ON NUMERICAL APPROXIMATION OF ATANGANA-BALEANU-CAPUTO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS UNDER UNCERTAINTY IN HILBERT SPACE

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Abstract. Many dynamic systems can be modeled by fractional differential equations in which some external parameters occur under uncertainty. Although these parameters increase the complexity, they present more acceptable solutions. With the aid of Atangana-Baleanu-Caputo (ABC) fractional differential operator, an advanced numerical-analysis approach is considered and applied in this work to deal with different classes of fuzzy integrodifferential equations of fractional order fitted with uncertain constraints conditions. The fractional derivative of ABC is adopted under the generalized H-differentiability (g-HD) framework, which uses the Mittag-Leffler function as a nonlocal kernel to better describe the timescale of the fuzzy models. Towards this end, applications of reproducing kernel algorithm are extended to solve classes of linear and nonlinear fuzzy fractional ABC Volterra-Fredholm integrodifferential equations. Based on the characterization theorem, preconditions are established under the Lipschitz condition to characterize the fuzzy solution in a coupled equivalent system of crisp ABC integrodifferential equations. Parametric solutions of the ABC interval are provided in terms of rapidly convergent series in Sobolev spaces. Several examples of fuzzy ABC Volterra-Fredholm models are implemented in light of g-HD to demonstrate the feasibility and efficiency of the designed algorithm. Numerical and graphical representations of both classical Caputo and ABC fractional derivatives are presented to show the effect of the ABC derivative on the parametric solutions of the posed models. The achieved results reveal that the proposed method is systematic and suitable for dealing with the fuzzy fractional problems arising in physics, technology, and engineering in terms of the ABC fractional derivative.

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1. INTRODUCTION

Modeling processes of natural phenomena create perceptions and general impressions about the dynamic behavior of any physical system that may involve uncertain parameters that result from many factors such as measurement errors, estimates, expectations, and deficient data. In this direction, fuzzy set theory has been established to describe the uncertainty that appears in mathematical modeling. In 1965, the standard theory was originally introduced by Zadeh [55]. Afterwards, Dubois and Prade [29] proposed the fuzzy real numbers notions along with some of the essential characteristics as well. In a later separate work, Kandel and Byatt [41] introduced the concept of fuzzy differential equations. Anyhow, there are several suggestions for defining fuzzy derivative operators as well as studying fuzzy differential equations, including Seikkala, Goetschel-Voxman, Hukuhara (H-differentiability), Puri-Ralescu, and strongly generalized differentiable concepts [3, 27, 30, 33]. The most popular approaches employing standard, strong, or generalized H-differentiability [11, 24, 36].

On the other hand, the theory of fractional calculus is an interesting topic, not only among mathematicians but also among physicists and engineers for its great importance applications in many fields of engineering and sciences [12, 13, 15, 20, 31, 34, 43]. It has been investigated extensively for describing memory and hereditary for various physical and engineering applications similar to rheology, continuum mechanics, entropy, electromagnetic problems, thermodynamics and so forth [2, 7, 19, 21, 25, 37, 42, 46]. Further, in the literature, there are many definitions of fractional derivatives, such as the concepts of Caputo, Erdelyi-Kober, Riemann-Liouville, Graunwald-Letnikov, Riesz, and Caputo-Fabrizio [17, 22, 39, 44, 49–54]. The most common are the concepts of Riemann-Liouville and Caputo which bring some privacy. Nevertheless, they involve a singular kernel function that may adversely affect a realistic understanding of real-world problems. Indeed, Caputo-Fabrizio concept proposed in [25] includes an exponential kernel function to describe variants and structures of different scales, which cannot be well formulated by standard local derivatives possessing single kernels such as Riemann-Liouville and Caputo. In this orientation, Atangana and Baleanu proposed a novel concept as a generalization of the Caputo-Fabrizio derivative in light of the generalized Mittag-Leffler function $E_\alpha(-z^\alpha) = \sum_{n=0}^{\infty} \frac{(-t)^{\alpha n}}{\Gamma(n\alpha+1)}$ to construct the non-singular as well as nonlocal kernel [21].

As a result of studying the fuzzy theory for fractional calculus, the term of fractional fuzzy differential equations (FFDEs) has been established in 2010 [1]. In [5], it has been proposed the fractional generalization by means of H-differentiable. Recently, the authors [4, 56] defined a novel operator of fractional derivatives based on Atangana-Baleanu-Caputo (ABC) in view of fuzzy valued function with form of parametric interval, called ABC gH-differentiability. In this paper, we intend to study the effect of ABC gH-differentiability on the solution of different types of fuzzy fractional integrodifferential equations (FFIDEs). More specifically, we consider the underlying Fredholm-Volterra FFIDE:

$$\begin{cases} ({}^{ABC}_{gH}D_{a+}^\beta \omega)(t) = f(t) + p(t)\omega(t) + \int_a^b H(t, \tau, \omega(\tau)) d\tau + \int_a^t K(t, \tau, \omega(\tau)) d\tau, \\ \omega(a) = \eta, \quad a \geq 0, \end{cases} \quad (1.1)$$

where $t \geq a$, $0 < \beta \leq 1$, η is a fuzzy number, $f(t)$ is a continuous fuzzy-valued function, $p(t)$ is a continuous real-valued function with nonnegative or nonpositive values on $[a, b]$, and ${}^{ABC}_{gH}D_{a+}^\beta(\cdot)$ denotes the ABC gH-derivative of order β in view of continuous kernel functions $H(t, \tau, \omega(\tau))$ and $K(t, \tau, \omega(\tau))$.

By and large, there are no conventional techniques for finding exact solutions for FFIDEs. Therefore, there is an urgent need for advanced numerical methods to obtain accurate approximate solutions to these equations. In this analysis, we modify a numerical method based on the reproducing kernel theory to obtain approximate solutions of equation (1.1). The current method has many positive advantages. For instance, but not limited to, it is reliable and accurate numerical results can be achieved easily, it can be applied directly without any further assumptions on the structure of specific physical problems, it is not affected by cumulative calculation errors, and it is universal in nature and has a high capacity for solving various nonlinear mathematical issues. Therefore, the RKHS method has received enough attention in the last decade [8–10, 14, 16, 28, 38].

Motivated by the aforementioned discussion, this numerical research aims to design a novel iterative algorithm to obtain solutions to fuzzy integrodifferential equations in terms of the new ABC-fractional concept containing nonsingular and nonlocal kernel under gH-differentiability in addition to studying the effect of ABC-fractional derivative on these solutions. To begin with, several kernel functions are created to establish a complete orthogonal system in the Hilbert space. For this purpose, a linear bounded and invertible fractional operator is defined to extend analytical solutions over a dense and compact interval. Based on reproducing kernel property, the approximate solutions converge uniformly to the analytical solutions. Eventually, some numerical examples are presented to illustrate the reliability and efficiency of the suggested algorithm. This paper is organized as follows: In Section 2, some basic concepts related to fuzzy calculus and fractional calculus are presented. In Section 3, the FFIDE converted into a fractional system of integrodifferential equations. The procedures of RKHSM for solving the general form of both linear and nonlinear FFIDEs are discussed in Section 4. Some numerical examples are carried out in Section 5. This paper ends with a conclusion in Section 6.

2. PRELIMINARIES AND MATHEMATICAL CONCEPTS

In this section, some necessary definitions and mathematical preliminaries of fuzzy calculus and fractional calculus are introduced. To this end, the main concepts used in this analysis are presented, namely, the strongly generalized differentiability, ABC gH-differentiability, and Riemann integrability. Anyhow, a fuzzy number is a generalization of a real number in the sense that it does not refer to a single value but rather to a set of possible values, each with a weight between 0 and 1. This weight is called the membership function.

Definition 2.1. [40] A fuzzy number η is a mapping $\eta: \mathbb{R} \rightarrow [0, 1]$ that satisfies the following properties:

- (1) η is normal. That is, there is $\xi \in \mathbb{R}$ with $\eta(\xi) = 1$.
- (2) η is fuzzy convex. That is, $\eta(\gamma\xi + (1 - \gamma)\zeta) \geq \min(\eta(\xi), \eta(\zeta))$ for all $\xi, \zeta \in \mathbb{R}$ and $\gamma \in [0, 1]$.
- (3) η is upper semi-continuous.
- (4) The set $\{\xi \in \mathbb{R} : \eta(\xi) > 0\}$ is bounded.

The set of all fuzzy numbers is denoted by $\mathbb{R}_{\mathcal{F}}$. An effective way to present a fuzzy number η is by using its r -cut representation which is given by $[\eta]^r = \{x \in \mathbb{R} : \eta(x) \geq r\}$ for $r \in (0, 1]$ and $[\eta]^0 = \overline{\{x \in \mathbb{R} : \eta(x) > 0\}}$. $[\eta]^0$ is called the support of A . The core of A is the crisp set of all points x in $\mathbb{R}$ such that $\eta(x) = 1$. Obviously, if η is a fuzzy number, then $[\eta]^r = [\eta_1(r), \eta_2(r)]$, where $\eta_1(r) = \eta_{1r} = \min\{x | x \in [\eta]^r\}$ and $\eta_2(r) = \eta_{2r} = \max\{x | x \in [\eta]^r\}$ $\forall r \in [0, 1]$. A common form of fuzzy numbers is the triangular fuzzy number, which is characterized below.

Definition 2.2. [45] A fuzzy number η is said to be a generalized triangular fuzzy number, which is denoted by $(a, b, c; \omega)$, if its membership function has the form:

$$\eta(x) = \begin{cases} \frac{x-a}{b-a}\omega, & \text{if } a \leq x < b, \\ \frac{c-x}{c-b}\omega, & \text{if } b \leq x \leq c, \\ 0 & \text{otherwise,} \end{cases}$$

where $a, b, c \in \mathbb{R}$ and $a < b < c$.

The r -cut representation for a generalized triangular fuzzy number $(a, b, c; \omega)$ is $[\eta]^r = \left[a + \frac{(b-a)r}{\omega}, c - \frac{(c-b)r}{\omega} \right]$ for any $r \in (0, 1)$. For more illustration, the underlying theorem gives the necessary conditions of any two real valued functions η_1 and η_2 defined on $[0, 1]$ to get the parametric form $[\eta_1(r), \eta_2(r)]$ of a fuzzy number η for each $r \in [0, 1]$.

Theorem 2.3. [33] Suppose that $\eta_1, \eta_2 : [0, 1] \rightarrow \mathbb{R}$ satisfy the conditions; First, η_1 is a bounded monotonic nondecreasing left continuous function $\forall r \in (0, 1]$ and right continuous for $r = 0$; Second, η_2 is a bounded monotonic nonincreasing left continuous function $\forall r \in (0, 1]$ and right continuous for $r = 0$; Third, $\eta_1(1) \leq \eta_2(1)$. Then, $\eta : \mathbb{R} \rightarrow [0, 1]$ which is defined by $\eta(x) = \text{Sup}\{r \mid \eta_1(r) \leq x \leq \eta_2(r)\}$ is a fuzzy number with parameterization $[\eta]^r = [\eta_1(r), \eta_2(r)]$. Moreover, if η is a fuzzy number with $[\eta]^r = [\eta_1(r), \eta_2(r)]$ (or simply, $[\eta_{1r}, \eta_{2r}]$), then the functions $\eta_1, \eta_2 : [0, 1] \rightarrow \mathbb{R}$ satisfy the above conditions. In this case, we can represent a fuzzy number by an ordered pair of functions (η_1, η_2) .

