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STABILITY ANALYSIS OF A STOCHASTIC UNEMPLOYMENT MODEL

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Abstract. This paper investigates the effect of stochastic perturbations on the deterministic UEV framework, which characterizes the problem of unemployment in poor countries. It examines how the system behaves and stays stable under random fluctuations. We study the existence and uniqueness of the non-negative solution. Furthermore, we examine the asymptotic behavior of this solution by analyzing the stability of the system at equilibrium points under some conditions. Finally, some numerical simulations are performed to verify the theoretical analysis using Matlab.

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1. Introduction

Unemployment refers to the percentage of the labor force that is jobless but looking for work. In an interview "*In Morocco, Youth Unemployment Is Driving Up Inequality*" Abdeslam Seddiki, Moroccan Minister of Labour and Social Affairs, discussed why creating jobs is a priority for the government. He stated: "Youth employment challenges are a global policy issue, but the situation is of serious concern in North Africa, which has one of the highest rates of youth unemployment in the world. Underemployment and job informality also affect young people.

In Morocco, four out of five unemployed people are aged 15 to 34. Although the unemployment rate has declined over the past decade, youth unemployment is still twice that of the total population.

There are variations according to gender, age, area of residence and education. Urban youth are more likely to be unemployed than rural youth. Girls and women are even worse off, even though Morocco is better than some neighbouring countries for female youth employment" [1].

The International Monetary Fund (IMF) World Economic Outlook predicts that Morocco's unemployment rate would average 10.7% in 2023, reflecting a slight decrease from 11.8% at the end of 2022. Accordingly, the analysis anticipates that employment in the North African nation will rebound following the hundreds of thousands of jobs lost as a result of the COVID-19-caused economic crisis.

Keywords and phrases: Brownien motion, stochastic unemployment model, asymptotic behavior, stochastic Lyapunov function.

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According to research published by Morocco's Higher Commission of Planning (HCP), the country's economy lost over 432 000 jobs in 2020 and 24 000 jobs in 2021 and 2022 [2].

Research involving modeling the unemployment problem mathematically has been investigated recently, in 2011 Misra and Singh [3] developed a nonlinear mathematical model that simulates the unemployment problem using three dynamic variables: the numbers of unemployed, temporarily employed, and regulary employed people. The same authors [4] in 2013, developed a mathematical model with delay for control by creating new jobs. Pathan and Bhathawala 2015 analyzed the effect of self-employment on the rate of unemployment, they considered the number of immigrants as a new variable [5]. In 2017, several researches interested in this problem are developed [6–8]. In [8], Pathan and Bhathawala extended their unemployment model to include four dynamic variables: the number of unemployed people, the number of migrant workers, the number of employed people, and the number of newly created vacancies by the government and private sectors. In 2018, Galindro and Torres [9] looked at a straightforward mathematical model that better captures real unemployment data from Portugal between 2004 and 2016. In 2020, the authors in [10] employed an expanded mathematical model to study the function that skill development plays in regulating unemployment and to illustrate the significance of both high and low-skilled individuals in the dynamics of unemployment. They looked at the contribution of highly qualified people with entrepreneurial abilities to the creation of jobs and the optimal strategies to reduce the unemployment rate. Al-maalwi, Ashi and Al-Sheikh [11] in 2021, investigated the impact of job scarcity on lowering the unemployment rate by modeling the circumstances in some nations where government assistance reaches a fixed point at which the rate of job creation ceases to be proportionate to the number of unemployed people owing to a lack of financial and economic resources. In 2022, El Yahyaoui and Amine in [12], presented and examined a brand new mathematical model of unemployment and examined it. There are two forms of unemployment that are relevant: cyclical and structural unemployment. It has been demonstrated that the suggested modeling technique can assist public authorities in simulating the impact of certain economic policies that aid in the recovery of those who are unemployed cyclically and keep them from becoming structurally unemployed.

In our 2023 work, we suggested and studied a mathematical model incorporating stochastic perturbation regarding unemployment by examining three primary factors. Specifically, those who have graduated or reached working age, recently unemployed people actively looking for work, salaried people and employees, and discouraged jobless people who have ceased actively looking for work because of unsatisfactory job prospects or other barriers [13]. A delayed mathematical model of unemployment with four dynamic variables has been created and studied by Rajpal, Bhatia, Goel, Kumar [14]. The population is divided into three classes, represented by three dynamic variables: the employed class, the jobless class, and the retired class. Additionally, a fourth dynamic variable in the model is taken into consideration for newly opened positions. The goal of the study [15] is to use truncated Spline nonparametric regression to assess the characteristics and identify the best model for the open unemployment rate (UR) in South Sulawesi Province. The Spline truncated technique, a model developed from nonparametric regression that can estimate data wherever the data pattern changes, was used in this study to measure the labor force participation rate, the percentage of individuals living in poverty, and the average duration of education. The goal of our study [16] is to look into the connection between internet freelancing and unemployment. Online freelancing can therefore lead to the creation of jobs in poor countries. In order to explain the dynamics of the unemployment issue with an online freelancing intervention, we proposed a mathematical model. The model suggests an optimal control problem and numerous reasonable and suitable control strategies.

This work is seen as a continuation of these earlier investigations and works. The researchers of [17] concentrated on modeling the issue of unemployment in poor countries using a nonlinear mathematical model. Unfortunately, the shortage of financial resources in these poor countries has limited the number of jobs available. By considering the system of differential equations which considers three dynamic variables: the proportions of unemployment, employees and available vacancies, and consider the following

UEV model:

$$\begin{cases} \dfrac{\mathrm{d}U(t)}{\mathrm{d}t} &= A - kU(t)V(t) + \beta E(t) - \mu U(t), \\ \dfrac{\mathrm{d}E(t)}{\mathrm{d}t} &= kU(t)V(t) - \beta E(t) - \alpha E(t) - \mu E(t), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} &= \alpha E(t) - \delta V(t). \end{cases} \tag{1.1}$$

