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MODELLING OPTIMAL PEST CONTROL OF NON-AUTONOMOUS PREDATOR-PREY INTERACTION *

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Abstract. An ecological system comprehended by a pest and its natural enemy, the predator, is considered. Parameters of system are time dependent in order to accompany their variations associated to climate evolutions. Combining the use of pesticides and of extra supply of food to predators, we propose the eradication of pest through optimal control having those two measures as controls. Is established that the resulting problem has a unique solution. Uniqueness is obtained on the whole interval using a recursive argument. The usefulness of model to tackle the pest population is backed by numerical simulation results.

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1. Introduction

The goods whose production depends on raising of crops constitute our main source of food. However, the land is limited and, in nature, there are many small animals that attack crops. There are also many fungi, bacteria and viruses that cause harm to crops by parasitizing plants, diminishing the production of plantations. Thus, pest control is one of the most interesting and challenging problems.

Mathematical models are an important tool in the analysis of the dynamics and the control of a pest. Venturino, see [14], presents a review of deterministic models of Ecoepidemiology. A considerable number of models are presented for the description of population interactions with pathogenic agents.

One of those models is generalized in this work, an eco-epidemic system suggested by [6], where the authors consider an ecological system where a given pest lives with its natural enemy, the predator. The role of infection to the pest population and the presence of some alternative source of food to predators is also considered (later designated g). The extra food supply is used mainly for biological conservation of predators, since the predator population suffer a reduction of his main source of food due the use of pesticides. The system is constituted by three compartments: the susceptible pest class S, the infected pest class I, and the predator population class Z. The eco-epidemic system is as follows:

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$$\begin{cases} \dot{S} =rS\left(1-\dfrac{S+\eta I}{K}\right)-\alpha SI-\beta\dfrac{SZ}{a+S} \\ \dot{I} =I\left(\alpha S-bZ-\sigma\right) \\ \dot{Z} =\ell\beta\dfrac{SZ}{a+S}+(m-n)bIZ+gZ\left(1-\dfrac{S+\eta I}{K}\right)-\mu Z-\delta Z^2 \end{cases} \tag{1}$$

subject to the initial conditions

$$S(0), I(0), Z(0) \geqslant 0. \tag{2}$$

The infected pest is also food for the predator population and therefore contribute to their reproduction at rate m, but also can harm them, causing an extra mortality rate n. Parameter β is a contact rate that represents the maximum capture rate of the susceptible pest. The description of the remaining parameters can be found in Table 1 (Section 7). For more details upon the model we refer the reader to [6] and the references therein.

In [6] the authors show that the optimal control, applied to the above system, is more effective in reduction of the pest than if we let the system work without pesticide control. They consider that the control is the level of pesticides to be used in the system.

The parameters of the above model are assumed to be constants, i.e, independent of time. This is not very realistic because it ignores that they may vary with time. Non-autonomous epidemic models are more rigorous, cause in it the parameters try to accompany certain evolution patterns. For example, [12], using three distinct contact rates, propose three scenarios for the spread of a "infection". For its part, [9] propose that not only the contact rate varies with time, but also the birth rate varies.

Along the raise of a given crop, the weather may favours the development of fungi and other pest that affect their growth. This conditions correspond, in general, to one season or to a period of time (months, weeks, ...). Hence, we consider that some of the parameters of our model are time dependent. One of our main objectives is to emphasize the importance of seasonal behaviour on the optimal strategy.

In what follows, an optimal control problem with state conditions (1)-(2) is analysed with general non autonomous parameters. The level of pesticide remains as control due to its widespread use. Besides that, the extra food supply is also a mean of control, because it influences the existing number of predators and preys. If a large amount of food supply is provided, then the number of predators and the number of preys (infected and non infected) increases since the predators do not have the necessity of feeding themselves. In this context, the preys have no predators. But if the extra food supply is low or even non existent, the number of infected predators will increase because they may have to feed themselves of the (infected) preys. Later, this leads to an increase of the number of preys.

This paper is organized as follows. In Section 2 the dynamics of the uncontrolled model is briefly discussed. Optimal control problem (OCP) associated to the predator-prey model is discussed in Section 3. The existence of solution for the OCP is studied in Section 4. Characterization of the controls is granted in Section 5. The uniqueness of solution for the OCP is investigated in Section 6. Numerical simulation is carried out in Section 7. Finally, the conclusions arise in Section 8.

2. Dynamics of the uncontrolled model

In this section we briefly state some results obtained by Jana and Kar [8] on the dynamics of the uncontrolled model. We stress that all results referred in this section are taken from [8].

Depending on the parameters, model (1) has all or some of the following equilibrium points:

- the trivial equilibrium $E_0 = (0, 0, 0)$;
- the boundary equilibrium $E_1 = (K, 0, 0)$;

- the pest free equilibrium $E_2 = (0, 0, (d - \mu)/\delta)$;
- the infection free equilibrium $E_3 = (S_1, 0, P_1)$, where S_1 is the positive root of a third order polynomial whose coefficients are determined by the parameters and P_1 is determined once we find S_1 (for more details see [8]);
- the predator free equilibrium $E_4 = (\sigma/\alpha, r(\alpha K - \sigma)/(\alpha(K\alpha + r\eta), 0)$; This equilibrium is feasible if $\sigma < \alpha K$, otherwise this equilibrium reduces to the boundary equilibrium $E_1(K, 0, 0)$;
- lastly the interior equilibrium $E^* = (S^*, I^*, P^*)$, where $S^*, I^*, P^* > 0$ have complicated expressions obtained in Appendix B of [8].

From the results obtained in [8] we have the following on the local stability of equilibria: the trivial equilibrium E_0 is always feasible and always unstable; the boundary equilibrium E_1 is always feasible and is locally asymptotically stable if $\sigma > K\alpha$ and $\mu > Kl\beta/(a+K)$; the pest free equilibrium E_2 is feasible if $d > \mu$, reduces to the trivial equilibrium E_0 if $d \leq \mu$, and is locally asymptotically stable if $d > \mu + \text{ar}\,\delta/\beta$; the infection free equilibrium E_3 and the predator free equilibrium E_4 aren't always feasible. Conditions for feasibility and local asymptotic stability are rather complicated (see [8] for additional details); finally, conditions for local asymptotic stability of the interior equilibrium E^* were obtained in Appendix B of [8].

Of particular importance for our work is the following result on the global stability of the interior equilibrium.

Theorem 2.1 (Theorem 2.4 of [8]). *The following two conditions are sufficient conditions for system* (1) *to be globally asymptotically stable around its interior equilibrium E^* are:*

(1) $F_1 = \frac{r}{K} + \frac{C^* d}{2K} - \frac{\beta(2P^* + alC^*)}{2a(a+S^*)} \geq 0$ *and* $F_2 = \delta + \frac{C^* d}{2K} - \frac{a\beta l C^*}{2a(a+S^*)} \geq 0$;

(2) $K\gamma m > d\eta + K\gamma n$

where $C^* = K\gamma/(\alpha(K\gamma(m-n) - d\eta))$.

In section 7, we use the result above to choose parameters that lead to a situation where we have persistence. Notice that we will consider in section 7 a nonautonomous problem but, nevertheless, the autonomous system corresponding to taking the average of the parameters can help us identify parameters for which the nonautonomous model is also persistent.

3. The optimal control problem

System (1) is modified with the inclusion of the quantity of pesticide that is used in the system to control the pest. That quantity is represented by the variable u. As a consequence, the quantity of susceptible and infected pest suffer a reduction of $\varepsilon_1 uS$ and $\varepsilon_2 uI$, respectively. It is known that efficacy of pesticides when applied to infected pest is not the same as when applied to susceptible healthy pests. Hence, we consider that the pesticide kills more units of infected pest than units of susceptible pest, i.e, we assume that $\varepsilon_1 < \varepsilon_2$. These chemical products, being some kind of poison, affects the entire ecosystem and hence the predators are also affected (cf. [8]). Due to the application of pesticide, at a level u, the predator population decreases at a rate ε_3.

