

INTERACTION OF HIGH-ORDER BREATHER, PERIODIC WAVE, LUMP, RATIONAL SOLITON SOLUTIONS AND MIXED SOLUTIONS FOR REDUCTIONS OF THE (4+1)-DIMENSIONAL FOKAS EQUATION

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Abstract. The interaction of high-order breather, periodic-wave, lump, rational soliton solutions and mixed solutions for reductions of the (4+1)-dimensional Fokas equation are investigated by means of the Kadomtsev-Petviashvili (KP) hierarchy reduction method. Through analyzing the structural characteristics of periodic wave solutions, we find that evolution of the breather is decided by two characteristic lines. Interestingly, growing-decaying amplitude periodic wave and amplitude-invariant periodic wave are given through some conditions posed on the parameters. Some fascinating nonlinear wave patterns composed of high-order breathers and high-order periodic waves are shown. Furthermore, taking the long wave limit on the periodic-wave solutions, the semi-rational solutions composed of lumps, moving solitons, breathers, and periodic waves are obtained. Some novel dynamical processes are graphically analyzed. Additionally, we provide a new method to derive periodic-wave and semi-rational solutions for the (3+1)-dimensional KP equation by reducing the solutions of the (4+1)-dimensional Fokas equation. The presented results might help to understand the dynamic behaviors of nonlinear waves in the fluid fields and may provide some new perspectives for studying nonlinear wave solutions of high dimensional integrable systems.

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1. INTRODUCTION

The research based on the nonlinear partial differential equations (NLPDEs) is a hot topic in properly understanding and explaining the characteristics of many phenomena in natural science, such as optical, plasmas, fluids fibers and other fields [15, 16, 31]. In fact, many physical phenomena can be proved or explained with nonlinear wave solutions of NLPDEs, and explicit solutions of the nonlinear waves are very valuable for analyzing the wave mechanism [37]. Therefore, scholars pay attention to the nonlinear wave solutions, for example breather, lump, soliton, and rogue wave solutions [2, 3, 11, 14, 22, 24, 35]. Various techniques exist to derive nonlinear wave solutions of NLPDEs, for instance, the bilinear method [23], Darboux transformation [39], Bäcklund transformation [19], Kadomtsev-Petviashvili (KP) hierarchy reduction method [25]. Due to its importance and

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wide application in the field of mathematical physics [26, 40], the research value of higher-dimensional nonlinear systems is immense as we all know. Furthermore, the higher-dimensional NLPDEs have richer explicit solutions, for example, breather, rogue wave, lump, quasi-periodic wave solutions and interaction solutions between above different kinds of solutions, which enrich the diversity of solitary waves [10, 21, 27, 36].

In this paper, we consider investigating the (4+1)-dimensional (4D) Fokas equation

$$4u_{x_1 t} - u_{x_1 x_1 x_1 x_2} + u_{x_1 x_2 x_2 x_2} + 6(u^2)_{x_1 x_2} - 6u_{y_1 y_2} = 0. \quad (1.1)$$

This equation was proposed as the higher-dimensional nonlinear wave equation by Fokas [12], which is an integrable extension of the KP and Davey-Stewartson (DS) systems. Due to the significant application of KP and DS systems in the field of mathematical physics [1, 9], the 4D Fokas system follows the physical nature of the KP and DS equations and may widely applied in fluid mechanics, ocean dynamics, etc. Recently, many researchers have investigated nonlinear wave solutions of the 4D Fokas equation by utilizing different methods. Soliton solutions have been obtained [28, 34, 38]; quasi-periodic waves, resonant solitary waves, breathers and lump-type solutions have been attained in [5, 6, 17, 33]; interactional solutions of lump with solitary wave have been discussed in [32, 41]; semi-rational solutions and reductions have been studied in [4].

Under the transformation $x = a_1 x_1 + a_2 x_2$, $y = a_3 y_1 + a_4 y_2$, the 4D Fokas equation can be changed into

$$4u_{xt} + a_2(a_2^2 - a_1^2)u_{xxxx} + 6a_2(u^2)_{xx} - \frac{6a_3a_4}{a_1}u_{yy} = 0. \quad (1.2)$$

Then, with the following transformation

$$u = (a_2^2 - a_1^2) \ln(f)_{xx}, \quad (1.3)$$

the 4D Fokas system (1.2)'s bilinear form is presented as

$$(D_x^4 + \frac{4}{a_2(a_2^2 - a_1^2)}D_x D_t - \frac{6a_3a_4}{a_1a_2(a_2^2 - a_1^2)}D_y^2)f \cdot f = 0, \quad (1.4)$$

in which D denotes the Hirota bilinear differential operator, defining as follows [13]:

$$D_t^m D_x^n a \cdot b = \frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial y^n} a(t + s, x + y) b(t - s, x - y)|_{s=0, y=0},$$

where m and n are nonnegative integers.

To our knowledge, the high-order breather, periodic-wave and rational soliton solutions of equation (1.1) have not been reported before. Then, we mainly focus on the high-order breather, periodic wave, rational solutions and discuss their interaction characteristics for the 4D Fokas equation. In this work, the KP hierarchy reduction and long-wave limit techniques can help achieve our goals. In Section 2, the determinant periodic-wave solutions for the 4D Fokas equation (1.1) are derived, whose corresponding properties are discussed graphically. In Section 3, we obtain mixed solutions of equation (1.1) composed of lumps, moving solitons, breathers and periodic waves by adopting the long-wave limit [30] on the periodic-wave solutions. In Section 4, we give a new way to derive the periodic-wave and hybrid solutions for the 3D KP equation. In Section 5, some conclusions are presented.