Arithmetic operations in $\mathbb{R}_{\mathcal{F}}$ can be defined as those on intervals of $\mathbb{R}$. So, for any $\gamma \in \mathbb{R}/\{0\}$, and $u, v \in \mathbb{R}_{\mathcal{F}}$ with $[u]^r = [u_{1r}, u_{2r}]$ and $[v]^r = [v_{1r}, v_{2r}]$, we have $[u + v]^r = [u]^r + [v]^r = [u_{1r} + v_{1r}, u_{2r} + v_{2r}]$, and $[\gamma u]^r = \gamma[u]^r = [\min\{\gamma u_{1r}, \gamma u_{2r}\}, \max\{\gamma u_{1r}, \gamma u_{2r}\}]$. Obviously, $\mathbb{R}_{\mathcal{F}}$ does not form a vector space with the zero element $\{0\}$. Hence, additive simplification is not valid, that is $u + v = u + w \nRightarrow v = w$ for fuzzy numbers u, v and w .

To overcome this situation, we have the Hukuhara difference (H-difference). The H-difference of $u, v \in \mathbb{R}_{\mathcal{F}}$, denoted by $u \ominus v = w$, is the fuzzy number that satisfies $u = v + w$. Its r -cut representation is $[u \ominus v]^r = [u_{1r} - v_{1r}, u_{2r} - v_{2r}]$. In [24], Bede and Stefanini introduced the gH-difference as follows

$$u \ominus_{gH} v = w \Leftrightarrow u = v + w \text{ or } v = u + (-1)w, \quad (2.1)$$

with r -cuts $[u \ominus_{gH} v]^r = [\min\{u_{1r} - v_{1r}, u_{2r} - v_{2r}\}, \max\{u_{1r} - v_{1r}, u_{2r} - v_{2r}\}]$.

Definition 2.4. [47] The Housdorff metric D on $\mathbb{R}_{\mathcal{F}}$ is defined by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}^+ \cup \{0\}$ such that $D(u, v) = \text{Sup}_{r \in [0, 1]} \max\{|u_{1r} - v_{1r}|, |u_{2r} - v_{2r}|\}$ for any fuzzy numbers $u = (u_1, u_2)$ and $v = (v_1, v_2)$.

A fuzzy function on an interval T is a mapping $F : T \rightarrow \mathbb{R}_{\mathcal{F}}$. If for fixed $t_0 \in T$ and $\varepsilon > 0$, $\exists \delta > 0$ such that $|t - t_0| < \delta \implies D(F(t), F(t_0)) < \varepsilon$, then we say that F is continuous at t_0 . If F is continuous $\forall t \in T$, then F is continuous on T [32]. The differentiability of a fuzzy function can be defined as follows.

Definition 2.5. [4] Let $F : (a, b) \rightarrow \mathbb{R}_{\mathcal{F}}$ and $t_0 \in (a, b)$. We say that F is generalized Hukuhara differentiable (gH-differentiable) at t_0 if there exists a fuzzy number ${}_gHDF(t_0)$ such that

$${}_gHDF(t_0) = \lim_{h \rightarrow 0^+} \frac{F(t_0 + h) \ominus_{gH} F(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{F(t_0) \ominus_{gH} F(t_0 + h)}{h},$$

where the limits here are taken in the metric space $(\mathbb{R}_{\mathcal{F}}, D)$.

With the help of gH-difference (2.1), the parametric interval form of the gH-derivative for a function $F(t)$ has only one of the following forms:

- Case 1: (1)-differentiability: $[{}_1-{}_gHDF(t)]^r = [F'_{1r}(t), F'_{2r}(t)]$, $0 \leq r \leq 1$.
- Case 2: (2)-differentiability: $[{}_2-{}_gHDF(t)]^r = [F'_{2r}(t), F'_{1r}(t)]$, $0 \leq r \leq 1$.

For integration of a fuzzy valued function, many approaches have been proposed such as Lebesgue integral [40] and fuzzy Riemann integral [6]. The Lebesgue integral ${}_gHDF(t)$ within interval parametric form is more convenient, which is defined level wise by

$$\left[\int_a^t {}_gHDF(t) dx \right]^r = \begin{cases} \left[\int_a^t F'_{1r}(x) dx, \int_a^t F'_{2r}(x) dx \right] & \text{for (1) - differentiability,} \\ \left[\int_a^t F'_{2r}(x) dx, \int_a^t F'_{1r}(x) dx \right] & \text{for (2) - differentiability.} \end{cases} \quad (2.2)$$

This analysis deals with ABC gH-derivative but before presenting its definition, it is worth recalling the Caputo and ABC fractional derivatives of crisp functions. Hereunder, the underlying notations will be used. $L_p[a, b] = \{f : [a, b] \rightarrow \mathbb{R}, \int_a^b |f(x)|^p dx < \infty, 1 \leq p < \infty\}$; $C^F[a, b]$ refers to the space of all continuous fuzzy

valued functions on $[a, b]$; $H^1(a, b) = \{F: F, F' \in L_2(a, b)\}$ refers to usual Sobolev space; $L_p^F[a, b] = \{F: [a, b] \rightarrow \mathbb{R}_F, F \text{ is measurable and } \int_a^b D(F(x), 0)^p dx < \infty, 1 \leq p < \infty\}$, as well as $AC[0, \infty) = \{f: [0, \infty) \rightarrow \mathbb{R} \text{ is absolutely continuous on } [0, \infty)\}$.

Definition 2.6. [42] The Caputo fractional derivative (CFD) operator of order $\beta > 0$ for a function $g(x) \in AC[a, b]$ is given as

$${}^C D_{a+}^\beta g(x) = \frac{1}{\Gamma(n-\beta)} \int_a^x \frac{g^{(n)}(\tau)}{(x-\tau)^{\beta-n+1}} d\tau,$$

where $n-1 < \beta \leq n$, $n \in \mathbb{N}$, $x > a$. Specifically, we have ${}^C D_{a+}^\beta g(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{g'(\tau)}{(x-\tau)^\beta} d\tau$ for $0 < \beta < 1$.

The CFD possesses a kernel with singularity that includes the memory effect, so this definition cannot describe the full effect of memory. Hereby, we present the Atangana-Baleanu definition of Caputo-type in which the singular kernel will be replaced by the Mittag-Leffler function.

Definition 2.7. [21] The Atangana-Baleanu fractional derivative of Caputo type (ABC) of order $\beta \in [0, 1]$ for a function $g(x) \in H^1(a, b)$ is given as

$${}^{ABC} D_a^\beta g(x) = \frac{\kappa(\beta)}{1-\beta} \int_a^x g'(\tau) E_\beta \left(-\beta \frac{(x-\tau)^\beta}{1-\beta} \right) d\tau = \frac{\kappa(\beta)}{1-\beta} g'(x) * E_\beta \left(\frac{-\beta x^\beta}{1-\beta} \right),$$

provided that $\kappa(\beta)$ is a normalization constant so that $\kappa(0) = \kappa(1) = 1$. Specifically, when $\kappa(\beta) = 1$, the ABC derivative can be presented as follows

$${}^{ABC} D_a^\beta g(x) = \frac{1}{1-\beta} \int_a^x g'(\tau) E_\beta \left(-\beta \frac{(x-\tau)^\beta}{1-\beta} \right) d\tau, \quad 0 < \beta \leq 1,$$

where the symbol $(*)$ denotes the convolution.

Definition 2.8. [21] The integral associated with the ABC derivative of order β is given as:

$${}^{AB} I_a^\beta (g(x)) = \frac{1-\beta}{\kappa(\beta)} g(x) + \frac{\beta}{\kappa(\beta)\Gamma(\beta)} \int_a^x g(\tau) (x-\tau)^{\beta-1} d\tau, \quad 0 < \beta \leq 1.$$

For $\beta = 1$, we obtain the classical integral as well as for $\beta = 0$, the ABC integral becomes the identity operator.

Note that the ABC derivative of any constant is zero, i.e., ${}^{ABC} D_a^\beta C = 0$ for any constant C . Further, the composition relation between the ABC derivative and integral [23] so that ${}^{AB} I_a^\beta \left({}^{ABC} D_a^\beta g(x) \right) = g(x) - g(a)$. Furthermore, in order to have an average equals to 1 in Definition 2.8, put $\frac{1-\beta}{\kappa(\beta)} + \frac{\beta}{\kappa(\beta)\Gamma(\beta)} = 1$ leads to $\kappa(\beta) = 1 - \beta + \frac{\beta}{\Gamma(\beta)}$. The generalized H-differentiability was used to expand the definitions of fractional derivatives in the crisp sense for the fuzzy space as follows.

Definition 2.9. [48] Let $0 < \beta < 1$, $F: [a, b] \rightarrow \mathbb{R}_F$ and $F \in C^F[a, b] \cap L^F[a, b]$. Then, F is said to be Caputo's gH-differentiable at x if $({}^C D_{a+}^\beta F)(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{F'(t)}{(x-t)^\beta} dt$ exists. We say that F is ${}^c[(1)-\beta]$ -differentiable if F is (1)-differentiable, and F is ${}^c[(2)-\beta]$ -differentiable if F is (2)-differentiable.

Theorem 2.10. [48] Let $0 < \beta < 1$, $F: [a, b] \rightarrow \mathbb{R}_F$ and $F \in AC^F[a, b]$ with $[F(x)]^r = [F_{1r}(x), F_{2r}(x)]$. Then, the fuzzy CFD exists almost everywhere on (a, b) so that

$$\begin{aligned}
- \left[({}_{1-gH}^C D_{a+}^\beta F)(x) \right]^r &= \left[({}^C D_{a+}^\beta F_{1r})(x), ({}^C D_{a+}^\beta F_{2r})(x) \right], \text{ if } F \text{ is } {}_{gH}^C[(1) - \beta]\text{-differentiable,} \\
- \left[({}_{2-gH}^C D_{a+}^\beta F)(x) \right]^r &= \left[({}^C D_{a+}^\beta F_{2r})(x), ({}^C D_{a+}^\beta F_{1r})(x) \right], \text{ if } F \text{ is } {}_{gH}^C[(2) - \beta]\text{-differentiable.}
\end{aligned}$$

Definition 2.11. [4] Let $0 < \beta < 1$, $F: [a, b] \rightarrow \mathbb{R}_F$ and $F \in C^F[a, b] \cap L^F[a, b]$ with $[F(x)]^r = [F_{1r}(x), F_{2r}(x)]$. Then, the fuzzy gH-Atangana-Baleanu-Caputo fractional differentiable fuzzy-valued function (ABC gH-differentiability) is defined by means of the underlying two cases:

$$\begin{aligned}
- \text{ If } {}_{gH}^{ABC}[(1) - \beta]\text{-differentiability: } & \left({}_{1-gH}^{ABC} D_{a+}^\beta F \right)(x) = \frac{\kappa(\beta)}{1-\beta} \int_a^x {}_{1-gH} D F(\tau) E_\beta \left(-\beta \frac{(x-\tau)^\beta}{1-\beta} \right) d\tau, \\
- \text{ If } {}_{gH}^{ABC}[(2) - \beta]\text{-differentiability: } & \left({}_{2-gH}^{ABC} D_{a+}^\beta F \right)(x) = \frac{\kappa(\beta)}{1-\beta} \int_a^x {}_{2-gH} D F(\tau) E_\beta \left(-\beta \frac{(x-\tau)^\beta}{1-\beta} \right) d\tau.
\end{aligned}$$