The extra food supply to predators, named g above, as impact over the number of predators and preys. Therefore, it is also a mean of control used in the model to manage the impact of the pest.

The initial model (1) is replaced by the following:

$$\begin{cases} \dot{S} = r(t)S\left(1 - \dfrac{S + \eta I}{K}\right) - \alpha SI - \beta \dfrac{SZ}{a+S} - \varepsilon_1 uS \\[2ex] \dot{I} = I(\alpha S - bZ - \sigma) - \varepsilon_2 uI \\[2ex] \dot{Z} = \ell\beta \dfrac{SZ}{a+S} + (m-n)bIZ + gZ\left(1 - \dfrac{S+\eta I}{K}\right) - \mu Z - \delta Z^2 - \varepsilon_3 uZ \end{cases} \tag{3}$$

where $r(t)$ is a smooth function.

The optimal control problem, we consider, aims the minimization of the objective functional

$$J(S, I, u, g) = \int_0^{t_f} \left(k_1 I + k_2 S + k_3 u^2 + k_4 g^2\right) dt, \tag{4}$$

where k_1 and k_2 are small positive constants that balance the dimension of variables I and S, respectively. While the positive constants k_3 and k_4 balance the size of quadratic control terms.

The first term in the objective, $k_1 I$, represents the number of individuals of a pest that are infected. The second term, $k_2 S$, represents the number of individuals of a pest that are susceptible. The term $k_3 u^2$ represent the cost of pesticide treatment. Finally, the term $k_4 g^2$ is the cost of the food supply to the population of predators. The controls are considered quadratics since the costs associated to them are assumed not to be linearly proportional to an increase of pesticide administration or to extra food supply, but rather a cost that increases when reaching higher percentages of the population involved.

When pesticides are used in high quantities, they may make the crops poisonous and useless. Hence, the quadratic functional (4) is most appropriated for this model since the term $k_3 u^2$ reflect such severity of the side effects of the pesticides (cf. [7]). We assume that there are obvious limitations on the maximum rate at which a pest can be controlled by pesticide and limitations on the maximum rate at which predators are fed. We define, therefore, positive upper-bounds, $u_{\max}$ and $g_{\max}$, for such rates.

In Theorem 4.2 (Section 4) we prove that the optimal control solution exists. Moreover, in Theorem 6.1 (Section 6) we prove that there is only one solution.

4. Existence of solution for the optimal control problem

Problem (3)-(4) is an optimal control problem in the Lagrange form

$$\mathcal{J}(x, u) = \int_{t_0}^{t_f} \mathbb{L}(t, x(t), u(t))\, dt \longrightarrow \min,$$

$$\begin{cases} x'(t) &= f(t, x(t), u(t)), \quad \text{a.e. } t \in [t_0, t_f], \\ x(t_0) &= x_0, \end{cases} \tag{5}$$

$$x(t) \in AC([0, t_f]; \mathbb{R}^n) \text{ and } u(t) \in L^1([0, t_f]; U).$$

where $U = [0, u_{1,\max}] \times \ldots \times [0, u_{m,\max}]$ $(u_{i,\max} \leqslant 1, i = 1, \ldots, m)$, AC stands for *Absolutely Continuous*, $x(t) = (x_1(t), \ldots, x_n(t))$ and the control $u(t)$ is $u(t) = (u_1(t), \ldots, u_m(t))$ for some naturals n and m.

In this context, a pair $(x, u) \in AC([0, t_f]; \mathbb{R}^n) \times L^1([0, t_f]; U)$ is feasible if it satisfies the control problem considered in (5). As usual, the set of all feasible pairs is denoted by $\mathcal{F}$.

The following theorem, that we will use to prove existence of solution, is contained in Theorem III.4.1 and Corollary III.4.1 in [3].

Theorem 4.1 (Existence of solutions for control problems). *Suppose that f and $\mathbb{L}$ are continuous and that there exist positive constants C_1 and C_2 such that, for $t \in \mathbb{R}$, $x, x_1, x_2 \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ we have:*

1. $\|f(t, x, u)\| \leqslant C_1(1 + \|x\| + \|u\|)$;
2. $\|f(t, x_1, u) - f(t, x_2, u)\| \leqslant C_2 \|x_1 - x_2\|(1 + \|u\|)$.
3. $\mathcal{F}$ *is non-empty;*
4. U *is closed;*
5. *there is a compact set* S *such that* $x(t_1) \in S$ *for any state variable* x*;*
6. U *is convex,* $f(t, x, u) = \alpha(t, x) + \beta(t, x)u$ *and* $\mathbb{L}(t, x, \cdot)$ *is convex on* U*;*
7. $L(t, x, u) \geqslant c_1 |u|^\beta - c_2$, *for some* $c_1 > 0$, $\beta > 1$.

Then there exist (x_0^*, u^*) minimizing J on $\mathcal{F}$.

Set $U = [0, u_{\max}] \times [0, g_{\max}]$ with $u_{\max}, g_{\max} \leqslant 1$. Applying Theorem 4.1 to our problem we obtain the following result:

Theorem 4.2. *There exists an optimal control pair (u^*, g^*) and a corresponding solution, (S^*, I^*, Z^*), of the initial value problem determined by (3) with initial condition $(S(0), I(0), Z(0)) = (S_0, I_0, Z_0)$, that minimizes the cost functional*

$$\mathcal{J}_2 = \int_0^{t_f} \left(k_1 I + k_2 S + k_3 u^2 + k_4 g^2\right) dt$$

over $L^1\left([0, t_f]; [0, u_{\max}] \times [0, g_{\max}]\right)$.

Proof. To apply Theorem 4.1 to our problem, we set $U = [0, u_{max}] \times [0, g_{\max}]$ and $[t_0, t_1] = [0, t_f]$. To keep the expressions in the proofs short we omit the dependency on time of the parameters. Adding the first two equations in (3), we get

$$\dot{S} + \dot{I} = r(t) S \left(1 - \frac{S + \eta I}{K}\right) - \beta \frac{SZ}{a + S} - \varepsilon_1 u S - I\left(bZ + \sigma\right) - \varepsilon_2 u I. \tag{6}$$

Writing (6) as

$$\dot{S} + \dot{I} = r(t) S \left(1 - \frac{S + I}{K}\right) - \frac{\eta - 1}{K} r S I - \beta \frac{SZ}{a + S} - \varepsilon_1 u S - I\left(bZ + \sigma\right) - \varepsilon_2 u I,$$

if $\eta \geqslant 1$, we conclude that $S(t) + I(t) \leqslant \max\{S_0 + I_0, K/\eta\}$ (since $S(t) + I(t)$ is decreasing if $S(t) + I(t) > K$). On the other hand, if $\eta < 1$, writing (6) as

$$\dot{S} + \dot{I} \leqslant r_{\max} S \left(1 - \eta \frac{S + I}{K}\right) - \beta \frac{SZ}{a + S} - \varepsilon_1 u S - I\left(bZ + \sigma\right) - \varepsilon_2 u I,$$

where $r_{\max} = \max\{r(t) : t \in [0, t_f]\}$. We conclude that $S(t) + I(t) \leqslant \max\{S_0 + I_0, K\}$ (since $S(t) + I(t)$ is decreasing if $S(t) + I(t) > K$). Thus we conclude that $S(t) + I(t) \leqslant \max\{S_0 + I_0, K, K/\eta\} := M_1$.