The parameters (all positive constants) have the following meaning: A is the rate at which the percentage of the unemployed is rising, k is the rate at which the percentage of jobless individuals who find employment changes, the percentage of employed individuals who quit, or were fired from their jobs is β, both the migration rate and the death rate of the unemployed are represented by μ, α is the rate at which people migrate, retire, or pass away while working and δ is the decline of available vacancies as a result of a shortage of government funding. This examines a compartmental model that is almost identical to epidemiological systems, in which the components of disease propagation are reflected in the unemployment factors, which are modeled by interaction-driven motions. While outpourings $(\alpha E, \delta V)$ parallel recovery and expulsion forms, the frequency of work kUV refers to work coordination between unemployed persons (U) and vacancies (V), in comparison to contamination rates in disease models. We draw attention to this methodological decision because it is systematically tractable and effectively captures nonlinear labor showcase components (such as competition for jobs). The connection provides an accepted method for modeling flow-based frameworks, with parameters like k calibrated to experimental enlistment information to ensure authenticity, although it abstracts from human decision-making. It seeks to find a threshold parameter known as the basic reproduction number that determines the stability of equilibrium. In system (1.1) the threshold is $R_0 = \dfrac{\alpha k A}{\mu\delta(\alpha+\beta)}$. It always has an employment free equilibrium $Q_0 = \left(\dfrac{A}{\mu}, 0, 0\right)$. When $R_0 < 1$, the employment free equilibruim Q_0 is globally asymptotically stable, and therefore, when $R_0 > 1$, Q_0 looses its stability and there is a positive equilibrium $Q^* = (U^*, E^*, V^*) = \left(\dfrac{A}{\mu R_0}, \dfrac{A}{\alpha}\left[1 - \dfrac{1}{R_0}\right], \dfrac{A}{\delta}\left[1 - \dfrac{1}{R_0}\right]\right)$ which is globally asymptotically stable.

Incorporating stochastic effects into the model enables us to describe unemployment dynamics more realistically, as most real-world situations are not deterministic and are therefore characterized by unpredictability and stochasticity. Random effects can be incorporated into deterministic models in a variety of ways. Here, we primarily discuss two methods [18]. The first is to substitute the corresponding stochastic counterparts for one or more of the deterministic models' parameters [19, 20]. Making positive, robust equilibria of deterministic models is the second method [21, 22]. The first option is what we do in this study, such that Model (1.2) introduces the random perturbation as an extension of system (1.1) by substituting the parameter λ, where $\dot{B}$ is the white noise, specifically $B(t)$ is a one-dimensional standard, and $\lambda \to \lambda + \sigma\dot{B}$, $B(0) = 0$ is the Brownian motion, and σ is the intensity of the white noise. The model takes the following form:

$$\begin{cases} \mathrm{d}U(t) &= [A - kU(t)V(t) + \beta E(t) - \mu U(t)]\mathrm{d}t - \sigma U(t)E(t)\mathrm{d}B(t), \\ \mathrm{d}E(t) &= [kU(t)V(t) - \beta E(t) - \alpha E(t) - \mu E(t)]\mathrm{d}t + \sigma U(t)E(t)\mathrm{d}B(t), \\ \mathrm{d}V(t) &= [\alpha E(t) - \delta V(t)]\mathrm{d}t. \end{cases} \tag{1.2}$$

The reason given is that the process of moving from unemployment to employment naturally involves randomness and is influenced by how these two groups interact; the noise term represents random fluctuations in the economy, like quick shifts in business confidence, new policy changes, or outside events. The impact of these changes depends on the size of the groups looking for work (U) and those seeking to hire (E), meaning economic uncertainty has a bigger effect when there are many workers and many job openings, and a smaller effect when each group is smaller.

This article is organized as follows. In Section 2, the global existence and positivity of the solution to system (1.2) are investgate. We discuss the behavior of Q_0, and around Q^* in Section 3. In Section 4 numerical simulations are presented to illustrate our results. In Section 5, we conclude the paper.

2. Existence and uniqueness of the nonnegative solution

2.1. Preliminary

Let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a fitration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual conditions ($i.e$ it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Also let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n, x_i > 0$ for all $1 \leq i \leq n\}$ and $x(t) = (U(t), E(t), V(t))^T$.

The auxiliary statements that are introduced in [23] are displayed below.

Examine the stochastic differential equation in d dimensions.

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t), t \geq t_0. \tag{2.1}$$

Denote by $C^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$ the family of all nonnegative functions $\mathcal{V}(x,t)$ defined on $\mathbb{R}^d \times [t_0, \infty]$ such that they are continuously twice differentiable in `x` and once in `t`.

The differential operator $\mathcal{L}$ of equation (2.1) is defined [23] by formula

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d}[g(x,t)g^T(x,t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_i}.$$

If $\mathcal{L}$ acts on a function $\mathcal{V} \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$, then

$$\mathcal{L}\mathcal{V}(x,t) = \mathcal{V}_t(x,t) + \mathcal{V}_x(x,t)f(x,t) + \frac{1}{2}trace[g^T(x,t)\mathcal{V}_{xx}(x,t)g(x,t)],$$

where $\mathcal{V}_t = \dfrac{\partial \mathcal{V}}{\partial t}$, $\mathcal{V}_x = \left(\dfrac{\partial \mathcal{V}}{\partial x_1}, \ldots, \dfrac{\partial \mathcal{V}}{\partial x_d}\right)$ and $\mathcal{V}_{xx} = \left(\dfrac{\partial^2}{\partial x_i \partial x_i}\right)_{d\times d}$. By Itô's formula,

$$\mathrm{d}\mathcal{V}(\mathrm{x(t),t}) = \mathcal{L}\mathcal{V}(\mathrm{x(t),t})\mathrm{dt} + \mathcal{V}_{\mathrm{x}}(\mathrm{x(t),t})\mathrm{g(x(t),t)dB(t)}.$$

2.2. Existence of unique positive solution

Since $U(t)$, $E(t)$ and $V(t)$ denote the proportions of the unemployed, employed people and available vacancies, it is reasonable to consider the solution of the stochastic model (1.2) in the set $\Omega = \left\{U(t), E(t), V(t) \in \mathbb{R}^3_+ : 0 \leq U(t) + E(t) \leq \dfrac{A}{\mu}, 0 \leq V(t) \leq \dfrac{\alpha A}{\mu\delta}\right\}$.

