*})^{2}(\cdot)\mathrm{d}x} \leq \frac{\mathcal{L}_{\xi}}{\int_{\Omega}\beta(\cdot)(\psi^{*})^{2}(\cdot)\mathrm{d}x}\int_{\Omega}\zeta_{\sigma}(|s|)\sum_{i,j}\frac{|s_{i}s_{j}|^{2}}{|s|^{2}}\left(\int_{0}^{1}\int_{0}^{1}r^{2}|\partial_{ij}\psi^{*}(x+r\sigma s)|^{2}\,\mathrm{d}\sigma\,\mathrm{d}r\right)\,\mathrm{d}s\mathrm{d}x} \qquad (3.25)$$

$$+\frac{(1+\xi)}{\int_{\Omega}\beta(\cdot)(\psi^{*})^{2}(\cdot)\mathrm{d}x}\frac{\int_{\omega}\zeta_{\sigma}(|s|)\frac{|\psi^{*}(x+s)-\psi_{*}(\cdot)|^{2}}{|s|^{2}}\,\mathrm{d}s\,\mathrm{d}x+\int_{\Omega}\alpha(\cdot)(\psi^{*})^{2}(\cdot)\mathrm{d}x}{\int_{\Omega}\beta(\cdot)(\psi^{*})^{2}(\cdot)\mathrm{d}x}-\xi\frac{\int_{\Omega}\alpha(\cdot)(\psi_{*})^{2}(\cdot)\mathrm{d}x}{\int_{\Omega}\beta(\cdot)\psi^{*}(\cdot)\mathrm{d}x},$$

therefore, we obtain

$$\frac{\frac{\mathbb{P}^{2}(J)}{2N}\int_{\Omega}|\nabla\psi^{*}(\cdot)|^{2}dx + \int_{\Omega}\alpha(\cdot)(\psi^{*})^{2}(\cdot)dx}{\int_{\Omega}\beta(\cdot)(\psi^{*})^{2}(\cdot)dx} \leq \mathbb{G}(\sigma) + \frac{(1+\xi)}{\int_{\Omega}\beta(\cdot)(\psi^{*})^{2}(\cdot)dx} \frac{\frac{-d_{2}}{\sigma^{2}}\int_{\omega}\mathcal{L}_{\sigma}(x-y)\left(\psi_{*}(y)-\psi_{*}(\cdot)|^{2}\right)ds\,dx + \int_{\Omega}\alpha(\cdot)(\psi^{*})^{2}(\cdot)dx}{\int_{\Omega}\beta(\cdot)(\psi^{*})^{2}(\cdot)dx} - \xi\frac{\int_{\Omega}\alpha(\cdot)(\psi^{*})^{2}(\cdot)dx}{\int_{\Omega}\beta(\cdot)(\psi^{*})^{2}(\cdot)dx}, \tag{3.26}$$

with

$$\mathbb{G}(\sigma) = \frac{C_{\xi} d_2 \mathbb{D}}{2 \int_{\Omega} \beta(\cdot) (\psi^*)^2(\cdot) \mathrm{d}x} \int_{\Omega} \zeta_{\sigma}(|s|) \sum_{i,j} \frac{|s_i s_j|^2}{|s|^2} \left(\int_0^1 \int_0^1 r^2 |\partial_{ij} \psi^*(x + r\sigma s)|^2 \, \mathrm{d}\sigma \, \mathrm{d}r \right) \mathrm{d}\sigma \, \mathrm{d}x,$$

by definition, we have $\partial_{ij}\psi^*(x+r\sigma s) = \partial_{ij}\nu_{\sigma}*\psi_*(x+r\sigma s)$. Thus, by applying Fubini's Theorem and employing standard convolution estimates, we derive the following result for small σ ,

$$\mathbb{G}(\sigma) \leq \frac{C_{\xi} d_{2} \mathbb{D}}{2 \int_{\Omega} \beta(\cdot) (\psi^{*})^{2}(\cdot) \mathrm{d}x} \sum_{i,j} \int_{|s| \leq 1} \int_{[0,1]^{2}} \nu_{\sigma}(|s|) \frac{|s_{i}s_{j}|^{2}}{|s|^{2}} t^{2} \left(\int_{\Omega} |\partial_{ij} \nu_{\sigma} * \psi_{\sigma}(x + r\sigma s)|^{2} \mathrm{d}x \right) \mathrm{d}r \, \mathrm{d}\sigma \, \mathrm{d}s,
\leq \frac{C_{\xi} d_{2} \mathbb{D}}{2 \int_{\Omega} \beta(\cdot) (\psi^{*})^{2}(\cdot) \mathrm{d}x} \left(\int_{|s| \leq 1} \int_{[0,1]} \nu_{\sigma}(|s|) \sum_{i,j} \frac{|s_{i}s_{j}|^{2}}{|s|^{2}} r^{2} \, \mathrm{d}r \, \mathrm{d}s \right) \|\nabla^{2} \nu_{\sigma}\|_{L^{1}(\mathbb{R}^{N})} \|\psi_{\sigma}\|_{L^{2}(\mathbb{R}^{N})}^{2}, \qquad (3.27)$$

$$\leq \frac{2}{3} \|\nabla^{2} \nu_{\sigma}\|_{L^{1}(\mathbb{R}^{N})} \|\psi_{*}\|_{L^{2}(\mathbb{R}^{N})}^{2} \int_{|s| \leq 1} \nu_{\sigma}(|s|) |s|^{2} \mathrm{d}z,$$

by substituting (3.27) into (3.26), we find that

$$\begin{split} \lambda_* \left(\frac{d_2 \mathbb{D}(J) \Delta}{2 \mathbf{N}} \right) &\leq (1+\xi) \left(\lambda^*(d_2, m) + 2\delta \right) \\ &+ \frac{C_{\xi} d_2 \mathbb{D}(\mathcal{L}_{\sigma})}{2 \mathbf{N} \int_{\Omega} \beta(\cdot) (\psi^*)^2(\cdot) \mathrm{d}x} \left(\int_{|s| \leq 1} \int_{[0,1]} \nu_{\sigma}(|s|) \sum_{i,j} \frac{|s_i s_j|^2}{|s|^2} r^2 \,\mathrm{d}r \,\mathrm{d}s \right) \| \nabla^2 \nu_{\sigma} \|_{L^1(\mathbb{R}^N)} \| \psi_{\sigma} \|_{L^2(\mathbb{R}^N)}^2, \\ &\leq (1+\xi) \left(\lambda^*(d_2, m) + 2\delta \right) \\ &+ \frac{2}{3} \frac{C_{\xi} d_2 \mathbb{D}(\mathcal{L}_{\sigma})}{2 \mathbf{N} \int_{\Omega} \beta(\cdot) (\psi^*)^2(\cdot) \mathrm{d}x} \| \nabla^2 \nu_{\sigma} \|_{L^1(\mathbb{R}^N)} \| \psi_* \|_{L^2(\mathbb{R}^N)}^2 \int_{|s| \leq 1} \nu_{\sigma}(|s|) |s|^2 \mathrm{d}s, \\ &\leq (1+\xi) \left(\lambda^*(d_2, m) + 2\delta \right) + \frac{C_{\xi} d_2 \mathbb{D}(\mathcal{L}_{\sigma})}{2 \mathbf{N} \min_{x \in \Omega \beta(\cdot) \mathrm{d}x}} \| \nabla^2 \nu_{\sigma} \|_{L^1(\mathbb{R}^N)} \frac{\| \psi_* \|_{L^2(\mathbb{R}^N)}^2}{\| \psi^* \|_{L^2(\mathbb{R}^N)}^2} \int_{|s| \leq 1} \nu_{\sigma}(|s|) |s|^2 \mathrm{d}s \\ &\leq (1+\xi) \left(\lambda^*(d_2, m) + 2\delta \right) + \frac{C_{\xi} d_2 \mathbb{D}(\mathcal{L}_{\sigma})}{2 \mathbf{N} \min_{x \in \Omega \beta(\cdot) \mathrm{d}x}} \| \nabla^2 \nu_{\sigma} \|_{L^1(\mathbb{R}^N)} \int_{|s| \leq 1} \nu_{\sigma}(|s|) |s|^2 \mathrm{d}s, \end{split}$$

applying the fact that $\int_{|s|\leq 1} \nu_{\sigma}(|s|) |s|^2 ds \leq \sigma$, and letting $\sigma \to 0$, we obtain that

$$\lambda_* \left(\frac{d_2 \mathbb{D}(J) \Delta}{2\mathbf{N}} \right) \le (1 + \xi) \left(\lambda^*(d_2, m) + 2\delta \right),$$

since the last inequality it is holds for all ξ , we deduce that

$$\lambda_*\left(\frac{d_2\mathbb{D}(J)\Delta}{2\mathbf{N}}\right) \le \left(\lambda^*(d_2,m) + 2\delta\right).$$

4. The threshold behavior

In this section, we investigate the existence of the infection-free equilibrium (IFE) and analyze the behavior of the model (1.2) when $\lambda^* (d_2, m) \leq 0$, which corresponds to $\mathbb{R}_0 < 1$. We begin by demonstrating the existence and uniqueness of the IFE, denoted by $\mathbb{E}^0 = \left(\frac{\mathbf{N}}{|\Omega|}, 0, 0\right)$.

Proposition 4.1. The model (1.2) has an unique IFE $\mathbb{E}^0 = \left(\frac{\mathbf{N}}{|\Omega|}, 0, 0\right)$.

Proof. From the model (1.2), \mathbb{E}^0 verifies the following equation,

$$\frac{d_2}{\sigma} \int_{\Omega} \mathcal{L}_{\sigma} \left(x - y \right) \left[S^0 \left(y \right) - S^0 \left(x \right) \right] \mathrm{d}y = 0, \quad x \in \Omega, \tag{4.1}$$

by applying the same approach as [40], we can deduce that S^0 is a constant. By applying the second assumption (2.1), we obtain that

$$S^0(\cdot) = \frac{\mathbf{N}}{|\Omega|}.$$

To show uniqueness, we take that any solution $S^0(x)$ must be constant, with the constant fixed by a population constraint. Define the operator:

$$\mathfrak{A}(S)(x) = \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \left[S(y) - S(x) \right] \, \mathrm{d}y.$$

then,

$$\mathfrak{A}(S^0)(x) = 0, \quad \forall x \in \Omega,$$

we assume that there exists two solutions $S_1^0(x)$ and $S_2^0(x)$ satisfy $\mathfrak{A}(S_1^0) = \mathfrak{A}(S_2^0) = 0$, we put $w(x) = S_1^0(x) - S_2^0(x)$, it then follows that

$$\mathfrak{A}(w)(x) = \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \left[w(y) - w(x) \right] \, \mathrm{d}y = 0,$$

so,

$$\int_{\Omega} \mathcal{L}_{\sigma}(x-y)w(y) \, \mathrm{d}y = w(x) \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \, \mathrm{d}y.$$

Let $k(x) = \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \, dy$, if we assume that w(x) is not constant, with a maximum at $x_0 \in \Omega$, so $w(y) - w(x_0) \leq 0$, we obtain

$$\mathfrak{A}(w)(x_0) = \int_{\Omega} \mathcal{L}_{\sigma}(x_0 - y) \left[w(y) - w(x_0) \right] \, \mathrm{d}y \le 0.$$

Since $\mathfrak{A}(w)(x_0) = 0$, and $\mathcal{L}_{\sigma} \geq 0$, the integrand $\mathcal{L}_{\sigma}(x_0 - y) [w(y) - w(x_0)] \leq 0$ must be zero almost everywhere. If $\mathcal{L}_{\sigma}(x_0 - y) > 0$ for y in a set of positive measure, then $w(y) = w(x_0)$. We suppose the existence of the connectivity (e.g., $\mathcal{L}_{\sigma}(x - y) > 0$ for $|x - y| < \delta$), this equality propagates across Ω , implying w(x) = constant.