2. PERIODIC-WAVE SOLUTIONS FOR THE 4D FOKAS EQUATION

2.1. Determinant periodic-wave solutions

In this section, we will derive determinant periodic-wave solutions to the 4D Fokas equation (1.1) by applying the KP hierarchy reduction technique, which was initially developed by the Kyoto School [8, 29] as a powerful

approach to generate solitary wave solution of NLPDEs. The radical idea of KP hierarchy reduction method is to regard the bilinear form of the govern equation as the KP hierarchy reduction. Hence, the following tau function of bilinear equations is introduced from the KP hierarchy [29]

$$\begin{aligned}\tau_n &= \det_{1 \leq r, j \leq N} \left(\delta_{rj} + \frac{p_r + q_r}{p_r + q_j} e^{\xi_r + \eta_j} \right), \\ \xi_r &= p_r z_1 + p_r^2 z_2 + p_r^3 z_3 + \xi_{r,0}, \quad \eta_j = q_j z_1 - q_j^2 z_2 + q_j^3 z_3 + \eta_{j,0}.\end{aligned}\tag{2.1}$$

Here, p_r , q_j , $\xi_{r,0}$, $\eta_{j,0}$ are complex constants, δ_{rj} is the Kronecker symbol, z_l ($l = 1, 2, 3$) are variables. The above tau functions implement the following bilinear equations at the KP hierarchy:

$$(D_{z_1}^4 + 3D_{z_2}^2 - 4D_{z_1}D_{z_3})\tau_n \cdot \tau_n = 0.\tag{2.2}$$

Further, by taking the change of variable as $z_1 = ix$, $z_2 = \sqrt{\frac{a_1 a_2 (a_2^2 - a_1^2)}{2a_3 a_4}} iy$, $z_3 = ia_2(a_2^2 - a_1^2)t$, equation (2.2) can be reduced to the bilinear equation (1.4) with $\tau_0 = f$. Summarizing the above results, bilinear form (1.4)'s determinant solutions are made as:

$$f = |\delta_{rj} + \frac{p_r + q_r}{p_r + q_j} e^{\xi_r + \eta_j}|_{N \times N},\tag{2.3}$$

where

$$\begin{aligned}\xi_r &= ia_1 p_r x_1 + ia_2 p_r x_2 + i \sqrt{\frac{a_1 a_2 a_3 (a_2^2 - a_1^2)}{2a_4}} p_r^2 y_1 + i \sqrt{\frac{a_1 a_2 a_4 (a_2^2 - a_1^2)}{2a_3}} p_r^2 y_2 + ia_2 (a_2^2 - a_1^2) p_r^3 t + \xi_{r,0}, \\ \eta_j &= ia_1 q_j x_1 + ia_2 q_j x_2 - i \sqrt{\frac{a_1 a_2 a_3 (a_2^2 - a_1^2)}{2a_4}} q_j^2 y_1 - i \sqrt{\frac{a_1 a_2 a_4 (a_2^2 - a_1^2)}{2a_3}} q_j^2 y_2 + ia_2 (a_2^2 - a_1^2) q_j^3 t + \eta_{j,0}.\end{aligned}\tag{2.4}$$

Next, we consider constructing the periodic-wave solutions of equation (1.1) by taking even-th order ($N = 2M$) and parametric constraints in (2.3) as

$$\begin{aligned}p_{2k-1} &= -\Omega_k + \frac{\omega_k}{2}, \quad q_{2k-1} = \Omega_k + \frac{\omega_k}{2}, \quad p_{2k} = -\Omega_k^* - \frac{\omega_k}{2}, \quad q_{2k} = \Omega_k^* - \frac{\omega_k}{2}, \\ \xi_{2k-1,0} &= \xi_{2k,0}^* = \xi_{k,0}, \quad \eta_{2k-1,0} = \eta_{2k,0}^* = \eta_{k,0}, \quad 1 \leq k \leq M,\end{aligned}\tag{2.5}$$

where k , M are the positive integers, ω_k denote the real constants, Ω_k represent the complex constants. Therefore, the function f in (2.3) can be verified as nonzero real function by using a similar method in [18].

Under the above conditions, the M th order periodic-wave solutions of equation (1.1) are achieved as

$$u = (a_2^2 - a_1^2)(\ln(f))_{xx}, \quad f = E \begin{vmatrix} F_{11} & \cdots & F_{1M} \\ \vdots & \ddots & \vdots \\ F_{M1} & \cdots & F_{MM} \end{vmatrix},\tag{2.6}$$

where

$$\begin{aligned}
F_{kk} &= \begin{pmatrix} \frac{1+e^{-(\zeta_k+c_k)}}{\omega_k} & \frac{1}{\Omega_k^*-\Omega_k} \\ \frac{1}{\Omega_k^*-\Omega_k} & \frac{1+e^{(\zeta_k^*+c_k^*)}}{\omega_k} \end{pmatrix}, \quad F_{kj} = \begin{pmatrix} \frac{2}{2\Omega_j-2\Omega_k+\omega_k+\omega_j} & \frac{2}{2\Omega_j^*-2\Omega_k+\omega_k-\omega_j} \\ \frac{2}{2\Omega_k^*-2\Omega_j+\omega_k-\omega_j} & \frac{2}{2\Omega_k^*-2\Omega_j+\omega_k+\omega_j} \end{pmatrix}, \\
\zeta_k &= i\omega_k(a_1x_1 + a_2x_2 - \Omega_k(\sqrt{\frac{2a_1a_2a_3(a_2^2-a_1^2)}{a_4}}y_1 + \sqrt{\frac{2a_1a_2a_4(a_2^2-a_1^2)}{a_3}}y_2) + (3\Omega_k^2 + \frac{1}{4}\omega_k^2)a_2(a_2^2-a_1^2)t), \\
E &= e^{\sum_{k=1}^M(\zeta_k+\zeta_k^*+c_k+c_k^*)} \prod_{k=1}^M \omega_k^2, \quad c_k = \xi_{k0} + \eta_{k0}, \quad 1 \leq k, j \leq M, \quad k \neq j.
\end{aligned} \tag{2.7}$$

2.2. High-order breather and periodic wave

By taking $M = 1$ into Solution (2.6), we acquire

$$\begin{aligned}
f &= (1 - \frac{\omega_1^2}{(\Omega_1 - \Omega_1^*)^2})e^{\zeta_1+\zeta_1^*+c_1+c_1^*} + e^{\zeta_1+c_1} + e^{\zeta_1^*+c_1^*} + 1, \\
\zeta_1 &= i\omega_1(a_1x_1 + a_2x_2 - \Omega_1(\sqrt{\frac{2a_1a_2a_3(a_2^2-a_1^2)}{a_4}}y_1 + \sqrt{\frac{2a_1a_2a_4(a_2^2-a_1^2)}{a_3}}y_2) + (3\Omega_1^2 + \frac{1}{4}\omega_1^2)a_2(a_2^2-a_1^2)t).
\end{aligned} \tag{2.8}$$