Consequently, the interval parametric forms in the two cases can be deduced from Lebesgue integral in (2.2) as follows:

$$\begin{aligned}
- \left[\left({}_{1-gH}^{ABC} D_{a+}^\beta F \right)(x) \right]^r &= \left[\left({}^{ABC}_1 D_{a+}^\beta F_{1r} \right)(x), \left({}^{ABC}_1 D_{a+}^\beta F_{2r} \right)(x) \right], \\
- \left[\left({}_{2-gH}^{ABC} D_{a+}^\beta F \right)(x) \right]^r &= \left[\left({}^{ABC}_2 D_{a+}^\beta F_{2r} \right)(x), \left({}^{ABC}_2 D_{a+}^\beta F_{1r} \right)(x) \right],
\end{aligned}$$

where

$$\begin{aligned}
\left({}^{ABC}_i D_{a+}^\beta F_{1r} \right)(x) &= \frac{\kappa(\beta)}{1-\beta} \int_a^x F'_{1r}(\tau) E_\beta \left(-\beta \frac{(x-\tau)^\beta}{1-\beta} \right) \Big|_{E_\beta \geq 0} d\tau + \frac{\kappa(\beta)}{1-\beta} \int_a^x F'_{2r}(\tau) E_\beta \left(-\beta \frac{(x-\tau)^\beta}{1-\beta} \right) \Big|_{E_\beta < 0} d\tau, \\
\left({}^{ABC}_i D_{a+}^\beta F_{2r} \right)(x) &= \frac{\kappa(\beta)}{1-\beta} \int_a^x F'_{2r}(\tau) E_\beta \left(-\beta \frac{(x-\tau)^\beta}{1-\beta} \right) \Big|_{E_\beta \geq 0} d\tau + \frac{\kappa(\beta)}{1-\beta} \int_a^x F'_{1r}(\tau) E_\beta \left(-\beta \frac{(x-\tau)^\beta}{1-\beta} \right) \Big|_{E_\beta < 0} d\tau,
\end{aligned}$$

for $i = 1, 2$. Moreover, the integral associated with the ABC gH-derivative can be formulated in interval representation as

$$\begin{aligned}
\left[\left({}^{ABC} I_{a+}^\beta F \right)(x) \right]^r &= \left[\left({}^{ABC} I_{a+}^\beta F_{1r} \right)(x), \left({}^{ABC} I_{a+}^\beta F_{2r} \right)(x) \right] \\
&= \left[\frac{1-\beta}{\kappa(\beta)} F_{1r}(x) + \frac{\beta}{\kappa(\beta) \Gamma(\beta)} \int_a^x F_{1r}(\tau) (x-\tau)^{\beta-1} d\tau, \frac{1-\beta}{\kappa(\beta)} F_{2r}(x) \right. \\
&\quad \left. + \frac{\beta}{\kappa(\beta) \Gamma(\beta)} \int_a^x F_{2r}(\tau) (x-\tau)^{\beta-1} d\tau \right].
\end{aligned}$$

It is worth to be mentioned that it was shown in [4] that for $0 \leq r \leq 1$,

$${}^{ABC} I_{a+}^\beta \left({}^{ABC} D_{a+}^\beta F \right)(x) = (F(x) - F(0))|_{E_\beta \geq 0} - (-1)(F(x) - F(0))|_{E_\beta < 0}.$$

3. ABC gH-DIFFERENTIABILITY FORMULATION OF FFIDES

In this section, we adopt the interval ABC gH-derivative to handle a general form of Fredholm-Volterra FFIDE (1.1). Without loss of generality, we assume that the kernels are separable such that $H(t, \tau, x(\tau)) = h(t, \tau)x(\tau)$ for $\tau \in [a, b]$ and $K(t, \tau, x(\tau)) = k(t, \tau)x(\tau)$ for $\tau \in [a, t]$ with $h(t, \tau) \geq 0$ for $\tau \in [a, c_1]$, $h(t, \tau) \leq 0$ for $\tau \in [c_1, b]$, $k(t, \tau) \geq 0$ for $\tau \in [a, c_2]$ and $k(t, \tau) \leq 0$ for $\tau \in [c_2, t]$. Moreover, we suppose that $p(t) \geq 0$ on $[a, b]$.

To perform the procedure, write $f(t)$ and $\omega(t)$ in term of their r -cut representations: $[f(t)]^r = [f_{1r}(t), f_{2r}(t)]$, $[\omega(t)]^r = [\omega_{1r}(t), \omega_{2r}(t)]$ and $[\omega(a)]^r = [\omega_{1r}(a), \omega_{2r}(a)] = [\alpha_{1r}, \alpha_{2r}]$. Hence, the interval parametric form of FFIDE

(1.1) is

$$\left[({}^{ABC}_{gH}D_{a+}^{\beta}\omega)(t)\right]^r = p(t)[\omega(t)]^r + \int_a^b [h(t,\tau)\omega(\tau)]^r d\tau + \int_a^t [k(t,\tau)\omega(\tau)]^r d\tau + [f(t)]^r, [\omega(a)]^r = [\alpha]^r.$$

Thus, the FFIDE (1.1) can be translated into one of the underlying systems:

- In the case of ${}^{ABC}_{gH}[(1) - \beta]$ -differentiability, we have

$$\begin{aligned} & \frac{\kappa(\beta)}{1-\beta} \int_a^x \left(\omega'_{1r}(\tau) E_{\beta} \left(-\beta \frac{(x-\tau)^{\beta}}{1-\beta} \right) \Big|_{E_{\beta} \geq 0} + \omega'_{2r}(\tau) E_{\beta} \left(-\beta \frac{(x-\tau)^{\beta}}{1-\beta} \right) \Big|_{E_{\beta} < 0} \right) d\tau \\ &= p(t) \omega_{1r}(t) + \int_a^{c_1} h(t,\tau) \omega_{1r}(t) d\tau + \int_{c_1}^b h(t,\tau) \omega_{2r}(t) d\tau \\ & \quad + \int_a^{c_2} k(t,\tau) \omega_{1r}(t) d\tau + \int_{c_2}^t k(t,\tau) \omega_{2r}(t) d\tau + f_{1r}(t), \\ & \frac{\kappa(\beta)}{1-\beta} \int_a^x \left(\omega'_{2r}(\tau) E_{\beta} \left(-\beta \frac{(x-\tau)^{\beta}}{1-\beta} \right) \Big|_{E_{\beta} \geq 0} + \omega'_{1r}(\tau) E_{\beta} \left(-\beta \frac{(x-\tau)^{\beta}}{1-\beta} \right) \Big|_{E_{\beta} < 0} \right) d\tau \\ &= p(t) \omega_{2r}(t) + \int_a^{c_1} h(t,\tau) \omega_{2r}(t) d\tau + \int_{c_1}^b h(t,\tau) \omega_{1r}(t) d\tau \\ & \quad + \int_a^{c_2} k(t,\tau) \omega_{2r}(t) d\tau + \int_{c_2}^t k(t,\tau) \omega_{1r}(t) d\tau + f_{2r}(t) \end{aligned} \quad (3.1)$$

- In the case of ${}^{ABC}_{gH}[(2) - \beta]$ -differentiability, we have

$$\begin{aligned} & \frac{\kappa(\beta)}{1-\beta} \int_a^x \left(\omega'_{1r}(\tau) E_{\beta} \left(-\beta \frac{(x-\tau)^{\beta}}{1-\beta} \right) \Big|_{E_{\beta} \geq 0} + \omega'_{2r}(\tau) E_{\beta} \left(-\beta \frac{(x-\tau)^{\beta}}{1-\beta} \right) \Big|_{E_{\beta} < 0} \right) d\tau \\ &= p(t) \omega_{2r}(t) + \int_a^{c_1} h(t,\tau) \omega_{2r}(t) d\tau + \int_{c_1}^b h(t,\tau) \omega_{1r}(t) d\tau \\ & \quad + \int_a^{c_2} k(t,\tau) \omega_{2r}(t) d\tau + \int_{c_2}^t k(t,\tau) \omega_{1r}(t) d\tau + f_{2r}(t), \\ & \frac{\kappa(\beta)}{1-\beta} \int_a^x \left(\omega'_{2r}(\tau) E_{\beta} \left(-\beta \frac{(x-\tau)^{\beta}}{1-\beta} \right) \Big|_{E_{\beta} \geq 0} + \omega'_{1r}(\tau) E_{\beta} \left(-\beta \frac{(x-\tau)^{\beta}}{1-\beta} \right) \Big|_{E_{\beta} < 0} \right) d\tau \\ &= p(t) \omega_{1r}(t) + \int_a^{c_1} h(t,\tau) \omega_{1r}(t) d\tau + \int_{c_1}^b h(t,\tau) \omega_{2r}(t) d\tau \\ & \quad + \int_a^{c_2} k(t,\tau) \omega_{1r}(t) d\tau + \int_{c_2}^t k(t,\tau) \omega_{2r}(t) d\tau + f_{1r}(t); \end{aligned} \quad (3.2)$$

along with the following initial conditions

$$\omega_{1r}(a) = \alpha_{1r} \text{ and } \omega_{2r}(a) = \alpha_{2r}. \quad (3.3)$$

Algorithm 3.1. Let $\omega(t)$ be the unique analytical solution of the FFIDE (1.1). To obtain an interval approach for the approximate fuzzy solution of FFIDE (1.1) in the sense of ABC gH-differentiability using RKHS technique, perform the underlying steps:

Step 1: Take the r -cut for both sides of equation (1.1) in light of assuming $^{ABC}_{gH}[(i) - \beta]$ -differentiability, $i = 1, 2$.

Step 2: Convert equation (1.1) into the equivalent system of parametric ABC fractional IDEs (3.1) or (3.2) equipped with conditions (3.3).

Step 3: Use RKHS technique to approximate the parametric solutions for system (3.1) or (3.2) along with (3.3).

4. ABC FFIDES IN TERMS OF RKHS ALGORITHM

In this section, we present a quick review to some basic definitions and theorems concerning the reproducing kernel Theory. For this purpose, we recall the definition of direct sum of Hilbert spaces $W_2^i[a, b] \oplus W_2^i[a, b]$; $i = 1, 2$. Subsequently, RKHS algorithm will be implemented to solve the ABC fuzzy fractional IDEs (3.1) or (3.2) along with (3.3). For more details, theorems, and applications for the reproducing kernel technique, the reader may refer to [18, 26, 35].

Definition 4.1. [18] Let $\mathcal{H}$ be Hilbert space over the set of complex numbers $\mathbb{C}$. A function $\mathcal{K} : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{C}$ is called a reproducing kernel of the Hilbert space $\mathcal{H}$ if and only if $\forall m \in \mathcal{M}$ and $\forall \mathfrak{h} \in \mathcal{H}$, we have $\mathcal{K}(\cdot, m) \in \mathcal{H}$ and $(\mathfrak{h}(\cdot), \mathcal{K}(\cdot, m)) = \mathfrak{h}(m)$.

The property of reproducing any element of $\mathcal{H}$ by the function $\mathcal{K}$ is called a reproducing property, in which the space $\mathcal{H}$ of this case is called a reproducing Kernel Hilbert space (RKHS). Some of the desirable complete RKHSs [26] are constructed as follows.