Thus, $S_1^0(x) = S_2^0(x) + c$. A population constraint, such as:

$$\int_{\Omega} S^0(x) \, \mathrm{d}x = \mathbf{N},$$

fixes the constant. We have that $\int_{\Omega} S_1^0 dx = \int_{\Omega} S_2^0 dx = \mathbf{N}$, then c = 0, ensuring $S_1^0 = S_2^0$. Hence, the solution is **unique**.

In the next Theorem, we show the stability of IFE for $\mathbb{R}_0 = 1$.

Theorem 4.2. Let $\mathbb{R}_0 = 1$, then the equilibrium IFE is globally asymptotically stable.

Proof. By using (3.1) and the fact that $\mathbb{R}_0 = 1$, we find $\lambda_{\sigma,m} = 0$, we define the Lyapunov function as follows

$$V(t) = \int_{\Omega} \psi(x) \mathcal{I}(t, x) \,\mathrm{d}x,$$

where $\psi(x) > 0$ represents the principal eigenfunction of (3.6). Next, we compute the time derivative of the Lyapunov function

$$\frac{dV}{dt} = \int_{\Omega} \psi(x) \frac{\partial \mathcal{I}}{\partial t} dx,$$

$$= \int_{\Omega} \psi(x) \left[\frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{I}(t,y) - \mathcal{I}(t,x)] dy + \frac{\beta(x)S\mathcal{I}}{S+\mathcal{I}+\mathcal{R}} - \alpha(x)\mathcal{I} \right] dx,$$

$$= \underbrace{\frac{d_2}{\sigma^m} \int_{\Omega} \psi(x) \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{I}(t,y) - \mathcal{I}(t,x)] dy dx}_{\text{Term 1: Diffusion}} + \underbrace{\int_{\Omega} \psi(x) \left[\frac{\beta(x)S\mathcal{I}}{S+\mathcal{I}+\mathcal{R}} - \alpha(x)\mathcal{I} \right] dx}_{\text{Term 2: Infection and Recovery}}.$$
(4.2)

We rewrite the following term,

$$\int_{\Omega} \psi(x) \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{I}(t,y) - \mathcal{I}(t,x)] \, \mathrm{d}y \, \mathrm{d}x = \int_{\Omega} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \psi(x) \mathcal{I}(y) \, \mathrm{d}y \, \mathrm{d}x - \int_{\Omega} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \psi(x) \mathcal{I}(x) \, \mathrm{d}y \, \mathrm{d}x.$$

Interchange x and y in the first integral, using symmetry $(\mathcal{L}_{\sigma}(x-y) = \mathcal{L}_{\sigma}(y-x))$, we get

$$\int_{\Omega} \int_{\Omega} \mathcal{L}_{\sigma}(x-y)\psi(x)\mathcal{I}(y) \,\mathrm{d}y \,\mathrm{d}x = \int_{\Omega} \int_{\Omega} \mathcal{L}_{\sigma}(y-x)\psi(y)\mathcal{I}(t,x) \,\mathrm{d}y \,\mathrm{d}x = \int_{\Omega} \int_{\Omega} \mathcal{L}_{\sigma}(x-y)\psi(y)\mathcal{I}(t,x) \,\mathrm{d}y \,\mathrm{d}x,$$

it then follows,

$$\frac{d_2}{\sigma^m} \int_{\Omega} \psi(x) \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{I}(t,y) - \mathcal{I}(t,x)] \, \mathrm{d}y \, \mathrm{d}x = \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{I}(t,x) \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\psi(y) - \psi(x)] \, \mathrm{d}y \, \mathrm{d}x, \qquad (4.3)$$

by using the eigenfunction equation (3.6), we get

$$\frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\psi(y) - \psi(x)] \, \mathrm{d}y = -[\beta(x) - \alpha(x)] \psi(x), \tag{4.4}$$

by multiplying (4.4) by $\mathcal{I}(t, x)$ and integrating over Ω , we have

$$\frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{I}(t,x) \left[\int_{\Omega} \mathcal{L}_{\sigma}(x-y)\psi(x) \,\mathrm{d}y - \int_{\Omega} \mathcal{L}_{\sigma}(x-y)\psi(y) \,\mathrm{d}y \right] \mathrm{d}x = -\int_{\Omega} [\beta(x) - \alpha(x)]\psi(x)\mathcal{I}(t,x) \,\mathrm{d}x,$$

by using (4.3), the first term becomes,

$$\frac{d_2}{\sigma^m} \int_{\Omega} \psi(x) \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{I}(t,y) - \mathcal{I}(t,x)] \, \mathrm{d}y \, \mathrm{d}x = -\int_{\Omega} [\beta(x) - \alpha(x)] \psi(x) \mathcal{I}(t,x) \, \mathrm{d}x, \tag{4.5}$$

we also have,

$$\int_{\Omega} \psi(x) \left[\frac{\beta(x)\mathcal{S}\mathcal{I}}{\mathcal{S} + \mathcal{I} + \mathcal{R}} - \alpha(x)\mathcal{I} \right] dx = \int_{\Omega} \psi(x)\mathcal{I}(t,x) \left[\beta(x) \frac{\mathcal{S}}{\mathcal{S} + \mathcal{I} + \mathcal{R}} - \alpha(x) \right] dx,$$
(4.6)

we rewrite the term (4.6) as follows,

$$\int_{\Omega} \psi(x) \left[\frac{\beta(x)\mathcal{S}\mathcal{I}}{\mathcal{S} + \mathcal{I} + \mathcal{R}} - \alpha(x)\mathcal{I} \right] dx = \int_{\Omega} \psi(x)\mathcal{I}(t,x) \left[\beta(x) \left(\frac{\mathcal{S}}{\mathcal{S} + \mathcal{I} + \mathcal{R}} - 1 \right) + \beta(x) - \alpha(x) \right] dx, \quad (4.7)$$

by combining (4.5) and (4.7), we get

$$\frac{dV}{dt} = \int_{\Omega} \psi(x) \mathcal{I}(t, x) \left[\beta(x) \left(\frac{S}{S + \mathcal{I} + \mathcal{R}} - 1 \right) \right] \mathrm{d}x,$$

Since $\frac{S}{S+I+R} = 1 - \frac{I+R}{S+I+R}$, then

$$\frac{\mathcal{S}}{\mathcal{S} + \mathcal{I} + \mathcal{R}} - 1 = -\frac{\mathcal{I} + \mathcal{R}}{\mathcal{S} + \mathcal{I} + \mathcal{R}} \le 0,$$

thus,

$$\int_{\Omega} \psi(x) \mathcal{I}\beta(x) \left(-\frac{\mathcal{I} + \mathcal{R}}{\mathcal{S} + \mathcal{I} + \mathcal{R}} \right) \mathrm{d}x \le 0,$$

thus, we find $\frac{dV}{dt} \leq 0$, and strictly negative unless $\mathcal{I} = 0$, solutions converge to the invariant set where $\mathcal{I} = 0$. To confirm $\mathcal{R} \to 0$, we consider the Lyapunov function $W(t) = \frac{1}{2} \int_{\Omega} \mathcal{R}(t, x)^2 dx$, we compute the time derivative of W, we have

$$\frac{dW}{dt} = \int_{\Omega} \mathcal{R}\left[\frac{d_3}{\sigma^n} \int_{\Omega} J_{\sigma}(x-y) [\mathcal{R}(t,y) - \mathcal{R}(t,x)] \, \mathrm{d}y + \alpha(x)\mathcal{I} - \gamma(x)\mathcal{R}\right] \mathrm{d}x,$$

the first term gives,

$$\frac{d_3}{\sigma^n} \int_{\Omega} \mathcal{R}(t,x) \int_{\Omega} J_{\sigma}(x-y) [\mathcal{R}(t,y) - \mathcal{R}(t,x)] \,\mathrm{d}y \,\mathrm{d}x, \tag{4.8}$$

we rewrite (4.8), we have

$$\frac{d_3}{\sigma^n} \int_{\Omega} \int_{\Omega} J_{\sigma}(x-y) \mathcal{R}(t,x) \mathcal{R}(t,y) \, \mathrm{d}y \, \mathrm{d}x - \frac{d_3}{\sigma^n} \int_{\Omega} \mathcal{R}(t,x)^2 \int_{\Omega} J_{\sigma}(x-y) \, \mathrm{d}y \, \mathrm{d}x, \tag{4.9}$$

by applying the following,

$$-2\mathcal{R}(t,x)\mathcal{R}(t,y) = [\mathcal{R}(t,y) - \mathcal{R}(t,x)]^2 - \mathcal{R}(t,y)^2 - \mathcal{R}(t,x)^2,$$

we obtain,

$$\int_{\Omega} \int_{\Omega} J_{\sigma}(x-y) \mathcal{R}(t,x) \mathcal{R}(t,y) \, \mathrm{d}y \, \mathrm{d}x = -\frac{1}{2} \int_{\Omega} \int_{\Omega} J_{\sigma}(x-y) [\mathcal{R}(t,y) - \mathcal{R}(t,x)]^2 \, \mathrm{d}y \, \mathrm{d}x + \int_{\Omega} \mathcal{R}(t,x)^2 \int_{\Omega} J_{\sigma}(x-y) \, \mathrm{d}y \, \mathrm{d}x,$$
(4.10)

by replacing (4.10) into (4.9), we get

$$-\frac{d_3}{2\sigma^n}\int_{\Omega}\int_{\Omega}J_{\sigma}(x-y)[\mathcal{R}(t,y)-\mathcal{R}(t,x)]^2\,\mathrm{d}y\,\mathrm{d}x\leq 0,$$

we conclude that

$$\frac{dW}{dt} \le -2\underline{\gamma}W(t),$$

thus, $\mathcal{R} \to 0$ in $L^2(\Omega)$, and smoothing ensures $C(\overline{\Omega})$ convergence. Hence, we get that $W(t) \to 0$, which gives $\mathcal{R} \to 0$ in $L^1(\Omega)$, and nonlocal smoothing ensures $\mathcal{R} \to 0$ in $C(\overline{\Omega})$. Since $\mathcal{I}, \mathcal{R} \to 0$, the first equation of (1.2) reduces to $\frac{\partial S}{\partial t} \approx \frac{d_1}{\sigma^m} \int_{\Omega} J_{\sigma}(x-y) [\mathcal{S}(t,y) - \mathcal{S}(t,x)] \, dy$, driving $\mathcal{S} \to S^0$.

Before presenting the main result of this section, we need to establish the following proposition.

Proposition 4.3. Let us consider $(\mathcal{S}(t, \cdot), \mathcal{I}(t, \cdot), \mathcal{R}(t, \cdot))$ be the solution of (1.2), then if $S_0(\cdot) \leq S^0(\cdot)$ for $on\Omega$, we get that $\mathcal{S}(t, \cdot) \leq S^0(\cdot)$, $\forall x \in \Omega$.