Through simple algebra calculation, the first-order periodic-wave solution can be presented as

$$\begin{aligned}
u &= (a_2^2 - a_1^2)(\ln(f))_{xx} = (a_1^2 - a_2^2)\omega_1^2 \frac{1 + \sqrt{A}\cos(\zeta_{1I} + c_{1I})\cosh(\zeta_{1R} + c_{1R} + \frac{1}{2}\ln A)}{(\sqrt{A}\cosh(\zeta_{1R} + c_{1R} + \frac{1}{2}\ln A) + \cos(\zeta_{1I} + c_{1I}))^2}, \\
\zeta_{1R} &= \omega_1(\Omega_{1I}(\sqrt{\frac{2a_1a_2a_3(a_2^2-a_1^2)}{a_4}}y_1 + \sqrt{\frac{2a_1a_2a_4(a_2^2-a_1^2)}{a_3}}y_2) - 6\Omega_{1R}\Omega_{1I}a_2(a_2^2-a_1^2)t), \\
\zeta_{1I} &= \omega_1(a_1x_1 + a_2x_2 - \Omega_{1R}(\sqrt{\frac{2a_1a_2a_3(a_2^2-a_1^2)}{a_4}}y_1 + \sqrt{\frac{2a_1a_2a_4(a_2^2-a_1^2)}{a_3}}y_2) + (3(\Omega_{1R}^2 - \Omega_{1I}^2) \\
&\quad + \frac{1}{4}\omega_1^2)a_2(a_2^2-a_1^2)t), \\
A &= 1 - \frac{\omega_1^2}{(\Omega_1 - \Omega_1^*)^2}, \quad e^{\zeta_1} = e^{\zeta_{1R}}(\cos(\zeta_{1I}) + i\sin(\zeta_{1I})), \\
\Omega_1 &= \Omega_{1R} + i\Omega_{1I}, \quad \zeta_1 = \zeta_{1R} + i\zeta_{1I}, \quad c_1 = c_{1R} + ic_{1I}.
\end{aligned} \tag{2.9}$$

The above solution contains trigonometric function $\cos(\zeta_{1I} + c_{1I})$ and hyperbolic function $\cosh(\zeta_{1R} + c_{1R} + \frac{1}{2}\ln A)$, which can be considered as a nonlinear superposition of the periodic wave and solitary wave components. Thus, the periodic-wave solution is localized along the line $L_1 : \zeta_{1R} + c_{1R} + \frac{1}{2}\ln A = 0$ and periodic along the line $L_2 : \zeta_{1I} + c_{1I} = 0$. It is apparent that evolution of the solutions is controlled by those two characteristic lines L_1, L_2 . Without losing generality, we take $c_1 = 0$, $a_1 = 1$, $a_3 = 1$, $a_4 = 1$, $a_2 = 1.2$ in this section. Spontaneously, solitary wave component only has velocity in the y_1, y_2 -directions as $\frac{\sqrt{9.504}\Omega_{1R}}{\omega_1}$ but the periodic wave component keep velocity in the x_1, x_2 -directions and y_1, y_2 -directions as $\frac{1.584(\Omega_{1I}^2 - \Omega_{1R}^2) + 0.132\omega_1^2}{\omega_1}$, $\frac{1.32(\Omega_{1I}^2 - \Omega_{1R}^2) + 0.11\omega_1^2}{\omega_1}$, and $\frac{\sqrt{2.376}(\Omega_{1I}^2 - \Omega_{1R}^2) + \sqrt{0.0165}\omega_1^2}{\Omega_{1R}\omega_1}$. According to the cases of $\Omega_{1R} = 0$ and $\Omega_{1R} \neq 0$, respectively, we derive two kinds of behaviors:

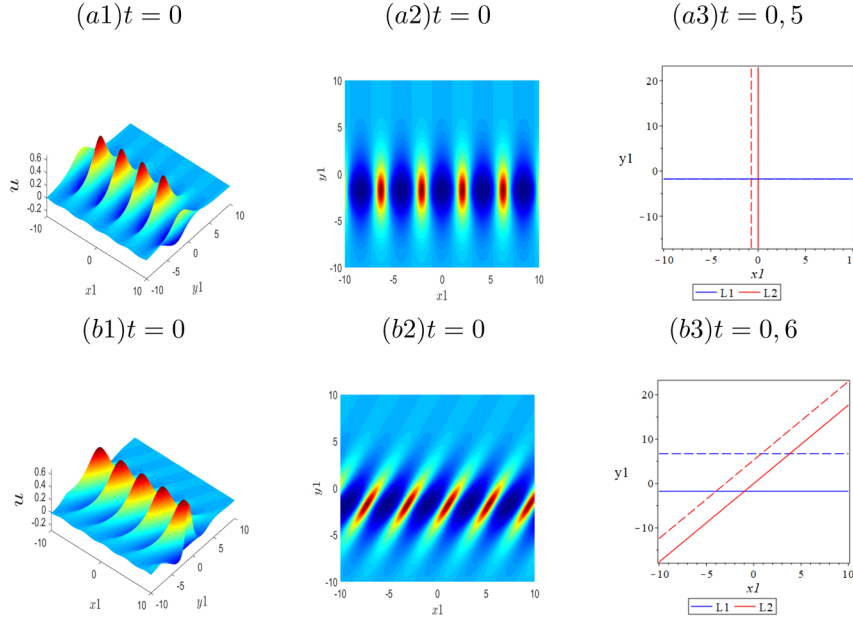


FIGURE 1. Two routes of propagation of the breathers and characteristic lines with $x_2 = 0$, $y_2 = 0$, $\omega_1 = 1.5$ and (a1) – (a3) : $\Omega_1 = \frac{1}{3}i$; (b1) – (b3) : $\Omega_1 = 0.55 + \frac{1}{3}i$.