Therefore, we must demonstrate that the stochastic model (1.2) has a unique global solution and that the solution will always stay inside Ω whenever it begins there in order for it to make sense.

Theorem 2.1. *For any initial value* $(U_0, E_0, V_0) \in \Omega$*, there is a unique solution* $(U(t), E(t), V(t)) \in \Omega$ *of system* (1.2) *on* $t \geq 0$*, with probability 1, that is*

$$\mathbb{P}\{U(t), E(t), V(t) \in \Omega, \forall t \geq 0\} = 1.$$

Proof. For any initial value $(U_0, E_0, V_0) \in \Omega$, the equation's coefficients are locally Lipschitz continuous, hence there is unique local solution $(U(t), E(t), V(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [24]. We must demonstrate that $\tau_e = \infty$ a.s. in order to demonstrate that this solution is global. Let $\kappa_0 \geq 1$ be sufficiently large, for $U(0)$, $E(0)$ and $V(0)$ lying with the interval $\left[\frac{1}{\kappa_0}, \kappa_0\right]$, for each integer $\kappa \geq \kappa_0$ define the stop-time as:

$$\tau_\kappa = \inf\{t \in [0, \tau_e] : \min\{U(t), E(t), V(t)\} \leq \frac{1}{\kappa} \text{ or } \max\{U(t), E(t), V(t)\} \geq \kappa\}.$$

Setting $\inf \emptyset = \infty$ ($\emptyset$ represents the empty set), τ_κ is increasing as $\kappa \to \infty$, set $\tau_\infty = \lim_{\kappa\to\infty} \tau_\kappa$, when $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s, then $\tau_e = \infty$ and $(U(t), E(t), V(t)) \in \Omega$ for all $t \geq 0$, to complete the proof all we need to show is that $\tau_\infty = \infty$.

If there is a violation of this statement, there exists a constant $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon.$$

Consequently, there exists an integer $\kappa_1 \geq \kappa_0$ such that:

$$\mathbb{P}\{\tau_\kappa \leq T\} \geq \epsilon, \kappa \geq \kappa_1. \tag{2.2}$$

Define a C^2-function $\mathcal{V} : \mathbb{R}^3_+ \to \mathbb{R}_+$ by

$$\mathcal{V}(U, E, V) = (U - 1 - logU) + (E - 1 - logE) + (V - 1 - logV).$$

This function is nonnegative, as shown by $s - 1 - logs \geq 0$, $\forall s > 0$.

Applying the formula for Itô, we obtain

$$\begin{aligned}
\mathrm{d}\mathcal{V} &= [\left(1 - \frac{1}{U}\right)(A - kUV + \beta E - \mu U) + \left(1 - \frac{1}{E}\right)(kUV - \beta E - \alpha E) \\
&\quad + \left(1 - \frac{1}{V}\right)(\alpha E - \delta V) + \frac{\sigma^2}{2}E^2 + \frac{\sigma^2}{2}U^2]\mathrm{d}t + \left(1 - \frac{1}{U}\right)\sigma UE\mathrm{d}B(t) \\
&\quad + \left(1 - \frac{1}{E}\right)\sigma UE\mathrm{d}B(t) \\
&= (A + kV + \mu + \beta + \alpha + \delta - \frac{1}{U}(A + \beta E) - \frac{kUV}{E} - \delta V - \frac{\alpha E}{V} \\
&\quad + \frac{\sigma^2}{2}E^2 + \frac{\sigma^2}{2}U^2)\mathrm{d}t + \sigma((E - 1)U - E(U - 1))\mathrm{d}B(t) \\
&\leq \left(A + kV + \mu + \beta + \alpha + \delta + \sigma^2\frac{E^2 + U^2}{2}\right)\mathrm{d}t + \sigma(E - U)\mathrm{d}B(t) \\
&\leq \left(A + k\frac{\alpha A}{\mu\delta} + \mu + \beta + \alpha + \delta + \sigma^2\left(\frac{A}{\mu}\right)^2\right)\mathrm{d}t + \sigma(E - U)\mathrm{d}B(t) \\
&:= C\mathrm{d}t + \sigma(E - U)\mathrm{d}B(t),
\end{aligned}$$

where $C = A + k\frac{\alpha A}{\mu\delta} + \mu + \beta + \alpha + \delta + \sigma^2\left(\frac{A}{\mu}\right)^2$.

Consequently, if $t_1 \leq T$,

$$\int_0^{\tau_\kappa\wedge t_1} \mathrm{d}\mathcal{V}(U(t), E(t), V(t)) \leq \int_0^{\tau_\kappa\wedge t_1} C\mathrm{d}t + \int_0^{\tau_\kappa\wedge t_1} \sigma[U(t)(E(t) - 1) - E(t)(U(t) - 1)]\mathrm{d}B(t).$$

This suggests that,

$$\mathbb{E}[\mathcal{V}(U(\tau_\kappa \wedge t_1), E(\tau_\kappa \wedge t_1), V(\tau_\kappa \wedge t_1))] \le \mathcal{V}(U_0, E_0, V_0) + \mathbb{E}\int_0^{\tau_\kappa \wedge t_1} C \mathrm{d}t$$
$$\le \mathcal{V}(U_0, E_0, V_0) + CT. \tag{2.3}$$

Set $\psi_\kappa = \{\tau_\kappa \le T\}$ for $\kappa \ge \kappa_1$ and by (2.2), $\mathbb{P}(\psi_\kappa) \ge \epsilon$.
For every $\psi \in \psi_\kappa$, there is at least one of $U(\tau_\kappa, \psi)$, $E(\tau_\kappa, \psi)$ and $V(\tau_\kappa, \psi)$ equals either κ or $\frac{1}{\kappa}$; and hence, $\mathcal{V}(U(\tau_\kappa, \psi), E(\tau_\kappa, \psi), V(\tau_\kappa, \psi))$ is no less than

$$\kappa - 1 - log\kappa \,\text{or}\, \frac{1}{\kappa} - 1 - log\frac{1}{\kappa} = \frac{1}{\kappa} - 1 + log\kappa.$$

Consequently

$$\mathcal{V}(U(\tau_\kappa, \psi), E(\tau_\kappa, \psi), V(\tau_\kappa, \psi)) \ge min\{\kappa - 1 - log\kappa, \frac{1}{\kappa} - 1 + log\kappa\}.$$