Proof. By subtracting the equation (4.1) and the first equation of (1.2) and we set that $S(t, x) = S(t, x) - S^{0}(x)$, we obtain that

$$\frac{\partial \mathcal{S}}{\partial t} = \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma} \left(x - y \right) \left[\mathcal{S} \left(t, y \right) - \mathcal{S} \left(t, x \right) \right] dy - \frac{\beta(\cdot) \mathcal{S}(t, \cdot) \mathcal{I}(t, \cdot)}{S + I + R} + \gamma(\cdot) \mathcal{R}(t, \cdot), \quad \forall \text{on}\Omega,
\frac{\partial \mathcal{S}}{\partial t} = \mathbb{A}_S^{\sigma} \mathcal{S} - \frac{\beta(\cdot) \mathcal{S}(t, \cdot) \mathcal{I}(t, \cdot)}{S + I + R} + \gamma(\cdot) \mathcal{R}(t, \cdot), \quad \forall x \in \Omega,$$
(4.11)

since the operator \mathbb{A}_{S}^{σ} generates a positive semi-group $\{\mathbb{T}_{S}\}_{t>0}$ by solving the last equation (4.11), we find that

$$\mathcal{S}(t,\cdot) = \mathbb{T}_{\mathcal{S}}(t)\mathcal{S}_{0}(\cdot) - \int_{0}^{t} \mathbb{T}_{\mathcal{S}}(t-s) \left[\frac{\beta(\cdot)\mathcal{S}(\tau,\cdot)\mathcal{I}(\tau,\cdot)}{\mathcal{S}+\mathcal{I}+\mathcal{R}}\right] \mathrm{d}\tau,$$
(4.12)

given that $S_0(\cdot) \leq 0$, it follows that $S \leq 0$, which implies that $S(t, \cdot) \leq S^0(\cdot)$ for all $x \in \Omega$.

Now, we are ready to establish the global stability of IFE (GAS).

Theorem 4.4. Let $\mathbb{R}_0 < 1$, then the IFE is GAS.

Proof. By applying Proposition (4.3), we have that $S(t, \cdot) \leq S^0(\cdot)$, $\forall \text{on}\Omega, t \geq 0$, and if we suppose that $(\mathcal{I}, \mathcal{R}) \leq (U_1, U_2)$ using the comparison principle on the second equation of (1.2), we have

$$\frac{\partial U_1}{\partial t} = \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma} \left(x - y \right) \left[\mathcal{I}(t, y) - \mathcal{I}(t, \cdot) \right] \mathrm{d}y + \frac{\beta(\cdot) S^0(\cdot) + \varepsilon U_1}{S^0(\cdot) + \varepsilon + U_1 + U_2} - \alpha(\cdot) U_1,$$
$$\frac{\partial U_2}{\partial t} = \frac{d_3}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma} \left(x - y \right) \left[\mathcal{R}(t, y) - \mathcal{R}(t, \cdot) \right] \mathrm{d}y - \gamma(\cdot) U_2,$$

from the fact that $\frac{S^0(\cdot) + \varepsilon}{S^0(\cdot) + \varepsilon + U_1 + U_2} \leq 1$, by using again the comparison principle, we obtain that

$$\frac{\partial V_1}{\partial t} = \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma} \left(x - y \right) \left[V_1(t, y) - V_1(t, \cdot) \right] dy + \beta(\cdot) V_1 - \alpha(\cdot) V_2, \in \Omega,
\frac{\partial V_2}{\partial t} = \frac{d_3}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma} \left(x - y \right) \left[V_2(t, y) - V_2(t, \cdot) \right] dy - \gamma(\cdot) V_2 + \alpha(\cdot) V_1,$$
(4.13)

we define the semi-group denoted as $\mathcal{T}_{I,R}$ associated to the linear problem (4.13) such that

$$\|\mathcal{T}_{I,R}\| \leq \mathbf{C}_* \mathrm{e}^{\lambda^*(d_2,m)t},$$

by applying the proposition (3.1) since $\mathbb{R}_0 < 1$, which gives that $\lambda^* < 0$. Consequently, we get that $(V_1, V_2) \rightarrow (0, 0)$, $\operatorname{on}\Omega, \operatorname{ast} \rightarrow \infty$. Furthermore, we obtain that $(U_1, U_2) \rightarrow (0, 0)$, $\operatorname{on}\Omega, \operatorname{ast} \rightarrow \infty$. Thus, $(I, R) \rightarrow (0, 0)$, $\operatorname{on}\Omega, \operatorname{ast} \rightarrow \infty$. By using the first equation of (1.2), we need to show that $\mathcal{S}(t, \cdot) \rightarrow S^0(\cdot)$, uniformly for $\operatorname{on}\Omega$, as $t \rightarrow \infty$. Since $(I, R) \rightarrow (0, 0)$, as $t \rightarrow \infty$, the first equation of model (1.2) can be written as follows,

$$\begin{cases} \frac{\partial S}{\partial t} = \frac{d_1}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma} \left(x - y \right) \left[\mathcal{S}(t, y) - \mathcal{S}(t, \cdot) \right] \mathrm{d}y, \\ \int_{\Omega} S^0(\cdot) \mathrm{d}x = \mathbf{N}, \end{cases}$$

Now, we set that $s(t, x) = \mathcal{S}(t, \cdot) - S^0(\cdot)$, with $s(t, \cdot)$ satisfies the following equation,

$$\frac{\partial s}{\partial t} = \frac{d_1}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \left[s(t,y) - s(t,\cdot) \right] \mathrm{d}y - \frac{\beta(\cdot)SI}{S+I+R} + \gamma(\cdot)\mathcal{R}(t,\cdot), \tag{4.14}$$

we know that $\beta(\cdot), \alpha(\cdot)$ are a holder continuous then there exists a constant $\mathcal{C} > 0$ such that

$$|\gamma(\cdot)\mathcal{R}(t,\cdot) - \frac{\beta(\cdot)\mathcal{S}(t,\cdot)\mathcal{I}(t,\cdot)}{\mathcal{S}(t,\cdot) + \mathcal{I}(t,\cdot) + \mathcal{R}(t,\cdot)}| \le \mathbf{C}\mathrm{e}^{-\frac{1}{4}\lambda^*(d_1,m)t},$$

we also define $\mathcal{V}(t) = \int_{\Omega} s^2(t, \cdot) \, \mathrm{d}x$. By computing its derivative, we obtain,

$$\frac{\mathrm{d}\mathcal{V}(t)}{\mathrm{d}t} = 2 \int s(t,\cdot) \frac{\partial s(t,\cdot)}{\partial t} \mathrm{d}x$$

$$= 2 \int s(t,\cdot) \left(\frac{d_2}{\sigma^m} \int \mathcal{L}_{\sigma}(x-y) \left[s(t,y) - s(t,\cdot)\right] \mathrm{d}y - \frac{\beta(\cdot)SI}{S+I+R} + \gamma(\cdot)\mathcal{R}(t,\cdot)\right)$$

$$= 2 \frac{d_2}{\sigma^m} \left(\int \int \mathcal{L}_{\sigma}(x-y) \left(s(t,y) - s(t,\cdot)\right)^2 - \int s^2(t,y) \mathrm{d}y\right)$$

$$+ 2 \int_{\Omega} s(t,\cdot) \left(\gamma(\cdot)\mathcal{R}(t,\cdot) - \frac{\beta(\cdot)SI}{S+I+R}\right) \mathrm{d}x$$

$$\leq -2 \frac{d_2}{\sigma^m} \mathcal{C}_* \mathcal{V}(t) + 2\mathbf{CN} \mathrm{e}^{-\frac{1}{4}\lambda^*(d_1,m)t},$$
(4.15)

by integrating (4.15), we get that

$$\mathcal{V}(t) \le \mathcal{V}(0) \mathrm{e}^{-2\frac{d_2}{\sigma^m}\mathcal{C}_* t} + 2\mathrm{e}^{-2\frac{d_2}{\sigma^m}\mathcal{C}_* t} \mathbf{C}_* \mathbf{N} \int_0^t \mathrm{e}^{\left(2\frac{d_2}{\sigma^m}\mathcal{C} - \frac{1}{4}\lambda^*(d_1, m)\right)\sigma} \mathrm{d}\sigma,$$
(4.16)

we put $\mathcal{A} = -2 \frac{d_2}{\sigma^m} \mathcal{C}_*, \ \mathcal{B} = 2\mathbf{C}_*\mathbf{N}$, and $\mathcal{C} = \frac{1}{4}\lambda^* (d_1, m)$. Thus, by a simple computation, we get

$$\mathcal{V}(t) \leq \begin{cases} (\mathcal{V}(0) + 2\mathcal{B}t) e^{\mathcal{A}t} & \text{if } \mathcal{C} = \frac{\mathcal{A}}{2}, \\ \left(\mathcal{V}(0) e^{\frac{\mathcal{A}}{2}t} + \frac{2\mathcal{B}}{\mathcal{C} - \frac{\mathcal{A}}{2}} e^{\mathcal{C}t}\right) & \mathcal{C} \neq \frac{\mathcal{A}}{2}, \end{cases}$$
(4.17)

using (4.14), we obtain

$$s(t,\cdot) = s_0(\cdot) \mathrm{e}^{-\frac{d_2}{\sigma^m}t} + \mathrm{e}^{-\frac{d_2}{\sigma^m}t} \int_0^t \mathrm{e}^{\frac{d_2}{\sigma^m}\tau} \left(\int_\Omega \mathcal{L}_\sigma(x-y)s(\tau,y)\,\mathrm{d}y + \gamma(\cdot)\mathcal{R}(\tau,x) - \frac{\beta(\cdot)SI}{S+I+R} \right) \mathrm{d}\tau,$$

it then follows from the Hölder's inequality, we obtain

$$\int_{\Omega} \mathcal{L}_{\sigma}(x-y) s(\tau, y) \, \mathrm{d}y \leq \mathbf{C}_{**} \left| \mathbf{V}(t) \right|_{L^2}$$

for $C_{**} > 0$, we get

$$|s(t,\cdot)| \leq \begin{cases} \alpha_1 \mathrm{e}^{-\frac{d_1}{\sigma^m}t} + (\alpha_2 t + \alpha_3)\mathrm{e}^{\frac{A}{2}t} + \alpha_4 \mathrm{e}^{\mathcal{C}t} & \text{if } \mathcal{C} = \frac{A}{2}, \\ \alpha_5 \mathrm{e}^{-\frac{d_1}{\sigma^m}t} + \alpha_6 \mathrm{e}^{\frac{A}{2}t} + \alpha_7 \mathrm{e}^{\mathcal{C}t} & \text{if } \mathcal{C} \neq \frac{A}{2}, \end{cases}$$

with α_i (for i = 1, ..., 7) represent a positive constants. In fact that $s(t, \cdot) \in C(\overline{\Omega} \times (0, \infty))$, it follows that

$$s(t, \cdot) \to 0$$
 uniformly on Ω as $t \to \infty$.

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Remark 4.5. In the preceding theorem, we demonstrated that, in practice, the epidemic will eventually be eradicated from our community. This approach serves as a valuable method for eliminating the transmission of

infectious diseases among the population. Additionally, it provides effective tools for controlling and halting the spread of epidemics.

