

New Aspects of Fractional Bloch Model Associated with Composite Fractional Derivative

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Abstract. This paper studies a fractional Bloch equation pertaining to Hilfer fractional operator. Bloch equation is broadly applied in physics, chemistry, nuclear magnetic resonance (NMR), magnetic resonance imaging (MRI) and many more. The sumudu transform technique is applied to obtain the analytic solutions for nuclear magnetization $M = (M_x, M_y, M_z)$. The general solution of nuclear magnetization M is shown in the terms of Mittag-Leffler (ML) type function. The influence of order and type of Hilfer fractional operator on nuclear magnetization M is demonstrated in graphical form. The study of Bloch equation with composite fractional derivative reveals the new features of Bloch equation. The discussed fractional Bloch model provides crucial and applicable results to introduce novel information in scientific and technological fields.

Keywords: Fractional order Bloch model; Nuclear magnetic resonance; Magnetization; Hilfer derivative; Sumudu transform; Mittag-Leffler function.

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1. Introduction

Nuclear magnetic resonance (NMR) is a physical occurrence broadly employed in physics, chemistry, medical science, and engineering for studying the complex materials. The Bloch equations are a collection of the macroscopic equations which are applied to compute nuclear magnetization $M = (M_x, M_y, M_z)$ as a function of time during relaxation times ρ_1 and ρ_2 are presented. Bloch equations were proposed by Felix Bloch in the year 1946 and they are employed for a representation of the NMR. NMR is controlled with the aid of the Bloch equation, which connects a macroscopic system of magnetization to the employed radiofrequency, static magnetic fields and gradient. Many researchers studied the characteristic and application of Bloch equation, for more details see [1-7]. Fractional calculus is a particular outlook of applied mathematics that related with derivatives and integrals of arbitrary orders. Recently, remarkably amount of research work in this special branch has been appeared in different scientific, technological, financial and several other field. In fact, novel features of FC are affected by different useful applications such as mathematical biology, fluid flow problems, turbulence, electrochemistry, controlled thermonuclear fusion, astrophysics, image processing, plasma physics, stochastic dynamical processes, control theory and several others. In view of the above considered details, it is worthy that FC has emerged as a crucial new mathematical key for the solution of distinct issues in the field of science and engineering. The principal benefit of calculus of arbitrary order is to model real world problems with complete memory influence. Many researchers have studied principal outcomes in the structure of monogram, books and quality research articles connected to applications of FC in distinct important directions, for further information; see [8-28]. Generally physical problems related to fractional operators capable to represent the

effective physical interpretation of the model in comparison to integer order derivatives and integrals. The non-integer order model describes the physical system having more accuracy, with high-order dynamics and with complex nonlinear phenomena. It occurs because of two reasons such that (i) choice to choose the order of fractional order derivatives while it is not employable to derivative of integer order (ii) as derivative of integer order is local in nature, so it does not narrate complete history and physical nature of system while derivative of arbitrary order has a non-local characteristic, therefore, it yields the whole history and physical aspects of the real world problem.

To manage above analyzed facts, we examine the physical model having arbitrary order derivative. Physical models of arbitrary orders have been nicely controlled. It has been shown that the physical models demonstrate the novel characteristic with full history of the problem, for complete study and implementation of the outcomes, see [8-28]. Several mathematicians suggested new operators of fractional nature time to time, after working on these operators it was analyzed by several researchers that these operators hold few restrictions. In the similar manner, Hilfer [8] suggested a composite derivative of fractional order which is a modified and crucial derivative of arbitrary order. Here, we use this notable derivative of fractional order to Bloch equation. The main objective of this paper is to describe the novel characteristic of Bloch equation associated to Hilfer fractional derivative. Further, we demonstrate the influences of parameters of fractional nature on behavior of solution of Bloch equation. Actually the capacity for modeling the differential equations of fractional order as a function of time is an appealing in addition entrancing fact for researchers, mathematician and scientist as well. The NMR is a physical occurrence broadly employed in physics, medical science, chemical sciences and engineering to study the complex problems. By analyzing the all above considered physical importance of Bloch equation, powerful memory nature of

composite fractional derivative, the authors are encouraged to study the proposed research work. The foundation of the article is laid on in six sections like: In second section, we discussed some important mathematical formulas and definition. In Section third, Bloch equation pertaining to Hilfer fractional derivative is constructed. Approximate analytical results of fractional Bloch equation is expressed by applying Sumudu transform method, in section fourth. Section fifth provides the numerical outcomes and graphical behavior of solution of the non-integer order Bloch equation. Lastly, in Section fifth the conclusion moreover future physical results are pointed out.

2. Mathematical Overture

Some principal mathematical postulate which are applied in the present paper are presented as

Sumudu Transform (ST)

The ST [29] for a function $h(\xi)$ on a set is represented as

$$S[h(\xi), q] = \int_0^{\infty} h(q\xi) e^{-\xi} d\xi, q \in (-\gamma_1, \gamma_2). \quad (1)$$

The basic characteristic and outcomes associated to sumudu transform are discussed by several authors, see [30-38].

Riemann- Liouville integral operator

For a function $h(\xi)$, the Riemann- Liouville (RL) type integral operator [11] of order $\gamma > 0$ is given as

$$I_{b^+}^{\gamma} h(\xi) = \frac{1}{\Gamma(\gamma)} \int_b^{\xi} \frac{h(u)}{(\xi - u)^{1-\gamma}} du, \gamma \in \mathbb{C}. \quad (2)$$

RL fractional derivative

Suppose $\gamma > 0$, additionally $\gamma, \xi \in \mathbb{R}$ then RL derivative of non-integer order [11] is represented in the following manner

$${}_0D_\xi^\gamma h(\xi) = \frac{1}{\Gamma(n-\lambda)} \frac{d^n}{d\xi^n} \int_0^\xi \frac{h(u)}{(\xi-u)^{\gamma+1-n}} du, n-1 < \gamma < n, n \in N. \quad (3)$$

$$= \frac{d^n h(u)}{d\xi^n}, \text{ if } \gamma = n \in N. \quad (4)$$

Caputo fractional operator

Suppose $\gamma > 0$, and $\gamma, \xi \in R$ then Caputo type operator of arbitrary order [10] is given as

$${}_0D_\xi^\gamma h(\xi) = \frac{1}{\Gamma(n-\lambda)} \int_0^\xi \frac{h^{(n)}(\xi)}{(\xi-u)^{\gamma+1-n}} du, n-1 < \gamma < n, n \in N. \quad (5)$$

Hilfer derivative of fractional order

Suppose $h : [e, f] \rightarrow R, 0 < \gamma \leq 1, 0 \leq \eta \leq 1$, then Hilfer derivative of non-integer order [8]

of a function $h(\xi)$ is given in the following way

$${}_0D_{b^+}^{\gamma, \eta} h(\xi) = \left(I_{b^+}^{\eta(1-\gamma)} \frac{d}{d\xi} \left(I_{b^+}^{(1-\eta)(1-\gamma)} h(\xi) \right) \right). \quad (6)$$

If set $\eta = 0$ and $\eta = 1$, in these cases, Hilfer derivative of fractional order converts to the RL derivative of arbitrary order described as Eq. (4) and in Caputo derivative of fractional order given as Eq. (5), respectively.

ST of Hilfer derivative of fractional order

The ST of Hilfer derivative of fractional order [36, 37] is expressed as

$$S \left[{}_0D_\xi^{\gamma, \eta} h(\xi); q \right] = q^{-\gamma} \bar{h}(q) - q^{-\eta(\gamma-1)-1} I_{0^+}^{(1-\eta)(1-\gamma)} h(0^+). \quad (7)$$

To analyze the more characteristic of Hilfer derivative, see [9].