- $W_2^1[a, b] = \{\mathfrak{p} : [a, b] \rightarrow \mathbb{R} : \mathfrak{p} \in AC[a, b], \mathfrak{p}' \in L_2[a, b]\}$ embedded with $\langle \mathfrak{p}, \mathfrak{q} \rangle_{W_2^1} = \mathfrak{p}(a)\mathfrak{q}(a) + \int_a^b \mathfrak{p}'(t)\mathfrak{q}'(t)dt$ and $\|\mathfrak{p}\|_{W_2^1} = \sqrt{\langle \mathfrak{p}(t), \mathfrak{p}(t) \rangle_{W_2^1}}$; $\mathfrak{p}, \mathfrak{q} \in W_2^1[a, b]$. The reproducing function is $R_t(s) = \begin{cases} 1 - a + t, & s \leq t, \\ 1 - a + s, & s > t. \end{cases}$
- $W_2^2[a, b] = \{\mathfrak{p} : \mathfrak{p}, \mathfrak{p}' \in AC[a, b], \mathfrak{p}'' \in L_2[a, b], \mathfrak{p}(a) = 0\}$ embedded with $\langle \mathfrak{p}, \mathfrak{q} \rangle_{W_2^2} = \mathfrak{p}'(a)\mathfrak{q}'(a) + \int_a^b \mathfrak{p}''(t)\mathfrak{q}''(t)dt$ and $\|\mathfrak{p}\|_{W_2^2} = \sqrt{\langle \mathfrak{p}(t), \mathfrak{p}(t) \rangle_{W_2^2}}$; $\mathfrak{p}, \mathfrak{q} \in W_2^2[a, b]$. The reproducing function is $G_t(s) = \begin{cases} \frac{1}{6}(s-a)(2a^2-s^2+3t(2+s)-a(6+3t+s)), & s \leq t, \\ \frac{1}{6}(t-a)(2a^2-t^2+3s(2+t)-a(6+3s+t)), & s > t. \end{cases}$

Remark 4.2. [18] The image of the interval $[a, b]$ under the mappings $R : [a, b] \times [a, b] \rightarrow \mathbb{R}$ and $G : [a, b] \times [a, b] \rightarrow \mathbb{R}$ is $1 + [0, b - a]$ and $\frac{1}{6}(b - a)^2[-b - 5a, 3(a + b) + 6]$, respectively.

Furthermore, the direct sums $W_2^i[a, b] \oplus W_2^i[a, b]$, $i = 1, 2$ of these RKHS's are formulated as

- $\Theta[a, b] = W_2^1[a, b] \oplus W_2^1[a, b] = \left\{ [\mathfrak{p}_1(t), \mathfrak{p}_2(t)]^T : \mathfrak{p}_1, \mathfrak{p}_2 \in W_2^1 \right\}$ equipped with $\langle \mathfrak{p}(t), \mathfrak{q}(t) \rangle_{\Theta} = \sum_{i=1}^2 \langle \mathfrak{p}_i(t), \mathfrak{q}_i(t) \rangle_{W_2^1}$ and $\|\mathfrak{p}\|_{\Theta} = \sqrt{\sum_{i=1}^2 \|\mathfrak{p}_i\|_{W_2^1}^2}$; $\mathfrak{p}(t) = [\mathfrak{p}_1(t), \mathfrak{p}_2(t)]^T, \mathfrak{q} = [\mathfrak{q}_1(t), \mathfrak{q}_2(t)]^T \in \Theta[a, b]$.
- $\Upsilon[a, b] = W_2^2[a, b] \oplus W_2^2[a, b] = \left\{ [\mathfrak{p}_1(t), \mathfrak{p}_2(t)]^T : \mathfrak{p}_1, \mathfrak{p}_2 \in W_2^2 \right\}$ equipped with $\langle \mathfrak{p}(t), \mathfrak{q}(t) \rangle_{\Upsilon} = \sum_{i=1}^2 \langle \mathfrak{p}_i(t), \mathfrak{q}_i(t) \rangle_{W_2^2}$ and $\|\mathfrak{p}\|_{\Upsilon} = \sqrt{\sum_{i=1}^2 \|\mathfrak{p}_i\|_{W_2^2}^2}$; $\mathfrak{p}(t) = [\mathfrak{p}_1(t), \mathfrak{p}_2(t)]^T, \mathfrak{q} = [\mathfrak{q}_1(t), \mathfrak{q}_2(t)]^T \in \Upsilon[a, b]$.

In the following, the RKHS method is implemented to obtain approximate and analytic solutions of FFIDE (1.1) in interval parametric form under assuming of $^{ABC}_{gH}[(1) - \beta]$ -differentiability. While the same procedures can be employed for $^{ABC}_{gH}[(2) - \beta]$ -differentiability. Towards this end, define the integrodifferential operators $l_{ij} : W_2^2[a, b] \rightarrow W_2^1[a, b]$, $i, j = 1, 2$, such that

$$l_{ij}z(t) = \begin{cases} \left. \frac{\kappa(\beta)}{1-\beta} \int_a^t z'(\tau) E_{\beta} \left(-\beta \frac{(t-\tau)^{\beta}}{1-\beta} \right) d\tau - p(t)z(t) - \int_a^{c_1} h(t, \tau) z(\tau) d\tau - \int_a^{c_2} k(t, \tau) z(\tau) d\tau, \right|_{E_{\beta} \geq 0}, & i = j, \\ \left. \frac{\kappa(\beta)}{1-\beta} \int_a^t z'(\tau) E_{\beta} \left(-\beta \frac{(t-\tau)^{\beta}}{1-\beta} \right) d\tau - \int_{c_1}^b h(t, \tau) z(\tau) d\tau - \int_{c_2}^t k(t, \tau) z(\tau) d\tau, \right|_{E_{\beta} < 0}, & i \neq j. \end{cases} \quad (4.1)$$

In this direction, put $F_r = \begin{bmatrix} f_{1r} \\ f_{2r} \end{bmatrix}$, $\omega_r = \begin{bmatrix} \omega_{1r} \\ \omega_{2r} \end{bmatrix}$, $\alpha_r = \begin{bmatrix} \alpha_{1r} \\ \alpha_{2r} \end{bmatrix}$, and $L = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$ so that $L : \Upsilon[a, b] \rightarrow \Theta[a, b]$, in which the system (3.1) along with conditions (3.3) can be rewritten as $L \omega_r(t) = F_r(t)$, $\omega_r(0) = \alpha_r$. With the aid of transform $\xi_r(t) = \begin{bmatrix} \xi_{1r}(t) \\ \xi_{2r}(t) \end{bmatrix} = \omega_r(t) - \alpha_r$ to homogenize the conditions (3.1), it yields

$$\begin{cases} (L \xi_r)(t) = Q_r(t) = \begin{bmatrix} q_{1r}(t) \\ q_{2r}(t) \end{bmatrix}, \\ \xi_r(a) = 0, \end{cases} \quad (4.2)$$

where

$$\begin{aligned} q_{1r}(t) &= \alpha_{1r} \left(p(t) + \int_a^{c_1} h(t, \tau) d\tau + \int_a^{c_2} k(t, \tau) d\tau \right) + \alpha_{2r} \left(\int_{c_1}^b h(t, \tau) + \int_{c_2}^t k(t, \tau) d\tau \right) + f_{1r}(t), \\ q_{2r}(t) &= \alpha_{2r} \left(p(t) + \int_a^{c_1} h(t, \tau) d\tau + \int_a^{c_2} k(t, \tau) d\tau \right) + \alpha_{1r} \left(\int_{c_1}^b h(t, \tau) + \int_{c_2}^t k(t, \tau) d\tau \right) + f_{2r}(t). \end{aligned}$$

Theorem 4.3. *The operator $L : \Upsilon[a, b] \rightarrow \Theta[a, b]$ is linear and bounded.*

Proof. We firstly prove that the operators $l_{ij} : W_2^2[a, b] \rightarrow W_2^1[a, b]$, $i, j = 1, 2$, are bounded and linear. The linearity is clear so for boundedness, $\forall \xi_{jr} \in W_2^2[a, b]$, $j = 1, 2$, we have

$$\|l_{ij}\xi_{jr}\|_{W_2^1}^2 = \langle l_{ij}\xi_{jr}, l_{ij}\xi_{jr} \rangle_{W_2^1} = [(l_{ij}\xi_{jr})(a)]^2 + \int_a^b [(l_{ij}\xi_{jr})'(t)]^2 dt.$$

By the reproducing property of $G_t(s)$, one can write $\xi_{jr}(t) = \langle \xi_{jr}(s), G_t(s) \rangle_{W_2^2}$ so that

$$\begin{aligned} (l_{ij}\xi_{jr})(t) &= \langle \xi_{jr}(s), (l_{ij}G_t(s)) \rangle_{W_2^2}, \\ (l_{ij}\xi_{jr})'(t) &= \left\langle \xi_{jr}(s), \frac{d}{dt}(l_{ij}G_t(s)) \right\rangle_{W_2^2}. \end{aligned}$$

Using Schwarz Inequality and the continuity of $G_t(s)$ over the closed interval $[a, b]$, we get

$$\begin{aligned} |(l_{ij}\xi_{jr})(t)| &= |\langle \xi_{jr}(s), (l_{ij}G_t(s)) \rangle_{W_2^2}| \leq \|\xi_{jr}\|_{W_2^2} \|(l_{ij}G_t(s))\|_{W_2^1} \leq M_{ij} \|\xi_{jr}\|_{W_2^2}, \\ |(l_{ij}\xi_{jr})'(t)| &= |\langle \xi_{jr}(s), \frac{d}{dt}(l_{ij}G_t(s)) \rangle_{W_2^2}| \leq \|\xi_{jr}\|_{W_2^2} \left\| \frac{d}{dt}(l_{ij}G_t(s)) \right\|_{W_2^1} \leq N_{ij} \|\xi_{jr}\|_{W_2^2}, \end{aligned}$$

where $M_{ij}, N_{ij} \in \mathbb{R}^+$.

Therefore, $\|l_{ij}\xi_{jr}\|_{W_2^1}^2 \leq (M_{ij}^2 + N_{ij}^2(b-a)) \|\xi_{jr}\|_{W_2^2}^2$, which leads to $\|l_{ij}\xi_{jr}\|_{W_2^1} \leq T_{ij} \|\xi_{jr}\|_{W_2^2}$, where $T_{ij} = \sqrt{(M_{ij}^2 + N_{ij}^2(b-a))}$. Hence, l_{ij} , $i, j = 1, 2$, are bounded operators.

For the boundedness of $\mathfrak{L}$, let $\xi_r \in \Upsilon[a, b]$, we have

$$\begin{aligned} \|\xi_r\|_{\Theta} &= \left\| \begin{bmatrix} \sum_{j=1}^2 l_{1j}\xi_{jr} \\ \sum_{j=1}^2 l_{2j}\xi_{jr} \end{bmatrix} \right\|_{\Theta} = \sqrt{\sum_{i=1}^2 \left(\left\| \sum_{j=1}^2 l_{ij}\xi_{jr} \right\|_{W_2^1} \right)^2} \leq \sqrt{\sum_{i=1}^2 \left(\sum_{j=1}^2 \|l_{ij}\| \|\xi_{jr}\|_{W_2^2} \right)^2} \\ &\leq \sqrt{\sum_{i=1}^2 \left(\sum_{j=1}^2 \|l_{ij}\|^2 \right) \left(\sum_{j=1}^2 \|\xi_{jr}\|_{W_2^2}^2 \right)} \leq \sqrt{\sum_{i=1}^2 \sum_{j=1}^2 \|l_{ij}\|^2} \|\xi_r\|_{\Upsilon}. \end{aligned}$$

Therefore, since l_{ij} , $j = 1, 2$ is bounded, then L is so. $\square$

To construct a complete function system on the space $\Upsilon[a, b]$ in the view of l_{ij} , $i, j = 1, 2$, pick a dense countable set $\{t_k\}_{k=1}^\infty$ in $[a, b]$ and let l_{ij}^* be the adjoint operator of l_{ij} so that $[l_{ij}^* R_{t_k}](t) = \langle [l_{ij}^* R_{t_k}](s), G_t(s) \rangle_{W_2^2} = \langle R_{t_k}(s), [l_{ij} G_t(s)] \rangle_{W_2^1} = l_{ij} G_t(s)|_{s=t_k}$. Meanwhile, we define $\Psi_{kj}(t) = [l_{j1} G_t(s)|_{s=t_k}, l_{j2} G_t(s)|_{s=t_k}]^T$, $k = 1, 2, 3, \dots$, $j = 1, 2$.