5. Positive endemic equilibrium states (PEES)

In this section, we aim to establish both the existence and uniqueness of the PEES, along with the corresponding asymptotic profiles related to this phenomenon. Specifically, we will examine scenarios in which the parameter σ influences the spread of disease within the community. The PEES satisfies the following system,

$$\begin{cases} 0 = \frac{d_1}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{S}(y) - \mathcal{S}(\cdot)] \, \mathrm{d}y - \frac{\beta(\cdot)\mathcal{S}(\cdot)\mathcal{I}(\cdot)}{S+I+R} + \gamma(\cdot)\mathcal{R}(\cdot), & \mathrm{on}\Omega, \\ 0 = \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{I}(y) - \mathcal{I}(\cdot)] \, \mathrm{d}y + \frac{\beta(\cdot)\mathcal{S}(\cdot)\mathcal{I}(\cdot)}{S+I+R} - \alpha(\cdot)\mathcal{I}(\cdot), & \mathrm{on}\Omega, \\ 0 = \frac{d_3}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) [\mathcal{R}(y) - \mathcal{R}(\cdot)] \, \mathrm{d}y - \gamma(\cdot)\mathcal{R}(\cdot) + \alpha(\cdot)\mathcal{I}(\cdot), & \mathrm{on}\Omega. \end{cases}$$
(5.1)

Furthermore, by summing the first and second equations of (1.2) and subsequently integrating the resulting equation over Ω , we obtain the following result,

$$\frac{\partial}{\partial t} \int_{\Omega} \left[\mathcal{S}(t, \cdot) + \mathcal{I}(t, \cdot) + \mathcal{R}(t, \cdot) \right] \mathrm{d}x = 0, \quad \forall t \ge 0.$$
(5.2)

which implies that

$$\int_{\Omega} [\mathcal{S}(t,\cdot) + \mathcal{I}(t,\cdot) + \mathcal{R}(t,\cdot)] \,\mathrm{d}x = \mathbf{N}, \quad \forall t \ge 0.$$
(5.3)

Thus, we can deduce that $(\mathcal{S}(\cdot), \mathcal{I}(\cdot), \mathcal{R}(\cdot))$ verifies the following equation,

$$\int_{\Omega} \left[\mathcal{S}(\cdot) + \mathcal{I}(\cdot) + \mathcal{R}(\cdot) \right] \mathrm{d}x = \mathbf{N}, \tag{5.4}$$

by adding the three equations of (5.1), we get

$$\int_{\Omega} \mathcal{L}_{\sigma}(x-y) \left[(d_1 S^*(y) + d_2 I(y) + d_3 R^*) - \left(d_1 \hat{S}(\cdot) + d_2 \hat{I}(\cdot) + d_3 \hat{R} \right) \right] = 0,$$
(5.5)

by using the same arguments as the Proposition (3.3) in [40], we obtain that

$$(d_1 S^*(\cdot) + d_2 I^*(\cdot) + d_3 R^*(\cdot)) = \mathbf{C}^*,$$
(5.6)

then, by setting that $\tilde{S}^* = \frac{S^*}{\mathbf{C}^*}$, $\tilde{I}^* = \frac{d_2 I^*}{\mathbf{C}^*}$, $\tilde{R}^* = \frac{d_3 R^*}{\mathbf{C}^*}$, it then follows that

$$(d_1 S^*(\cdot) + I^*(\cdot) + R^*(\cdot)) = 1$$
, on $\in \Omega$.

We proceed to show the existence at least of (PEES), denoted by $E^* = (S^*, I^*, R^*)$. This analysis utilizes Theorem 3 from [41], as detailed in the following theorem. **Theorem 5.1.** We assume that $\mathcal{R}_0 > 1$, it then follows as there exists an at least solution $(\hat{S}^*, \hat{I}^*, \hat{R}^*)$ of model (5.1), and the model (1.2) is uniformly persistent which means that there exist $\delta > 0$ such that

$$\liminf_{t \to \infty} \mathcal{S}(t,.) > \delta, \quad \liminf_{t \to \infty} \mathcal{I}(t,.) > \delta, \quad \liminf_{t \to \infty} \mathcal{R}(t,.) > \delta$$

Furthermore, there exists at least at least of (PEES), \tilde{E}^* .

Proof. To proceed, we must confirm that all claims of Theorem 4.2 in [35] are satisfied. Let us define

$$\mathcal{X}_{0} := \{ \psi = (\psi_{1}, \psi_{2}, \phi_{3}) \in \mathbb{X}^{+} : \psi_{2}(\cdot) \neq 0 \text{ and } \psi_{3}(\cdot) \neq 0 \}, \\ \partial \mathcal{X}_{0} = \{ \psi = (\psi_{1}, \psi_{2}, \psi_{3}) \in \mathbb{X}^{+} : \psi_{2}(\cdot) \equiv 0 \text{ or } \psi_{3}(\cdot) \equiv 0 \}, \\ \mathcal{M} = \{ (S_{0}, I_{0}, R_{0}) \in \mathbb{X}^{+} : (\mathcal{S}, \mathcal{I}, \mathcal{R}) \in \partial \mathbb{X}_{0} \}.$$
(5.7)

To address **the first claim**, we define the following sets: $\omega(S_0, I_0, R_0)$, representing the limit set of the orbit $\{\Phi_t(S_0, I_0, R_0) : t \ge 0\}$, and $W^s(S^0, 0, 0)$, which denotes the stable set of the IFE. Moreover, we need to prove that $\Phi_t \mathcal{X}_0 \subset \mathcal{X}_0$, $\forall t \ge 0$. We consider $(\psi_1, \psi_2, \psi_3) \in \mathcal{X}_0$, by applying the variation constant method into the model (1.2), we get

$$\begin{split} \mathcal{S}(t,\cdot) &= \phi_{1} \mathrm{e}^{-\int_{0}^{t} \frac{d_{1}}{\sigma^{m}} \int_{\Omega}; /\mathcal{L}_{\sigma}\left(x-y\right) \mathrm{d}y + \frac{\beta(\cdot)\mathcal{S}(\sigma,\cdot)\mathcal{I}(\sigma,\cdot)}{S+I+R} \mathrm{d}\sigma} \\ &+ \int_{0}^{t} \left(\frac{d_{1}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y)\mathcal{S}(\sigma,y) \,\mathrm{d}y + \gamma(\cdot)\mathcal{R}(\sigma,\cdot)\right) \mathrm{e}^{-\int_{\sigma}^{t} \frac{d_{1}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}\left(x-y\right) \mathrm{d}y + \frac{\beta(\cdot)\mathcal{S}(\sigma,\cdot)\mathcal{I}(\sigma,\cdot)}{S+I+R} \,\mathrm{d}\sigma} \mathrm{d}\sigma, \\ \mathcal{I}(t,\cdot) &= \phi_{2} \mathrm{e}^{-\left(\frac{d_{2}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}\left(x-y\right) \mathrm{d}y + \alpha(\cdot)\right)} \mathrm{d}t \\ &+ \int_{0}^{t} \left(\frac{d_{2}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}\left(x-y\right)\mathcal{I}(\sigma,y) \,\mathrm{d}y + \frac{\beta(\cdot)\mathcal{S}(\sigma,\cdot)\mathcal{I}(\sigma,\cdot)}{S+I+R}\right) \mathrm{e}^{-\left(\frac{d_{2}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}\left(x-y\right) \mathrm{d}y + \alpha(\cdot)\right)^{(t-\sigma)}} \mathrm{d}\sigma, \\ \mathcal{R}(t,\cdot) &= \phi_{3} \mathrm{e}^{-\left(\frac{d_{3}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}\left(x-y\right) \mathrm{d}y + \gamma(\cdot)\right)t} \\ &+ \int_{0}^{t} \left(\frac{d_{3}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}(x-y)\mathcal{R}(\sigma,y) \,\mathrm{d}y + \alpha(\cdot)\mathcal{I}(\sigma,\cdot)\right) \mathrm{e}^{-\left(\frac{d_{3}}{\sigma^{m}} \int_{\Omega} \mathcal{L}_{\sigma}\left(x-y\right) \mathrm{d}y + \gamma(\cdot)\right)^{(t-\sigma)}} \mathrm{d}\sigma, \end{split}$$

$$(5.8)$$

Since this implies that $\mathcal{S}(t,\cdot), \mathcal{I}(t,\cdot), \mathcal{R}(t,\cdot) > 0$ for all $t \ge 0$ and $x \in \Omega$, we can conclude that $(\mathcal{S}(t,\cdot), \mathcal{I}(t,\cdot), \mathcal{R}(t,\cdot)) \in \mathcal{X}_0$, with the property that $\Phi_t \mathcal{X}_0 \subset \mathcal{X}_0$.

In the second claim, it remains to establish that $\omega(\phi) = \{S^0(\cdot), 0, 0\}$ for every initial condition $(S_0, I_0, R_0) \in \mathbf{M}$. We suppose by contradiction that $\mathbf{M} \subset \{\psi_1 \in C (\mathbb{R}^+, \overline{\Omega}) : (\psi_1, 0, 0)\}$. Indeed, we have two cases $I \equiv 0, R \neq \equiv 0$ and $I \neq \equiv 0, R \equiv 0$. In the first case, we assume that $R \neq \equiv 0$, since we have $\Phi_t \mathcal{X}_0 \subset \mathcal{X}_0$ then we obtain that $\mathcal{I}(t, \cdot) > 0, \forall on\Omega, t > 0$, which contradicts with the fact that $I \in \mathbf{M}$. In the second case, by (5.8), we obtain that $\mathcal{I}(t, \cdot) > 0, \mathcal{R}(t, \cdot) \forall t \geq 0, \forall on\Omega$, then we conclude that $\Phi_t(\psi) \in \mathcal{X}_0$, which implies that $\psi \in \mathbf{M}$. Moreover, we have $\omega(\psi) = (S^0(\cdot), 0, 0), \forall x \in \Omega$.

In the last claim , we focus to investigate that $E^{0}(\cdot) = (S^{(\cdot)}, 0, 0)$ is a uniform weak repeller which means there exists $\varepsilon > 0$, such that

$$\limsup_{t \to \infty} \left\| \Phi_t \left(S_0, I_0, R_0 \right) - \left(S^0(\cdot), 0, 0 \right) \right\| \ge \varepsilon, \forall \left(\mathcal{S}, \mathcal{I}, \mathcal{R} \right) \in \mathcal{X}_0.$$

We suppose by contradiction that there exists a $\epsilon > 0$ such that

$$\limsup_{t \to \infty} \left\| \Phi_t \left(S_0, I_0, R_0 \right) - \left(S^0(\cdot), 0, 0 \right) \right\| < \varepsilon, \forall \left(\mathcal{S}, \mathcal{I}, \mathcal{R} \right) \in \mathcal{X}_0.$$

Moreover, we can find a $\tilde{t} > 0$, with $\forall t \geq \tilde{t}$, such that:

$$S^{0}(\cdot) - \varepsilon \leq \mathcal{S}(t, \cdot) \quad 0 \leq \mathcal{I}(t, \cdot) \leq \overline{\delta}, \quad 0 \leq \mathcal{I}(t, \cdot) \leq \varepsilon, \quad 0 \leq \mathcal{R}(t, \cdot) \leq \varepsilon, \quad x \in \overline{\Omega}.$$

Since $\mathbb{R}_0 > 1$, applying Proposition (3.1) implies that $\lambda(\varepsilon) > 0$ for sufficiently small values of ε . Based on this, we propose a model where I serves as an upper solution to the following model

The system under consideration is given by:

$$\begin{cases} \frac{\partial U}{\partial t} = \frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma} \left(x - y \right) \left(U(t, y) - U(t, \cdot) \right) \mathrm{d}y + \beta(\cdot) \frac{S^0(\cdot) - \varepsilon}{S^0(\cdot) + 2\varepsilon} U(t, \cdot) - \alpha(\cdot) U(t, \cdot), & t > \tilde{t}, \text{ on}\Omega, \\ \frac{\partial V}{\partial t} = \frac{d_3}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma} \left(x - y \right) \left(V(t, y) - V(t, \cdot) \right) \mathrm{d}y + \gamma(\cdot) V - \alpha(\cdot) U, & \text{on}\Omega, \\ U(\tilde{t}, \cdot) = \mathcal{I}(\tilde{t}, \cdot) > 0, & \text{on}\Omega, \\ V(\tilde{t}, \cdot) = \mathcal{R}(\tilde{t}, \cdot) > 0, & \text{on}\Omega, \end{cases}$$

by selecting a sufficiently small $\varepsilon > 0$ such that $\varepsilon (\phi_2^{\varepsilon}, \phi_3^{\varepsilon}) \leq (\mathcal{I}(\tilde{t}, \cdot : \phi_2), \mathcal{R}(\tilde{t}, \cdot : \phi_3))$ in $\bar{\Omega}$, it follows that $(U, V) = \varepsilon e^{\lambda^*(\varepsilon)} (\phi_1^{\varepsilon}(t-\tilde{t}), \phi_2^{\varepsilon}(\cdot)(t-\tilde{t}))$ serves as a subsolution to the system. Since $\lambda(\varepsilon) > 0$, which yields that $U \to \infty$, $V \to \infty$, as $t \to \infty$. Additionally, we obtain that $I \to \infty$ and $R \to \infty$ as $t \to \infty$, which gives a contradiction against Theorem (2.2).