Next, note that

$$\begin{aligned}
\dot{Z} &= \ell\beta \frac{SZ}{a + S} + (m - n) b I Z + g Z \left(1 - (S + \eta I)/K\right) - \mu Z - \delta Z^2 - \varepsilon_3 u Z \\
&= Z \left(\ell\beta \frac{S}{a + S} + (m - n) b I + g \left(1 - (S + \eta I)/K\right) - \mu - \delta Z - \varepsilon_3 u\right) \\
&\leqslant Z \left(\ell\beta + |m - n| b M_1 + g - \delta Z\right)
\end{aligned} \tag{7}$$

and thus Z is decreasing if $Z(t) > (\ell\beta + |m - n| b M_1 + g)/\delta$. Thus we conclude that $Z(t) \leqslant \max\{Z_0, (\ell\beta + |m - n| b M_1 + g)/\delta\} := M_2$. Using the bounds above, we immediately obtain 1. and 2.

Conditions 3. and 4. are immediate from the definition of $\mathcal{F}$ and since $U = [0, u_{max}] \times [0, g_{\max}]$.

We conclude that all the state variables are in the compact set

$$\{(x, y, z) \in (\mathbb{R}_0^+)^3 : 0 \leqslant x + y + z \leqslant M_1 + M_2\}$$

and condition 5. follows.

Since the state equations are linearly dependent on the controls and L is quadratic in the controls, we obtain 6. Finally,

$$L = k_1 I + k_2 S + k_3 u^2 + k_4 g^2 \geqslant \min\{k_3, k_4\}(u^2 + g^2) \geqslant \min\{k_3 + k_4\} \|(u, g)\|^2$$

and we establish 7. with $c_1 = \min\{k_3, k_4\}$.

Having checked all the hypothesis, the result follows from Theorem 4.1. □

5. Characterization of the controls

First-order necessary conditions for optimality of a controlled trajectory are given by the *Pontryagin Maximum Principle* (cf. [11]; for a formulation adapted to a Minimization problem see [2]).

Since we have a Minimization problem, using the adjoint variable $p = (p_1, p_2, p_3) \in \mathbb{R}^3$, the Hamiltonian for the objective functional (4) and the control system (3) is given by

$$
\begin{aligned}
\mathcal{H} &= \sum_{i=1}^{3} p_i\, f_i(S, I, Z) + k_1 I + k_2 S + k_3 u^2 + k_4 g^2 \\
&= p_1 \left[r(t) S \left(1 - \frac{S + \eta I}{K} \right) - \alpha S I - \beta \frac{SZ}{a+S} - \varepsilon_1 u S \right] \\
&\quad + p_2 \left[I(\alpha S - bZ - \sigma) - \varepsilon_2 u I \right] \\
&\quad + p_3 [\ell \beta \frac{SZ}{a+S} + (m-n) b I Z + g Z \left(1 - \frac{S + \eta I}{K} \right) \\
&\quad - \mu Z - \delta Z^2 - \varepsilon_3 u Z] + k_1 I + k_2 S + k_3 u^2 + k_4 g^2.
\end{aligned} \tag{8}
$$

The co-state variables are given by

$$
p_1(t) = -\frac{\partial \mathcal{H}}{\partial S},\; p_2(t) = -\frac{\partial \mathcal{H}}{\partial I} \text{ and } p_3(t) = -\frac{\partial \mathcal{H}}{\partial Z},
$$

that is

$$
\begin{aligned}
\dot{p}_1 &= -k_2 - \alpha I p_2 - p_3 \left[-\frac{gZ}{K} + \frac{\ell a \beta Z}{(a+S)^2} \right] + \\
&\quad - p_1 \left[-\varepsilon_1 u - \alpha I + r(t) \left(1 - \frac{2S + \eta I}{K} \right) - \frac{a \beta Z}{(a+S)^2} \right],
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\dot{p}_2 &= -k_1 - p_1 \left[-\alpha S - \frac{r(t) \eta S}{K} \right] - p_2 \left[-\sigma - \varepsilon_2 u + \alpha S - bZ \right] \\
&\quad - p_3 \left[-\frac{g \eta Z}{K} + b(m-n) Z \right],
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\dot{p}_3 &= b I p_2 + \frac{\beta p_1 S}{a+S} - p_3 [-\mu - \varepsilon_3 u + b(m-n) I \\
&\quad + \ell \frac{\beta S}{a+S} + g \left(1 - \frac{S + \eta I}{K} \right) - 2 \delta Z].
\end{aligned} \tag{11}
$$

Since the terminal state, $(S(t_f), I(t_f), Z(t_f))$, is free, the transversality conditions are

$$
p_1(t_f) = p_2(t_f) = p_3(t_f) = 0. \tag{12}
$$

Using the *Pontryagin Maximum Principle* (PMP), we characterize the optimal controls u^* and g^* in the following Theorem.

Theorem 5.1. *The optimal control pair is given by*

$$
u^* = \min \left\{ \max \left\{ 0, \frac{\varepsilon_1 S^* p_1^* + \varepsilon_2 I^* p_2^* + \varepsilon_3 Z^* p_3^*}{2 k_3} \right\}, u_{\max} \right\} \tag{13}
$$

and

$$g^* = \min\left\{\max\left\{0, \left(\frac{\eta I^* + S^*}{K} - 1\right)\frac{Z^* p_3^*}{2k_4}\right\}, g_{\max}\right\} \tag{14}$$

where S^*, I^*, Z^*, p_1^*, p_2^* and p_3^* are the optimal variables.

Proof. The optimality conditions dictate that $\frac{\partial \mathcal{H}}{\partial u} = 0$ and $\frac{\partial \mathcal{H}}{\partial g} = 0$, that is,

$$u = \frac{\varepsilon_1 S p_1 + \varepsilon_2 I p_2 + \varepsilon_3 Z p_3}{2k_3} \quad \text{and} \quad g = \left(\frac{\eta I + S}{K} - 1\right)\frac{Z p_3}{2k_4}$$

for the optimal controls u^* and g^* on the interior of the control set. Due to the bounds on the controls the optimal control are given by equations (13)-(14). □

Note that, as a consequence of the transversality conditions, the optimal controls u and g will be null at the end of the intervention period. Having obtaining the solution of the problem, this condition allow us to check rapidly if we have the correct one.

6. Uniqueness of solution for the optimal control problem

In this section we consider a slightly distinct version of our problem where the control space will be a smaller space (where we can apply a version of PMP for bounded controls). Let Ω_1 as the set of left continuous piecewise functions defined in $[0, t_f]$ and with values in $[0, u_{max}] \times [0, g_{max}]$.

Next, we will show that when there is a solution for the optimality system in Ω_1 – defined by the state equations, the initial conditions, the adjoint equations and the transversality conditions – this solution is unique. The proof of this result is done in two steps. The first step consists in obtaining uniqueness of the optimality system in a sufficiently small interval and is inspired on [4] where a similar strategy was used to obtain uniqueness of the optimality system for an autonomous model in a sufficiently small interval and in [9] that adapted that strategy to obtain uniqueness of the optimality system for a non-autonomous SEIRS model. The second step extend the result to the whole interval and the argument is borrowed from [9].