3. Construction of fractional Bloch Equation

The replacement of time derivative by fractional derivative in Bloch equation recommends a number of attractive and useful possibilities concerning magnetization relaxation and spin dynamics. In the present paper, we consider the standard Bloch equations [1, 4].

$$\begin{aligned}
\frac{dM_z(\xi)}{d\xi} &= \frac{M_0 - M_z(\xi)}{\rho_1}, \\
\frac{dM_x(\xi)}{d\xi} &= \mu_0 M_y(\xi) - \frac{M_x(\xi)}{\rho_2}, \\
\frac{dM_y(\xi)}{d\xi} &= -\mu_0 M_x(\xi) - \frac{M_y(\xi)}{\rho_2}.
\end{aligned} \tag{8}$$

“ The nomenclature of model (8) is describes as

$M_x(\xi)$ represents the system magnetization of x component, $M_y(\xi)$ describes the system magnetization of y component, $M_z(\xi)$ indicates the system magnetization of z component. M_0 denotes the equilibrium magnetization, ρ_1 shows spin-lattice relaxation time, ρ_2 indicates spin-spin relaxation time, in addition μ_0 represents resonant frequency specified by Larmor relationship $\mu_0 = \alpha C_0$, where C_0 indicates static magnetic field of z -component and $\frac{\alpha}{2\pi}$ designates the gyromagnetic ratio.”

As above mention in section 1, the principal characteristic of FC and to describe the efficient physical aspects of system (8), in this study we change the classical time derivative by a fractional operator in Hilfer sense. In this case, the newly modified fractional Bloch equation model is given in the following manner

$$\begin{aligned}
\lambda_1^{\gamma-1} {}_0 D_\xi^{\gamma,\eta} M_z(\xi) &= \frac{M_0 - M_z(\xi)}{\rho_1}, \\
\lambda_2^{\gamma-1} {}_0 D_\xi^{\gamma,\eta} M_x(\xi) &= \mu_0 M_y(\xi) - \frac{M_x(\xi)}{\rho_2}, \\
\lambda_2^{\gamma-1} {}_0 D_\xi^{\gamma,\eta} M_y(\xi) &= -\mu_0 M_x(\xi) - \frac{M_y(\xi)}{\rho_2}.
\end{aligned} \tag{9}$$

The time constants λ_1 and λ_2 of fractional nature are required on to the left hand side of model (10) to control the consistent set of units for magnetization. Model (9) can also be represented in the following way

$$\begin{aligned} {}_0D_{\xi}^{\gamma,\eta}M_z(\xi) &= \frac{M_0 - M_z(\xi)}{\rho_1}, \\ {}_0D_{\xi}^{\gamma,\eta}M_x(\xi) &= \bar{\mu}_0 M_y(\xi) - \frac{M_x(\xi)}{\rho_2}, \\ {}_0D_{\xi}^{\gamma,\eta}M_y(\xi) &= -\bar{\mu}_0 M_x(\xi) - \frac{M_y(\xi)}{\rho_2}, \end{aligned} \quad (10)$$

where $\bar{\mu}_0 = \frac{\mu_0}{\lambda_2^{\gamma-1}}$, $\frac{1}{\rho_1} = \frac{\lambda_1^{1-\gamma}}{\rho_1}$ and $\frac{1}{\rho_2} = \frac{\lambda_2^{1-\gamma}}{\rho_2}$ each components holds the units of $(\text{sec})^{-\gamma}$.

The initial conditions (ICs) as

$$\begin{aligned} I_{0+}^{(1-\eta)(1-\gamma)}M_x(0) &= c_1 = 0, \\ I_{0+}^{(1-\eta)(1-\gamma)}M_z(0) &= c_2 = 0, \\ I_{0+}^{(1-\eta)(1-\gamma)}M_y(0) &= c_3. \end{aligned} \quad (11)$$

4. Analytical results of fractional Bloch Equation associated with Sumudu transform method

Here to analyze the new physical aspects and to evaluate the consequences of fractional parameters on the solution of discussed model (10), we solve system (10) by employing a powerful in addition efficient ST method. Primarily we use ST on model (10), then we get

$$\begin{aligned} q^{-\gamma}S[M_z(\xi)] - q^{-\eta(\gamma-1)-1}I_{0+}^{(1-\eta)(1-\gamma)}M_z(0) &= S\left[\frac{M_0 - M_z(\xi)}{\rho_1}\right], \\ q^{-\gamma}S[M_x(\xi)] - q^{-\eta(\gamma-1)-1}I_{0+}^{(1-\eta)(1-\gamma)}M_x(0) &= S\left[\bar{\mu}_0 M_y(\xi) - \frac{M_x(\xi)}{\rho_2}\right], \end{aligned} \quad (12)$$

$$q^{-\gamma} S[M_y(\xi)] - q^{-\eta(\gamma-1)-1} I_{0+}^{(1-\eta)(1-\gamma)} M_y(0) = S \left[-\bar{\mu}_0 M_x(\xi) - \frac{M_y(\xi)}{\rho_2} \right].$$

Now by utilizing Eq. (11) in to Eq. (12) and on rationalization, we get

$$S[M_z(\xi)] = \frac{\rho_1}{q^{-\gamma} \rho_1 + 1} p^{-\eta(\gamma-1)-1} M_z(0) + \frac{M_0}{q^{-\gamma} \rho_1 + 1}. \quad (13)$$

Further, by using the inverse ST technique on both sides Eq. (13) and applying outcomes of Chaurasia and Singh [38] moreover on rationalizing the deriving equations, we obtain

$$M_z(\xi) = I_0^{(1-\eta)(1-\gamma)} M_z(0) \xi^{\gamma-\eta(\gamma-1)-1} E_{\gamma, \gamma-\eta(\gamma-1)} \left(\frac{-\xi^\gamma}{\rho_1} \right) + \frac{M_0}{\rho_1} \xi^\gamma E_{\gamma, \gamma+1} \left(\frac{-\xi^\gamma}{\rho_1} \right). \quad (14)$$

The solutions for $M_x(\xi)$ and $M_y(\xi)$ can be obtained by solving the corresponding differential equations of arbitrary order. It is supposed that

$$M_+(\xi) = M_x(\xi) + iM_y(\xi). \quad (15)$$

$$\text{with } I_0^{(1-\eta)(1-\gamma)} M_+(0) = I_0^{(1-\eta)(1-\gamma)} M_x(0) + i I_0^{(1-\eta)(1-\gamma)} M_y(0), \quad (16)$$

Now we can merge two equations for x component and y component of the magnetization, discussed above to provide

$${}_0 D_\xi^{\gamma, \eta} (M_x(\xi) + iM_y(\xi)) = \bar{\mu}_0 (M_y(\xi) - iM_x(\xi)) - \frac{1}{\rho_2} (M_x(\xi) + iM_y(\xi)) \quad (17)$$

By using Eq. (15), we have

$${}_0 D_\xi^{\gamma, \eta} (M_+(\xi)) = -i\bar{\mu}_0 M_+(\xi) - \frac{1}{\rho_2} M_+(\xi), \quad (18)$$

Taking ST on both sides of Eq. (18), we get

$$S[M_+(\xi)] = \frac{q^{-\eta(\gamma-1)-1} I_0^{(1-\eta)(1-\gamma)} M_+(0)}{q^{-\gamma} + a}, \quad (19)$$

$$\text{where } a = - \left(i\bar{\mu}_0 + \frac{1}{\rho_2} \right).$$

Now taking inverse ST on both the sides of Eq. (19) and utilizing Eq. (15), we have

$$M_x(\xi) + iM_y(\xi) = I_0^{(1-\eta)(1-\gamma)} \left(M_x(0) + iM_y(0) \right) E_{\gamma, \gamma-\eta(\gamma-1)}(-a\xi^\gamma). \quad (20)$$

On comparing the real and imaginary parts, we get

$$M_x(\xi) = I_0^{(1-\eta)(1-\gamma)} M_x(0) E_{\gamma, \gamma-\eta(\gamma-1)}(-a\xi^\gamma), \quad (21)$$

and

$$M_y(\xi) = I_0^{(1-\eta)(1-\gamma)} M_y(0) E_{\gamma, \gamma-\eta(\gamma-1)}(-a\xi^\gamma). \quad (22)$$

4.1 Particular cases of fundamental outcomes

Hilfer derivative of fractional order has two very special cases as follows

(i) If set put $\eta=1$ then, Hilfer operator reduces into Caputo derivative and the fractional Bloch equation (10) reduces into fractional Bloch equation pertaining to Caputo fractional derivative.