Next, get from (2.2) and (2.3)

$$\mathcal{V}(U(0), E(0), V(0)) + CT \ge \mathbb{E}\left[1_{\psi_\kappa}\mathcal{V}(U(\tau_\kappa, \psi), E(\tau_\kappa, \psi), V(\tau_\kappa, \psi))\right]$$
$$\ge \epsilon \min\{\kappa - 1 - log\kappa, \frac{1}{\kappa} - 1 + log\kappa\},$$

where 1_{ψ_κ} is the indicator function of ψ_κ.
Letting $\kappa \to \infty$ leads to the contradiction $\infty > \mathcal{V}(U_0, E_0, V_0) + CT = \infty$. So we must have $\tau_\infty = \infty$ a.s. □

3. Stability analysis

In this section, we will examine the system's dynamic behavior by examining its stability at equilibrium points Q_0 and Q^*. The presence of two equilibrium points is dependent on the basic reproduction number $R_0 = \dfrac{\alpha k A}{\mu\delta(\alpha + \beta)}$. The next-generation matrix approach is used to determine it [25], which is the average number of new job openings created by one existing job opening in a group of unemployed individuals. The stochastic reproduction number $R_1 = \dfrac{A^2\sigma^2 + kA\mu}{2\mu^2(\beta + \alpha)}$ is built through stochastic stability analysis; it is the average number of new jobs created by one person who is working at a time when the economy is unstable and job hiring happens unpredictably.

3.1. Asymptotic behavior of the employment free equilibrium

This section primarily uses the stochastic Lyapunov function to determine the stability of the employment free equilibrium Q_0. Consider equation (2.1), assume $f(0, t) = 0$ and $g(0, t) = 0$ for all $t \ge t_0$. So $x(t) \equiv 0$ is a solution to equation (2.1), called the equilibrium position. In terms of Lyapunov functions, the following theorems provide requirements for the stability of equation (2.1)'s equilibrium position.

Lemma 3.1. *[23] The trivial solution of equation (2.1) is stochastically asymptotically stable in the large if there is a positive-definite decrescent radially unbounded function $\mathcal{V}(x, t) \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty); \overline{\mathbb{R}}_+)$ such that $\mathcal{L}\mathcal{V}(x, t)$ is negative-definite.*

Lemma 3.2. *[23] Suppose that a function $\mathcal{V}(x,t) \in \mathbb{C}^{2,1}(\mathbb{R}^d \times [t_0,\infty)\overline{\mathbb{R}}_+)$ and positive constants b_1, b_2, b_3, exist to the extent that*

$$b_1|x|^p \leq \mathcal{V}(x,t) \leq b_2|x|^p \; and \, \mathcal{L}\mathcal{V}(x,t) \leq -b_3\mathcal{V}(x,t),$$

for all $(x,t) \in \mathbb{R}^d \times [t_0,\infty)$. Then the trivial solution of (2.1) is exponentially mean-square stable.

Theorem 3.3. *If $R_1 < 1$ and $\delta \geq bk\dfrac{A}{2\mu} + \dfrac{\alpha}{2}$, then the solution $\left(\dfrac{A}{\mu}, 0, 0\right)$ of system (1.2) is stochatically asymptotically stable in the large.*

Proof. Let $x = U - \dfrac{A}{\mu}$, $y = E$, $z = V$, then $x \leq 0$, $y \geq 0$, $z \geq 0$ and system (1.2) can be written as

$$\begin{cases} \mathrm{d}x(t) &= [-kxz - k\dfrac{A}{\mu}z + \beta y - \mu x]\mathrm{d}t - \sigma\left(x + \frac{A}{\mu}\right)y\mathrm{d}B(t), \\ dy(t) &= [kxz + k\frac{A}{\mu}z - \beta y - \alpha y]\mathrm{d}t + \sigma\left(x + \frac{A}{\mu}\right)ydB(t), \\ dz(t) &= [\alpha y - \delta z]\mathrm{d}t. \end{cases} \tag{3.1}$$

Define the stochastic Lypunov function $\mathbb{R}^3 \to \overline{\mathbb{R}}_+$:

$$\mathcal{V}(x,y,z) = (x+y)^2 + by^2 + z^2,$$

in which b is a positive constant that will be found later.
Since

$$\begin{aligned}(x+y)^2 + by^2 + z^2 &= x^2 + (1+b)y^2 + 2xy + z^2 \\ &\leq 2x^2 + (2+b)y^2 + z^2,\end{aligned}$$

and

$$\begin{aligned}(x+y)^2 + by^2 + z^2 &= x^2 + (1+b)y^2 + 2xy + z^2 \\ &= x^2 + (1+b)y^2 + 2\left(\frac{1}{\sqrt{1+b/2}}x\right)\left(\sqrt{1+b/2}y\right) + z^2 \\ &\geq x^2 + (1+b)y^2 - \frac{1}{1+b/2}x^2 - (1+b/2)\,y^2 + z^2 \\ &= \frac{b}{2+b}x^2 + \frac{b}{2}y^2 + z^2.\end{aligned}$$

Then $\mathcal{V}(x,y,z)$ is positive-definite, decrescent and radially unbounded.

$\mathcal{L}$ is the differential operator of system (3.1), then

$$\mathcal{L}\mathcal{V} = 2(x+y)(-\mu x - \alpha y) + 2by\left[(kx + k\frac{A}{\mu})z - (\beta+\alpha)y\right]$$

$$
\begin{aligned}
&+(2+b)\sigma^2\left(x+\frac{A}{\mu}\right)^2 y^2+2z(\alpha y-\delta z)\\
=&-2\mu x^2-2xy(\alpha+\mu)-2y^2(\alpha+b(\beta+\alpha))+2bkxyz+2yz\left(bk\frac{A}{\mu}\right)\\
&+(2+b)\sigma^2\left(x+\frac{A}{\mu}\right)^2 y^2-2\delta z^2.
\end{aligned}
$$