• Here, we first discuss the dynamical behavior of the breather solutions. When $\Omega_{1R} = 0$, L_1 is not depending on t and L_2 is not depending on y_1, y_2 . Solution (2.9) will leads to a breather which only travels along the x_1 direction in (x_1, y_1) -plane with velocity $\frac{1.728(3\Omega_{1I}^2 - \frac{1}{4}\omega_1^2)}{\omega_1}$, as seen in Figure 1(a1)-(a3). While $\Omega_{1R} \neq 0$, L_1 and L_2 respect to variable x_1, x_2, y_1, y_2, t , the breather propagates along the x_1 and y_1 directions on the (x_1, y_1) -plane will be obtained as shown in Figure 1(b1)-(b3). And the evolution of the breather is demonstrated by the evolution of two characteristic lines (L_1 and L_2) in Figure 1(a3) and Figure 1(b3) (with the dotted line meaning $t = 5$ and $t = 6$, respectively). In fact, breather on the other planes obtained *via* Solution (2.9) is similar in the (x_1, y_1) -plane, thus, it doesn't adopt additional discussion here.

• Similarly, we also consider the periodic wave solutions in the (x_1, t) -plane since L_1 can be regarded as a constant on the (x_1, t) -plane. Then the amplitude-growing-decaying and invariant periodic wave are formed in Solution (2.9) by selecting the appropriate parameters. When $\Omega_{1R} = 0$, L_1 being independent of t , Figure 2(a) illustrate the evolution dynamics property with different t . It is obvious that the amplitude of periodic waves does not alter over time. And each single periodic wave is one-dimensional, it can be viewed as a superposition of overlapping solitary waves placed one period apart. While $\Omega_{1R} \neq 0$, we see that the wave travel pattern of the wave along the $x_1 = 0$ axis when $y_1 = 0$ in Figure 2(b). With the increase of time, the amplitude of periodic waves also changes and comes up to maximum value when $t = 0$. The characteristics of this periodic wave solution are similar to the so called growing-decaying periodic wave in [40] and we also call it.

With $M = 2$ in Solution (2.6), the second-order periodic-wave solutions produce more abundant waveform structures. Based on the above analyses of breathers, amplitude-growing-decaying and invariant periodic wave, the mixed solutions that consist of the periodic waves and breathers are constructed by choosing the necessary appropriate parameters. Then in Figure 3, three typical dynamics of those interaction solutions are illustrated and analyzed in detail. As shown in Figure 3(a1)-(a5), two breathers are parallel with each other in (x_1, y_1) -plane. And the breather with large period ($\Omega_{1R} \neq 0$) passes through the one with small period ($\Omega_{2R} = 0$), then the distance is getting farther and farther. The velocity, shape and density remain the same during the propagation, which is an elastic interaction. While $\Omega_{1R} \neq 0$, $\Omega_{2R} = 0$, Figure 3(b1)-(b5) present the interaction of one breather and one growing-decaying periodic wave in (x_1, t) -plane. Interestingly, changes in the amplitude

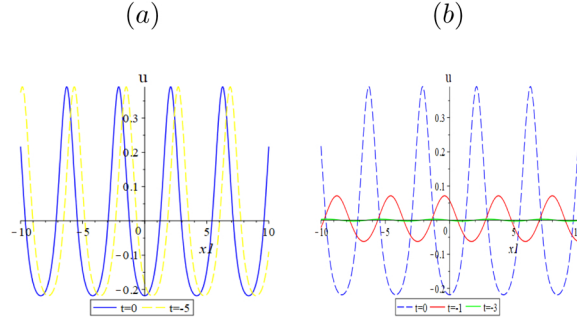


FIGURE 2. Amplitude-invariant and growing-decaying periodic wave with $y_1 = 0$, $x_2 = 0$, $y_2 = 0$, $\omega_1 = 1.5$ and (a) : $\Omega_1 = \frac{1}{3}i$; (b) : $\Omega_1 = 1 + \frac{1}{3}i$.