(ii) If we set $\eta=0$ then, Hilfer derivative converts into RL derivative and fractional Bloch equation (10) reduces into fractional Bloch equation with RL arbitrary order derivative. It should be noted that for RL approach the integral type ICs $I_{0+}^{(1-\gamma)} M_x(0) = c_1 = 0$, $I_{0+}^{(1-\gamma)} M_z(0) = c_2 = 0$ and $I_{0+}^{(1-\gamma)} M_y(0) = c_3$ should be taken [8, 39].

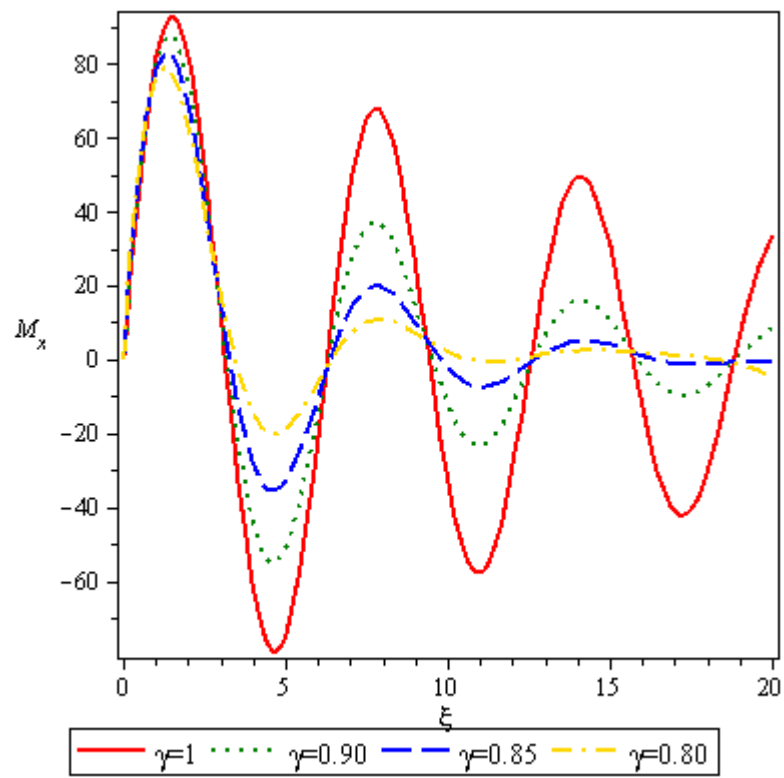
The same condition happens for $0 < \eta < 1$ and $0 < \gamma < 1$, where the nonlocal ICs of the type $I_{0+}^{(1-\eta)(1-\gamma)} M_x(0) = c_1 = 0$, $I_{0+}^{(1-\eta)(1-\gamma)} M_z(0) = c_2 = 0$ and $I_{0+}^{(1-\eta)(1-\gamma)} M_y(0) = c_3$ is observed [8, 39].

5. Numerical simulation and discussion for fractional Bloch model

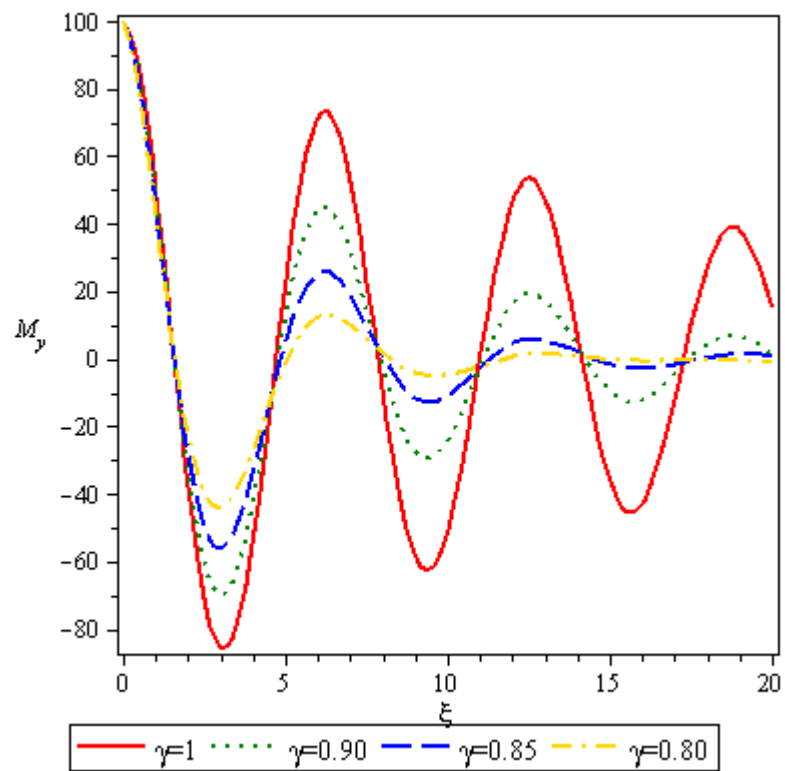
Here, we discuss about Bloch equation pertaining to Hilfer operator. The analytical expressions for the magnetization components (M_x, M_y, M_z) of Bloch equation are analyzed by applying a powerful technique i.e. ST method. The general solutions of the magnetization components (M_x, M_y, M_z) are described in type of ML function. The

affect of Hilfer non-integer derivative on magnetization components (M_x, M_y, M_z) can be seen simply via Figs. 1-7.