Theorem 4.4. *Let $\{t_k\}_{k=1}^\infty$ be dense on $[a, b]$ and assume that the system (3.1) along with ICs (3.3) has a unique solution, then $\{\Psi_{kj}(t)\}_{(k,j)=(1,1)}^{(\infty,2)}$ forms a complete function system of $\Upsilon[a, b]$.*

Proof. First of all, we show that the set $\{\Psi_{kj}(t)\}_{(k,j)=(1,1)}^{(\infty,2)}$ is linearly independent in $\Upsilon[a, b]$, then we prove the completeness of such function system. To achieve this, assume on the contrary that $\{\Psi_{k_1 j_1}(t), \Psi_{k_2 j_2}(t), \Psi_{k_3 j_3}(t), \dots, \Psi_{k_m j_m}(t)\}$ is a linearly dependent subset of $\{\Psi_{kj}(t)\}_{(k,j)=(1,1)}^{(\infty,2)}$. Let $r = \max\{k_1, k_2, k_3, \dots, k_m\}$ and put $v_k(s) = R_{t_k}(s)$. So, $\{\Psi_{kj}(t)\}_{(k,j)=(1,1)}^{(\infty,2)}$ is linearly dependent, i.e., there are scalars c_{kj} , $j = 1, 2$, $k = 1, 2, \dots, r$, not all are zeros such that $\sum_{j=1}^2 \sum_{k=1}^r c_{kj} \Psi_{kj}(t) = 0$. For $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \Upsilon[a, b]$, we have

$$\begin{aligned} \left\langle LY(t), \begin{bmatrix} \sum_{k=1}^r c_{k1} v_k(t) \\ \sum_{k=1}^r c_{k2} v_k(t) \end{bmatrix} \right\rangle_{\Theta} &= \left\langle \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \begin{bmatrix} \sum_{k=1}^r c_{k1} v_k(t) \\ \sum_{k=1}^r c_{k2} v_k(t) \end{bmatrix} \right\rangle_{\Theta} \\ &= \left\langle \begin{bmatrix} \sum_{j=1}^2 l_{1j} y_j(t) \\ \sum_{j=1}^2 l_{2j} y_j(t) \end{bmatrix}, \begin{bmatrix} \sum_{k=1}^r c_{k1} v_k(t) \\ \sum_{k=1}^r c_{k2} v_k(t) \end{bmatrix} \right\rangle_{\Theta} \\ &= \left\langle \sum_{j=1}^2 l_{1j} y_j(t), \sum_{k=1}^r c_{k1} v_k(t) \right\rangle_{W_2^1} + \left\langle \sum_{j=1}^2 l_{2j} y_j(t), \sum_{k=1}^r c_{k2} v_k(t) \right\rangle_{W_2^1} \\ &= \sum_{j=1}^2 \sum_{k=1}^r \langle l_{1j} y_j(t), c_{k1} v_k(t) \rangle_{W_2^1} + \sum_{j=1}^2 \sum_{k=1}^r \langle l_{2j} y_j(t), c_{k2} v_k(t) \rangle_{W_2^1} \\ &= \sum_{j=1}^2 \sum_{k=1}^r \langle y_j(t), l_{1j}^* c_{k1} v_k(t) \rangle_{W_2^2} + \sum_{j=1}^2 \sum_{k=1}^r \langle y_j(t), l_{2j}^* c_{k2} v_k(t) \rangle_{W_2^2} \\ &= \left\langle \begin{bmatrix} \sum_{j=1}^2 y_j(t) \\ \sum_{j=1}^2 y_j(t) \end{bmatrix}, \begin{bmatrix} \sum_{k=1}^r l_{1j}^* c_{k1} v_k(t) \\ \sum_{k=1}^r l_{2j}^* c_{k2} v_k(t) \end{bmatrix} \right\rangle_{\Upsilon} \\ &= \left\langle \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \begin{bmatrix} \sum_{j=1}^2 \sum_{k=1}^r l_{1j}^* c_{k1} v_k(t) \\ \sum_{j=1}^2 \sum_{k=1}^r l_{2j}^* c_{k2} v_k(t) \end{bmatrix} \right\rangle_{\Upsilon} \\ &= \left\langle Y(t), \sum_{j=1}^2 \sum_{k=1}^r c_{kj} \Psi_{kj}(t) \right\rangle_{\Upsilon} = 0. \end{aligned}$$

Consequently, $\left\langle LY(t), \begin{bmatrix} \sum_{k=1}^r c_{k1} v_k(t) \\ \sum_{k=1}^r c_{k2} v_k(t) \end{bmatrix} \right\rangle_{\Theta} = 0$. Due to Theorem 4.3, that is, L is bounded, and by the uniqueness of the solution, it yields $\begin{bmatrix} \sum_{k=1}^r c_{k1} v_k(t) \\ \sum_{k=1}^r c_{k2} v_k(t) \end{bmatrix} = 0$.

Now, define $w_k(t) \in W_2^1[a, b]$ by $w_r(t) = \begin{cases} 1, & t = t_r, \\ 0, & t = t_k, k = 1, 2, \dots, r, k \neq r, \\ \text{linear function,} & \text{otherwise.} \end{cases}$

Thus, $c_{ij} = \langle w_i(t), \sum_{k=1}^r c_{kj} v_k(t) \rangle_{W_2^1} = 0$, $j = 1, 2$, $i = 1, 2, \dots, r$, which contradicts the fact that not all c_{ij} are zeros. Therefore, $\{\Psi_{kj}(t)\}_{(k,j)=(1,1)}^{(\infty,2)}$ is linearly independent.

To complete the proof, let $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \Upsilon[a, b]$ with $\langle Y(t), \Psi_{kj}(t) \rangle_{\Upsilon} = 0$ for each $k = 1, 2, 3, \dots, j = 1, 2$, and use the linearly independence of $\{\Psi_{kj}(t)\}_{(k,j)=(1,1)}^{(\infty,2)}$ to get

$$\begin{aligned} 0 = \langle Y(t), \Psi_{kj}(t) \rangle_{\Upsilon} &= \left\langle \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \begin{bmatrix} l_{j1} G_t(s)|_{s=t_k} \\ l_{j2} G_t(s)|_{s=t_k} \end{bmatrix} \right\rangle_{\Upsilon} \\ &= \langle y_1(t), l_{j1} G_t(s)|_{s=t_k} \rangle_{W_2^2} + \langle y_2(t), l_{j2} G_t(s)|_{s=t_k} \rangle_{W_2^2} \\ &= \langle y_1(t), [l_{j1}^* R_{t_k}](t) \rangle_{W_2^2} + \langle y_2(t), [l_{j2}^* R_{t_k}](t) \rangle_{W_2^2} \\ &= l_{j1} y_1(t_k) + l_{j2} y_2(t_k). \end{aligned}$$

Since $\{t_k\}_{k=1}^{\infty}$ is dense on $[a, b]$, we deduce that $l_{11} y_1(t) + l_{12} y_2(t) = 0$ and $l_{21} y_1(t) + l_{22} y_2(t) = 0$. Equivalently, $\begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$. Again, from the uniqueness of the solution, it follows $Y(t) = 0$. Hence, $\{\Psi_{kj}(t)\}_{(k,j)=(1,1)}^{(\infty,2)}$ is the complete function system of $\Upsilon[a, b]$. $\square$

As a next step, we obtain the orthonormal function system for $\Upsilon[a, b]$, namely $\{\bar{\Psi}_{kj}(t)\}_{(k,j)=(1,1)}^{(\infty,2)}$, of $\{\Psi_{kj}(t)\}_{(k,j)=(1,1)}^{(\infty,2)}$ such that $\bar{\Psi}_{kj}(t) = \sum_{m=1}^k \lambda_{kjm} \Psi_{mj}(t)$, $k = 1, 2, 3, \dots, j = 1, 2$. Here, λ_{kjm} are the Gram-Schmidt orthogonalization coefficients. Afterwards, we use Fourier expansion to express the solution of equation (4.2) in a series form as in the underlying theorem.

Theorem 4.5. *Let $\{t_k\}_{k=1}^{\infty}$ be dense on $[a, b]$ and $\xi_r(t) = \begin{bmatrix} \xi_{1r}(t) \\ \xi_{2r}(t) \end{bmatrix} \in \Theta[a, b]$ be the unique solution of equation (4.2). Then, $\xi_r(t)$ has the analytic series representation:*

$$\xi_r(t) = \sum_{j=1}^2 \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{kjm} q_{jr}(t_m) \bar{\Psi}_{kj}(t). \quad (4.3)$$

Proof. Assume that $\{t_k\}_{k=1}^{\infty}$ is dense on $[a, b]$ and expand the solution $\xi_r(t)$ in Fourier series about the complete function system $\{\bar{\Psi}_{kj}(t)\}_{(k,j)=(1,1)}^{(\infty,2)}$ of $\Theta[a, b]$ as follows

$$\begin{aligned} \xi_r(t) &= \sum_{j=1}^2 \sum_{k=1}^{\infty} \langle \xi_r(t), \bar{\Psi}_{kj}(t) \rangle_{\Upsilon} \bar{\Psi}_{kj}(t) \\ &= \sum_{j=1}^2 \sum_{k=1}^{\infty} \left\langle \xi_r(t), \sum_{m=1}^k \lambda_{kjm} \Psi_{mj}(t) \right\rangle_{\Upsilon} \bar{\Psi}_{kj}(t) \\ &= \sum_{j=1}^2 \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{kjm} \langle \xi_r(t), \Psi_{mj}(t) \rangle_{\Upsilon} \bar{\Psi}_{kj}(t) \\ &= \sum_{j=1}^2 \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{kjm} \left\langle \begin{bmatrix} \xi_{1r}(t) \\ \xi_{2r}(t) \end{bmatrix}, \begin{bmatrix} l_{j1} G_t(s)|_{s=t_m} \\ l_{j2} G_t(s)|_{s=t_m} \end{bmatrix} \right\rangle_{\Upsilon} \bar{\Psi}_{kj}(t) \\ &= \sum_{j=1}^2 \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{kjm} \left(\langle \xi_{1r}(t), l_{j1} G_t(s)|_{s=t_m} \rangle_{W_2^2} + \langle \xi_{2r}(t), l_{j2} G_t(s)|_{s=t_m} \rangle_{W_2^2} \right) \bar{\Psi}_{kj}(t) \\ &= \sum_{j=1}^2 \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{kjm} \left(\langle \xi_{1r}(t), l_{j1}^* R_{t_m}(t) \rangle_{W_2^2} + \langle \xi_{2r}(t), l_{j2}^* R_{t_m}(t) \rangle_{W_2^2} \right) \bar{\Psi}_{kj}(t) \\ &= \sum_{j=1}^2 \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{kjm} \left(\langle l_{j1} \xi_{1r}(t), R_{t_m}(t) \rangle_{W_2^1} + \langle l_{j2} \xi_{2r}(t), R_{t_m}(t) \rangle_{W_2^1} \right) \bar{\Psi}_{kj}(t) \\ &= \sum_{j=1}^2 \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{kjm} (l_{j1} \xi_{1r}(t_m) + l_{j2} \xi_{2r}(t_m)) \bar{\Psi}_{kj}(t) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{k1m} (l_{11}\xi_{1r}(t_m) + l_{12}\xi_{2r}(t_m)) \bar{\Psi}_{k1}(t) + \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{k2m} (l_{21}\xi_{1r}(t_m) + l_{22}\xi_{2r}(t_m)) \bar{\Psi}_{k2}(t) \\
&= \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{k1m} q_{1r}(t_m) \bar{\Psi}_{k1}(t) + \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{k2m} q_{2r}(t_m) \bar{\Psi}_{k2}(t) \\
&= \sum_{j=1}^2 \sum_{k=1}^{\infty} \sum_{m=1}^k \lambda_{kjm} q_{jr}(t_m) \bar{\Psi}_{kj}(t).
\end{aligned}$$