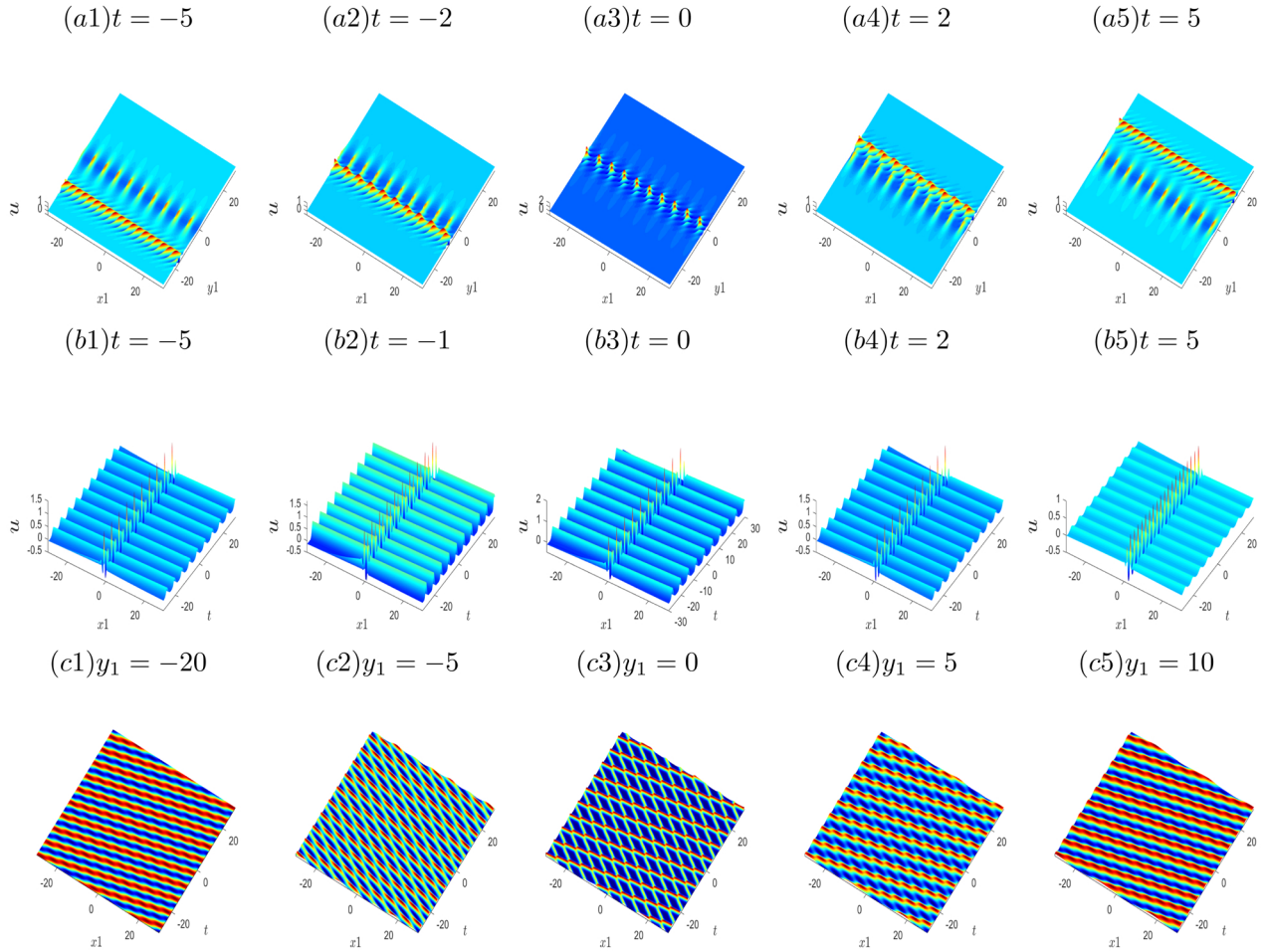


FIGURE 3. The three different interaction solutions for equation (1.1) with $c_1 = 0$, $c_2 = 0$, $x_2 = 0$, $y_2 = 0$, $\omega_1 = 2$, $\omega_2 = 1$ and (a1) – (a5) : $\Omega_1 = 1 + \frac{1}{3}i$, $\Omega_2 = \frac{1}{3}i$; (b1) – (b5) : $\Omega_1 = 1 + \frac{1}{3}i$, $\Omega_2 = \frac{1}{2}i$; (c1) – (c5) : $\Omega_1 = \frac{1}{3}i$, $\Omega_2 = \frac{1}{2}i$.

of growing-decaying periodic wave and breather seem to follow the same trend. So that is a type of inelastic interaction. When $\Omega_{1R} = 0$, $\Omega_{2R} = 0$, as observed in Figures 3(c1)–(c5), the two periodic waves at a non-zero angle collide with each other which lead periodic along the different axes. And it seems that evolve a network structure at the middle time attaining much higher amplitudes. Finally, one of the two periodic waves will integrates in constant background, the other still exists.

3. MIXED SOLUTIONS TO THE 4D FOKAS EQUATION

Taking the long wave limit $\omega_{k_1} \rightarrow 0$ in Solution (2.6), the exponential functions with regard to ω_{k_1} are transformed into the polynomial functions. So that the M_1 -th order principal minor concerning the ω_{k_1} in (2.6) leads to M_1 -th order rational solution for the 4D Fokas equation such as lump, moving soliton and rogue wave. Then, setting $M = M_1 + M_2$ and letting $\omega_{k_1} \rightarrow 0$, $1 \leq k_1 \leq M_1$ to Solution (2.6), the mixed solutions comprising M_1 -th order lumps (moving solitons, rogue waves) and M_2 -th order breathers (periodic waves) to equation (1.1) are obtained as follows:

$$u = (a_2^2 - a_1^2)(\ln(f))_{xx}, \quad f = E \begin{vmatrix} \dot{F} & \bar{F} \\ \check{F} & \bar{F} \end{vmatrix}, \quad (3.1)$$

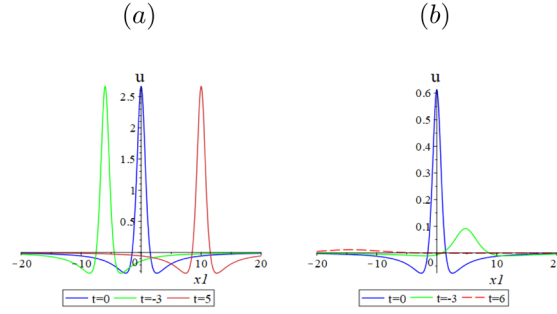
where $\dot{F}$ is $M_1 \times M_1$ matrices, $\check{F}$ is $M_2 \times M_2$ matrices, $\check{F}$ is $M_2 \times M_1$ matrices, $\bar{F}$ is $M_1 \times M_2$ matrices, whose elements are

$$\begin{aligned} F'_{kk} &= \begin{pmatrix} \theta_k & \frac{1}{\Omega_k^* - \Omega_k} \\ \frac{1}{\Omega_k^* - \Omega_k} & \theta_k^* \end{pmatrix}, \quad F'_{kj} = \begin{pmatrix} \frac{1}{\Omega_j - \Omega_k} & \frac{1}{\Omega_j^* - \Omega_k} \\ \frac{1}{\Omega_k^* - \Omega_j} & \frac{1}{\Omega_k^* - \Omega_j^*} \end{pmatrix}, \\ \dot{F}_{kk} &= \begin{pmatrix} \frac{1+e^{-(\zeta_k+c_k)}}{\omega_k} & \frac{1}{\Omega_k^* - \Omega_k} \\ \frac{1}{\Omega_k^* - \Omega_k} & \frac{1+e^{-(\zeta_k^*+c_k^*)}}{\omega_k} \end{pmatrix}, \quad \dot{F}_{kj} = \begin{pmatrix} \frac{2}{2\Omega_j - 2\Omega_k + \omega_k + \omega_j} & \frac{2}{2\Omega_j^* - 2\Omega_k + \omega_k - \omega_j} \\ \frac{2}{2\Omega_k^* - 2\Omega_j + \omega_k - \omega_j} & \frac{2}{2\Omega_k^* - 2\Omega_j + \omega_k + \omega_j} \end{pmatrix}, \\ \check{F}_{kl} &= \begin{pmatrix} \frac{2}{2\Omega_j - 2\Omega_k + \omega_k} & \frac{2}{2\Omega_j^* - 2\Omega_k + \omega_k} \\ \frac{2}{2\Omega_k^* - 2\Omega_j + \omega_k} & \frac{2}{2\Omega_k^* - 2\Omega_j + \omega_k} \end{pmatrix}, \quad \bar{F}_{kl} = \begin{pmatrix} \frac{2}{2\Omega_j - 2\Omega_k + \omega_j} & \frac{2}{2\Omega_j^* - 2\Omega_k - \omega_j} \\ \frac{2}{2\Omega_k^* - 2\Omega_j - \omega_j} & \frac{2}{2\Omega_k^* - 2\Omega_j + \omega_j} \end{pmatrix}, \\ \theta_k &= i(a_1x_1 + a_2x_2 - \Omega_k(\sqrt{\frac{2a_1a_2a_3(a_2^2 - a_1^2)}{a_4}}y_1 + \sqrt{\frac{2a_1a_2a_4(a_2^2 - a_1^2)}{a_3}}y_2) + 3\Omega_k^2a_2(a_2^2 - a_1^2)t), \\ \zeta_k &= i\omega_k(a_1x_1 + a_2x_2 - \Omega_k(\sqrt{\frac{2a_1a_2a_3(a_2^2 - a_1^2)}{a_4}}y_1 + \sqrt{\frac{2a_1a_2a_4(a_2^2 - a_1^2)}{a_3}}y_2) + (3\Omega_k^2 + \frac{1}{4}\omega_k^2)a_2(a_2^2 - a_1^2)t), \\ E &= e^{\sum_{k=1}^M (\zeta_k + \zeta_k^* + c_k + c_k^*)} \prod_{k=1}^M \omega_k^2, \quad c_k = \xi_{k0} + \eta_{k0}, \quad k \neq j, \quad 1 \leq k, j \leq M, \end{aligned} \quad (3.2)$$