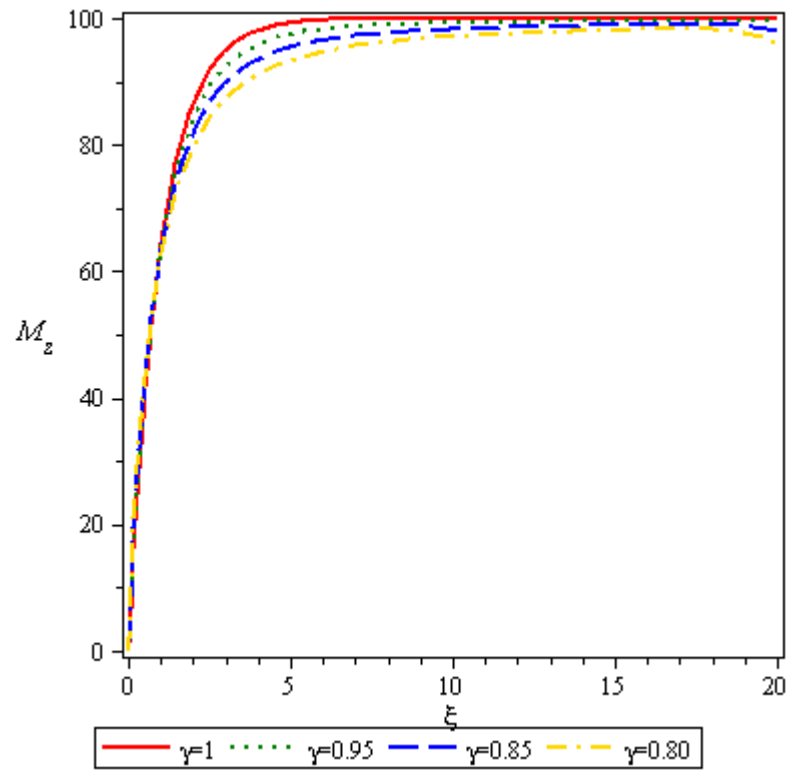
We demonstrate the numerical results of magnetization components (M_x, M_y, M_z) at distinct values of γ and η . In Figs. 1(a)-(c), we take $\eta = 1$ and $\gamma = 1, 0.95, 0.90, 0.85$. In Figs. 2(a)-(b) we take $\eta = 0$ and $\gamma = 1, 0.95, 0.90, 0.85$. In Figs. 3(a)-(b), we put the values as $\gamma = \eta = 1, \gamma = \eta = 0.95, \gamma = \eta = 0.90, \gamma = \eta = 0.85$. In Figs. 4(a)-4(d), represents the dynamic interrelation between the components $M_x(\xi)$ and $M_y(\xi)$ for $\eta = 1, \gamma = 1, 0.95, 0.90, 0.85$. In Figs. 5(a)-5(c), shows the dynamic interrelation between the components $M_x(\xi)$ and $M_y(\xi)$ for $\eta = 1, \gamma = 0.95, 0.90, 0.85$. In Figs. 6(a)-6(c), expresses dynamic relationship into components $M_x(\xi)$ and $M_y(\xi)$ for $\gamma = \eta = 0.95, \gamma = \eta = 0.90, \gamma = \eta = 0.85$. Figs 7(a)-7(d) shows the whole trajectory of magnetization in three dimension (3 D) for $\gamma = 1, 0.95, 0.90, 0.85$. The solution of fraction Bloch model is attained by employing ST having the initial condition as $M_x(0) = 0, M_y(0) = 100, M_z(0) = 0$ and particular parameters as $\bar{\mu}_0 = 1, \rho_1 = 1(s)^\gamma, \rho_2 = 20(ms)^\gamma, M_0 = 100$. Figs. 1-7 demonstrates the important characteristic of magnetization components of fractional Bloch equation w.r.t. time ξ at distinct order as well as type of Hilfer fractional operator. Further, the Figs. 1-7 reveals that discussed system is strongly dependent on the order as well as type of Hilfer operator. The crucial facts and novel nature of magnetization components (M_x, M_y, M_z) can be easily observe at the various values of γ and η .



(a)

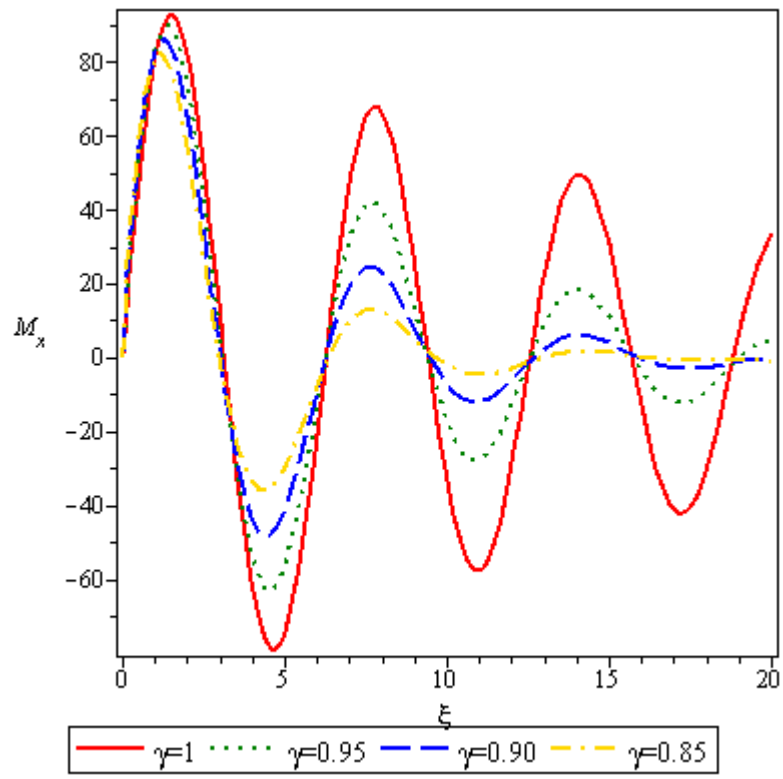


(b)

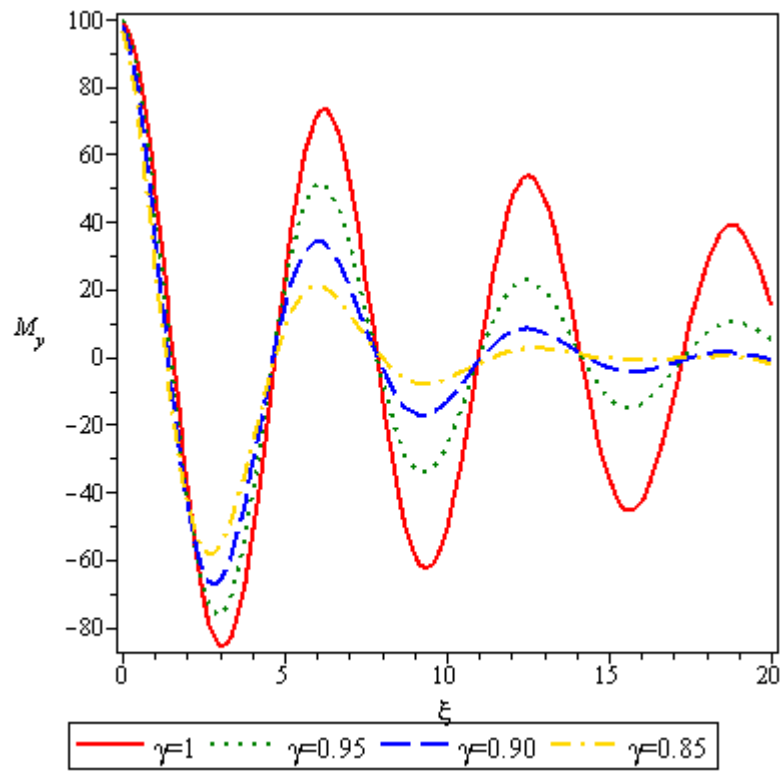


(c)

Figure 1. Plots of system magnetization with respect to ξ at $\eta = 1$ for distinct values of γ (a) Characteristic of $M_x(\xi)$; (b) Behavior of $M_y(\xi)$; (c) Nature of $M_z(\xi)$.



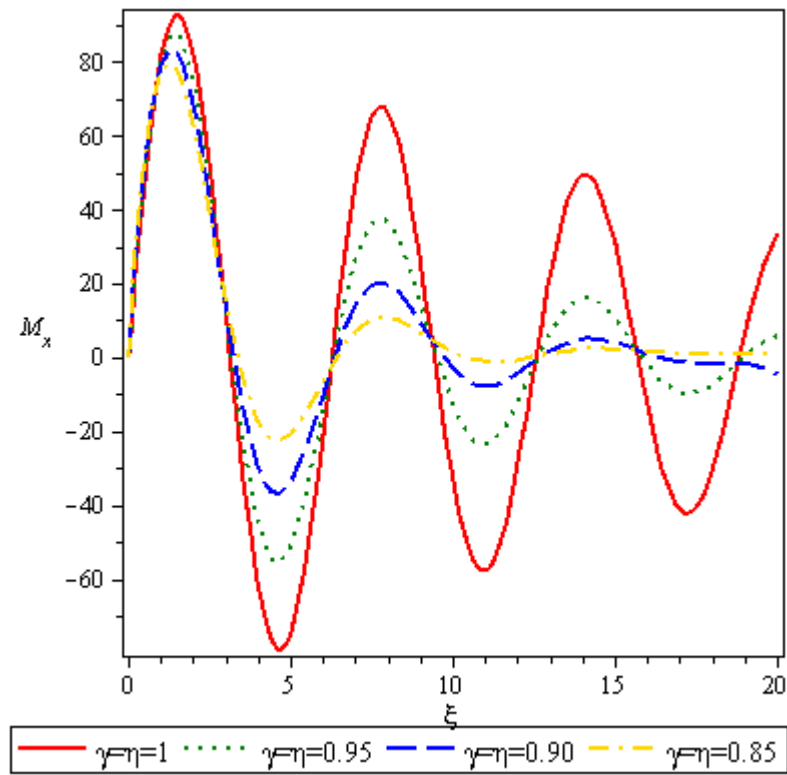
(a)



(b)

Figure 2. Plots of system magnetization with respect to ξ at $\eta = 0$ for several values

of γ (a) Characteristic of $M_x(\xi)$; (b) Behavior of $M_y(\xi)$.