Next, we use Theorem 3 in [41], we set the following function $\Xi : \mathbb{X}^+ \to \mathbb{R}^+$ such that $\Xi(\phi) := \min \{\min_{x \in \Omega} \phi_2(\cdot), \min_{x \in \Omega} \phi_3(\cdot)\}, \quad \phi \in \mathbb{X}^+$, with $\phi_2 = I$, $\phi_3 = R$, we have that $\Xi(\Phi_t(\phi)) > 0$, $\forall t > 0$, or $\Xi(\Phi_t(\phi)) = 0, \forall t > 0$, it then follows that Ξ generalized a distance of Φ_t . By using the first claim, we can deduce that

$$\bigcup_{\phi \in \partial \mathcal{X}_0} \omega(\phi) = \{E_0\}, \text{ and } W^s(E_0) \cap \Xi^{-1}(0, \infty) = \emptyset,$$

where $W^s(E_0)$ represents the stable set of E_0 , then $\{E_0\}$ is an isolated invariant set in X_H , and no subset of $\{Q_0\}$ forms a cycle in ∂X_{H_0} . Therefore, by [41], Theorem 3, there exists $\varsigma > 0$ such that

$$\min_{\phi \in \mathcal{L}} \Xi(\phi) > \varsigma,$$

where $\mathcal{L} \subset \mathcal{X}_0 \setminus \{E_0\}$ is any compact chain-transitive set. This result implies that

$$\limsup_{t \to \infty} \mathcal{I}(t,.) \ge \varsigma, \quad \limsup_{t \to \infty} \mathcal{R}(t,.) \ge \varsigma,$$

for any non-zero initial value $\begin{pmatrix} I_0 \\ R_0 \end{pmatrix} \in \mathcal{X}_0.$

By combining this with [42], Theorem 4.7, we conclude that system (1.2) possesses at least one PEES E_* , in \mathcal{X}_0 .

6. Asymptotic profiles of PEES

In this section, we examine the asymptotic behavior of PEES, demonstrating that the model converges to PEES under the condition of sufficiently small of σ .

Theorem 6.1. Assuming that $\mathcal{R}_0 > 1$, and we suppose that m = 2, and $\sigma \to 0^+$, then we have

$$(\mathcal{S}(\cdot), \mathcal{I}(\cdot), \mathcal{R}(\cdot)) \to (S^*(\cdot), I^*(\cdot), R^*(\cdot))$$

where $(S^*(\cdot), I^*(\cdot), R^*(\cdot))$ represents a unique positive smooth solution to the following system:

$$\begin{cases} \frac{d_1\mathbb{D}_2(\mathcal{J})}{2\mathbf{N}}\Delta S^*(\cdot) - \frac{\beta(\cdot)S^*(\cdot)I^*(\cdot)}{S^*(\cdot)+I^*(\cdot)+R^*(\cdot)} + \gamma(\cdot)R^*(\cdot) = 0, \quad on\Omega, \\\\ \frac{d_2\mathbb{D}_2(\mathcal{J})}{2\mathbf{N}}\Delta I^*(\cdot) + \frac{\beta(\cdot)S^*(\cdot)I^*(\cdot)}{S^*(\cdot)+I^*(\cdot)+R^*(\cdot)} - \alpha(\cdot)I^*(\cdot) = 0, \quad on\ \Omega, \\\\ \frac{d_3\mathbb{D}_2(\mathcal{J})}{2\mathbf{N}}\Delta R^*(\cdot) + \alpha(\cdot)I^*(\cdot) - \gamma(\cdot)R^*(\cdot) = 0, \quad on\Omega, \\\\ \frac{\partial S^*}{\partial n} = \frac{\partial I^*}{\partial n} = \frac{\partial R^*}{\partial n} = 0, \qquad x \in \partial\Omega, \end{cases}$$

where $\partial/\partial n$ represents the normal derivative on the boundary $\partial \Omega$.

Proof. For m = 2, and by applying (3) in Theorem (3.2), we know that

$$\lambda^*(\frac{d_2\mathbb{D}_2}{2\mathbf{N}}\Delta) := \inf_{\psi \in H^1_0(\Omega), \psi \neq 0} \frac{\int_\Omega \left(\frac{\mathbb{D}^2(J)}{2\mathbf{N}} |\nabla \varphi|^2(\cdot) \,\mathrm{d}x\right)}{\int_\Omega \beta(\cdot) \psi^2(\cdot) \,\mathrm{d}x} + \frac{\int_\Omega \alpha(\cdot) \psi^2(\cdot) \,\mathrm{d}x}{\int_\Omega \beta(\cdot) \psi^2(\cdot) \,\mathrm{d}x}$$

since $\mathcal{R}_0 > 1$, it then follows from (5.1) and (5.6), we have that

$$||S||_{L^{\infty}} \le \mathbf{C}^{1}_{*}, \; ||I||_{L^{\infty}} \le \mathbf{C}^{2}_{*}, \; ||R||_{L^{\infty}} \le \mathbf{C}^{3}_{*}, \tag{6.1}$$

and we set the following functions $\mathbf{f}(\cdot) = -\frac{\beta(\cdot)SI}{S+I+R} + \gamma(\cdot)R$, on Ω , and $\mathbf{g}(\cdot) = \frac{\beta(\cdot)SI}{S+I+R} - \alpha(\cdot)I$, on Ω , $\mathbf{h} = \alpha(\cdot)I - \gamma(\cdot)R$, on Ω , by (6.1) and since β, α, γ are a Hölder continuous, then we obtain

$$||\mathbf{f}||_{L^{\infty}(\Omega)} \leq \mathbf{C}^{3}_{*}, ||\mathbf{g}||_{L^{\infty}(\Omega)} \leq \mathbf{C}^{4}_{*}, ||\mathbf{h}||_{L^{\infty}(\Omega)} \leq \mathbf{C}^{5}_{*}$$

$$(6.2)$$

where $\mathbf{C}^3_*, \mathbf{C}^4_*, \mathbf{C}^5_*$ are a positive constants.

By multiplying the first, second and third equations of (5.1) by S_n, I_n, R_n , respectively, we get that

$$\frac{d_1 \mathbb{D}_2(\mathcal{L}_{\sigma})}{2\sigma_m^n} \int_{\Omega} \int_{\Omega} J_{\sigma_n}(x-y) \left[S_n(\cdot) - S_n(y) \right]^2 \, \mathrm{d}y \, \mathrm{d}x = \int_{\Omega} S_n(\cdot) \mathbf{f}_n(\cdot) \, \mathrm{d}x,$$

$$\frac{d_2 \mathbb{D}_2(\mathcal{L}_{\sigma})}{2\sigma_m^n} \int_{\Omega} \int_{\Omega} J_{\sigma_n}(x-y) \left[I_n(\cdot) - I_n(y) \right]^2 \, \mathrm{d}y \, \mathrm{d}x = \int_{\Omega} I_n(\cdot) \mathbf{g}_n(\cdot) \, \mathrm{d}x.$$

$$\frac{d_3 \mathbb{D}_2(\mathcal{L}_{\sigma})}{2\sigma_m^n} \int_{\Omega} \int_{\Omega} \int_{\Omega} J_{\sigma_n}(x-y) \left[R_n(\cdot) - R_n(y) \right]^2 \, \mathrm{d}y \, \mathrm{d}x = \int_{\Omega} R_n(\cdot) \mathbf{h}_n(\cdot) \, \mathrm{d}x,$$
(6.3)

we have that $\zeta_{\sigma} = \frac{1}{\sigma^2 \mathbb{D}_2(\mathcal{L}_{\sigma})} J_{\sigma_n}(z) |z|^2$, by using (6.3)-(5.6) and (6.2), we obtain that

$$\begin{cases} \frac{\mathbb{D}(\mathcal{L}_{\sigma})}{2} \int_{\Omega} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{[S_{n}(\cdot) - S_{n}(y)]^{2}}{|z|^{2}} \, \mathrm{d}y \, \mathrm{d}x &\leq \frac{\sigma^{m-2}\mathbf{C}_{*}^{3}}{d_{1}}, \\ \frac{\mathbb{D}(\mathcal{L}_{\sigma})}{2} \int_{\Omega} \int_{\Omega} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{[I_{n}(\cdot) - I_{n}(y)]^{2}}{|z|^{2}} \, \mathrm{d}y \, \mathrm{d}x &\leq \frac{\sigma^{m-2}\mathbf{C}_{*}^{4}}{d_{2}}, \\ \frac{\mathbb{D}(\mathcal{L}_{\sigma})}{2} \int_{\Omega} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{[R_{n}(\cdot) - R_{n}(y)]^{2}}{|z|^{2}} \, \mathrm{d}y \, \mathrm{d}x &\leq \frac{\sigma^{m-2}\mathbf{C}_{*}^{5}}{d_{3}}, \end{cases}$$

since m = 2, we obtain that

$$\begin{cases} \frac{\mathbb{D}(\mathcal{L}_{\sigma})}{2} \int_{\Omega} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{[S_{n}(\cdot) - S_{n}(y)]^{2}}{|z|^{2}} \, \mathrm{d}y \, \mathrm{d}x &\leq \frac{\mathbf{C}_{*}^{3}}{d_{1}}, \\ \frac{\mathbb{D}(\mathcal{L}_{\sigma})}{2} \int_{\Omega} \int_{\Omega} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{[I_{n}(\cdot) - I_{n}(y)]^{2}}{|z|^{2}} \, \mathrm{d}y \, \mathrm{d}x &\leq \frac{\mathbf{C}_{*}^{4}}{d_{2}}. \\ \frac{\mathbb{D}(\mathcal{L}_{\sigma})}{2} \int_{\Omega} \int_{\Omega} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{[R_{n}(\cdot) - R_{n}(y)]^{2}}{|z|^{2}} \, \mathrm{d}y \, \mathrm{d}x &\leq \frac{\mathbf{C}_{*}^{5}}{d_{3}}. \end{cases}$$

Now, by employing the same arguments as Theorems 1.2 and 1.3 in [43], we find that a subsequence $(S_{n_k}, I_{n_k}, R_{n_k})$, such that

$$(S_{n_k}(\cdot), I_{n_k}(\cdot), R_{n_k}) \leftarrow (S^*(\cdot), I^*(\cdot), R^*(\cdot)), \in L^2(\Omega), \text{ as } k \to \infty,$$

where $(S^*(\cdot), I^*(\cdot), R^*(\cdot)) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$.