Theorem 6.1. *Assuming that there is a solution of the optimal control problem in Ω_1, the optimality system is unique in the interval $[0, t_f]$.*

Proof. First we prove the uniqueness of the solution to the OCP problem, (S, I, Z, p_1, p_2, p_3) on some interval $[0, T]$ for some $T \in \mathbb{R}^+$, eventually less than t_f. We assume that we have two optimality systems corresponding to trajectories and state equations (S, I, Z), (p_1, p_2, p_3) and $(\bar{S}, \bar{I}, \bar{Z})$, $(\bar{p}_1, \bar{p}_2, \bar{p}_3)$ and we will show that the two coincide in some small interval. Consider the change of variables

$$S(t) = e^{\theta t} s(t), \quad I(t) = e^{\theta t} \iota(t), \quad Z(t) = e^{\theta t} z(t)$$

and

$$p_1(t) = e^{-\theta t}\phi_1(t), \quad p_2(t) = e^{-\theta t}\phi_2(t), \quad p_3(t) = e^{-\theta t}\phi_3(t).$$

By the first equation in (3) we get

$$\theta e^{\theta t} s + e^{\theta t}\dot{s} = r(t) s e^{\theta t}(1 - e^{\theta t}(s + \eta\iota)/K) - \alpha e^{2\theta t} s\iota - \beta e^{2\theta t} sz/(\alpha + e^{\theta t} s) - \varepsilon_1 e^{\theta t} us$$

and thus

$$\theta s + \dot{s} = r(t) s(1 - e^{\theta t}(s + \eta\iota)/K) - \alpha s\iota - \beta sz/(\alpha e^{-\theta t} + s) - \varepsilon_1 us.$$

Multiplying by $(s-\bar{s})$, integrating from 0 to T and noting that $s(0)=\bar{s}(0)$ we have

$$\begin{aligned}
&\theta \int_0^T (s-\bar{s})^2 dt + \frac{1}{2}(s(T)-\bar{s}(T))^2 \\
&= r(t) \int_0^T (s-\bar{s})^2 dt - \int_0^T e^{\theta t}(s^2+\eta \iota s - \bar{s}^2 - \eta\bar{\iota}\bar{s})(s-\bar{s})/K dt - \alpha \int_0^T (s\iota - \bar{s}\bar{\iota})(s-\bar{s})\ dt \\
&\quad - \beta \int_0^T \left[sz/(\alpha e^{-\theta t}+s) - \bar{s}\bar{z}/(\alpha e^{-\theta t}+\bar{s})\right](s-\bar{s})\ dt - \varepsilon_1 \int_0^T (us-\bar{u}\bar{s})(s-\bar{s})\ dt.
\end{aligned}$$

Recall that

$$(x-\bar{x})(y-\bar{y}) \leqslant \frac{1}{2}[(x-\bar{x})^2+(y-\bar{y})^2] \tag{15}$$

and that, for each $x, y, w, \bar{x}, \bar{y}, \bar{w} > 0$, there is $C>0$ (depending on $x, y, w, \bar{x}, \bar{y}$ and $\bar{w}$) such that

$$(xy-\bar{x}\bar{y})(w-\bar{w}) = C[(x-\bar{x})^2+(y-\bar{y})^2+(w-\bar{w})^2]. \tag{16}$$

Using (15), (16) and

$$\begin{aligned}
\int_0^T \left[\frac{sz}{\alpha e^{-\theta t}+s} - \frac{\bar{s}\bar{z}}{\alpha e^{-\theta t}+\bar{s}}\right]\ dt &= \int_0^T \frac{sz(\alpha e^{-\theta t}+\bar{s}) - \bar{s}\bar{z}(\alpha e^{-\theta t}+s)}{(\alpha e^{-\theta t}+s)(\alpha e^{-\theta t}+\bar{s})}(s-\bar{s})\ dt \\
&\leqslant \int_0^T \frac{\alpha e^{-\theta t}(sz-\bar{s}\bar{z})(s-\bar{s}) + s\bar{s}(z-\bar{z})(s-\bar{s})}{(\alpha e^{-\theta t}+s)(\alpha e^{-\theta t}+\bar{s})}\ dt \\
&\leqslant \frac{Ce^{\theta T}}{\alpha} \int_0^T (z-\bar{z})^2 + 2(s-\bar{s})^2\ dt \\
&+ \frac{1}{2}\int_0^T (z-\bar{z})^2 + (s-\bar{s})^2\ dt,
\end{aligned}$$

there is $C_1, \tilde{C_1} > 0$ (depending on the state variables and adjoint equations) such that

$$\theta \int_0^T (s-\bar{s})^2 dt + \frac{1}{2}(s(T)-\bar{s}(T))^2 = (C_1+\tilde{C_1}e^{\theta T}) \int_0^T (s-\bar{s})^2 + (\iota-\bar{\iota})^2 + (z-\bar{z})^2 + (u-\bar{u})^2 dt. \tag{17}$$

By the second equation in (3) we get

$$\theta e^{\theta t}\iota + e^{\theta t}\dot{\imath} = \iota e^{\theta t}(\alpha e^{\theta t}s - be^{\theta t}z - \sigma) - \varepsilon_2 u e^{\theta t}\iota$$

and thus

$$\theta\iota + \dot{\imath} = \iota(\alpha e^{\theta t}s - be^{\theta t}z - \sigma) - \varepsilon_2 u\iota.$$

Multiplying by $(\iota-\bar{\iota})$, integrating from 0 to T and noting that $s(0)=\bar{s}(0)$ we have

$$\begin{aligned}
&\theta \int_0^T (\iota-\bar{\iota})^2 dt + \frac{1}{2}(\iota(T)-\bar{\iota}(T))^2 \\
&= \int_0^T \alpha e^{\theta t}(s\iota-\bar{s}\bar{\iota})(\iota-\bar{\iota}) - be^{\theta t}(z\iota-\bar{z}\bar{\iota})(\iota-\bar{\iota}) - \sigma(\iota-\bar{\iota})^2\ dt - \varepsilon_2 \int_0^T (u\iota-\bar{u}\bar{\iota})(\iota-\bar{\iota})\ dt
\end{aligned}$$

and, using (15) and (16), we conclude that there is $C_2, \tilde{C_2} > 0$ (depending on the state variables and adjoint equations) such that

$$\theta \int_0^T (\iota-\bar{\iota})^2 dt + \frac{1}{2}(\iota(T)-\bar{\iota}(T))^2 = (C_2+\tilde{C_2}e^{\theta T}) \int_0^T (s-\bar{s})^2 + (\iota-\bar{\iota})^2 + (z-\bar{z})^2 + (u-\bar{u})^2 dt. \tag{18}$$

By the third equation in (3) we get

$$\begin{aligned}\theta e^{\theta t}z + e^{\theta t}\dot{z} &= \ell\beta\frac{sze^{2\theta t}}{\alpha + e^{\theta t}s} + (m-n)be^{2\theta t}z\iota \\ &\quad + ge^{\theta t}z - \frac{1}{K}ge^{2\theta t}z(s+\eta\iota) - \mu e^{\theta t}z - \delta e^{2\theta t}z^2 - \varepsilon_3 e^{\theta t}uz\end{aligned}$$

and thus

$$\theta z + \dot{z} = \ell\beta\frac{sze^{\theta t}}{\alpha + e^{\theta t}s} + (m-n)be^{\theta t}z\iota + gz - \frac{1}{K}ge^{\theta t}z(s+\eta\iota) - \mu z - \delta e^{\theta t}z^2 - \varepsilon_3 uz.$$

Proceeding similarly, using (15) and (16), we conclude that there are $C_3, \tilde{C}_3 > 0$ (depending on the state variables and adjoint equations) such that

$$\theta\int_0^T (z-\bar{z})^2dt + \frac{1}{2}(z(T)-\bar{z}(T))^2 = (C_3 + \tilde{C}_3e^{\theta T})\int_0^T (s-\bar{s})^2 + (\iota-\bar{\iota})^2 + (z-\bar{z})^2 + (u-\bar{u})^2dt. \tag{19}$$