Note that

$$
\begin{gathered}
2bkxyz\leq 0\\
\textit{and}\\
(2+b)\sigma^2\left(x+\frac{A}{\mu}\right)^2 y^2\leq(2+b)\sigma^2\left(\frac{A}{\mu}\right)^2 y^2,
\end{gathered}
$$

then,

$$
\begin{aligned}
\mathcal{LV}\leq&-2\mu x^2-2(\alpha+\mu)xy-2(\alpha+b(\beta+\alpha))y^2+2yz\left(bk\frac{A}{\mu}+\alpha\right)-2\delta z^2\\
&+(2+b)\sigma^2\left(\frac{A}{\mu}\right)^2 y^2\\
=&-2\mu x^2-2(\alpha+\mu)xy-2\left[\alpha+b(\beta+\alpha)-\frac{b}{2}\sigma^2\left(\frac{A}{\mu}\right)^2\right]y^2+2yz\left(bk\frac{A}{\mu}+\alpha\right)\\
&-2\delta z^2+2\sigma^2\left(\frac{A}{\mu}\right)^2 y^2\\
\leq&-2\mu x^2+\mu x^2+\frac{(\alpha+\mu)^2y^2}{\mu}-2\left[\alpha+b(\beta+\alpha)-\frac{b}{2}\sigma^2\left(\frac{A}{\mu}\right)^2\right]y^2+\left(bk\frac{A}{\mu}+\alpha\right)y^2\\
&-\left[2\delta-\left(bk\frac{A}{\mu}+\alpha\right)\right]z^2+2\sigma^2\left(\frac{A}{\mu}\right)^2 y^2\\
=&-\mu x^2-\left[\alpha+2b(\beta+\alpha)-b\sigma^2\left(\frac{A}{\mu}\right)^2-bk\frac{A}{\mu}\right]y^2+\frac{(\alpha+\mu)^2}{\mu}y^2\\
&+2\sigma^2\left(\frac{A}{\mu}\right)^2 y^2-\left[2\delta-\left(bk\frac{A}{\mu}+\alpha\right)\right]z^2\\
=&-\mu x^2-\left[\alpha+2b(1-R_1)(\beta+\alpha)-\frac{(\alpha+\mu)^2}{\mu}-2\sigma^2\left(\frac{A}{\mu}\right)^2\right]y^2\\
&-\left[2\delta-\left(bk\frac{A}{\mu}+\alpha\right)\right]z^2,
\end{aligned}
$$

where $R_1=\dfrac{A^2\sigma^2+kA\mu}{2\mu^2(\beta+\alpha)}$.

Let $2b(1-R_1)(\beta+\alpha) - \frac{(\alpha+\mu)^2}{\mu} - 2\sigma^2\left(\frac{A}{\mu}\right)^2 = 0$, that is, $b = \frac{\mu(\alpha+\mu)^2 + 2\sigma^2 A^2}{2\mu^2(1-R_1)(\beta+\alpha)}$, then

$$\mathcal{L}\mathcal{V} \leq -\mu x^2 - 2\alpha y^2 - \left[2\delta - \left(bk\frac{A}{\mu} + \alpha\right)\right] z^2, \tag{3.2}$$

which is negative-definite. Accordingly, lemma 3.1 implies that if $R_1 < 1$ and under the condition that indicates when a high unemployment rate can remain steady, if there is a significant loss of jobs, the economy is trapped with long-term unemployment. The economy's capacity to create and fill new jobs must continue to be outpaced by this job loss. As a result, the economy continues to experience significant unemployment. The trivial solution of system (3.1) is stochastically asymptotically stable in the large, indicating that system (1.2)'s solution $\left(\frac{A}{\mu}, 0, 0\right)$ is also stochastically asymptotically stable in the large. □

Theorem 3.4. *The solution* $\left(\frac{A}{\mu}, 0, 0\right)$ *of system* (1.2) *is exponentially mean-square stable if* $R_1 < 1$ *and* $\delta \geq bk\frac{A}{2\mu} + \frac{\alpha}{2}$.

Proof. We also select the stochastic Lyapunov function $\mathbb{R}^3 \to \overline{\mathbb{R}}_+$, as we did in the proof of Theorem 3.3:

$$\mathcal{V}(x,y,z) = (x+y)^2 + by^2 + z^2,$$

where b, as defined in the proof of Theorem 3.3, is a positive constant.
Theorem 3.3's proof reveals

$$\mathcal{V}(x,y,z) \geq \frac{b}{2+b}x^2 + \frac{b}{2}y^2 + z^2 \geq \frac{b}{2+b}(x^2+y^2+z^2)$$

and

$$\mathcal{V}(x,y,z) \leq 2x^2 + (2+b)y^2 + z^2 \leq (2+b)(x^2+y^2+z^2),$$

that is,

$$\frac{b}{2+b}(x^2+y^2+z^2) \leq \mathcal{V}(x,y,z) \leq (2+b)(x^2+y^2+z^2). \tag{3.3}$$

Additionally, (3.2) suggests that

$$\begin{aligned}\mathcal{L}\mathcal{V} &\leq -\mu x^2 - \alpha y^2 - \left[2\delta - \left(bk\frac{A}{\mu} + \alpha\right)\right] z^2 \\ &\leq -min\left\{\mu, \alpha, \left[2\delta - \left(bk\frac{A}{\mu} + \alpha\right)\right]\right\}(x^2+y^2+z^2) \\ &\leq -\frac{min\left\{\mu, \alpha, \left[2\delta - \left(bk\frac{A}{\mu} + \alpha\right)\right]\right\}}{2+b}\mathcal{V}(x,y,z).\end{aligned} \tag{3.4}$$

Lemma 3.2, (3.3), and (3.4) thus lead us to the conclusion that system (3.1)'s trivial solution is exponentially mean-square stable. The employment free equilibrium $(\frac{A}{\mu}, 0, 0)$ of system (1.2) is therefore exponentially mean-square stable. □

3.2. Asymptotic behavior around the positive equilibrium Q^* of deterministic model

Since there is no employment equilibrium of the system (1.2), which is the perturbed system of the system (1.1), we examine the behavior of the deterministic model (1.1) around the employment equilibrium Q^* in this section. Employment growth is calculated by dividing the rate of worker loss *via* migration, retirement, or death by the number of open positions, adjusted for reductions brought on by a lack of government financing. This balance provides a useful perspective on how employment options fit the requirements and limitations of the workforce. We get the following result.