Next, we apply the Proposition 5.4 in [22], we obtain the following result

$$\frac{d_{1}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2} \int_{|z|<\delta} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{1}{|z|^{2}} S_{n}(\cdot) \left[\phi_{\mathcal{S}}(x-z) - 2\phi_{\mathcal{S}}(\cdot) + \phi_{\mathcal{S}}(x+z)\right] \mathrm{d}x \mathrm{d}z - \int_{\Omega} \phi_{\mathcal{S}}(\cdot) \mathbf{f}_{n}(\cdot) \mathrm{d}x = 0,$$

$$\frac{dID^{2}(J)}{2} \int_{|z|<\delta} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{1}{|z|^{2}} I_{n}(\cdot) \left[\phi_{\mathcal{I}}(x-z) - 2\phi_{\mathcal{I}}(\cdot) + \phi_{\mathcal{I}}(x+z)\right] \mathrm{d}x \mathrm{d}z - \int_{\Omega} \mathbf{g}_{n}(\cdot) \varphi_{\mathcal{I}}(\cdot) \mathrm{d}x = 0,$$

$$\frac{d_{3}\mathbb{D}_{2}(J)}{2} \int_{|z|<\delta} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{1}{|z|^{2}} R_{n}(\cdot) \left[\phi_{\mathcal{R}}(x-z) - 2\phi_{\mathcal{R}}(\cdot) + \phi_{\mathcal{R}}(x+z)\right] \mathrm{d}x \mathrm{d}z - \int_{\Omega} \mathbf{h}_{n}(\cdot) \phi_{\mathcal{R}}(\cdot) \mathrm{d}x = 0,$$

with the functions \mathbf{f}_n , \mathbf{g}_n , and \mathbf{h}_n are as previously defined in equation (6.3). By adding and substraction the term $\zeta_{\sigma_n}(z) \frac{1}{|z|^2} S_n(\cdot) \left[z D^2 \phi_S z^T \right]$, with $D^2 = \{\partial_{ij}\}_{i,j}$, we find that

$$\frac{d_{1}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2} \int_{|z|<\delta} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{1}{|z|^{2}} S_{n}(\cdot) \left[zD^{2}\phi_{S}z^{T}\right] dxdz - \int_{\Omega} \mathbf{f}_{n}(\cdot)\phi_{S}(\cdot)dx$$

$$= -\frac{d_{1}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2} \int_{|z|<\delta} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{1}{|z|^{2}} S_{n}(\cdot) \left[\phi_{S}(x-z) - 2\phi_{S}(\cdot) + \phi_{S}(x+z) - zD^{2}\phi_{S}z^{T}\right] dxdz,$$

$$\frac{d_{2}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2} \int_{|z|<\delta} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{1}{|z|^{2}} I_{n}(\cdot) [zD^{2}\phi_{S}z^{T}] dxdz + \int_{\Omega} \mathbf{g}_{n}(\cdot)\phi_{\mathcal{I}}(\cdot)dx$$

$$= -\frac{d_{2}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2} \int_{|z|<\delta} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{1}{|z|^{2}} I_{n}(\cdot) [\phi_{\mathcal{I}}(x-z) - 2\phi_{\mathcal{I}}(\cdot) + \phi_{\mathcal{I}}(x+z) - zD^{2}\phi_{I}z^{T}] dxdz,$$

$$\frac{d_{3}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2} \int_{|z|<\delta} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{1}{|z|^{2}} I_{n}(\cdot) [zD^{2}\varphi_{I}z^{T}] dxdz + \int_{\Omega} \mathbf{h}_{n}(\cdot)\phi_{\mathcal{I}}(\cdot)dx$$

$$= -\frac{d_{3}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2} \int_{|z|<\delta} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{1}{|z|^{2}} R_{n}(\cdot) [\phi_{\mathcal{R}}(x-z) - 2\phi_{\mathcal{R}}(\cdot) + \phi_{\mathcal{R}}(x+z) - zD^{2}\phi_{R}z^{T}] dxdz,$$
(6.4)

since the function ζ_{σ_n} is radially symmetric, by apply the same approach as the proof of Theorem 1.4 in [22], we can rewrite the following formulas:

$$\frac{d_1 \mathbb{D}_2(\mathcal{L}_{\sigma})}{2} \int_{|z| < \delta} \int_{\Omega} \zeta_{\sigma_n}(z) \frac{1}{|z|^2} S_n(\cdot) [zD^2 \phi_S z^T] dx dz = \frac{d_1 \mathbb{D}_2(\mathcal{L}_{\sigma}) \mathbb{K}_{2,N}}{2} \int_{\Omega} S_n(\cdot) \Delta \phi_S(\cdot) dx,$$

$$\frac{d_2 \mathbb{D}_2(\mathcal{L}_{\sigma})}{2} \int_{|z| < \delta} \int_{\Omega} \zeta_{\sigma_n}(z) \frac{1}{|z|^2} I_n(\cdot) [zD^2 \varphi_I z^T] dx dz = \frac{d_2 \mathbb{D}_2(\mathcal{L}_{\sigma}) \mathbb{K}_{2,N}}{2} \int_{\Omega} I_n(\cdot) \Delta \phi_{\mathcal{I}}(\cdot) dx,$$

$$\frac{d_3 \mathbb{D}_2(\mathcal{L}_{\sigma})}{2} \int_{|z| < \delta} \int_{\Omega} \zeta_{\sigma_n}(z) \frac{1}{|z|^2} I_n(\cdot) [zD^2 \phi_R z^T] dx dz = \frac{d_3 \mathbb{D}_2(\mathcal{L}_{\sigma}) \mathbb{K}_{2,N}}{2} \int_{\Omega} R_n(\cdot) \Delta \phi_{\mathcal{R}}(\cdot) dx,$$
(6.5)

where $\mathbb{K}_{2,N} = \int_{S^{N-1}} (s \cdot e_1)^2 ds = \frac{1}{N}$. By substituting (6.5) into (6.4), we have

$$\begin{split} &\frac{d_{1}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2N}\int_{\Omega}S_{n}(\cdot)\Delta\phi_{\mathcal{S}}(\cdot)\mathrm{d}x + \int_{\Omega}\mathbf{f}_{n}(\cdot)\phi_{\mathcal{S}}(\cdot)\mathrm{d}x, \\ &= -\frac{d_{1}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2}\int_{|z|<\delta}\int_{\Omega}\zeta_{\sigma_{n}}(z)\frac{1}{|z|^{2}}S_{n}(\cdot)\left[\phi_{\mathcal{S}}(x-z) - 2\phi_{\mathcal{S}}(\cdot) + \phi_{\mathcal{S}}(x+z) - zD^{2}\phi_{\mathcal{S}}z^{T}\right]\mathrm{d}x\mathrm{d}z, \\ &\frac{d_{2}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2N}\int_{\Omega}I_{n}(\cdot)\Delta\varphi_{\mathcal{I}}(\cdot)\mathrm{d}x + \int_{\Omega}\mathbf{g}_{n}(\cdot)\phi_{\mathcal{I}}(\cdot)\mathrm{d}x, \\ &= -\frac{d_{2}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2}\int_{|z|<\delta}\int_{\Omega}\zeta_{\sigma_{n}}(z)\frac{1}{|z|^{2}}I_{n}(\cdot)\left[\phi_{\mathcal{I}}(x-z) - 2\phi_{\mathcal{I}}(\cdot) + \phi_{\mathcal{I}}(x+z) - zD^{2}\phi_{I}z^{T}\right]\mathrm{d}x\mathrm{d}z, \\ &\frac{d_{2}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2N}\int_{\Omega}R_{n}(\cdot)\Delta\phi_{\mathcal{R}}(\cdot)\mathrm{d}x + \int_{\Omega}\mathbf{h}_{n}(\cdot)\phi_{\mathcal{I}}(\cdot)\mathrm{d}x, \\ &= -\frac{d_{3}\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2}\int_{|z|<\delta}\int_{\Omega}\zeta_{\sigma_{n}}(z)\frac{1}{|z|^{2}}I_{n}(\cdot)\left[\phi_{\mathcal{I}}(x-z) - 2\phi_{\mathcal{I}}(\cdot) + \phi_{\mathcal{I}}(x+z) - zD^{2}\phi_{I}z^{T}\right]\mathrm{d}x\mathrm{d}z, \end{split}$$

given that $(S_n, I_n, R_n) \to (S^*(\cdot), I^*, R^*), \in L^2(\Omega)$, we obtain

$$\int_{\Omega} \mathbf{f}_{n}(\cdot)\phi_{\mathcal{S}}(\cdot)\mathrm{d}x \to \int_{\Omega} \mathbf{f}(\cdot)\phi_{\mathcal{S}}(\cdot)\mathrm{d}x = \int_{\Omega} -\left[\beta(\cdot)\frac{S^{*}I^{*}}{S^{*}+I^{*}+R^{*}} + \gamma(\cdot)R^{*}\right]\phi_{S}\mathrm{d}x,$$
$$\int_{\Omega} \mathbf{g}_{n}(\cdot)\varphi_{\mathcal{I}}(\cdot)\mathrm{d}x \to \int_{\Omega} \mathbf{g}(\cdot)\varphi_{\mathcal{I}}(\cdot)\mathrm{d}x = \int_{\Omega} \left[\beta(\cdot)\frac{S^{*}I^{*}}{S^{*}+I^{*}+R^{*}} - \alpha(\cdot)I^{*}\right]\phi_{I}\mathrm{d}x$$
$$\int_{\Omega} \mathbf{h}_{n}(\cdot)\varphi_{\mathcal{R}}(\cdot)\mathrm{d}x \to \int_{\Omega} \mathbf{h}(\cdot)\phi_{\mathcal{R}}(\cdot)\mathrm{d}x = \int_{\Omega} -\left[\alpha(\cdot)I^{*}-\gamma(\cdot)R^{*}\right]\phi_{R}\mathrm{d}x.$$