(a)

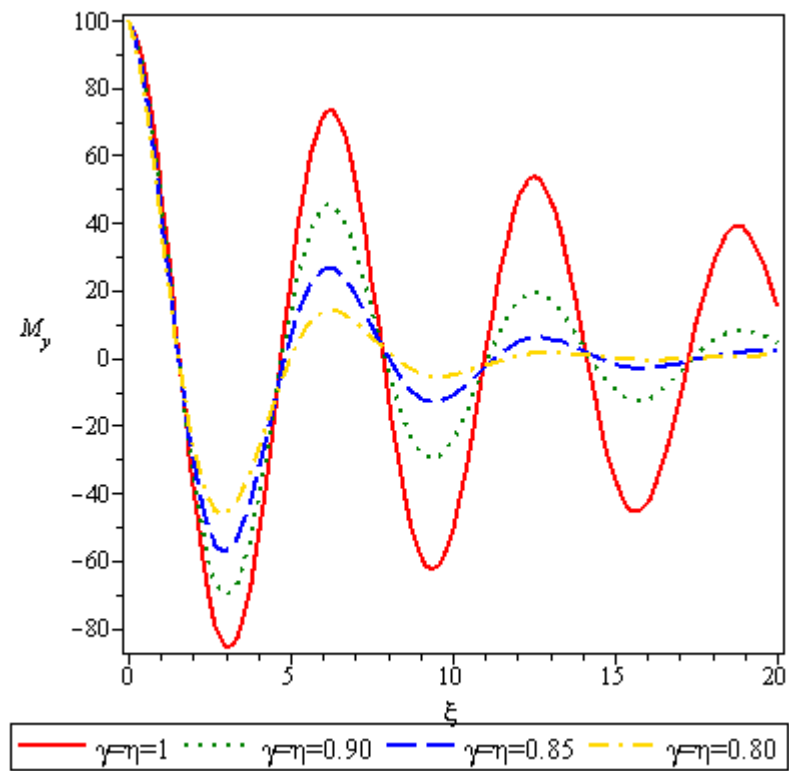
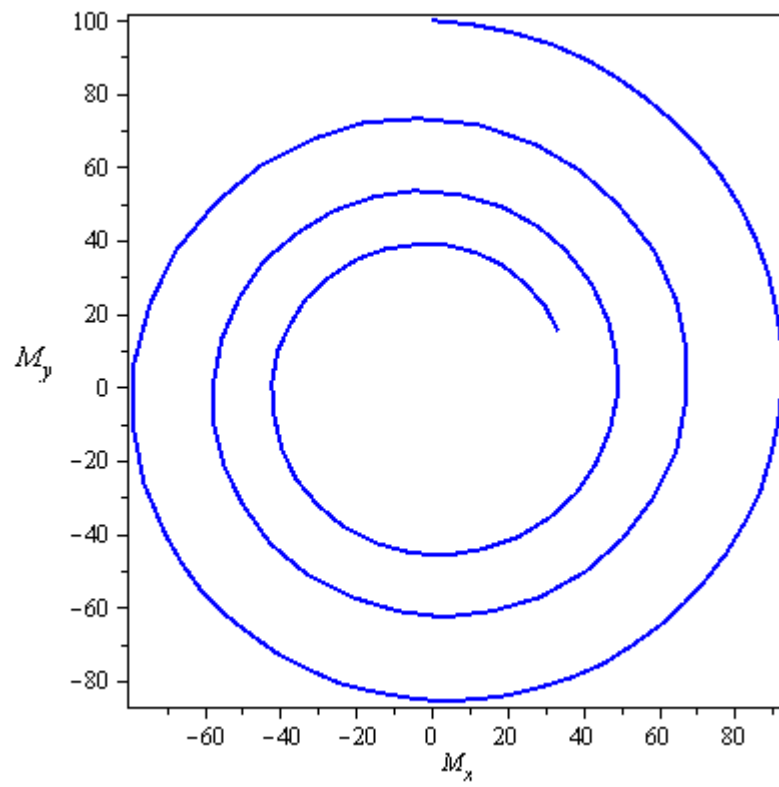
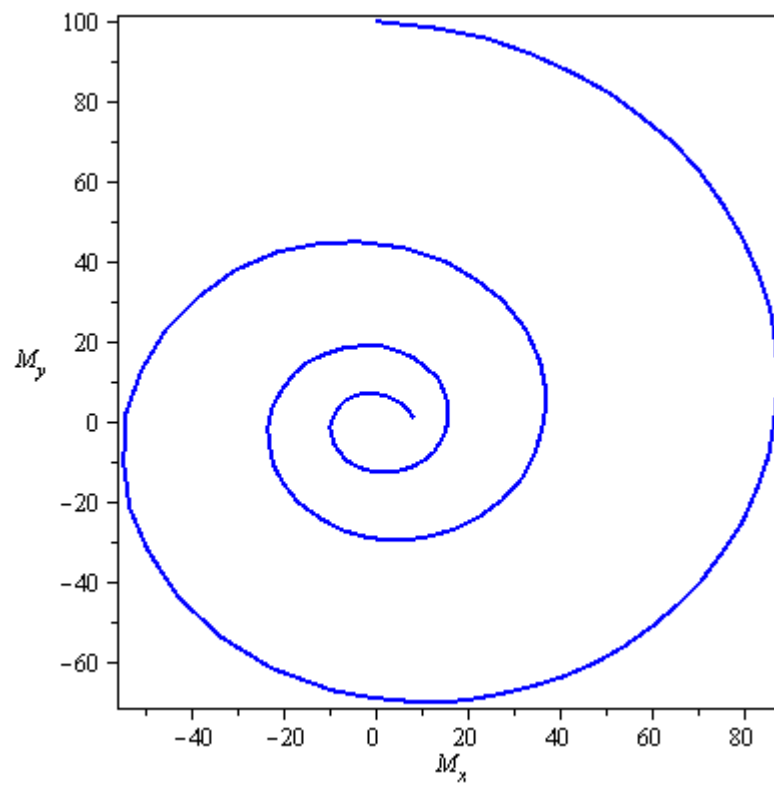