Lemma 3.5. *[23] (Strong Law of Large Numbers). Let $M = \{M_t\}_{t\geq 0}$ be a real-valued continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t\to\infty} \langle M, M\rangle_t = \infty \; a.s. \Rightarrow \lim_{t\to\infty} \frac{M_t}{\langle M, M\rangle_t} = 0 \; a.s.$$

And also

$$\lim_{t\to\infty} sup \frac{\langle M, M\rangle_t}{t} < \infty \; a.s. \Rightarrow \lim_{t\to\infty} \frac{M_t}{t} = 0 \; a.s.$$

Theorem 3.6. *With any initial value $(U_0, E_0, V_0) \in \overline{\mathbb{R}}^3_+$, let $(U(t), E(t), V(t))$ be the solution of system (1.2). If $R_0 > 1$ and $E \geq \dfrac{\delta}{\alpha} V$, then we have*

$$\lim_{t\to\infty} sup \frac{1}{t}\int_0^t \left[(U(u) - U^*)^2 + (E(u) - E^*)^2 + (V(u) - V^*)^2\right] du \leq N\sigma^2 \; a.s.$$

where (U^*, E^*, V^*) is the employement equilibrium of system (1.1).

$$R_0 = \frac{\alpha k A}{\mu\delta(\alpha + \beta)} \; and\, N = \frac{\left(\frac{A}{\mu}\right)^2 (\left(\frac{A}{\mu}\right)^2 + b_1 E^*)}{min\{2\mu, k\alpha U^*, b_2\delta\}}.$$

Proof. Define

$$\mathcal{V} = (U - U^* + E - E^*)^2 + b_1\left(E - E^* - E^* log\frac{E}{E^*}\right) + b_2(V - V^*)^2,$$

where the positive constants b_1 and b_2 are to be found later.
Using Itô's formula, we get

$$\begin{aligned} d\mathcal{V} &= [2(U - U^* + E - E^*)(A - \mu U - \alpha E) + b_1(E - E^*)(k\frac{UV}{E} - \beta - \alpha) \\ &\quad +2b_2(V - V^*)(\alpha E - \delta V) + 2\sigma^2 U^2 E^2 + \frac{b_1 E^*}{2}\sigma^2 U^2]dt + b_1(E - E^*)\sigma U dB(t). \\ &:= \mathcal{L}\mathcal{V}dt + b_1(E - E^*)\sigma U dB(t), \end{aligned}$$

where $\mathcal{L}$ is system (1.2)'s differential operator, and

$$\begin{aligned}
\mathcal{L}\mathcal{V} = & 2(U-U^*+E-E^*)(A-\mu U-\alpha E)+b_1(E-E^*)(k\frac{UV}{E}-\beta-\alpha) \\
& +2b_2(V-V^*)(\alpha E-\delta V)+2\sigma^2U^2E^2+\frac{b_1E^*}{2}\sigma^2U^2 \\
& \leq 2(U-U^*+E-E^*)(A-\mu U-\alpha E)+b_1(E-E^*)(k\frac{\alpha}{\delta}U-\beta-\alpha) \\
& +2b_2(V-V^*)(\alpha E-\delta V)+2\sigma^2U^2E^2+\frac{b_1E^*}{2}\sigma^2U^2 \\
= & 2(U-U^*+E-E^*)[-\mu(U-U^*)-\alpha(E-E^*)]+k\frac{\alpha}{\delta}b_1(E-E^*)(U-U^*) \\
& +2b_2(V-V^*)[\alpha(E-E^*)-\delta(V-V^*)]+2\sigma^2U^2E^2+\frac{b_1E^*}{2}\sigma^2U^2 \\
= & -2\mu(U-U^*)^2-2\alpha(E-E^*)^2-2b_2\delta(V-V^*)^2+(U-U^*)(E-E^*)[-2\alpha-2\mu \\
& +k\frac{\alpha}{\delta}b_1]+2\alpha b_2(V-V^*)(E-E^*)+2\sigma^2U^2E^2+\frac{b_1E^*}{2}\sigma^2U^2.
\end{aligned}$$

Configuring $-2\alpha-2\mu+k\frac{\alpha}{\delta}b_1=0$, we acquire $b_1=\frac{2\delta}{\alpha k}(\alpha+\mu)$ and

$$\begin{aligned}
\mathcal{L}\mathcal{V} = & -2\mu(U-U^*)^2-2\alpha(E-E^*)^2-2b_2\delta(V-V^*)^2+2\alpha b_2(V-V^*)(E-E^*) \\
& +2\sigma^2U^2E^2+\frac{b_1E^*}{2}\sigma^2U^2 \\
\leq & -2\mu(U-U^*)^2-2\alpha(E-E^*)^2-2b_2\delta(V-V^*)^2+b_2\delta(V-V^*)^2 \\
& +\frac{b_2\alpha^2}{\mu}(E-E^*)^2+2\sigma^2U^2E^2+\frac{b_1E^*}{2}\sigma^2U^2 \\
= & -2\mu(U-U^*)^2-\left(2\alpha-\frac{b_2\alpha^2}{\mu}\right)(E-E^*)^2-b_2\delta(V-V^*)^2 \\
& +2\sigma^2U^2E^2+\frac{b_1E^*}{2}\sigma^2U^2.
\end{aligned}$$

Choose $b_2=\frac{\mu}{\alpha}(2-\delta)-\frac{\delta\mu}{\alpha^2}\beta$ such that $2\alpha-\frac{b_2\alpha^2}{\mu}-k\alpha U^*=0$,
then

$$\begin{aligned}
\mathcal{L}\mathcal{V} & \leq -2\mu(U-U^*)^2-k\alpha U^*(E-E^*)^2-b_2\delta(V-V^*)^2+2\sigma^2U^2E^2+\frac{b_1E^*}{2}\sigma^2U^2 \\
& \leq -\left[2\mu(U-U^*)^2+k\alpha U^*(E-E^*)^2+b_2\delta(V-V^*)^2-\left(\frac{A}{\mu}\right)^2\left(2\sigma^2E^2+\frac{b_1}{2}E^*\sigma^2\right)\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathrm{d}\mathcal{V} \leq & -\left[2\mu(U-U^*)^2+k\alpha U^*(E-E^*)^2+b_2\delta(V-V^*)^2-\left(\frac{A}{\mu}\right)^2\left(2\sigma^2E^2+\frac{b_1}{2}E^*\sigma^2\right)\right]\mathrm{d}t \\
& +b_1\sigma U(E-E^*)\mathrm{d}B(t) \\
& \leq -[min\{2\mu,k\alpha U^*,b_2\delta\}\left((U-U^*)^2+(E-E^*)^2+(V-V^*)^2\right)
\end{aligned}$$

$$-\left(\frac{A}{\mu}\right)^2\left(2\sigma^2E^2+\frac{b_1}{2}E^*\sigma^2\right)]\mathrm{d}t+b_1\sigma U(E-E^*)\mathrm{d}B(t).$$