with $\theta_k = \lim_{\omega_k \rightarrow 0} \frac{1+e^{-\zeta_k}}{\omega_k}$, $c_k = i\pi$.

For example, the first-order rational solution is constructed by taking $M = M_1 = 1$ in Solution (3.1), which is written as

$$\begin{aligned} u &= (a_2^2 - a_1^2)(\ln(f))_{xx} = 2((a_2^2 - a_1^2)) \frac{\theta_{1R}^2 - \theta_{1I}^2 + \frac{1}{\Omega_{1I}^2}}{(\theta_{1R}^2 + \theta_{1I}^2 + \frac{1}{\Omega_{1I}^2})^2}, \\ \theta_1 &= i(a_1x_1 + a_2x_2 - \Omega_1(\sqrt{\frac{2a_1a_2a_3(a_2^2 - a_1^2)}{a_4}}y_1 + \sqrt{\frac{2a_1a_2a_4(a_2^2 - a_1^2)}{a_3}}y_2) + 3\Omega_1^2a_2(a_2^2 - a_1^2)t), \end{aligned} \quad (3.3)$$

FIGURE 4. Moving soliton and rogue wave *via* Solutions (3.3).

where $\theta_1 = \theta_{1R} + i\theta_{1I}$. We also denote the line $L_3 : \theta_{1R} = 0$ and $L_4 : \theta_{1I} = 0$. Then, the properties of the above rational solution can be analyzed in detail on the basis of different parameters constraints. It gives birth to a rogue wave and moving soliton in (x_1, t) -plane by selecting appropriate parameters. The trajectories of moving soliton at different times are shown in Figure 4. When $a_1 = 1$, $a_3 = 1$, $a_4 = 1$, $a_2 = 2$, $\Omega_1 = \frac{1}{3}i$, different from that of the soliton in [34], we obtain a soliton keeping one peak and two valleys exhibited on Figure 4(a), whose waveform does not vary with time propagating. It can be seen as a moving soliton. Besides, rogue wave occur in the (x_1, t) -plane when $a_1 = 1$, $a_3 = 1$, $a_4 = 1$, $a_2 = 1.3$, $\Omega_1 = 1 + \frac{1}{3}i$. Figure 4(b) illustrates that the rogue waves rise and then decay into zero background with a short period of time. Interestingly, dynamic behaviors of moving soliton and rogue wave are very similar to the amplitude-invariant and growing-decaying periodic waves. Besides, we can also give birth to one lump solution in (x_1, y_1) -plane. And its characteristic lines L_3 and L_4 determine the evolution properties of solution. In fact, we can still get higher-order rogue waves and lumps through Solution (3.1), where we omit its analysis.

When we take $M_1 = M_2 = 1$ in Solution (3.1), the fundamental mixed solution can be derived. Based on the above periodic-wave and rational solutions analysis, we can find interaction phenomenon between the lump(moving soliton) and breather(periodic wave) from the mixed solution. With different constraints in the parameters Ω_1 , Ω_2 , the collision phenomenon of the lump and breather in (x_1, y_1) -plane are very interesting. When Ω_1 and Ω_2 both are complex constants, we can observe a head-on collision among one lump and one breather from Figure 5(a1)-(a5). Indeed, the lump and the breather travel in opposite directions, such that the lump will pass through the breather and eventually split, forming an elastic interaction. While Ω_1 and Ω_2 as pure imaginary constants, the lump and the breather only moves uniformly along x_1 axis keeping propagation velocity with $v_{lump} = \frac{3\Omega_{1I}^2 a_2 (a_2^2 - a_1^2)}{a_1}$ and $v_{breather} = \frac{(3\Omega_{2I}^2 - \frac{1}{4}\omega_1^2) a_2 (a_2^2 - a_1^2)}{a_1}$ respectively. Therefore, the lump propagates faster than the breather and will not collide with it, as depicted in Figure 5(b1)-(b5). And the distance between the breather and lump can be affected *via* altering parameter ω_1 . We also explore the situation in which the shape changes of the line wave after collision, that is, inelastic collision. When Ω_1 denotes pure imaginary and Ω_2 denotes complex constant, the interactions among one breather and one line wave on the (x_1, t) -plane are shown in Figure 5(c1)-(c5) with the increase of y_1 . We observe that the amplitude of line wave differ on both sides of the breather when it moves along the coordinate plane. Afterwards, the amplitude of the linear wave increases and then attenuation variation with the value of y_1 . Furthermore, when we take $M_1 > 1$, $M_2 > 1$ in Solution (3.1), the mixed solutions possess similar properties and the much complex coherent structures, such as hybrid of the higher-order breathers, higher-order lumps, higher-order rouges and higher-order periodic waves.