Since $\phi_S, \phi_I, \phi_R \in C_c^{\infty}(\Omega)$, there exist constants $\mathbf{C}(\phi_S)$, $\mathbf{C}(\phi_I)$, and $\mathbf{C}(\phi_R)$. We also define the characteristic functions $\infty_{\mathbf{B}(\phi_S)}, \infty_{\mathbf{B}(\phi_I)}$, and $\infty_{\mathbf{B}(\phi_R)}$ corresponding to the sets $\mathbf{B}(\phi_S)$, $\mathbf{B}(\phi_I)$, and $\mathbf{B}(\phi_R)$, respectively, such that the following inequalities are satisfied,

$$\begin{aligned} \left|\phi_{\mathcal{S}}(x-z) - 2\phi_{\mathcal{S}}(\cdot) + \phi_{\mathcal{S}}(x+z) - zD^{2}\phi_{S}z^{T}\right| &\leq \mathbf{C}(\phi_{S})|z|^{3}\infty_{\mathbf{B}(\phi_{S})}, \\ \left|\phi_{\mathcal{I}}(x-z) - 2\phi_{\mathcal{I}}(\cdot) + \phi_{\mathcal{I}}(x+z) - zD^{2}\phi_{I}z^{T}\right| &\leq \mathbf{C}(\phi_{I})|z|^{3}\infty_{\mathbf{B}(\phi_{S})}, \\ \left|\phi_{\mathcal{S}}(x-z) - 2\phi_{\mathcal{S}}(\cdot) + \phi_{\mathcal{S}}(x+z) - zD^{2}\phi_{S}z^{T}\right| &\leq \mathbf{C}(\phi_{R})|z|^{3}\infty_{\mathbf{B}(\phi_{R})}, \end{aligned}$$
(6.6)

we have that $S_n(\cdot), I_n(\cdot), R_n(\cdot)$ are uniformly bounded with respect to n, it then follows as

$$\begin{aligned} &\left| \frac{d_1 \mathcal{D}_2(\mathcal{L}_{\sigma})}{2} \int_{|z| < \delta} \int_{\Omega} \zeta \sigma_n(z) |z|^2 S_n(\cdot) \left[\phi_{\mathcal{S}}(x-z) - 2\phi_{\mathcal{S}}(\cdot) + \phi_{\mathcal{S}}(x+z) - z D^2 \phi_{\mathcal{S}} z^T \right] \mathrm{d}x \, \mathrm{d}z \\ &\leq \|S_n\|_{L^{\infty}(\Omega)} \mathbf{C}(\phi_{\mathcal{S}}) \int_{\Omega} \zeta_{\sigma_n}(z) |z| \, \mathrm{d}z \\ &\leq \mathbf{C}_1 \sigma_n, \end{aligned} \end{aligned}$$

 $\quad \text{and} \quad$

$$\begin{aligned} &\left| \frac{d_2 \mathcal{D}_2(\mathcal{L}_{\sigma})}{2} \int_{|z| < \delta} \int_{\Omega} \rho \sigma_n(z) |z|^2 \overline{I}_n(\cdot) \left[\phi_{\mathcal{I}}(x-z) - 2\phi_{\mathcal{I}}(\cdot) + \phi_{\mathcal{I}}(x+z) - z D^2 \phi_I z^T \right] \mathrm{d}x \, \mathrm{d}z \\ &\leq \|I_n\|_{L^{\infty}(\Omega)} \mathbf{C}(\phi_I) \int_{\Omega} \zeta_{\sigma_n}(z) |z| \, \mathrm{d}z \\ &\leq \mathbf{C}_2 \sigma_n, \end{aligned}$$

and

$$\left| \frac{d_{3}\mathcal{D}_{2}(\mathcal{L}_{\sigma})}{2} \int_{|z|<\delta} \int_{\Omega} \zeta_{\sigma_{n}}(z) |z|^{2} R_{n}(\cdot) \left[\phi_{\mathcal{R}}(x-z) - 2\phi_{\mathcal{R}}(\cdot) + \phi_{\mathcal{R}}(x+z) - zD^{2}\phi_{I}z^{T} \right] dx dz \right|$$

$$\leq \|R_{n}\|_{L^{\infty}(\Omega)} \mathbf{C}(\phi_{R}) \int_{\Omega} \zeta_{\sigma_{n}}(z) |z| dz$$

$$\leq \mathbf{C}_{3}\sigma_{n}, \qquad (6.7)$$

as $n \to \infty$, we obtain $\sigma_n \to 0$, which leads to the following system of equations,

$$\frac{d_{1}\mathbb{D}(\mathcal{L}_{\sigma})}{2\mathbf{N}} \int_{\Omega} \mathcal{S}(\cdot)\Delta\phi_{\mathcal{S}}(\cdot) \,\mathrm{d}x + \int_{\Omega} \mathbf{f}_{n}(\cdot)\phi_{\mathcal{S}}(\cdot) \,\mathrm{d}x = 0,$$

$$\frac{d_{2}\mathbb{D}(\mathcal{L}_{\sigma})}{2\mathbf{N}} \int_{\Omega} \mathcal{I}(\cdot)\Delta\phi_{\mathcal{I}}(\cdot) \,\mathrm{d}x + \int_{\Omega} \mathbf{g}_{n}(\cdot)\varphi_{\mathcal{I}}(\cdot) \,\mathrm{d}x = 0,$$

$$\frac{d_{3}\mathbb{D}(\mathcal{L}_{\sigma})}{2\mathbf{N}} \int_{\Omega} \mathcal{I}(\cdot)\Delta\phi_{\mathcal{R}}(\cdot) \,\mathrm{d}x + \int_{\Omega} \mathbf{h}_{n}(\cdot)\varphi_{\mathcal{I}}(\cdot) \,\mathrm{d}x = 0,$$
(6.8)

which implies that

$$\frac{d_{1}\mathbb{D}(\mathcal{L}_{\sigma})}{2\mathbf{N}}\Delta S^{*}(\cdot) - \beta(\cdot)\frac{S^{*}(\cdot)I^{*}(\cdot)}{S^{*}(\cdot) + I^{*}(\cdot) + R^{*}(\cdot)} + \gamma(\cdot)R^{*}(\cdot) = 0, \text{ on } \in \Omega,$$

$$\frac{d_{2}\mathbb{D}(\mathcal{L}_{\sigma})}{2\mathbf{N}}\Delta I^{*}(\cdot) + \beta(\cdot)\frac{S^{*}(\cdot)I^{*}(\cdot)}{S^{*}(\cdot) + I^{*}(\cdot) + R^{*}(\cdot)} - \alpha(\cdot)\mathcal{I}(\cdot) = 0, \text{ on } \in \Omega,$$

$$\frac{d_{3}\mathbb{D}(\mathcal{L}_{\sigma})}{2\mathbf{N}}\Delta R^{*}(\cdot) - \gamma(\cdot)R^{*}(\cdot) + \alpha(\cdot)\mathcal{I}(\cdot) = 0, \text{ on } \in \Omega,$$
(6.9)

with Neumann conditions:

$$\frac{\partial S^*}{\partial n} = \frac{\partial I^*}{\partial n} = \frac{\partial R^*}{\partial n} = 0, \quad \text{on } \in \partial \Omega.$$

Finally, to show the regularity of the PEES $(S^*(\cdot), I^*(\cdot), R^*(\cdot))$, we have that these functions are already known to be bounded. By invoking elliptic regularity theory, we conclude that $(S^*(\cdot), I^*(\cdot), R^*(\cdot))$ are in fact smooth. Moreover, since $\sigma \to 0$ (with σ being arbitrary), it follows that

$$\left(\mathcal{S}(\cdot), \mathcal{I}(\cdot), \mathcal{R}(\cdot)\right) \to \left(S^*(\cdot), I^*(\cdot), R^*(\cdot)\right).$$

The following theorem investigates the asymptotic behavior of model (5.1) for m > 2.

Theorem 6.2. Assume that $\mathbb{R}_0 > 1$ and $\int_{\Omega} \beta(\cdot) - \alpha(\cdot) dx > 0$. Additionally, suppose that m > 2 and the kernel \mathcal{L}_{σ} is radially symmetric. Then, the solution $(\mathcal{S}(\cdot), \mathcal{I}(\cdot), \mathcal{R}(\cdot))$ of model (5.1) converges to $(S^*(\cdot), I^*(\cdot), \mathbb{R}^*(\cdot))$ as $\sigma \to 0$ with $(S^*(\cdot), I^*(\cdot), \mathbb{R}^*(\cdot))$ verifies

$$\left(\frac{\mathbf{N}}{|\Omega|} - \frac{\mathbb{C}_2}{|\Omega|} \left(1 + \int_{\Omega} \frac{\alpha(\cdot)}{\gamma(\cdot)} \mathrm{d}x\right), \frac{\mathbf{N}}{|\Omega|} \frac{\int_{\Omega} (\beta(\cdot) - \alpha(\cdot)) \,\mathrm{d}x}{\left(1 + \int_{\Omega} \frac{\alpha(\cdot)}{\gamma(\cdot)} \mathrm{d}x\right) \left(\frac{\int_{\Omega} \beta(\cdot) \mathrm{d}x}{|\Omega|} + \int_{\Omega} \alpha(\cdot) \mathrm{d}x \left(1 - \frac{1}{\Omega}\right)\right)}, \int_{\Omega} \frac{\alpha(\cdot)}{\gamma(\cdot)} \mathrm{d}x \mathbb{C}_2\right).$$

Proof. Since $\mathbb{R}_0 > 1$, by using Theorem (5.1), there exists at least one PEES \tilde{E}^* of the model (5.1). Next, we consider (S_n, I_n, R_n) the solutions of the model (5.1), and we consider $\{\sigma_n\}$ a sequence of σ such that $\sigma_n \to 0^+$ as $n \to \infty$.

Using a similar approach as in (6.3), we obtain

$$\begin{cases} \frac{\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2} \int_{\Omega} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{[S_{n}(\cdot) - S_{n}(y)]^{2}}{|z|^{2}} \, \mathrm{d}y \, \mathrm{d}x &\leq \frac{\sigma^{m-2} \mathbf{C}_{*}^{3}}{d_{1}}, \\ \frac{\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2} \int_{\Omega} \int_{\Omega} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{[I_{n}(\cdot) - I_{n}(y)]^{2}}{|z|^{2}} \, \mathrm{d}y \, \mathrm{d}x &\leq \frac{\sigma^{m-2} \mathbf{C}_{*}^{4}}{d_{2}}. \\ \frac{\mathbb{D}_{2}(\mathcal{L}_{\sigma})}{2} \int_{\Omega} \int_{\Omega} \zeta_{\sigma_{n}}(z) \frac{[R_{n}(\cdot) - R_{n}(y)]^{2}}{|z|^{2}} \, \mathrm{d}y \, \mathrm{d}x &\leq \frac{\sigma^{m-2} \mathbf{C}_{*}^{5}}{d_{3}}, \end{cases}$$
(6.10)

where $\zeta_{\sigma_n} = \frac{1}{\sigma^2 \mathbb{D}_2(\mathcal{L}_{\sigma})} J_{\sigma_n}(z) |z|^2$. By applying the same reasoning as in Theorems 1.2 and 1.3 of [43], we conclude the existence of a subsequence $(S_{n_k}, I_{n_k}, R_{n_k})$ such that

$$(S_{n_k}(\cdot), I_{n_k}(\cdot), R_{n_k}(\cdot)) \to (S^*(\cdot), I^*(\cdot), R^*(\cdot)) \text{ in } L^2(\Omega), \quad \text{as } k \to \infty,$$

with $(S^*(\cdot), I^*(\cdot), R^*(\cdot)) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$. We use (6.10), we can conclude

$$\int_{\Omega} |\nabla S^*|^2 \,\mathrm{d}x \le \frac{\sigma^{m-2} \mathbf{C}_*^3}{d_1},$$

$$\int_{\Omega} |\nabla I^*|^2 \,\mathrm{d}x \le \frac{\sigma^{m-2} \mathbf{C}_*^4}{d_2},$$

$$\int_{\Omega} |\nabla R^*|^2 \,\mathrm{d}x \le \frac{\sigma^{m-2} \mathbf{C}_*^5}{d_3}.$$
(6.11)

Letting $\sigma \to 0^+$ in the inequalities (6.11), we obtain

$$\int_{\Omega} |\nabla S^*|^2 \, \mathrm{d}x = 0,$$
$$\int_{\Omega} |\nabla I^*|^2 \, \mathrm{d}x = 0,$$
$$\int_{\Omega} |\nabla I^*|^2 \, \mathrm{d}x = 0,$$

which implies that S^*, I^*, R^* are constants, we denoted these constants by $\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3$, respectively, we deduce that

$$S_n \to \mathbb{C}_1, \ I_n \to \mathbb{C}_2, \ R_n \to \mathbb{C}_3, \quad \text{in } L^2(\Omega).$$
 (6.12)

Next, it is essential to show that $\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3$ are positive. The verification process is organized into distinct cases, outlined as follows:

Case 1: In this scenario, we assume that the constants are all zero, implying $\mathbb{C}_1 = \mathbb{C}_2 = \mathbb{C}_3 = 0$, which directly contradicts equation (5.3).