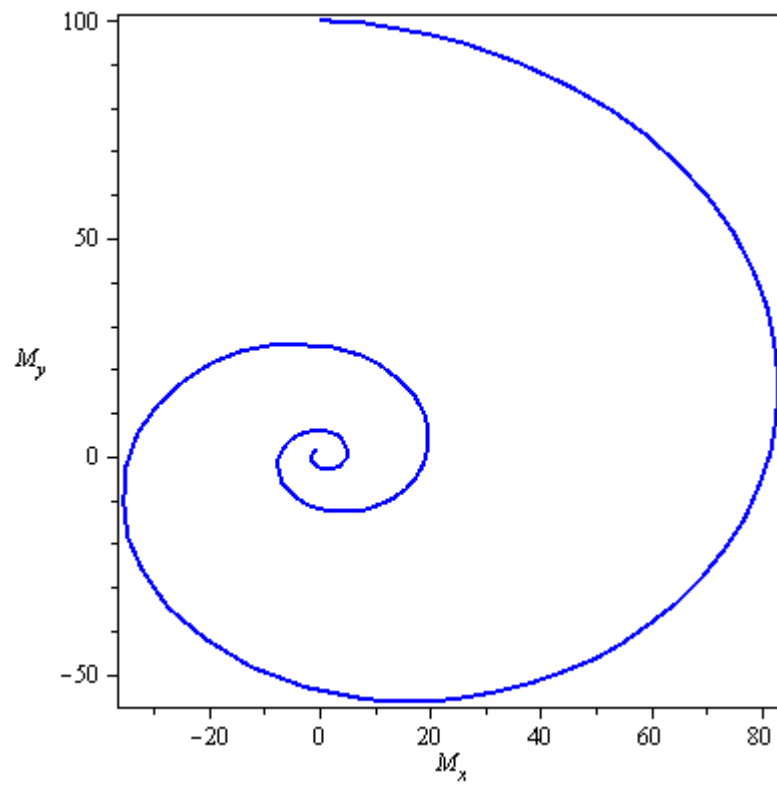
Figure 3. Plots of system magnetization with respect to ξ at for various values of γ and η (a) Characteristic of $M_x(\xi)$; (b) Behavior of $M_y(\xi)$.



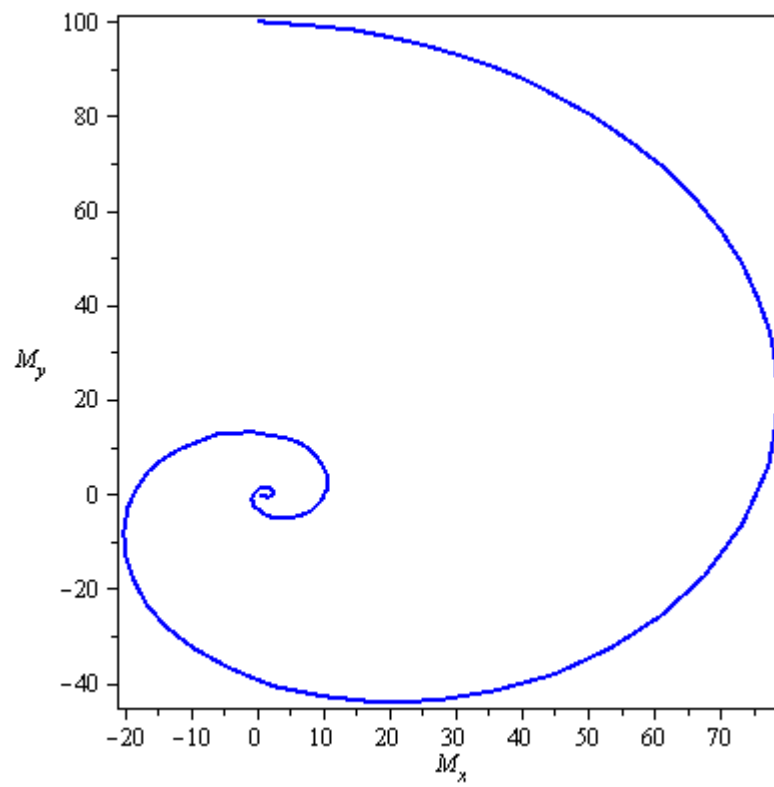
(a)



(b)



(c)

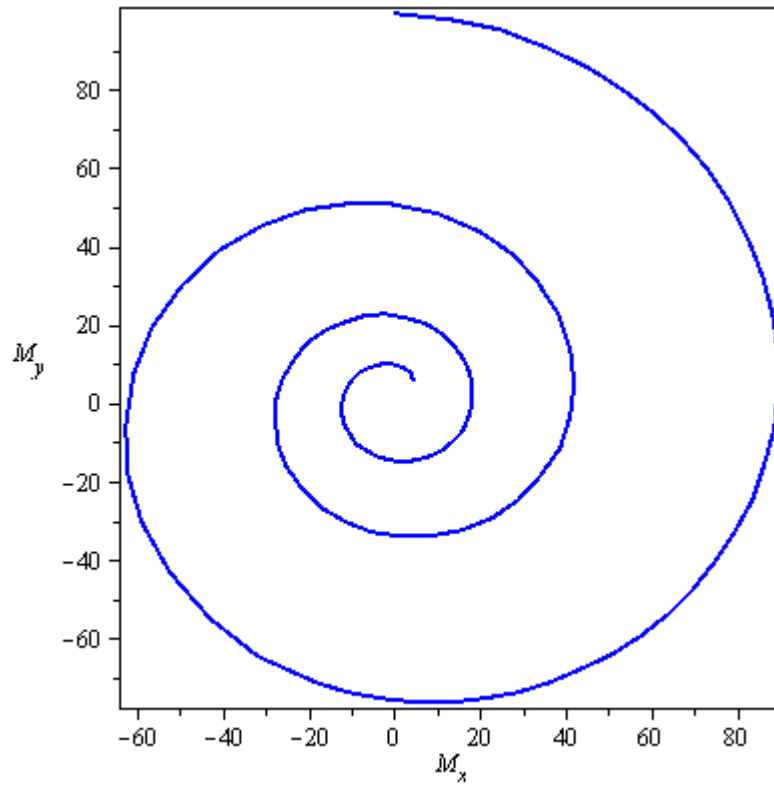


(d)

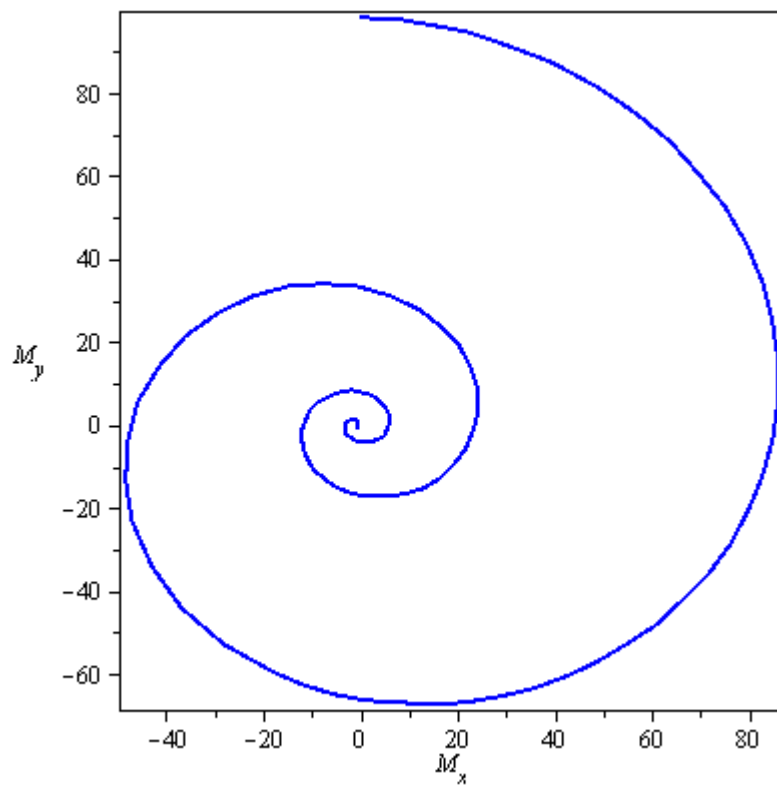
Figure 4. 2 D phase plane of solutions $M_x(\xi)$ and $M_y(\xi)$ at $\eta = 1$ having

$\rho_2' = 20(ms)^\gamma, \bar{\mu}_0 = 1$ with $M_x(0) = 0$ and $M_y(0) = 100$ for (a) $\gamma = 1$ (b) $\gamma = 0.95$ (c)