4. PERIODIC-WAVE AND MIXED SOLUTIONS ASSOCIATED WITH THE 3D KP EQUATION

This section studies the following 3D KP equation based on the application for solutions of the 4D Fokas equation

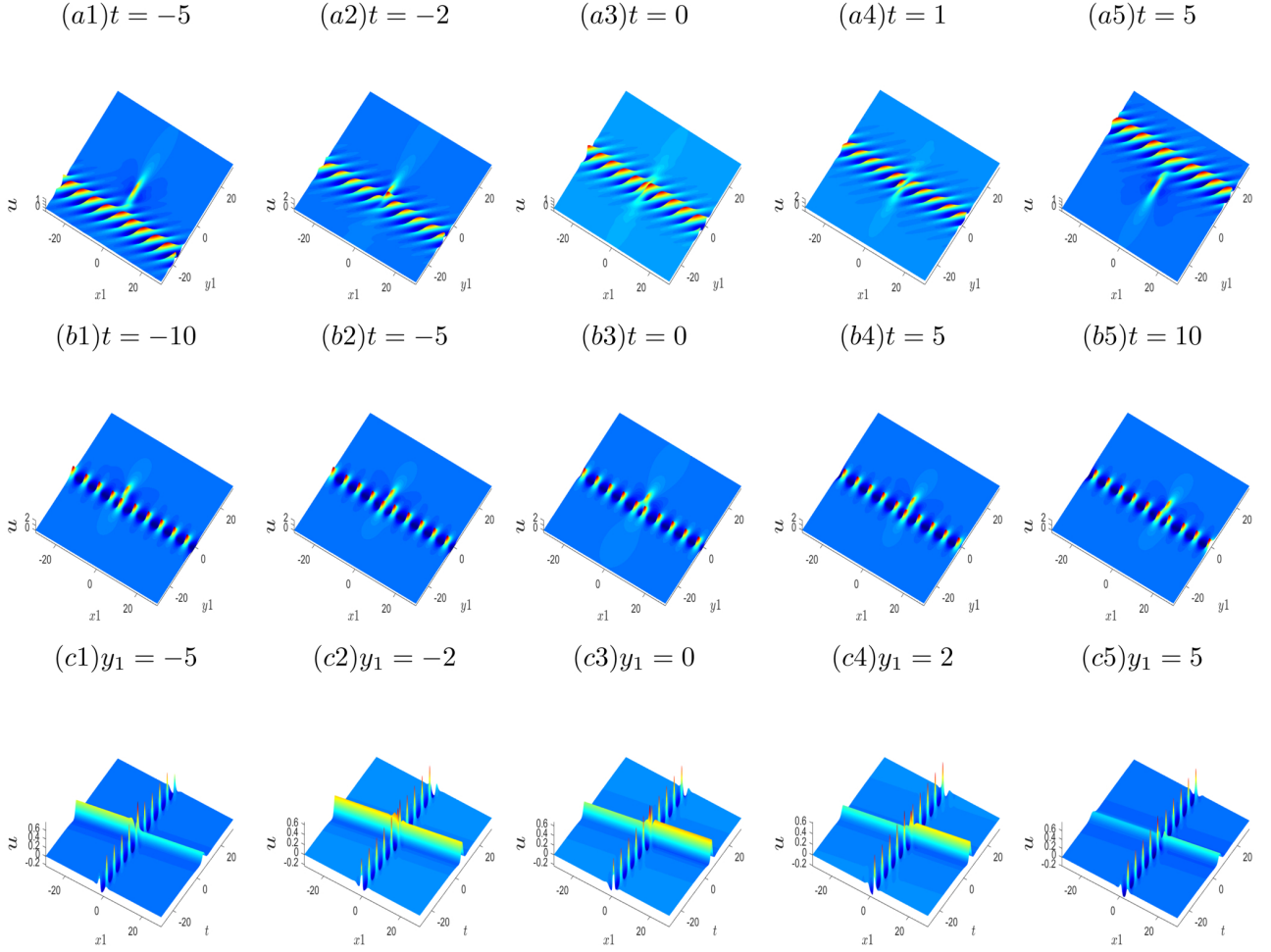


FIGURE 5. Plots of the three different types of interactions solutions for the 4D Fokas equation with $a_1 = 1$, $a_3 = 1$, $a_4 = 1$, $a_2 = 1.2$, $c_{2,0} = 0$, $\omega_2 = 1$ and (a1) – (a5) : $\Omega_1 = 1 + \frac{1}{3}i$, $\Omega_2 = 1 + \frac{1}{3}i$; (b1) – (b5) : $\Omega_1 = \frac{1}{3}i$, $\Omega_2 = \frac{1}{3}i$; (c1) – (c5) : $\Omega_1 = \frac{1}{3}i$, $\Omega_2 = 1 + \frac{1}{3}i$.

$$(W_\gamma + 6WW_\nu + W_{\nu\nu\nu})_\nu - \alpha(W_{\phi\phi}) + W_{\psi\psi} = 0, \quad (4.1)$$

which have been investigated both analytically and numerically [7, 20]. The periodic-wave solutions induced breathers or periodic waves in the determinant form for this equation are not studied. We can also achieve it through KP hierarchy reduction technique. The derivation process is similar to the 4D Fokas system and the detail is omitted. Then, the periodic-wave solutions for equation (4.1) can be represented as:

$$W = 2(\ln(f))_{\nu\nu}, \quad f = B \begin{vmatrix} H_{11} & \cdots & H_{1M} \\ \vdots & \ddots & \vdots \\ H_{M1} & \cdots & H_{MM} \end{vmatrix}, \quad (4.2)$$

where

$$\begin{aligned}
H_{kk} &= \begin{pmatrix} \frac{1+e^{-\rho_k}}{\omega_k} & \frac{1}{\Omega_k^*-\Omega_k} \\ \frac{1}{\Omega_k^*-\Omega_k} & \frac{1+e^{-\rho_k^*}}{\omega_k} \end{pmatrix}, \quad H_{kj} = \begin{pmatrix} \frac{2}{2\Omega_j-2\Omega_k+\omega_k+\omega_j} & \frac{2}{2\Omega_j^*-2\Omega_k+\omega_k-\omega_j} \\ \frac{2}{2\Omega_k^*-2\Omega_j+\omega_k-\omega_j} & \frac{2}{2\Omega_k^*-2\Omega_j+\omega_k+\omega_j} \end{pmatrix}, \\
\rho_k &= i\omega_k(\nu - \sqrt{2}\Omega_k(\phi + \psi) + 4(3\Omega_k^2 + \frac{1}{4}\omega_k^2)\gamma) + \rho_{k,o}, \\
B &= e^{\sum_{k=1}^M(\rho_k+\rho_k^*)} \prod_{k=1}^M \omega_k^2, \quad k \neq j, \quad 1 \leq k, j \leq M.
\end{aligned} \tag{4.3}$$