Considering now the equations for the adjoint equations and reasoning similarly we can conclude that there are constants $C_4, \tilde{C}_4, C_5, \tilde{C}_5, C_6, \tilde{C}_6 > 0$ (depending on the state variables and adjoint equations) such that

$$\begin{aligned}&\theta\int_0^T (\phi_1-\bar{\phi}_1)^2dt + \frac{1}{2}(\phi_1(0)-\bar{\phi}_1(0))^2 \\ &\leqslant (C_4+\tilde{C}_4e^{\theta T})\int_0^T (s-\bar{s})^2+(\iota-\bar{\iota})^2+(z-\bar{z})^2+(\phi_1-\bar{\phi}_1)^2+(\phi_2-\bar{\phi}_2)^2+(\phi_3-\bar{\phi}_3)^2+(u-\bar{u})^2dt,\end{aligned} \tag{20}$$

$$\begin{aligned}&\theta\int_0^T (\phi_2-\bar{\phi}_2)^2dt + \frac{1}{2}(\phi_2(0)-\bar{\phi}_2(0))^2 \\ &\leqslant (C_5+\tilde{C}_5e^{\theta T})\int_0^T (s-\bar{s})^2+(\iota-\bar{\iota})^2+(z-\bar{z})^2+(\phi_1-\bar{\phi}_1)^2+(\phi_2-\bar{\phi}_2)^2+(\phi_3-\bar{\phi}_3)^2+(u-\bar{u})^2dt,\end{aligned} \tag{21}$$

and

$$\begin{aligned}&\theta\int_0^T (\phi_3-\bar{\phi}_3)^2dt + \frac{1}{2}(\phi_3(0)-\bar{\phi}_3(0))^2 \\ &= (C_6+\tilde{C}_6e^{\theta T})\int_0^T (s-\bar{s})^2+(\iota-\bar{\iota})^2+(z-\bar{z})^2+(\phi_1-\bar{\phi}_1)^2+(\phi_2-\bar{\phi}_2)^2+(\phi_3-\bar{\phi}_3)^2+(u-\bar{u})^2dt.\end{aligned} \tag{22}$$

By (13), we obtain

$$u-\bar{u} = \frac{\varepsilon_1}{2k_3}(s\phi_1-\bar{s}\bar{\phi}_1) + \frac{\varepsilon_2}{2k_3}(\iota\phi_2-\bar{\iota}\bar{\phi}_2) + \frac{\varepsilon_3}{2k_3}(z\phi_3-\bar{z}\bar{\phi}_3)$$

and we conclude that there is $C_7 > 0$ (depending on the state variables and adjoint equations) such that

$$\begin{aligned}(u-\bar{u})^2 &= \left[\frac{\varepsilon_1}{2k_3}(s\phi_1-\bar{s}\bar{\phi}_1) + \frac{\varepsilon_2}{2k_3}(\iota\phi_2-\bar{\iota}\bar{\phi}_2) + \frac{\varepsilon_3}{2k_3}(z\phi_3-\bar{z}\bar{\phi}_3)\right]^2 \\ &= \left[\frac{\varepsilon_1}{2k_3}(s(\phi_1-\bar{\phi}_1)+(s-\bar{s})\bar{\phi}_1)+\frac{\varepsilon_2}{2k_3}(\iota(\phi_2-\bar{\phi}_2)+(\iota-\bar{\iota})\bar{\phi}_2)+\frac{\varepsilon_3}{2k_3}(z(\phi_3-\bar{\phi}_3)+(z-\bar{z})\bar{\phi}_3)\right]^2 \\ &\leqslant C_7\left[(s-\bar{s})^2+(\iota-\bar{\iota})^2+(z-\bar{z})^2+(\phi_1-\bar{\phi}_1)^2+(\phi_2-\bar{\phi}_2)^2+(\phi_3-\bar{\phi}_3)^2\right].\end{aligned} \tag{23}$$

and for the control g we have

$$
\begin{aligned}
(g-\bar{g})^2 &= \frac{1}{4k_4^2}\left(K\left(-p_3 z+\bar{p}_3\bar{z}\right)+e^{\theta t}\left(p_3 s z-\bar{p}_3\bar{s}\bar{z}\right)+e^{t\theta}\eta\left(p_3 z\iota-\bar{p}_3\bar{z}\bar{\iota}\right)\right)^2 \\
&\leqslant C_8\left(\left(\phi_3-\bar{\phi}_3\right)^2+(z-\bar{z})^2\right)+C_9K\eta e^{\theta T}\left(\left(\phi_3-\bar{\phi}_3\right)^2+(z-\bar{z})^2+(s-\bar{s})^2\right) \\
&+C_{10}e^{2\theta T}\left((\phi_3-\bar{\phi}_3)^2+(z-\bar{z})^2+(s-\bar{s})^2+(\iota-\bar{\iota})^2\right). \qquad (24)
\end{aligned}
$$

We have finally all we need to prove our result. Define

$$\Psi(t)=(s(t)-\bar{s}(t))^2+(y(t)-\bar{y}(t))^2+(z(t)-\bar{z}(t))^2$$

and

$$\Phi(t)=(\phi_1(t)-\bar{\phi}_1(t))^2+(\phi_2(t)-\bar{\phi}_2(t))^2+(\phi_3(t)-\bar{\phi}_3(t))^2.$$

and observe that $\Psi(t)\geqslant 0$ and $\Phi(t)\geqslant 0$ for all t.

Adding equations (17), (18), (19), (20), (21), (22) and (23), we obtain for the sum of left-hand sides

$$\frac{1}{2}\Psi(T)+\frac{1}{2}\Phi(0)+\theta\int_0^T \Psi(t)+\Phi(t)dt$$

and thus

$$\frac{1}{2}[\Psi(T)+\Phi(0)]+\theta\int_0^T \Psi(t)+\Phi(t)dt\leqslant (D+\tilde{D}e^{2\theta T})\int_0^T \Psi(t)+\Phi(t)dt,$$

where $D=\sum_{i=1}^{7}C_i$ and $\tilde{D}=\sum_{i=1}^{7}\tilde{C}_i$. Thus

$$\frac{1}{2}[\Psi(T)+\Phi(0)]+(\theta-D-\tilde{D}e^{2\theta T})\int_0^T \Psi(t)+\Phi(t)dt\leqslant 0. \qquad (25)$$

We now choose θ so that

$$\theta > D+\tilde{D}$$

and note that $\frac{\theta-D}{\tilde{D}}>1$. Subsequently, we choose T such that

$$T<\frac{1}{2\theta}\ln\left(\frac{\theta-D}{\tilde{D}}\right).$$

Then,

$$2\theta T<\ln\left(\frac{\theta-D}{\tilde{D}}\right) \quad\Rightarrow\quad e^{2\theta T}<\frac{\theta-D}{\tilde{D}}.$$

It follows that $\theta-D-\tilde{D}e^{\theta T}>0$, so inequality (25) can hold if and only if, for all $t\in[0,T]$, we have $s(t)=\bar{s}(t)$, $\iota(t)=\bar{\iota}(t)$, $z(t)=\bar{z}(t)$, $\phi_1(t)=\bar{\phi}_1(t)$, $\phi_2(t)=\bar{\phi}_2(t)$ and $\phi_3(t)=\bar{\phi}_3(t)$. But this is equivalent to $S(t)=\bar{S}(t)$, $I(t)=\bar{I}(t)$, $Z(t)=\bar{Z}(t)$, $p_1(t)=\bar{p}_1(t)$, $p_2(t)=\bar{p}_2(t)$ and $p_3(t)=\bar{p}_3(t)$.

With this, the uniqueness of the optimal control is established in a small interval $[0,T]$.