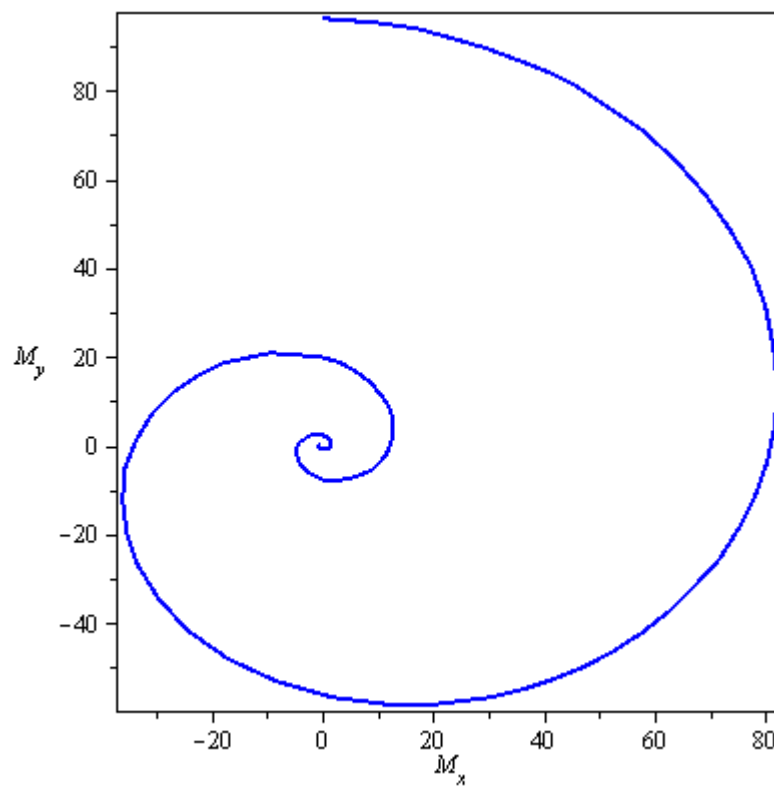
$\gamma = 0.90$ (d) $\gamma = 0.85$.



(a)



(b)

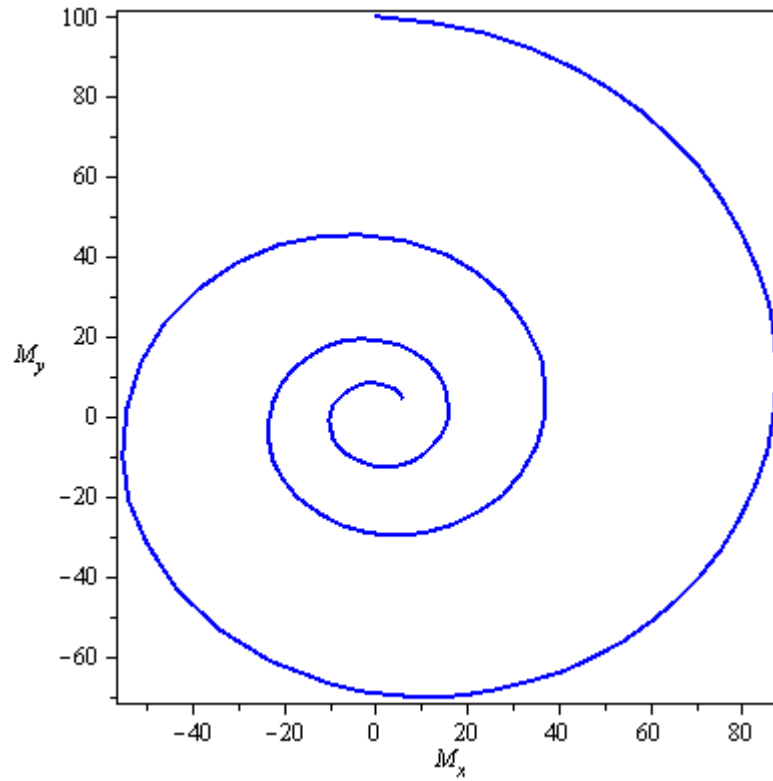


(c)

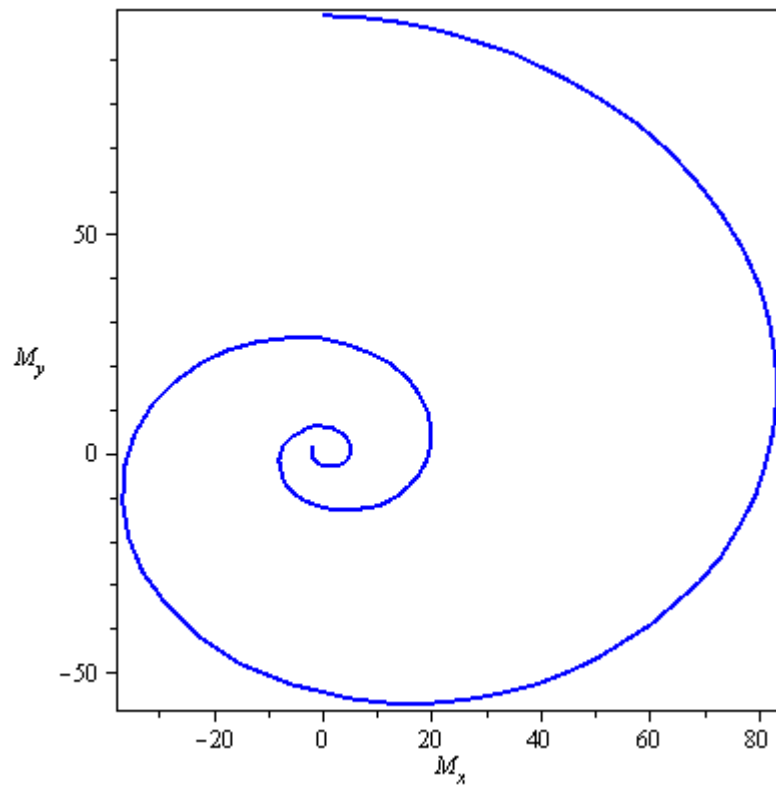
Figure 5. 2 D phase plane of solutions $M_x(\xi)$ and $M_y(\xi)$ at $\eta = 0$ having

$\rho_2' = 20(ms)^\gamma, \bar{\mu}_0 = 1$ with $M_x(0) = 0$ and $M_y(0) = 100$ for (a) $\gamma = 1$ (b) $\gamma = 0.95$ (c)

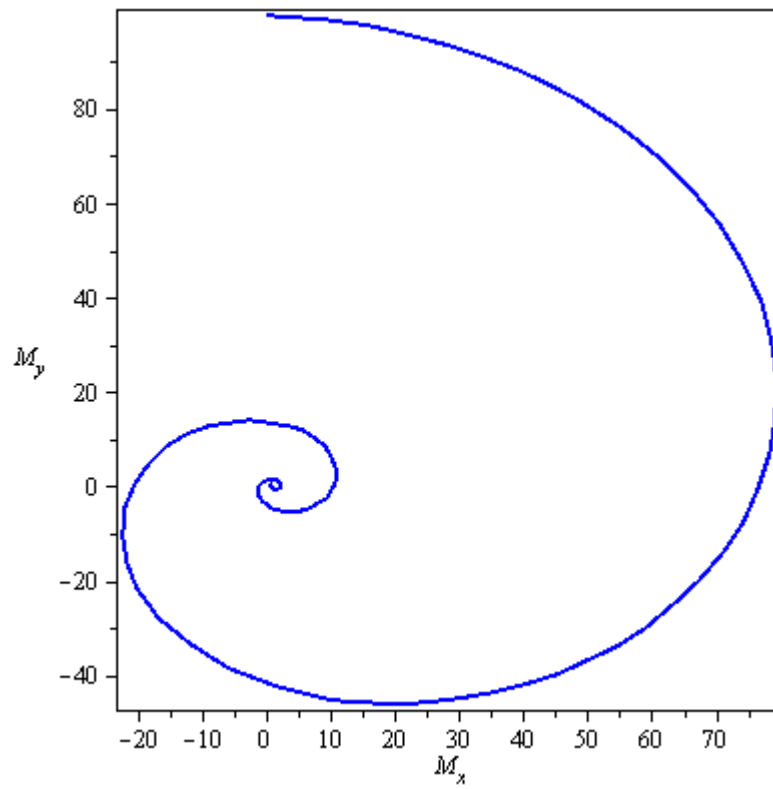
$\gamma = 0.90$ (d) $\gamma = 0.85$.