Integrating it from 0 to t, gives

$$\begin{aligned}\mathcal{V}(t)-\mathcal{V}(0)\leq -min\{2\mu,k\alpha U^*,b_2\delta\}\int_0^t\left[(U(u)-U^*)^2+(E(u)-E^*)^2+(V(u)-V^*)^2\right]du\\+\left(\frac{A\sigma}{\mu}\right)^2\left(2\left(\frac{A}{\mu}\right)^2+\frac{b_1E^*}{2}\right)t+b_1\sigma\int_0^t U(u)(E(u)-E^*)\mathrm{d}B(t)\end{aligned}$$

and

$$\begin{aligned}&\int_0^t\left[(U(u)-U^*)^2+(E(u)-E^*)^2+(V(u)-V^*)^2\right]du\\&\leq\frac{\mathcal{V}(0)}{min\{2\mu,k\alpha U^*,b_2\delta\}}+\frac{\left(\frac{A\sigma}{\mu}\right)^2\left(2E^2+\frac{b_1E^*}{2}\right)}{min\{2\mu,k\alpha U^*,b_2\delta\}}t\\&+\frac{b_1\sigma}{min\{2\mu,k\alpha U^*,b_2\delta\}}\int_0^t U(u)(E(u)-E^*)\mathrm{d}B(u).\end{aligned}\tag{3.5}$$

Allow

$$M(t)=\int_0^t U(u)(E(u)-E^*)\mathrm{d}B(u),$$

which is a continuous local martingal and $M(0)=0$. Additionally,

$$\begin{aligned}\langle M,M\rangle_t&=\left(\int_0^t U(u)(E(u)-E^*)\mathrm{d}B(u)\right)^2\\&=\int_0^t U^2(u)(E(u)-E^*)^2du\\&\leq 4\left(\frac{A}{\mu}\right)^4t,\end{aligned}$$

and

$$\lim_{t\to\infty}sup\frac{\langle M,M\rangle_t}{t}\leq 4\left(\frac{A}{\mu}\right)^4<\infty,\text{ a.s.}$$

When combined with lemma 3.5, this suggests

$$\lim_{t\to\infty}\frac{M(t)}{t}=\frac{\int_0^t U(u)(E(u)-E^*)\mathrm{d}B(u)}{t}=0\,\text{a.s.}\tag{3.6}$$

Thus, when we combine (3.6), with (3.5) we have

$$\lim_{t\to\infty} sup\frac{1}{t}\int_0^t \left[(U(u)-U^*)^2+(E(u)-E^*)^2+(V(u)-V^*)^2\right]du \leq N\sigma^2 \text{ a.s.}$$

Where

$$N=\frac{\left(\frac{A}{\mu}\right)^2(\left(\frac{A}{\mu}\right)^2+b_1E^*)}{min\{2\mu,k\alpha U^*,b_2\delta\}}$$

□

Remark 3.7. According to Theorem 3.6, for system (1.1) (*i.e.*, $\sigma=0$), we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \left[(U(u)-U^*)^2+(E(u)-E^*)^2+(V(u)-V^*)^2\right]du=0,$$

which suggests

$$\lim_{t\to\infty}U(t)=U^*, \lim_{t\to\infty}E(t)=E^* \text{ and } \lim_{t\to\infty}V(t)=V^*,$$

demonstrating that the solution tends to Q^*.

Furthermore, the difference between $(U(t),E(t),V(t))$, and Q^* is proportional to the intensity of white noise in the temporal average, according to the result of Theorem 3.6. The solution approaches Q^* more closely when white noise is reduced.

• The level of volatility needed to upset the equilibrium of unemployment isn't very big on its own, but it depends on how weak the economy really is. When an economy is already having trouble with a lot of people losing jobs and not enough new jobs being created, even a small amount of instability can push it past the point where things stay stable. Real-life situations like political confusion, fast changes in technology, or changes in trade rules often create enough instability on their own to tip the equilibrium. This shows that in many real cases, the deterministic equilibrium is highly vulnerable. The model also shows that not only can instability cause short-term job losses, but it can also lead to a long-term situation where unemployment stays high, which explains why joblessness can last a long time even after the first problems are gone.

4. Numerical simulations

This section includes some numerical simulations that illustrate our theoretical findings regarding the system (1.2) stability analysis. The Euler-Maruyama Method [26] was used to write and compile a code in Matlab. Some of the main parameters used in those simulations came from official reports. This included data from Morocco's High Commissioner for Planning (HCP) on trends in the job market, such as unemployment rates and changes in the workforce, from early 2023 [27]. The simulations also utilized Morocco's Finance Draft Law 2023 [28], which outlines the government's plan for the year, detailing its expenditure, revenue sources, and economic objectives.