To finish the proof, we observe that if $T\geqslant t_f$, then the result follows. Otherwise, we can obtain uniqueness on $[T,2T]$ for the optimal control problem whose initial conditions on time T coincide with the values of S, I and Z on T (note that, we still obtain the constants α, D and $\tilde{D}$ and thus we get the same T). Proceeding in the same way, we conclude, after a finite number of steps, that we have uniqueness on the interval $[0,t_f]$. The proof is complete. □

Name	Description	Value
η	impact of a predator individual on the per capita growth rate of a pest individual relative to the impact of a pest individual on its own per capita growth rate (cf. [1], [5], [15])	1
K	environmental carrying capacity	50
α	force of infection	0.3
a	half saturation constant	0.21
ε_1	death rate of susceptible pest due to pesticides	1
b	maximum capturing rate of infected pest	0.5
σ	natural death rate of infected pest	0.15
ε_2	death rate of infected pest due to pesticides	1.25
l	contribution of predator population from susceptible pest	0.5
m	contribution of predator population from infected pest	0.3
$-n$	negative effect to the biomass of predators due to the infection from infected pest	-0.15
μ	natural death rate of predators	0.21
δ	density dependent mortality of predators	0.11
ε_3	death rate of predator population due to pesticides	0.01
β	maximum capturing rate of susceptible pest	0.04

TABLE 1. Values of the models' parameters borrowed from [6] with the exception of β and parameters $\varepsilon_{i(i=1,2,3)}$, which are assumed.

7. NUMERICAL RESULTS FOR THE OPTIMAL CONTROL MODEL

The Pontryagin Maximum Principle is used to numerically solve the optimal control problem using a fourth order Runge–Kutta iterative method, coded in MATLAB. First we solve the system (3) with initial conditions for the state variables stated in Table 2 and a guess for the control over the time interval $[0, t_f]$, by the forward Runge–Kutta fourth order procedure, obtaining the values of the state variables S, I and Z. Using those values, then we solve the system (9)-(11) with the transversality conditions (12), by the backward fourth order Runge–Kutta procedure, and obtain the values of the co-state variables. The controls are updated by convex combination of the previous controls and the values from (13)-(14). The process is stopped when the values of the variables at the earlier iteration are indistinguishable to the ones at the current iteration.

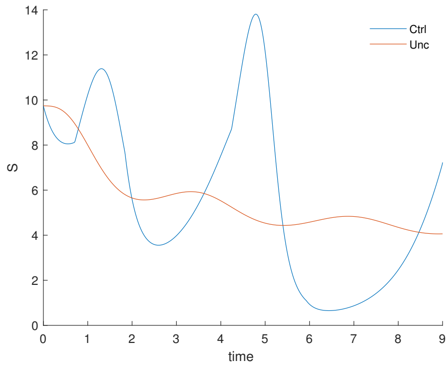
TABLE 2. Initial conditions for the optimal control problem presented in Section 3 with parameters given above in Table 1, corresponding to the non-trivial equilibrium for the system (1) with $d = 0.6$.

$S(0)$	$I(0)$	$Z(0)$
9.73862	4.87921	5.54317

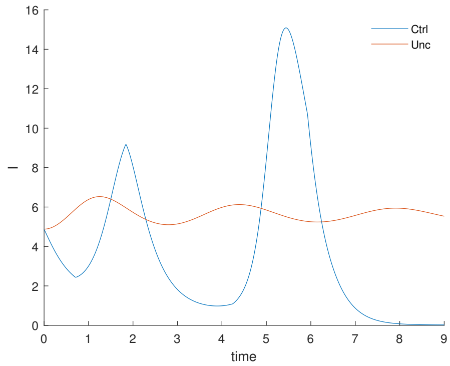
The configuration of the intrinsic growth rate of susceptible pest, $r(t)$, is presented in Figure 2. It is time dependent, periodic and is given by

$$r(t) = 2.1(1 + per \cdot \cos(\pi t/2 + \pi))$$

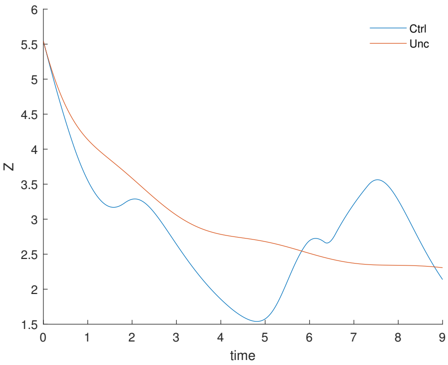
with $per \in [0, 1[$. Without loss of generality, the remaining parameters are invariant and their values are the ones that are presented in Table 1. Those values were borrowed from [6] with the exception of β and ε_1, ε_2 and



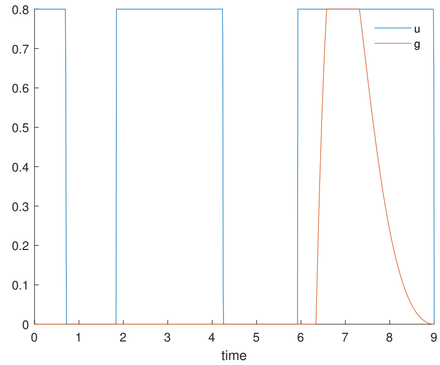
(A) Susceptible pest, S.



(B) Infected pest, I.



(C) Predator population, Z.



(D) Controls u and g.

FIGURE 1. Comparison of the state variables of the OCP (Ctrl) with the correspondents variables of the original model (Unc) – solution of (1) without g – for the autonomous model ($per = 0$) with parameter values from Table 1. Optimal controls of the autonomous OCP (*Figure (D)*). The weights of OCP objective functional are $k_1 = k_2 = 1$, $k_3 = 0.05$ and $k_4 = 0.15$.

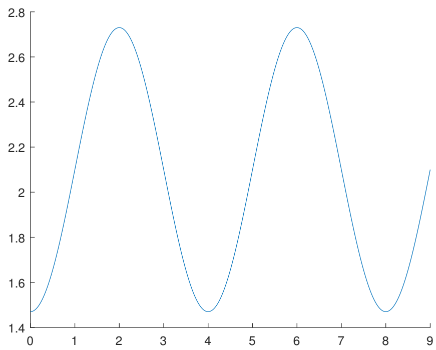


FIGURE 2. Configuration of the intrinsic growth rate of susceptible pest function, $r(t)$.

ε_3, whose values are considered lower than the homonymous values in the original model. Note that we are in the autonomous case when $per = 0$ and in a periodic scenario when $0 < per < 1$.

When in presence of extreme conditions, motivated by drastic changes in the weather in a relatively short period of time, the growth rate of susceptible pest may be function of a greater value of rate per ($per \gg 0$). In those circumstances, the environment may control the pest by itself. Usually this scenario, known as extreme, is limited in time and rare in nature. In what follows, we consider relatively small values for per.

In the numerical experiments we consider weights $k_1 = k_2 = 1$. Since nowadays is cheap to use pesticides to fight a pest, we take $k_3 = 0.05$. Maintain a population of predators alive and healthy by feeding it, in order to diminish the pest population, is not as cheap as using pesticides, hence we take $k_4 = 0.15$. We consider that the maximum value of the rate of pesticide (first control) and the maximum value of level of extra food supply to the predators (second control) are equals and $u_{\max} = g_{\max} = 0.8$ (see [10] for the upper bound of g). As we will see, the pest is extinguished after a certain time value, which is lower than 9. We noticed that this is verified by the solution of the problem even if we consider bigger values for the time interval. This is the reason why we consider $t_f = 9$ in what follows.

In section 2, we briefly presented several results concerning the equilibrium points of model (1). In order to perform the numerical simulation to the uncontrolled and controlled model, we consider the interior equilibrium point, provided by appendix B of [6]. With these initial conditions, the model is globally asymptotically stable around its interior equilibrium since conditions of Theorem 2.1 are satisfied. With these conditions $F_1 = 0.068009$, $F_2 = 0.242139$ and $K\gamma m - d\eta - K\gamma n = 3.15$.