The mixed solutions for the 3D KP equation composed of the breathers, periodic waves, lumps and rational solitons are expressed as

$$W = 2(\ln(f))_{\nu\nu}, \quad f = B \begin{vmatrix} \dot{D} & \bar{D} \\ \check{D} & \bar{D} \end{vmatrix}, \tag{4.4}$$

where $\dot{D}$ is $M_1 \times M_1$ matrices, $\check{D}$ is $M_2 \times M_2$ matrices, $\bar{D}$ is $M_2 \times M_1$ matrices, $\bar{D}$ is $M_1 \times M_2$ matrices, whose elements are

$$\begin{aligned}
\dot{D}_{kk} &= \begin{pmatrix} \chi_k & \frac{1}{\Omega_k^*-\Omega_k} \\ \frac{1}{\Omega_k^*-\Omega_k} & \chi_k^* \end{pmatrix}, \quad \dot{D}_{kj} = \begin{pmatrix} \frac{1}{\Omega_j-\Omega_k} & \frac{1}{\Omega_j^*-\Omega_k} \\ \frac{1}{\Omega_k^*-\Omega_j} & \frac{1}{\Omega_k^*-\Omega_j^*} \end{pmatrix}, \\
\check{D}_{kk} &= \begin{pmatrix} \frac{1+e^{-\rho_k}}{\omega_k} & \frac{1}{\Omega_k^*-\Omega_k} \\ \frac{1}{\Omega_k^*-\Omega_k} & \frac{1+e^{-\rho_k^*}}{\omega_k} \end{pmatrix}, \quad \check{D}_{kj} = \begin{pmatrix} \frac{2}{2\Omega_j-2\Omega_k+\omega_k+\omega_j} & \frac{2}{2\Omega_j^*-2\Omega_k+\omega_k-\omega_j} \\ \frac{2}{2\Omega_k^*-2\Omega_j+\omega_k-\omega_j} & \frac{2}{2\Omega_k^*-2\Omega_j+\omega_k+\omega_j} \end{pmatrix}, \\
\bar{D}_{kj} &= \begin{pmatrix} \frac{2}{2\Omega_j-2\Omega_k+\omega_k} & \frac{2}{2\Omega_j^*-2\Omega_k+\omega_k} \\ \frac{2}{2\Omega_k^*-2\Omega_j+\omega_k} & \frac{2}{2\Omega_k^*-2\Omega_j+\omega_k} \end{pmatrix}, \quad \bar{D}_{kj} = \begin{pmatrix} \frac{2}{2\Omega_j-2\Omega_k+\omega_j} & \frac{2}{2\Omega_j^*-2\Omega_k-\omega_j} \\ \frac{2}{2\Omega_k^*-2\Omega_j-\omega_j} & \frac{2}{2\Omega_k^*-2\Omega_j+\omega_j} \end{pmatrix}, \\
\chi_k &= i(\nu - \sqrt{2}\Omega_k(\phi + \psi) + 12\Omega_k^2\gamma), \quad \rho_k = i\omega_k(\nu - \sqrt{2}\Omega_k(\phi + \psi) + 4(3\Omega_k^2 + \frac{1}{4}\omega_k^2)\gamma) + \rho_{k,0}, \\
B &= e^{\sum_{k=1}^M(\rho_k+\rho_k^*)} \prod_{k=1}^M \omega_k^2, \quad k \neq j, \quad 1 \leq k, j \leq M.
\end{aligned} \tag{4.5}$$

Furthermore, it is noted that the 4D Fokas system is the meaningful multidimensional extension to the 3D KP system, then the solutions of the 4D Fokas system can be used to derive the solutions for the 3D KP system. Spontaneously, we find a new way for seeking the periodic-waves and mixed solutions to the 3D KP system. Based on the above analysis, it can be proved that the periodic-waves and mixed solutions of the 3D KP system are deduced from the periodic-waves and mixed solutions of the 4D Fokas system with the transformation

$$\begin{aligned}
W_{3DKP-I}(\nu, \phi, \psi, \gamma) &= \frac{2}{a_2^2 - a_1^2} u_{4DFokas}(x_1, x_2, y_1, y_2, t), \quad \alpha = 1, \\
a_1 x_1 + a_2 x_2 &= \nu, \quad a_3 y_3 + a_4 y_4 = \sqrt{\frac{2a_3 a_4}{a_1 a_2 (a_2^2 - a_1^2)}} (\phi + \psi), \quad t = \frac{4}{a_2 (a_2^2 - a_1^2)} \gamma.
\end{aligned} \tag{4.6}$$

5. CONCLUSION

In this paper, the interaction of high-order periodic waves, breathers, lumps, rational soliton solutions and mixed solutions of the reductions of 4D Fokas system are investigated by means of the KP hierarchy reduction technique. Firstly, the determinant periodic-wave solutions for the 4D Fokas system are constructed. *Via* analyzing the periodic-wave solution's coherent structure, one can find the evolution of breather is decided by two characteristic lines. Particularly, amplitude-growing-decaying and invariant periodic waves are obtained in different planes. The evolution dynamics for the solutions can be shown in Figures 1 and 2, respectively. Moreover, some interesting nonlinear wave patterns composed of high-order breathers and high-order periodic waves through some conditions posed on the parameters Ω_k , ω_k are shown in Figure 3. Further, rational solutions of the 4D Fokas equation are derived through adopting the long wave limit technique. To illustrate some typical dynamics of the rational solutions, we discuss the lump, moving soliton, and rogue wave solutions. And the corresponding generation mechanism and the dynamics have been analyzed in Figure 4. Then, we derive the mixed solutions composed of the breathers, periodic waves, lumps, moving solitons, and rogue waves for the 4D Fokas equation. With special parameters constraints, some interaction phenomena are simulated and discussed as shown Figure 5. Finally, using our periodic-wave and mixed solutions of the 4D Fokas system, we can derive the periodic-wave and mixed solutions for the 3D KP system. The presented results provide us with powerful scientific ideas in solving higher-dimensional soliton equations and some new perspectives for studying nonlinear wave solutions of higher-dimensional integrable systems.

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