□

The N th-term approximate solution of equation (4.2) can be obtained by truncating the finite sum $\xi_r^N(t)$ of the series solution (4.3) as follows.

$$\xi_r^N(t) = \sum_{j=1}^2 \sum_{k=1}^N \sum_{m=1}^k \lambda_{kjm} q_{jr}(t_m) \bar{\Psi}_{kj}(t). \quad (4.4)$$

Theorem 4.6. *The approximate solution $\xi_r^N(t)$ given in (4.4) converges uniformly to the analytic solution $\xi_r(t)$ presented in (4.3).*

Proof. By using the reproducing property, Schwarz inequality together with the fact that the reproducing kernel $\mathbb{A}_t(s)$ of the space $W_2^2[a, b]$ is bounded over $[a, b]$, we have

$$\begin{aligned}
\left| \xi_r(t) - \xi_r^N(t) \right| &= \left| \begin{bmatrix} \xi_{1r}(t) \\ \xi_{2r}(t) \end{bmatrix} - \begin{bmatrix} \xi_{1r}^N(t) \\ \xi_{2r}^N(t) \end{bmatrix} \right| = \left| \left\langle \begin{bmatrix} \xi_{1r}(s) \\ \xi_{2r}(s) \end{bmatrix} - \begin{bmatrix} \xi_{1r}^N(s) \\ \xi_{2r}^N(s) \end{bmatrix}, \begin{bmatrix} G_t(s) \\ G_t(s) \end{bmatrix} \right\rangle_{\Upsilon} \right| \\
&\leq \left\| \begin{bmatrix} \xi_{1r} \\ \xi_{2r} \end{bmatrix} - \begin{bmatrix} \xi_{1r}^N \\ \xi_{2r}^N \end{bmatrix} \right\|_{\Upsilon} \left\| \begin{bmatrix} G_t(s) \\ G_t(s) \end{bmatrix} \right\|_{\Upsilon} = \left\| \begin{bmatrix} \xi_{1r} - \xi_{1r}^N \\ \xi_{2r} - \xi_{2r}^N \end{bmatrix} \right\|_{\Upsilon} \sqrt{2(\|G_t(s)\|_{\Upsilon})^2} \\
&= \sqrt{2} \left\| \xi_r - \xi_r^N \right\|_{\Upsilon} \|G_t(s)\|_{W_2^2} \leq \rho \left\| \xi_r(t) - \xi_r^N(t) \right\|_{\Upsilon}.
\end{aligned}$$

From Remark 2.1, it yields $\rho = \sqrt{2}(b-a)\sqrt{\frac{1}{2}(a+b+2)}$. Meanwhile, $\left\| \xi_r - \xi_r^N \right\|_{\Upsilon} \rightarrow 0$ as $N \rightarrow \infty$. Therefore, $\xi_r^N(t)$ converges uniformly to $\xi_r(t)$. □

5. NUMERIC INVESTIGATION OF FFIDEs WITH ABC gH-DIFFERENTIABILITY

In this section, numerical solutions of various types FFIDEs with ABC gH-fractional derivative are investigated using RKHS algorithm. In specific, Volterra, Fredholm, and mixed Volterra-Fredholm FFIDEs are considered by taking different types of kernel functions, non-homogeneous terms, and fractional derivatives. The accuracy of our method is examined by comparing the exact, if exist, with RKHS approximate solutions. To see the effect of ABC gH-derivative on the behavior of the given FFIDEs, a comparison is also done between fuzzy approximate solutions under ABC gH-differentiability with those of classical Caputo gH-differentiability. In each example, the normalization coefficient is considered as $\kappa(\beta) = 1 - \beta + \frac{\beta}{\Gamma(\beta)}$. Our numerical results are carried out using Mathematica 10.

Example 5.1. Consider $H(t, \tau, \omega(\tau)) = 0$, $K(t, \tau, \omega(\tau)) = \frac{1}{\Gamma(\beta)}(t - \tau)^{\beta-1}\omega(\tau)$, $p(t) = 0$, and $f(t) = \eta(1 + t^{\beta})$, where η is the generalized triangular fuzzy number $(-1, 0, 1; 1)$, along with the fuzzy initial condition $\omega(0) = 0$. Under such circumstance, FFIDE (1.1) converts into the Volterra FFIDE as:

$$\left({}^{ABC}_{gH} D_{0+}^{\beta} \omega \right)(t) = \eta(1 + t^{\beta}) + \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} \omega(\tau) d\tau, \quad \omega(0) = 0, \quad 0 < \beta \leq 1. \quad (5.1)$$

- For ${}^{ABC}_{gH}[(1) - \beta]$ -differentiability, the Volterra FFIDE (5.1) will be equivalent to the underlying crisp system of Volterra FIDEs:

FIGURE 1. Comparison between the exact solution and its derivative (*above*) and approximate RKHS-solution (*below*) for $\beta = 1$ under ${}^{ABC}_{gH}[(1) - \beta]$ -differentiability of Example 5.1, case 1.

$$\begin{aligned} \left({}^{ABC}_1 D_{0+}^\beta \omega_{1r}\right)(t) &= (r-1) \left(1+t^\beta\right) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \omega_{1r}(\tau) \, d\tau, \quad \omega_{1r}(0) = 0, \\ \left({}^{ABC}_1 D_{0+}^\beta \omega_{2r}\right)(t) &= (1-r) \left(1+t^\beta\right) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \omega_{2r}(\tau) \, d\tau, \quad \omega_{2r}(0) = 0. \end{aligned} \quad (5.2)$$

If $\beta = 1$, then the exact solution of system (5.2) is $\omega(t) = \eta(e^t - 1)$.

- For ${}^{ABC}_{gH}[(2) - \beta]$ -differentiability, the Volterra FFIDE (5.1) will be equivalent to the underlying crisp system of Volterra FIDEs:

$$\begin{aligned} \left({}^{ABC}_2 D_{0+}^\beta \omega_{1r}\right)(t) &= (r-1) \left(1+t^\beta\right) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \omega_{2r}(\tau) \, d\tau, \quad \omega_{1r}(0) = 0, \\ \left({}^{ABC}_2 D_{0+}^\beta \omega_{2r}\right)(t) &= (1-r) \left(1+t^\beta\right) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \omega_{1r}(\tau) \, d\tau, \quad \omega_{2r}(0) = 0. \end{aligned} \quad (5.3)$$

If $\beta = 1$, then the exact solution of system (5.3) is $\omega(t) = \eta(\sin t - \cos t + 1)$.

Applying the RKHS method with $n = 40$, the achieved numerical outcomes are summarized in the form of tables and graphical representations as follows. The 3D surface plot for the exact and approximate RKHS solutions and its

FIGURE 2. Approximate core and support at different β values by means of $^{ABC}_{gH}[(1 - \beta)]$ -differentiability (above) and $^C_{gH}[(1 - \beta)]$ -differentiability (below) of Example 5.1, case 1.

FIGURE 3. Graphs of β -level curves for different β and $r = 0.25$ in terms of $^{ABC}_{gH}[(1 - \beta)]$ -differentiability (left) and in terms of $^C_{gH}[(1 - \beta)]$ -differentiability (right) of Example 5.1, case 1.

derivatives in view of $^{ABC}_{gH}[(1 - \beta)]$ -differentiable are given in Figure 1. While Figure 2 exhibits a comparison between the approximate cores and supports for the fuzzy solutions under $^{ABC}_{gH}[(1 - \beta)]$ - and $^C_{gH}[(1 - \beta)]$ -differentiabilities at different β indexes.

Figure 3 displays a comparison of approximate solutions for different β indexes in the light of $^{ABC}_{gH}[(1 - \beta)]$ - and $^C_{gH}[(1 - \beta)]$ -differentiabilities. In Figure 4, r -level curves are presented for different r values with $\beta = 0.9$ under gH -differentiability in Caputo and ABC sense. It is clear that the surface plots of fuzzy approximate solutions are constantly dependent on corresponding fractional values. Numerical approximations of Example 5.1 compared with the exact solutions at $\beta = 1$ and are summarized in Table 1 in view of $^{ABC}_{gH}[(1 - \beta)]$ - and $^C_{gH}[(1 - \beta)]$ -differentiabilities over the

FIGURE 4. Graphs of r -level curves for different r values at $\beta = 0.9$ by means of ${}^{ABC}_{gH}[(1) - \beta]$ -differentiability (left) and ${}^C_{gH}[(1) - \beta]$ -differentiability (right) for Example 5.1, case 1.

TABLE 1. Numerical results of Example 5.1 in terms of ${}^{ABC}_{gH}[(1) - \beta]$ - and ${}^C_{gH}[(1) - \beta]$ -differentiabilities.

t_i	Approximate RKHS solution $\omega_{1r}(t)$ with $r = 0.75$					Absolute Error (ABC operator) $\beta = 1$
	$\beta = 1$	${}^{ABC}[(1) - 0.95]$	${}^C[(1) - 0.95]$	${}^{ABC}[(1) - 0.9]$	${}^C[(1) - 0.9]$	
0.1	-0.0262927	-0.03311720	-0.03024964	-0.02707321	-0.02810861	1.3170346×10^{-6}
0.3	-0.0874647	-0.10040613	-0.09947497	-0.08915656	-0.09289777	2.1570681×10^{-6}
0.5	-0.1621800	-0.17685462	-0.18167977	-0.16441221	-0.17123478	1.9887312×10^{-6}
0.7	-0.2534380	-0.27180802	-0.28093880	-0.25677242	-0.26633358	6.9243846×10^{-7}
0.9	-0.3649010	-0.38923951	-0.40211111	-0.37028170	-0.38244686	1.9674212×10^{-6}
t_i	Approximate RKHS solution $\omega_{2r}(t)$ with $r = 0.75$					Absolute Error (ABC operator) $\beta = 1$
	$\beta = 1$	${}^{ABC}[(2) - 0.95]$	${}^C[(2) - 0.95]$	${}^{ABC}[(2) - 0.9]$	${}^C[(2) - 0.9]$	
0.1	0.0262927	0.03311720	0.03024964	0.02707321	0.02810861	1.3170346×10^{-6}
0.3	0.0874647	0.10040613	0.09947497	0.08915656	0.09289777	2.1570681×10^{-6}
0.5	0.1621800	0.17685462	0.18167977	0.16441221	0.17123478	1.9887312×10^{-6}
0.7	0.2534380	0.27180802	0.28093880	0.25677242	0.26633358	6.9243846×10^{-7}
0.9	0.3649010	0.38923951	0.40211111	0.37028170	0.38244686	1.9674212×10^{-6}

interval $[0.1]$ with step-size 0.2. Furthermore, some representative results of ${}^{ABC}_{gH}[(2) - \beta]$ -differentiability are presented in Figures 5–8 and Table 2 as well. From the figures, it can be seen that all the plots are nearly compatible, analogous, and similar in behavior as well as fully consistent with each other, especially when considering the fuzzy ABC derivative of integer order $\beta = 1$. Also, the solution is an interval at each instantaneous point of all levels of r -cut, which means that the solution is a fuzzy function at every point in the internal domain.