Case 2: We suppose that $\mathbb{C}_1 > 0$, $\mathbb{C}_2 = \mathbb{C}_3 = 0$, from the second equation of (5.1), we define \hat{I} as

$$\hat{I}_n := \frac{I_n}{||I_n||_{\infty}},$$

it is easy to get that $||\hat{I}_n||_{\infty} = 1, \forall n \in \mathbb{N}$, and \hat{I}_n satisfies the equation

$$\frac{d_2}{\sigma^m} \int_{\Omega} \mathcal{L}_{\sigma}(x-y) \left[\hat{I}_n(y) - \hat{I}_n(\cdot) \right] dy + \beta(\cdot) \frac{S_n(\cdot) \hat{I}_n(\cdot)}{S_n(\cdot) + I_n(\cdot) + R_n(\cdot)} - \alpha(\cdot) \hat{I}_n(\cdot) = 0, \text{ on } \in \Omega,$$
(6.13)

by using a method analogous to Theorems 1.2 and 1.3 of [43], it can be shown that $\lim_{n\to\infty} \hat{I}_n \to 1$.

By integrating both sides of (6.13) over Ω and taking the limit as $n \to \infty$ we can obtain $\int_{\Omega} \beta(\cdot) dx =$ $\int_{\Omega} \gamma(\cdot) \, \mathrm{d}x$, which gives a contradiction.

^{*J*_Ω}**Case 3:** In this scenario, we assume that $\mathbb{C}_1 > 0$, $\mathbb{C}_2 > 0$, $\mathbb{C}_3 = 0$, by intergrading the third equation of (5.1), we get that $\int_{\Omega} \alpha(\cdot) dx = 0$ which implies a contradiction.

Case 4: We assume that $\mathbb{C}_1 = \mathbb{C}_2 = 0$, $\mathbb{C}_3 > 0$, we integrate the first equation of (5.1), we obtain that $\int_{\Omega} \gamma(\cdot) dx = 0 \text{ which is also a contradiction.}$ Case 5: By assuming that $\mathbb{C}_1 = \mathbb{C}_3 = 0$, $\mathbb{C}_2 > 0$, by using also the second equation we obtain that

 $-\int_{\Omega} \alpha(\cdot) dx = 0$, which gives also a contradiction. **Case 6:** in this case, we assume that $\mathbb{C}_1 > 0$, $\mathbb{C}_2 > 0$, $\mathbb{C}_3 > 0$, by using the first-second and third equations of (5.1), we obtain that

$$\begin{cases} \int_{\Omega} (\mathbb{C}_{1} + \mathbb{C}_{2} + \mathbb{C}_{3}) &= \mathbf{N}, \\ \frac{\mathbb{C}_{1}\mathbb{C}_{2} \int_{\Omega} \beta(\cdot) \mathrm{d}x}{\mathbb{C}_{1} + \mathbb{C}_{2} + \mathbb{C}_{3}} &= \mathbb{C}_{2} \int_{\Omega} \alpha(\cdot) \mathrm{d}x, \\ \mathbb{C}_{2} \int_{\Omega} \alpha(\cdot) \mathrm{d}x &= \mathbb{C}_{3} \int_{\Omega} \gamma(\cdot) \mathrm{d}x, \end{cases}$$
(6.14)

which implies

$$\begin{cases} \mathbb{C}_1 &= \frac{\mathbf{N}}{|\Omega|} - \frac{\mathbb{C}_2}{|\Omega|} \left(1 + \int_{\Omega} \frac{\alpha(\cdot)}{\gamma(\cdot)} dx \right), \\ \mathbb{C}_2 &= \frac{\mathbf{N}}{|\Omega|} \frac{\int_{\Omega} (\beta(\cdot) - \alpha(\cdot)) dx}{\left(1 + \int_{\Omega} \frac{\alpha(\cdot)}{\gamma(\cdot)} dx \right) \left(\frac{\int_{\Omega} \beta(\cdot) dx}{|\Omega|} + \int_{\Omega} \alpha(\cdot) dx \left(1 - \frac{1}{\Omega} \right) \right)}, \\ \mathbb{C}_3 &= \int_{\Omega} \frac{\alpha(\cdot)}{\gamma(\cdot)} dx \mathbb{C}_2. \end{cases}$$

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7. Discussion and conclusion

In this work, we have established a SIRS epidemic model incorporating both nonlinear incidence which write as $\frac{\beta(\cdot)SI}{S+I+R}$ and the nonlocal diffusion with a scaling factor σ and a cost parameter m. The principal results outcomes are abstracted in the following points:

- Firstly, we have proved the existence-uniqueness of the solutions of (1.2) by constructing (1.2) as a Cauchy system. Furthermore, we employ the semi-group theories to demonstrate the global existence of solutions.
- We have shown \mathbb{R}_0 . By applying the same methodology akin to [3], we can get the same formula of (3.8) which have a relationship the principal eigenvalue λ^* , the difficult task in this model is to show that model (1.2) has an unique PEES. Specifically, if $\mathbb{R}_0 < 1$, then we conclude that the infectious is predicted to be extinct; otherwise, when $\mathbb{R}_0 > 1$, then the epidemic persists.
- We have showed the persistence of the model for $\mathbb{R}_0 > 1$, which can be very important in showing the existence of at least one PEES.
- The main objective of this paper was to provide the asymptotic dynamics of SIRS epidemic model when for a small σ and a nonlinear incidence rates.

The study of asymptotic profiles in epidemic models has garnered increasing attention in recent years (e.g., [3, 5, 17, 29]). The mobility of individuals plays a critical role in the spread of infectious diseases. Our analysis highlights that when the scaling parameter σ is very small, the coexistence of individuals is guaranteed, leading to the persistence of infections. Such findings have significant implications for mathematical modeling in epidemiology and other applications, including SIS epidemic model (see, for instance, [3, 5]).

In this work, we considered nonlinear incidence functions where the denominator is the total population, providing a more realistic representation of interactions between susceptible and infected individuals. Future research could extend this framework to incorporate generalized incidence functions of the form $\mathbf{f}(\mathcal{S}, \mathcal{I}, \mathcal{R})$. Exploring such extensions may yield new insights into the dynamics of epidemic models and offer a broader perspective on interaction mechanisms (see [5, 16, 44]).

Our results also emphasize the significant impact of nonlocal diffusion on the epidemic model (1.2). When the scaling factor σ is small, and the cost parameter m > 2, we identified high-risk regions defined by:

$$\left\{x\in\Omega:\int_\Omega(\beta(\cdot)-\alpha(\cdot))\,\mathrm{d} x>0\right\},$$

where the basic reproduction number $\mathbb{R}_0 > 1$. In such regions, the spread of the epidemic poses a severe threat, as susceptible individuals entering these areas are likely to contribute to further disease transmission. Furthermore, the model predicts that recovered individuals may lose their immunity, presenting additional challenges for disease control.

To mitigate these risks, it is crucial for governments to identify and control these high-risk regions. Measures such as restricting the movement of susceptible and recovered individuals into these areas through confinement, combined with public awareness campaigns *via* media, can be effective in reducing the spread of the epidemic. These strategies are vital for minimizing potential harm and ensuring public health safety.

It is worth emphasizing that the diffusion patterns of individuals are not uniform, and the scaling factors for dispersal kernels vary among different groups. It is more logical and reasonable to consider distinct dispersal kernels and scaling factors for different individuals, as this approach better reflects natural movement behaviors. This modeling perspective is both insightful and practical, offering a more realistic description of individual movement. In the future, exploring such differentiated movement dynamics could provide valuable insights and open new avenues for research.

For $\mathbb{R}_0 < 1$, we have demonstrated that the infection-free equilibrium (IFE) is globally asymptotically stable, indicating the eventual elimination of the epidemic. This aligns with the results obtained by Allen *et al.* [1] in the context of models involving local diffusion. However, our framework incorporates reinfection and spatial heterogeneity, which strengthens the conclusion that uniform intervention strategies remain effective across the spatial domain Ω . The proposed model broadens classical SIRS-type formulations by introducing nonlocal dispersal, spatial variability in parameters, and a nonlinear transmission term. This allows for a more nuanced representation of epidemic behavior in heterogeneous environments. Our results on the role of dispersal σ contribute new perspectives to the existing body of work. Notably, the analysis related to the dispersal scale provides important insights relevant to public health. In comparison with models based on local diffusion [1, 13– 15], metapopulation frameworks [45], and spatially homogeneous nonlocal systems [3], our study captures richer spatial effects that are particularly pertinent for emerging infectious diseases. From a practical standpoint, the result that $\mathbb{R}_0 < 1$ ensures disease elimination is encouraging. At the same time, the possibility of persistence for small dispersal values σ points to the threat of localized outbreaks.

These observations are consistent with previous studies such as [13, 14], which highlighted the importance of spatial factors in epidemic control, and Pan *et al.* [15], who examined the SIRS model in a heterogeneous setting with logistic growth. Overall, our work emphasizes the importance of large-scale public health actions, such as travel restrictions, in preventing the spatial propagation of infection.

In our research, we have constructed the asymptotic behavior of the model (1.2) based on the notion that the kernel scaling σ is very small which represents to a plan of dispersing many offspring on a local range, which is considered to be quite significant. We will, however, demonstrate the behavior for scaling in the future, which is quite enormous which considered to maximize the analysis of the environment at the expense of the number of hers dispersed. The asymptotic behavior of the model and the potential impact of environmental exploration on the number of scattered offspring may be demonstrated, in fact, by employing the identical arguments developed in the cited work [21]. Age structure has a significant effect in the influence of infectious diseases (see, for example, [44, 46]) on the model's behavior; we will investigate this strategy further in the future.

Our SIRS model with nonlocal diffusion and spatial heterogeneity establishes a robust framework for understanding infectious disease dynamics, as demonstrated by our results on well-posedness, global attractors, and the role of dispersal parameters. To further enhance the realism of our model, an future work is to incorporate **memory effects** to describe how individuals' knowledge of a disease effect its spread. As individuals acquire information about past infectious (*e.g.*, through media or personal experience), their behaviors, such as social distancing or vaccination uptake, may depend on historical disease states. Replacing the classical time derivatives in (1.2) with the **Hattaf mixed fractional derivative** or **fractal-fractional derivative**, as introduced in [47, 48], could model these memory effects. These advanced operators show the temporal nonlocality and fractal structures, potentially revealing delayed or persistent epidemic dynamics driven by biological memory. Such an extension would complement our nonlocal spatial diffusion, offering deeper insights into diseases like COVID-19, where public awareness significantly transmission. We aim to pursue these fractional approaches in future work strengthen the model's utility for public health strategies.

DATA AVAILABILITY STATEMENT

No new data/codes were created or analyzed in this study.

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