Our tests are broken up into two parts, each designed to show a particular aspect of the model. The goal of the first set of tests is about Theorem 3.3, where we expect that the employment free equilibrium $Q_0=\left(\frac{A}{\mu},0,0\right)$ of system (1.2) is globally asymptotically stable. In Figure 1, we choose parameters $A=0.02231$, $k=0.576$, $\sigma=0.8$, $\mu=0.058$, $\beta=0.129$, $\alpha=0.077$ and $\delta=1.01837$, the chosen initial proportion of unemployed people, employed individuals, and available vacancies is $U_0=0.4$, $E_0=0.6$, and $V_0=0.1$ over 100 days, such that

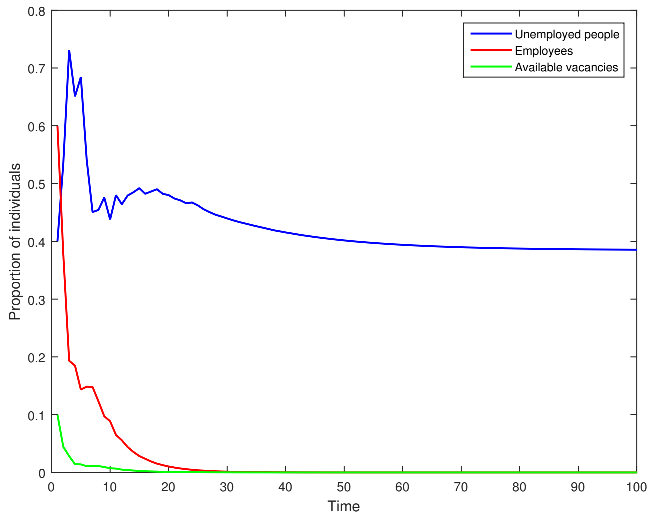

FIGURE 1. The convergence of the solutions to the equilibrium point Q_0.

$R_1 < 1$ and $\delta \geq ck\dfrac{A}{2\mu} + \dfrac{\alpha}{2}$, this inequality shows when a situation of high unemployment can stay stable. It means that the economy gets stuck with long-term unemployment, if the number of jobs being lost is very high. This loss of jobs has to be strong enough to keep beating the economy's ability to make and fill new jobs. Because of this, the economy stays in a state of high unemployment. In addition, as said in Theorem 3.4, the employment free equilibrium Q_0 of system (1.2) is exponentially mean-square stable. From Figure 1, we can see that $E(t)$ and $V(t)$ indeed tend to 0.

In the first quarter of 2023, from Figure 1, it seems that there are two parts before and after 25 days. The first one oscillates about the perturbed UEV model, allowing us to demonstrate the uncertainties of the labor market, such as unforeseen shifts in the quantity of jobs or governmental regulations. Second, after 25 days, it's observed that the proportion of unemployed people (U) stays stable in $A/\mu = 0.38465$, and the proportion of (E) and (V) converges to 0. It indicates the absence of available vacancies and no new employees.

Due to the lack of financial resources because of the lack of government funds, the parameter δ is getting worse to 0.0837, which is the goal of the second set of tests and is about Theorem 3.6. The three proportions of unemployed individuals (U), employed individuals (E), and available vacancies (V) go around $Q^* = (0.214, 0.0559, 0.1182)$ if $R_0 > 1$ and $E \geq \dfrac{\delta}{\alpha}V$, this stability condition means the workforce needs to be big enough to keep the economy's job structure in place, if there aren't enough workers to fill the available jobs, the system becomes unstable. It states that having a sufficient number of jobs is crucial for the job market to remain strong, without enough employment, the economy can't settle into a stable state. The parameters we choose in Figure 2 are $A = 0.02231$, $k = 0.676$, $\sigma = 0.8$, $\mu = 0.058$, $\beta = 0.129$, $\alpha = 0.177$, $U_0 = 0.1294$, $E_0 = 0.8705$, and $V_0 = 0.008356$.

We can conclude that the increase in the number of employed people is proportional to make raise the rate of available vacancies by the government, and decrease the rate of migration and retirement of employed people to get stability around the positive equilibrium Q^* of the deterministic model (1.1) for enhancing the number of employment and hence avoid the disaster of unemployment.

This study uses a stochastic framework to show how real-life uncertainties in the labor market, like sudden changes in job demand or shifts in government policies, affect the UEV model. This makes the model more realistic and requires using tools like Itô formula. In practice, the model performs well, according to the results: Figure 1 shows that even with random noise, the system still converges to the employment free equilibrium

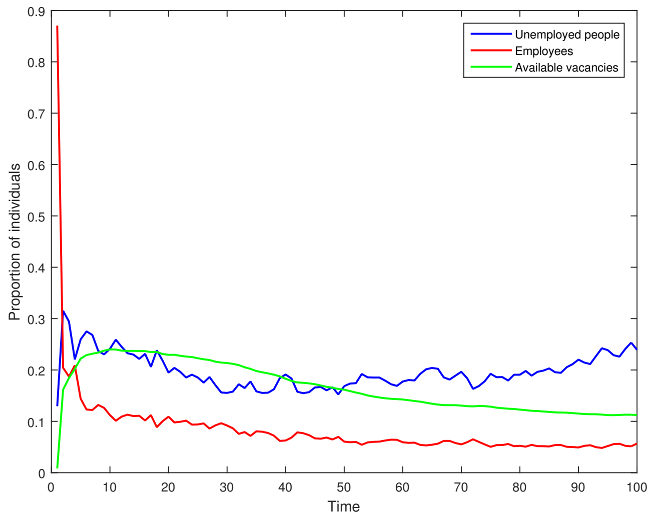

FIGURE 2. The convergence of the solutions around the equilibrium point Q^*.

Q_0, and Figure 2 shows that the system stays stable around the positive equilibrium Q^*, proving the model is strong even when there is randomness.

5. Conclusion

We develop a stochastic mathematical model in this work to explain the dynamics of the unemployment issue under random fluctuations. After analyzing the suggested model, we discovered that there is only one global positive solution for system (1.2). In addition, we explored the asymptotic behaviors of the employment equilibrium Q^* and the employment free equilibrium Q_0 by developing some relevant Lyapunov functionals also under some conditions. The government can overcome the phenomena of unemployment by managing three coefficients: lowering the rate of migration, retirement, or death of employed individuals in comparison to the rate of available positions as a result of a shortage of government funding and the quantity of open positions under this condition and if $R_0 < 1$ we can obtain that the solution of system (1.2) is going around Q^*. Our study suggests that a number of intriguing subjects should be considered for additional research, such as data constraints, including into this model an optimal control problem, and putting out other random models that address the topic of unemployment from various angles and viewpoints.

Conflicts of interest

The authors declare no competing interests.

Data availability statement

On behalf of all authors, the corresponding author declares that the data supporting the findings of this study are available within the paper.

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