Example 5.2. Consider $H(t, \tau, \omega(\tau)) = 0$, $K(t, \tau, \omega(\tau)) = -\omega$, $p(t) = 0$, and $f(t) = -2\eta \sin t$, along with the fuzzy initial condition $\omega(0) = \eta$, where η is the generalized triangular fuzzy number $(-2, 1, 2; 1)$. Herein, the FFIDE (1.1) converts into the underlying Volterra FFIDE:

$$\left({}^{ABC}_{gH}D_{0+}^{\beta}\omega\right)(t) = -2\eta \sin t - \int_0^t \omega(\tau) \, d\tau, \quad \omega(0) = \eta, 0 < \beta \leq 1. \quad (5.4)$$

FIGURE 5. Comparison between the exact solution with its derivative (*above*) and approximate RKHS-solution (*below*) at $\beta = 1$ by means of ${}^{ABC}_{gH}[(2) - \beta]$ -differentiability for Example 5.1, case 2.

TABLE 2. Numerical results for Example 5.1 in terms of ${}^{ABC}_{gH}[(2) - \beta]$ - and ${}^C_{gH}[(2) - \beta]$ -differentiabilities.

t_i	Approximate RKHS solution $\omega_{1r}(t)$ with $r = 0.75$					Absolute Error (ABC operator) $\beta = 1$
	$\beta = 1$	${}^{ABC}[(1) - 0.95]$	${}^C[(1) - 0.95]$	${}^{ABC}[(1) - 0.9]$	${}^C[(1) - 0.9]$	
0.1	-0.02620800	-0.03393330	-0.03018032	-0.027490546	-0.02807284	6.3902269×10^{-7}
0.3	-0.08504360	-0.09547365	-0.09679080	-0.086671534	-0.09095203	2.3318557×10^{-6}
0.5	-0.15045516	-0.15424662	-0.16366358	-0.149360350	-0.15726918	5.5832508×10^{-6}
0.7	-0.21983502	-0.21543716	-0.23015837	-0.215064637	-0.22536286	8.8530113×10^{-6}
0.9	-0.29041737	-0.27599123	-0.29427415	-0.281029870	-0.29283396	1.1868656×10^{-5}
t_i	Approximate RKHS solution $\omega_{2r}(t)$ with $r = 0.75$					Absolute Error (ABC operator) $\beta = 1$
	$\beta = 1$	${}^{ABC}[(2) - 0.95]$	${}^C[(2) - 0.95]$	${}^{ABC}[(2) - 0.9]$	${}^C[(2) - 0.9]$	
0.1	0.02620800	0.03393330	0.03018032	0.027490546	0.02807284	6.3902269×10^{-7}
0.3	0.08504360	0.09547365	0.09679080	0.086671534	0.09095203	2.3318557×10^{-6}
0.5	0.15045516	0.15424662	0.16366358	0.149360350	0.15726918	5.5832508×10^{-6}
0.7	0.21983502	0.21543716	0.23015837	0.215064637	0.22536286	8.8530113×10^{-6}
0.9	0.29041737	0.27599123	0.29427415	0.281029870	0.29283396	1.1868656×10^{-5}

FIGURE 6. Approximate core and support for different β values in view of $^{ABC}_{gH}[(2) - \beta]$ -differentiability (*above*) and in view of $^{C}_{gH}[(2) - \beta]$ -differentiability (*below*) of Example 5.1, case 2.

FIGURE 7. Graphs of β -level curves for different β indexes with $r = 0.25$ in terms of $^{ABC}_{gH}[(2) - \beta]$ -differentiability (*left*) and $^{C}_{gH}[(2) - \beta]$ -differentiability (*right*) for Example 5.1, case 2.

- For $^{ABC}_{gH}[(1) - \beta]$ -differentiability, the Volterra FFIDE (5.4) will be equivalent to the underlying crisp system of Volterra FIDEs:

$$\begin{aligned} \left({}^{ABC}_1 D_{0+}^\beta \omega_{1r} \right) (t) &= (2r - 4) \sin t - \int_0^t \omega_{2r}(\tau) \, d\tau, \quad \omega_{1r}(0) = 3r - 2, \\ \left({}^{ABC}_1 D_{0+}^\beta \omega_{2r} \right) (t) &= (4 - 6r) \sin t - \int_0^t \omega_{1r}(\tau) \, d\tau, \quad \omega_{2r}(0) = 2 - r. \end{aligned} \quad (5.5)$$

FIGURE 8. Graphs of r -level curves for different r values at $\beta = 0.9$ under $^{ABC}_{gH}[(2) - \beta]$ -differentiability (left) and under $^C_{gH}[(2) - \beta]$ -differentiability (right) for Example 5.1, case 2.

If $\beta = 1$, then the r -cut representations of exact solution of system (5.5) are given as

$$\begin{aligned}\omega_{1r}(t) &= -rt \sin t + (2-r) \cos t + 4(r-1) \cosh t, \\ \omega_{2r}(t) &= -rt \sin t + (3r-2) \cos t - 4(r-1) \cosh t\end{aligned}$$

- For $^{ABC}_{gH}[(2) - \beta]$ -differentiability, the Volterra FFIDE (5.4) will be equivalent to the underlying crisp system of Volterra FIDEs:

$$\begin{aligned}\left(^{ABC}_2 D_{0+}^\beta \omega_{1r}\right)(t) &= (4-6r) \sin t - \int_0^t \omega_{1r}(\tau) d\tau, \quad \omega_{1r}(0) = 3r-2, \\ \left(^{ABC}_2 D_{0+}^\beta \omega_{2r}\right)(t) &= (2r-4) \sin t - \int_0^t \omega_{2r}(\tau) d\tau, \quad \omega_{2r}(0) = 2-r.\end{aligned}\tag{5.6}$$

If $\beta = 1$, then the exact solution of system (5.6) is $\omega(t) = \eta(\cos t - t \sin t)$.

Applying the RKHS method with $n = 15$, numerical and graphical results for fuzzy approximate solutions in terms of $^{ABC}_{gH}[(1) - \beta]$ - and $^C_{gH}[(1) - \beta]$ -differentiabilities are shown in Figures 9 and 10 and in terms of $^{ABC}_{gH}[(2) - \beta]$ - and $^C_{gH}[(2) - \beta]$ -differentiabilities are shown in Figures 11 and 12.

Example 5.3. Consider $\beta = \frac{1}{2}$, $H(t, \tau, \omega(\tau)) = p(t) = 0$, $K(t, \tau, \omega(\tau)) = \frac{-1}{3} \tau t \omega(\tau)$, and $f(t) = \eta \zeta(t)$, where η is the generalized triangular fuzzy number $(1, 2, 5; 1)$, and $\zeta(t) = \frac{1}{15\sqrt{\pi}} 24\sqrt{t} (15 + 2t(5 + 2t)) - \sqrt{\pi} (180 + t(180 + 90t + t^5)) + 180e^t \sqrt{\pi} \text{Erfc}(z)$, in which $\text{Erfc}(z)$ refers to the complementary error function, equipped with the fuzzy initial condition $\omega(0) = 0$. Herein, the FFIDE (1.1) converts into the underlying Volterra FFIDE:

$$\left(^{ABC}_{gH} D_{0+}^{\frac{1}{2}} \omega\right)(t) = \eta \zeta(t) - \frac{1}{3} \int_0^t \tau t \omega(\tau) d\tau, \quad \omega(0) = 0.\tag{5.7}$$

By means of $^{ABC}_{gH}[(1) - \beta]$ -differentiability, the Volterra FFIDE (5.7) will be equivalent to the underlying crisp system of Volterra FIDEs:

$$\begin{aligned}\left(^{ABC}_1 D_{0+}^{\frac{1}{2}} \omega_{1r}\right)(t) &= (r+1) \zeta(t) - \frac{1}{3} \int_0^t \tau t \omega_{2r}(\tau) d\tau, \quad \omega_{1r}(0) = 0, \\ \left(^{ABC}_1 D_{0+}^{\frac{1}{2}} \omega_{2r}\right)(t) &= (5-3r) \zeta(t) - \frac{1}{3} \int_0^t \tau t \omega_{1r}(\tau) d\tau, \quad \omega_{2r}(0) = 0,\end{aligned}\tag{5.8}$$

FIGURE 9. Graphs of β -level curves for different β indexes at $r = 0.4$ in terms of $^{ABC}_{gH}[(1) - \beta]$ -differentiability (*left*) and in terms of $^C_{gH}[(1) - \beta]$ -differentiability (*right*) for Example 5.2, case 1.

FIGURE 10. Graphs of r -level curves for different r values at $\beta = 0.8$ under $^{ABC}_{gH}[(1) - \beta]$ -differentiability (*left*) and under $^C_{gH}[(1) - \beta]$ -differentiability (*right*) for Example 5.2, case 1.

FIGURE 11. Graphs of β -level curves for different β indexes at $r = 0.4$ in terms of $^{ABC}_{gH}[(2) - \beta]$ -differentiability (*left*) and in terms of $^C_{gH}[(2) - \beta]$ -differentiability (*right*) for Example 5.2, case 2.

FIGURE 12. Graphs of r -level curves for different r values at $\beta = 0.8$ under $^{ABC}_{gH}[(2) - \beta]$ -differentiability (left) and under $^C_{gH}[(2) - \beta]$ -differentiability (right) for Example 5.2, case 2.

FIGURE 13. Approximate core and support $^{ABC}_{gH}[(1) - 0.5]$ -differentiability of Example 5.3.

whose exact solution is $\omega(t) = \eta t^3$. Using the RKHSM with $n=25$, numerical results for Example 5.3 are given in Table 3 and Figures 15 and 16 in order to show the accuracy of the proposed method and to support the theoretical framework as well.

Example 5.4. Consider $\beta = \frac{1}{2}$, $K(t, \tau, \omega(\tau)) = p(t) = 0$, $H(t, \tau, \omega(\tau)) = \frac{1}{10}\tau^2 t \omega^2(\tau)$, and $f(t) = [f_1, f_2]$, where $f_1(t) = -\frac{1}{40}r^2 t^2 + 2r \left(-1 + \frac{2\sqrt{t}}{\sqrt{\pi}} + e^t \text{Erfc}[\sqrt{t}]\right)$ and $f_2(t) = \frac{1}{40\sqrt{\pi}}(r-2)((160\sqrt{t} + \sqrt{\pi}(-80 + (-2+r)t^2) + 80e^t \sqrt{\pi} \text{Erfc}[\sqrt{t}]))$ along with the fuzzy initial condition $\omega(0) = 0$. Herein, the FFIDE (1.1) converts into the underlying nonlinear Fredholm FFIDE:

$$\left(^{ABC}_{gH} D_{0+}^{\frac{1}{2}} \omega\right)(t) = f(t) + \frac{1}{10} \int_0^1 (\tau^2 t) (\omega(\tau))^2 d\tau, \quad \omega(0) = 0. \quad (5.9)$$

By means of $^{ABC}_{gH}[(1) - \beta]$ -differentiability, the Fredholm FFIDE (5.9) will be equivalent to the underlying nonlinear crisp system of Fredholm FIDEs:

$$\left(^{ABC}_1 D_{0+}^{\frac{1}{2}} \omega_{1r}\right)(t) = f_1(t) + \frac{1}{10} \int_0^1 (\tau^2 t) (\omega_{1r}(\tau))^2 d\tau, \quad \omega_{1r}(0) = 0,$$

TABLE 3. Numerical results of Example 5.3 under ${}^{ABC}_{gH}[(1) - \beta]$ -differentiability.