(a)



(b)

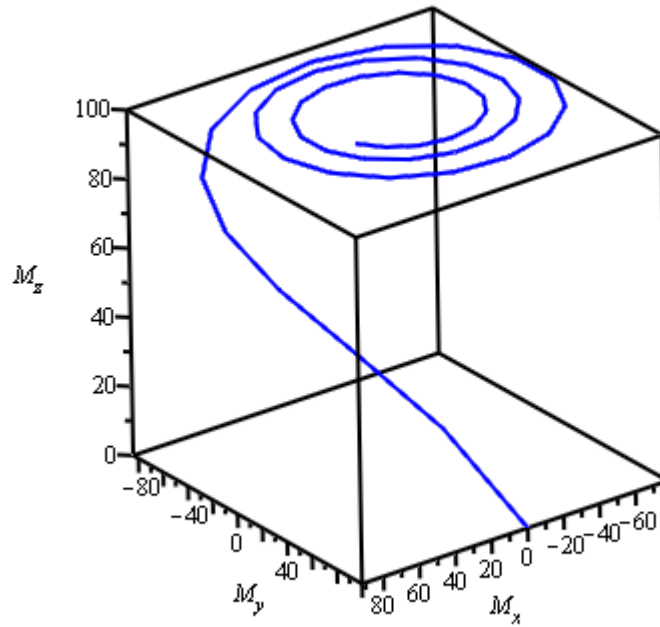


(c)

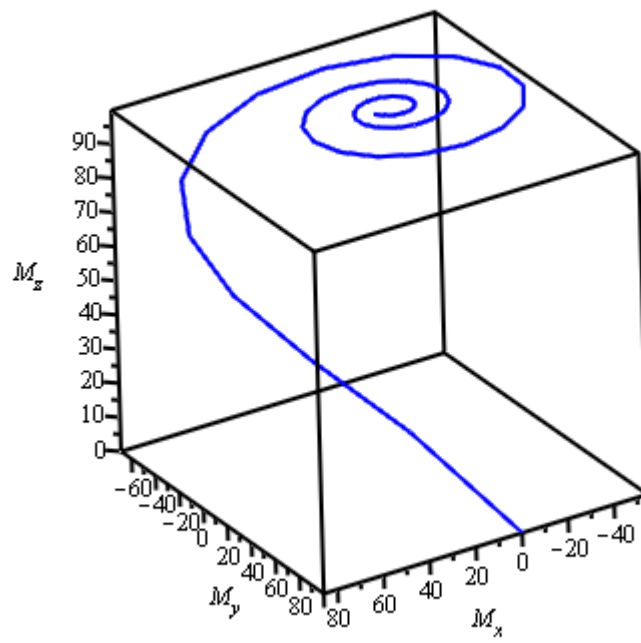
Figure 6. 2 D phase plane of solutions $M_x(\xi)$ and $M_y(\xi)$ at $\rho_2' = 20(ms)^\gamma, \bar{\mu}_0 = 1$

with $M_x(0) = 0$ and $M_y(0) = 100$ for (a) $\gamma = \eta = 0.95$ (b) $\gamma = \eta = 0.90$ (c)

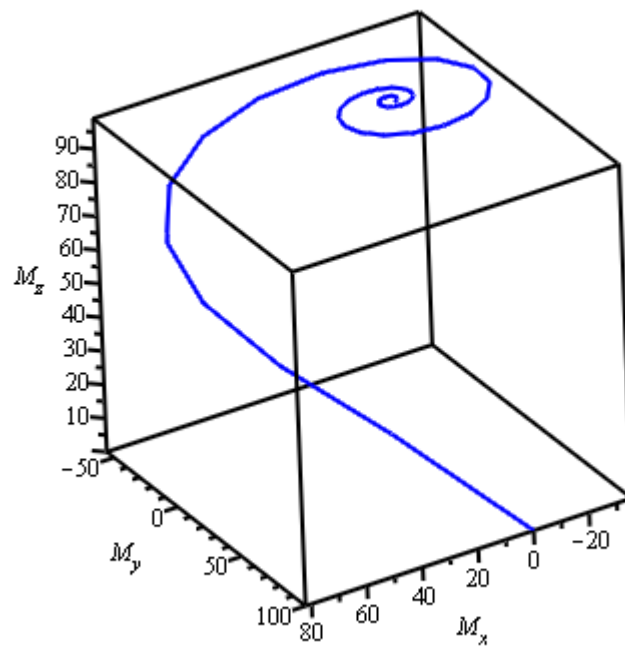
$\gamma = \eta = 0.85$.



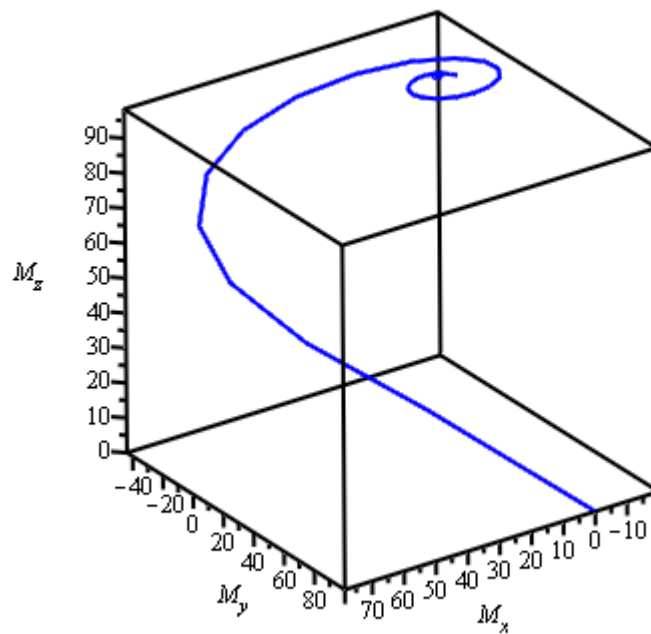
(a)



(b)



(c)



(d)

Figure 7. 3 D phase plane of $M = (M_x, M_y, M_z)$ having $\rho_2' = 20(ms)^\gamma$, $\rho_1' = 20(s)^\gamma$, $\bar{\mu}_0 = 1$ with $M_x(0) = 0$ in addition $M_y(0) = 100$ for (a) $\gamma = 1$ (b) $\gamma = 0.95$ (c) $\gamma = 0.90$ (d) $\gamma = 0.85$.

6. Concluding Remarks

Here, we have discussed new aspects of the Bloch equation pertaining to Hilfer fractional derivative. The solutions of the magnetization components (M_x, M_y, M_z) for fractional Bloch equation have been analyzed by applied an authentic Sumudu transform algorithm. The graphical behavior of magnetization components (M_x, M_y, M_z) have been shown on many arbitrary order values. The relative analysis for magnetization

components (M_x, M_y, M_z) shows novel aspects about the studied fractional Bloch equation that was never evaluated earlier for any type of Bloch equation.

Lastly, by observing all the key facts studied in all the sections and concluding statements, it would be very beneficial when researchers, mathematician would consider the attained outcomes of this article into cogitation when analyzing the results behind the Bloch equation details. The studied fractional Bloch equation system provides basic moreover beneficial consequences to introduce new details in physical sciences. Further, new investigations can be conducted to analyze its validity for more general and broader applications.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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