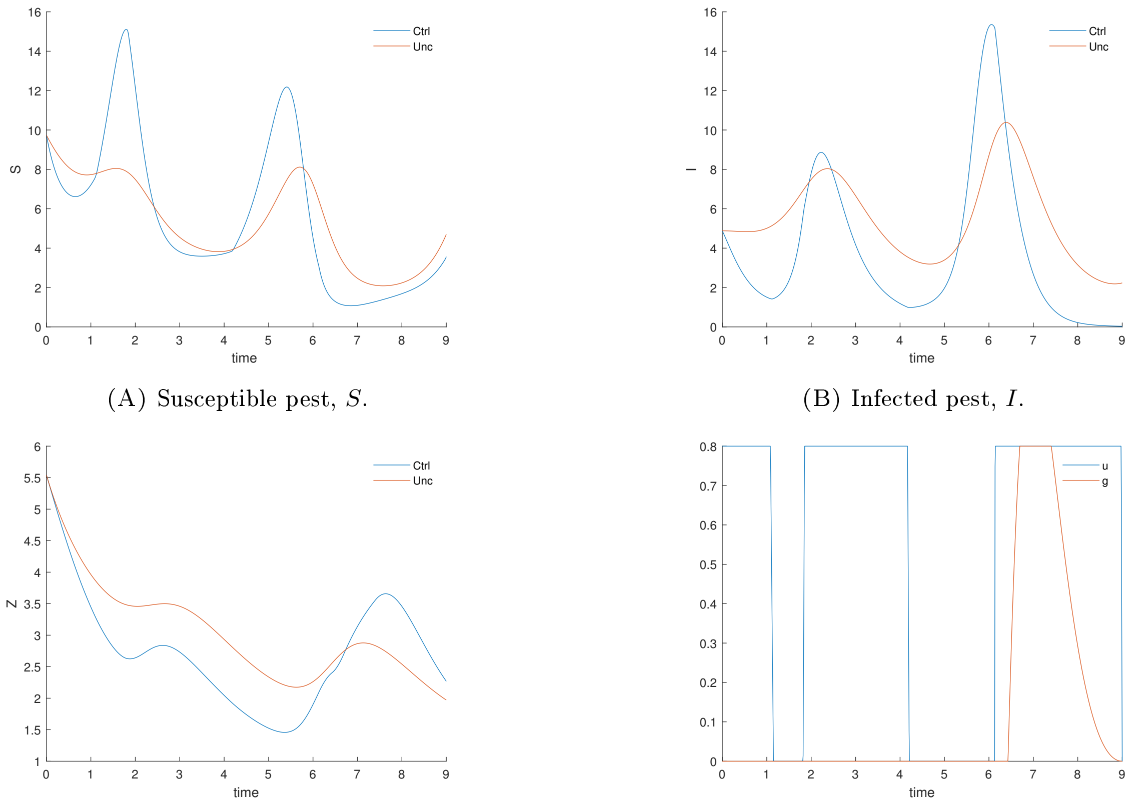
The solution for the autonomous optimal control problem (OCP) is illustrated in Figure 1. Rate of pesticide used in the control of pest must have – in most part of time interval – maximal intensity. Rate of extra food supply to predators must have maximal intensity during final part of time interval. In the absence of pesticide and food supply to predators, corresponding to almost one and to two units of time, respectively, we observe that the pest level grow considerably and surpass the respective level of the uncontrolled model. On the other hand, the combination of the two measures above makes that the infected pest level be rather low at the end of the time interval. We also observe that the population of predators grows considerably when food supply is introduced and diminishes after, in the final part of the time interval, coinciding with pest decrease and the reduction of food supply.

The solution for the non-autonomous OCP is exhibited in Figure 3. We can see that the pest level (susceptible and infected) at beginning decreases more than autonomous case and later surpasses it. Figure 4A confirms such conclusion. This behaviour is motivated by the evolution of the intrinsic growth rate of susceptible pest, $r(t)$, which is periodic and varies between a minimum of 1.47 and a maximum of 2.73. The evolution of $r(t)$ also influences the uncontrolled problem causing that the evolution of pest population is, in that case, more irregular than previously.

Autonomous and non-autonomous solutions of the OCP are compared in Figure 4. Both models require that pesticide shall be applied in most part of time interval, but the periodic model requires that pesticide shall be applied differently, it starts with a longer period and ends with a quite smaller period. On the other hand, we observe that the extra food supply to predators population in the periodic model shall be provided later in time, comparatively with regular model.

As a different scenario, we discuss hereafter the use of only one control instead of two. Figure 5A depicts the evolution of the pest population of the non-autonomous OCP model when only the control g is used. It can be seen that this strategy of control is also effective in reducing pest population. In this case, the control g is required in last part of time interval as shows Figure 5B.

Figure 6A depicts the evolution of pest population for the non-autonomous OCP model when only the control u is used. We can see that the pest population in the final of time interval reaches a rather small value. Compared with previous scenarios, it is only surpassed by the original strategy (see Figure 7). In this case, the evolution of control u is similar to the homonymous control in the problem with both controls, as shows Figure 6B.



(A) Susceptible pest, S.

(B) Infected pest, I.

(C) Predator population, Z.

(D) Controls u and g.

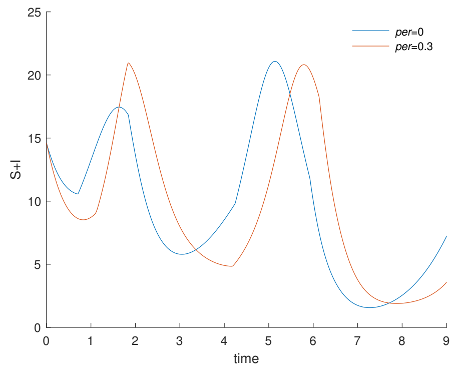
FIGURE 3. Comparison of the state variables of the OCP (Ctrl) with the correspondents variables of the original model (Unc), solution of (1) without g, for the non-autonomous model ($per = 0.3$) with parameter values according to Table 1. Optimal controls of the non-autonomous OCP (*Figure (D)*). The weights of OCP objective functional are $k_1 = k_2 = 1$, $k_3 = 0.05$ and $k_4 = 0.15$.

The control of the pest with the use of both controls, the use of only u or the use of only g is depicted in Figure 7. In it we see that pesticide treatment is a very effective control and the addition of food supply makes difference cause it makes the treatment even more effective.

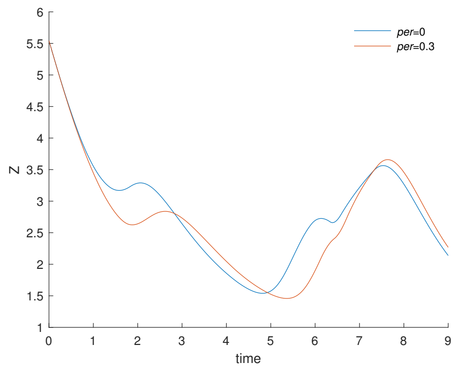
8. Conclusions

An optimal control problem for a non-linear system of ordinary differential equations, which models the dynamics of a pest in a ecological system where pest live together with its predator, is studied. Considering a more realistic model where parameters are time dependent, the authors look for the best strategy to control pest population. The existence and uniqueness of optimal solutions are established. Some simulation results of such model are presented and compared with the ones obtained for uncontrolled model showing the importance of chosen controls.

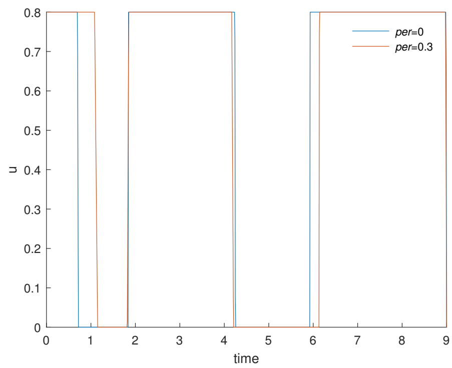
Numerical simulations show that the proposed Optimal Control problem (OCP) is effective in the control of pest population. Ignoring one of the controls, the level of pesticides or the extra supply of food to predators,



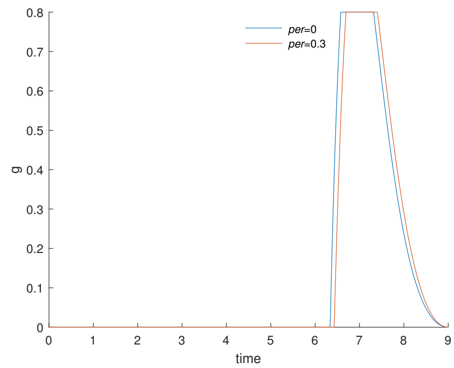
(A) Pest population, $S + I$.