t_k	r	$\omega_{1r}(t)$	$\omega_{2r}(t)$	Absolute Error
0.2	0.0	0.0080000107	0.0400000535	1.0695×10^{-8}
0.4		0.0640000186	0.3200000928	1.8557×10^{-8}
0.6		0.2160000273	1.0800001360	2.7263×10^{-8}
0.8		0.5120001372	2.5600006860	1.3721×10^{-7}
0.2	0.2	0.0096000128	0.0352000471	1.2834×10^{-8}
0.4		0.0768000223	0.2816000816	2.2268×10^{-8}
0.6		0.2592000327	0.9504001200	3.2716×10^{-8}
0.8		0.6144001646	2.2528006040	1.6465×10^{-8}
0.2	0.4	0.0112000149	0.0304000406	1.4973×10^{-8}
0.4		0.0896000259	0.2432000705	2.5979×10^{-8}
0.6		0.3024000382	0.8208001036	3.8168×10^{-8}
0.8		0.7168001921	1.9456005210	1.9209×10^{-8}
0.2	0.6	0.0128000171	0.0256000342	1.7112×10^{-8}
0.4		0.1024000297	0.2048000594	2.9691×10^{-8}
0.6		0.3456000436	0.6912000872	4.3621×10^{-8}
0.8		0.8192002195	1.6384004390	2.1953×10^{-7}
0.2	0.8	0.0144000193	0.0208000278	1.9251×10^{-8}
0.4		0.1152000334	0.1664000482	3.3402×10^{-8}
0.6		0.3888000491	0.5616000709	4.9074×10^{-8}
0.8		0.9216002470	1.3312003570	2.4697×10^{-7}
0.2	1.0	0.0160000214	0.0160000214	2.1391×10^{-8}
0.4		0.1280000371	0.1280000371	3.7113×10^{-8}
0.6		0.4320000545	0.4320000545	5.4526×10^{-8}
0.8		1.0240002740	1.0240002740	2.7441×10^{-7}

$$\left({}^{ABC}_1 D_{0+}^{\frac{1}{2}} \omega_{1r}\right)(t) = f_2(t) + \frac{1}{10} \int_0^1 \left(\tau^2 t\right) (\omega_{2r}(\tau))^2 d\tau, \quad \omega_{2r}(0) = 0, \quad (5.10)$$

whose exact solution is $\omega(t) = (r, 2-r)t$. Using the RKHSM with $n = 15$, the graphical results are reported in Figures 14 and 15.

Example 5.5. Consider $K(t, \tau, \omega(\tau)) = t$, $H(t, \tau, \omega(\tau)) = \tau$, $p(t) = -2e^t$, and $f(t) = \sinh(t)(1-t) + e^{2t} + e^{-1})\eta$, where η is a fuzzy number with $[\eta]^r = [-\sqrt{1-r}, \sqrt{1-r}]$, equipped with the initial condition $\omega(0) = 0$, $0 < \beta \leq 1$, $t \in [0, 1]$. Herein, the FFIDE (1.1) converts into the underlying Volterra-Fredholm FFIDE:

$$\left({}^{ABC}_{gH} D_{0+}^{\beta} \omega\right)(t) + 2e^t \omega(t) = \left(\sinh(t)(1-t) + e^{2t} + e^{-1}\right) \alpha + \int_0^1 \tau \omega(\tau) d\tau + \int_0^t t \omega(\tau) d\tau. \quad (5.11)$$

In terms of ${}^{ABC}_{gH}[(1) - \beta]$ -differentiability, the FFIDE (5.10) will be equivalent to the underlying crisp system of Volterra Fredholm FIDEs:

$$\begin{aligned} \left({}^{ABC}_1 D_{0+}^{\beta} \omega_{1r}\right)(t) &= -2e^t \omega_{1r}(t) - \left(\sinh(t)(1-t) + e^{2t} + e^{-1}\right) \sqrt{1-r} + \int_0^1 \tau \omega_{1r}(\tau) d\tau \\ &\quad + \int_0^t t \omega_{1r}(\tau) d\tau, \quad \omega_{1r}(0) = -\sqrt{1-r}, \\ \left({}^{ABC}_1 D_{0+}^{\beta} \omega_{2r}\right)(t) &= -2e^t \omega_{2r}(t) + \left(\sinh(t)(1-t) + e^{2t} + e^{-1}\right) \sqrt{1-r} + \int_0^1 \tau \omega_{2r}(\tau) d\tau \end{aligned}$$

FIGURE 14. Comparison between the exact solution with its derivative (*above*) and the approximate RKHS-solution (*below*) in terms of $^{ABC}_{gH}[(1) - 0.5]$ -differentiability for Example 5.4.

TABLE 4. Absolute errors of Example 5.5 in terms of $^{ABC}_{gH}[(1) - \beta]$ -differentiability with $\beta = 1$.

t_k	$r = 0$	$r = 0.25$	$r = 0.5$	$r = 0.75$	$r = 1$
0.1	1.6009667×10^{-7}	1.3864778×10^{-7}	1.1320544×10^{-8}	8.0048335×10^{-8}	0.00
0.3	1.0607738×10^{-7}	7.4350777×10^{-7}	7.5008040×10^{-8}	5.3038693×10^{-8}	0.00
0.5	7.0289852×10^{-7}	5.1089475×10^{-7}	4.9702431×10^{-7}	3.5144926×10^{-8}	0.00
0.7	5.2261785×10^{-7}	4.2906646×10^{-7}	3.6954662×10^{-7}	2.6130892×10^{-8}	0.00
0.9	5.2615182×10^{-7}	5.3071953×10^{-6}	3.7204551×10^{-7}	2.6307591×10^{-7}	0.00

$$+ \int_0^t t \omega_{2r}(\tau) d\tau, \quad \omega_{2r}(0) = \sqrt{1-r}, \quad (5.12)$$

whose exact solution for $\beta = 1$ is $\omega(t) = \eta \cosh(t)$.

Using this concept, Table 4 summarizes the error in approximating using 30 iterations in view of the RKHSM, while Figure 16 shows the core of fuzzy approximate solution for different values of ABC gH-derivatives such that $\beta \in \{1, 0.95, 0.9\}$.

Example 5.6. Consider $\beta = \frac{1}{2}$, $K(t, \tau, \omega(\tau)) = 3\omega(\tau)$, $p(t) = 0$, $H(t, \tau, \omega(\tau)) = t^2\omega(\tau)$, and $f(t) = \gamma\zeta(t)$, where γ is the generalized triangular fuzzy number $(0, 1, 2; 1)$ and $\zeta(t) = (\frac{1}{35\sqrt{\pi}}(8\sqrt{t}(525 + 2t(175 + 8t(7 + 2t))) - 7\sqrt{\pi}(300 +$

FIGURE 15. Graph of exact and approximate solutions: (a) core and support of exact, (b) core and support of RKHS; (c) upper solutions, (d) lower solutions at $r = 0.4$ under ${}^{ABC}_{gH}[(1) - 0.5]$ -differentiability of Example 5.4.

FIGURE 16. The cores and supports for fuzzy RKHS-solution of Example 5.5 under ${}^{ABC}_{gH}[(1) - \beta]$ -differentiability with $\beta = 1, 0.95, 0.9$, respectively.

$t(300 + t(126 + 55t + 3t^3))) + 2100e^t\sqrt{\pi}\text{Erfc}[\sqrt{t}])$, along with the initial condition $\omega(0) = 0$. Herein, the FFIDE (1.1) converts into the underlying mixed Volterra-Fredholm FFIDE:

$$\left({}^{ABC}_{gH}D_{0+}^{\frac{1}{2}}\omega\right)(t) = \gamma \zeta(t) + \int_0^1 t^2 \omega(\tau) d\tau + 3 \int_0^t \omega(\tau) d\tau, \omega(0) = 0. \quad (5.13)$$

TABLE 5. Numeric results of Example 5.6 in terms of ${}^{ABC}_{gH}[(1) - 0.5]$.

t_k	r	$\omega_{1r}(t)$	$\omega_{2r}(t)$	Absolute Error
0.2	0.0	0.0	0.24320	0.0
0.4		0.0	1.01120	0.0
0.6		0.0	2.41920	0.0
0.8		0.0	4.65920	0.0
1.0		0.0	8.00000	0.0
0.2	0.2	0.02432	0.21888	8.0880199×10^{-9}
0.4		0.10112	0.91008	1.4985391×10^{-9}
0.6		0.24192	2.17728	2.7681427×10^{-9}
0.8		0.46592	4.19328	5.1021274×10^{-8}
1.0		0.80000	7.20000	1.0701321×10^{-8}
0.2	0.4	0.04864	0.19456	1.6176039×10^{-9}
0.4		0.20224	0.80896	2.9970783×10^{-9}
0.6		0.48384	1.93536	5.5362855×10^{-8}
0.8		0.93184	3.72736	1.0204254×10^{-8}
1.0		1.60000	6.40000	2.1402643×10^{-8}
0.2	0.6	0.07296	0.17024	2.4264059×10^{-9}
0.4		0.30336	0.70784	4.4956175×10^{-8}
0.6		0.72576	1.69344	8.3044283×10^{-8}
0.8		1.39776	3.26144	1.5306382×10^{-8}
1.0		2.40000	5.60000	3.2103965×10^{-7}
0.2	0.8	0.09728	0.14592	3.2352079×10^{-9}
0.4		0.40448	0.60672	5.9941567×10^{-8}
0.6		0.96768	1.45152	1.1072571×10^{-8}
0.8		1.86368	2.79552	2.0408509×10^{-8}
1.0		3.20000	4.80000	4.2805287×10^{-7}

By means of ${}^{ABC}_{gH}[(1) - \beta]$ -differentiability, the mixed FFIDE (5.13) will be equivalent into the underlying crisp system of Volterra-Fredholm FIDEs:

$$\begin{aligned}
\left({}^{ABC}_1 D_{0+}^{\frac{1}{2}} \omega_{1r}\right)(t) &= r \zeta(t) + \int_0^1 t^2 \omega_{1r}(\tau) d\tau + 3 \int_0^t \omega_{1r}(\tau) d\tau, \quad \omega_{1r}(0) = 0, \\
\left({}^{ABC}_1 D_{0+}^{\frac{1}{2}} \omega_{2r}\right)(t) &= (2-r) \zeta(t) + \int_0^1 t^2 \omega_{2r}(\tau) d\tau + 3 \int_0^t \omega_{2r}(\tau) d\tau, \quad \omega_{2r}(0) = 0.
\end{aligned} \tag{5.14}$$

whose exact solution is $\omega(t) = (r, 2-r)(t^4 + 3t^2)$. Using the RKHSM with $n = 35$, the numerical outcomes of the proposed method are summarized in Table 5 for different r -levels.

6. CONCLUSIONS

In this work, a modified numerical algorithm has been profitably designed in light of RKHS method and employed to get approximate solutions of fuzzy fractional integrodifferential equations by means of Atangana-Baleanu-Caputo gH -differentiability. In this direction, characterization theorem was established for ABC-fractional order, in which the studied fuzzy fractional model was consequently transformed into a crisp system of fractional IVPs under fuzzy ABC calculus. The analytical solutions have been given in series form of the parametric interval of ABC in the space $W_2^2[a, b] \oplus W_2^2[a, b]$. The N th-term approximate solutions and its derivatives were uniformly convergent to the analytical solutions and its derivatives, respectively. The convergent analysis and error estimation of the proposed method have been discussed as

well. Several applications for both linear and nonlinear, Fredholm-Volterra FFIDEs have been presented to demonstrate the reliability and effectiveness of the RKHS method and to support the theoretical framework. With providing numerical examples, the accuracy of the analytical results has been illustrated. From the achieved results, it can be observed that the posed method yields accurate approximate solutions. Anyhow, the numerical results of ABC gH-differentiability have been compared with those of generalized Caputo derivative. Using ABC gH-differentiability, we conclude that the presented study can be effectively utilized as an extended planner in handling many kinds of fractional issues under uncertainty arising in engineering, physics, and natural sciences. Hopefully, the current analysis will be employed in the near future to study more uncertain fractional models by means of ABC gH-differentiability of higher fractional order.

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