(B) Predator population, Z.



(C) Level of pesticide treatment, u.



(D) Food supply to the population of predators, g.

FIGURE 4. Comparison of the solution of the autonomous OCP ($per = 0$) with the solution of the non-autonomous OCP ($per = 0.3$) with parameter values according to Table 1, weights $k_1 = k_2 = 1$, $k_3 = 0.05$ and $k_4 = 0.15$.

we obtain an OCP not as effective as the one with both controls. Therefore, we conclude that both controls are indispensable.

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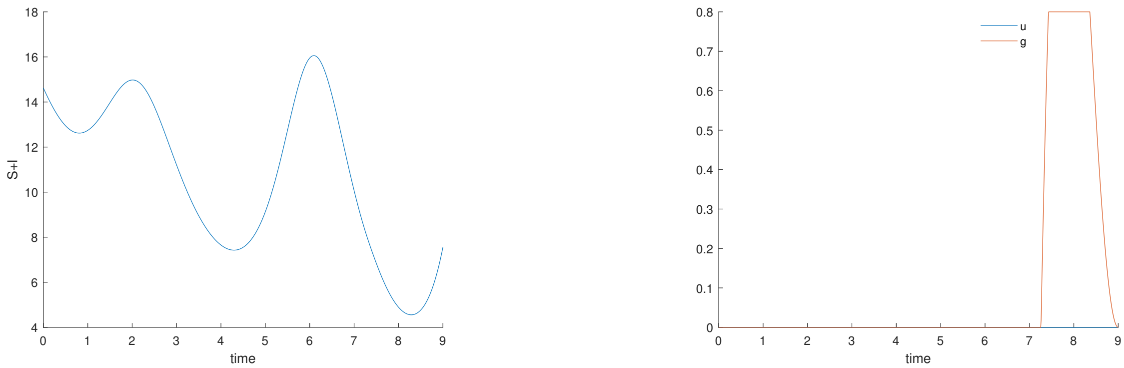
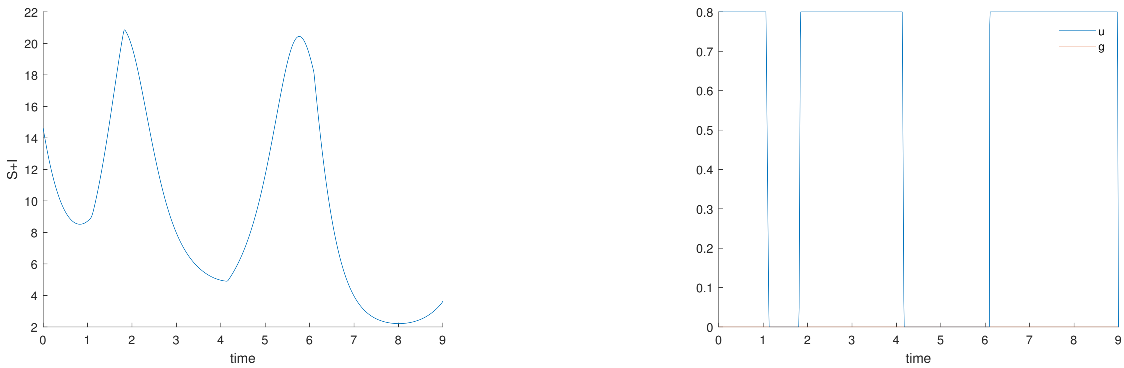
References

[1] Peter A Abrams and Carl J Walters. Invulnerable prey and the paradox of enrichment. *Ecology*, 77(4):1125–1133, 1996.
[2] Leonard David Berkovitz and Negash G Medhin, *Nonlinear optimal control theory*, Boca Raton, FL: CRC press, 2013.
[3] Wendell H Fleming and Raymond W Rishel. *Deterministic and stochastic optimal control*, volume 1. Springer Science & Business Media, 1975.
[4] Holly Gaff and Elsa Schaefer. Optimal control applied to vaccination and treatment strategies for various epidemiological models. *Mathematical biosciences and engineering: MBE*, 6(3):469–492, 2009.
[5] Paul Georgescu and Hong Zhang. An impulsively controlled predator–pest model with disease in the pest. *Nonlinear Analysis: Real World Applications*, 11(1):270–287, 2010.



(A) Pest population, $S + I$. (B) Controls u and g.

FIGURE 5. Evolution of pest population (*left*) and controls u and g (*right*). Only the control g is applied ($u = 0$).



(A) Pest population. (B) Controls u and g.

FIGURE 6. Evolution of pest population (*left*), $S + I$, and of controls u and g (*right*). Only the control u is applied ($g = 0$).

[6] Soovoojeet Jana and TK Kar. A mathematical study of a prey–predator model in relevance to pest control. *Nonlinear Dynamics*, 74(3):667–683, 2013.

[7] Hem Raj Joshi. Optimal control of an hiv immunology model. *Optimal control applications and methods*, 23(4):199–213, 2002.

[8] TK Kar, Abhijit Ghorai, and Soovoojeet Jana. Dynamics of pest and its predator model with disease in the pest and optimal use of pesticide. *Journal of theoretical biology*, 310:187–198, 2012.

[9] Joaquim Mateus, Paulo Rebelo, Silvério Rosa, Cesar Silva, and Delfim F. M. Torres. Optimal control of non-autonomous seirs models with vaccination and treatment. *Discrete & Continuous Dynamical Systems - S*, 11(06):1179–1199, 2018.

[10] UK Pahari and TK Kar. Conservation of a resource based fishery through optimal taxation. *Nonlinear Dynamics*, 72(3):591–603, 2013.

[11] Lev Semenovich Pontryagin. *Mathematical theory of optimal processes*. CRC Press, 1987.

[12] Silvério Rosa, Paulo Rebelo, César M Silva, Helena Alves, and Pedro G Carvalho. Optimal control of the customer dynamics based on marketing policy. *Applied Mathematics and Computation*, 330:42–55, 2018.

[13] Pauline van den Driessche and James Watmough. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences*, 180(1):29 – 48, 2002.

[14] Ezio Venturino. Ecoepidemiology: a more comprehensive view of population interactions. *Mathematical Modelling of Natural Phenomena*, 11(1):49–90, 2016.

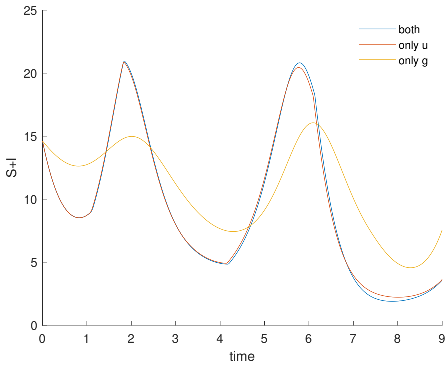


FIGURE 7. Dynamics of pest population, $S + I$, in three scenarios: 1) *both* controls are used; 2) *only control* u is used ($g = 0$); 3) *only control* g is used ($u = 0$).

[15] Xia Wang and Xinyu Song. Mathematical models for the control of a pest population by infected pest. *Computers & Mathematics with Applications*, 56(1):266–278, 2008.