Mathematical modelling in nonlocal Mindlin's strain gradient thermoelasticity with voids

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Abstract

A nonlocal theory for thermoelastic materials with voids based on Mindlin's strain gradient theory was derived in this paper with some qualitative properties. We have also established the size effect of nonlocal heat conduction with the aids of extended irreversible thermodynamics and generalized free energy. The obtained system of equations is a coupling of three equations with higher gradients terms due to the length scale parameters ϖ and l. This poses some new mathematical difficulties due to the lack of regularity. Based on nonlinear semigroups and the theory of monotone operators, we establish existence and uniqueness of weak and strong solutions to the one dimensional problem. By an approach based on the Gearhart-Herbst-Prüss-Huang theorem, we prove that the associated semigroup is exponentially stable; but not analytic.

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1 Introduction

Classical continuum mechanics theories are not able to predict and interpret size-dependent behaviors that occur in structures at a micron and submicron scale. This is due to the absence of the material constants related to structural scale parameters in the constitutive relations [31]. However, nonclassical continuum theories such as nonlocal and strain gradient theories are more suitable to study and explain some phenomena related to nanoscale size effect. Thus, there is a need to model the mechanical response of the new class of materials by bringing the length scales into the structural theories. The elastic materials with voids (called also porous elastic materials) applying to geological materials such as rocks or soils as well as to manufactured materials such as ceramics or pressed powders, fall within this scope.

Nonlocal continuum elastic stress field approach first proposed by Eringen [11, 12] and another dealing with the strain gradient theories [27, 28] are two convenient methodologies for considering the small scale effects that are exhibited by nanoscopic structures. In nonlocal theory of elasticity, the stress at any reference point

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$\mathbf{x}$ within a continuous body depends not only on the strain at that point, but also influenced by the strains at all other points $\mathbf{x}'$ in the domain through a nonlocal attenuation function in an integrated sense. Thus, the nonlocal stress forces act as a remote action forces, which are frequently encountered in atomic theory of lattice dynamics.

Nonlocal elastic models can only account for softening stiffness with increasing nanoscale parameter; however, the stiffness enhancement effect, noticed from experimental observation and as well as the gradient elasticity (or modified couple stress) theories cannot be characterized. The strain gradient elastic theories provide extensions of the classical equations of elasticity with additional higher-order strain gradient terms based on the assumption that the materials cannot be modeled as collections of points.

The classic heat conduction law is acceptable for engineering applications at macro-temporal and spatial scales. However, this law may not be applicable in nonlocal situations. Apparently, the size effect is not considered, which can become significant when the characteristic length scale of the process is comparable to the average free path of coolants. Chan *et al.* [9] demonstrated that the nonlocal effects due to a large temperature gradient, which exists over a small distance with respect to the mean free path of electrons, can describe the confinement of heat near the interface observed in their experience. Tzou [34] also suggested that thermal conduction at the micro-nano scale is essentially nonlocal. In this paper and based on these observations, the thermal conduction law is examined by introducing the characteristic length of the material.

In local models, Ieşan [19, 20] was recently extended the theory of elastic materials with voids in the frame of the second gradient theory. Aouadi *et al.* [1, 2] extended some local porous thermoelastic theory to Form II Mindlin's strain gradient where the characteristic length of the material structure is taken into account. In nonlocal models, Bachher and Sarkar [4] established a new nonlocal theory of generalized thermoelastic materials with voids. Recently, based on nonlocal strain gradient theory, Biswas studied [6, 7] the propagation of waves in different medium. Without trying to be exhaustive, let us refer to some studies carried out in nonlocal strain gradient theory [4, 6, 7, 32, 33, 37, 38] and in local porous thermoelasticity [1, 2, 8, 13, 22, 26] among others.

In this paper, we derive the constitutive relations and field equations for thermoelastic materials with voids based on the nonlocal heat conduction combined with nonlocal strain gradient theory. This theory intends to generalize the classical strain gradient local porous elasticity [1, 2, 19, 20] and the classical nonlocal porous elasticity [4, 6, 7] by introducing the nonlocal heat conduction together with two additional kinds of parameters, the strain gradient length scale parameter l and the elastic nonlocal parameter ϖ. We aim to show that both length scales describe two entirely different physical characteristics on well-posedness, stability and analyticity issues.

The rest of the paper is organized as follows. In Section 2 we derive the constitutive relations and the field equations for porous elastic materials based on the nonlocal strain gradient theory. Under quite general assumptions on sources terms, we prove in Section 3 the well-posedness of the one dimensional problem. In section 4, by an approach based on the Gearhart-Herbst-Prüss-Huang theorem, we prove that the associated semigroup is exponentially stable, but not analytic.

2 Derivation of nonlocal strain gradient equations

2.1 Review of nonlocal strain gradient elasticity theory

Based on elasticity theory, we consider a bounded elastic material having volume V. For classical strain gradient linear elastic solids, the equations of motion have the form

$$\sigma_{ij,j} = \rho(\ddot{u}_i - G_i), \tag{2.1}$$

where a comma $(,)$ in the subscript represents the spatial partial derivative, the dot above a letter means time derivative, ρ and G_i are, respectively, the mass density and the body (and/or applied) forces; u_i are the components of the displacement vector, and σ_{ij} is the stress tensor of the strain gradient elasticity defined by [27, 28]

$$\sigma_{ij} = \tau_{ij} - \nabla \cdot t_{ij}, \tag{2.2}$$

with $\tau_{ij}(\mathbf{x}, t)$ being the classical stress tensor at any point $\mathbf{x}$ in the body, and $t_{ij}(\mathbf{x}, t)$ the higher-order stress tensor. In elasticity, $\tau_{ij} = A_{ijkl}e_{kl}$, where $e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$ is the strain tensor and A_{ijkl} are the symmetric elastic modulus components. In the Eringen's nonlocal elasticity model [11, 12], the stress at a point $\mathbf{x}$ in an elastic continuum depends not only on the strain at the point but also on strains at all points of the body. In nonlocal context, the (classical) stress tensors τ_{ij} and t_{ij} become τ^1_{ij} and τ^2_{ij}, respectively, with the form

$$\tau^1_{ij} = \int_V \mathscr{G}_1(|\mathbf{x} - \mathbf{x}'|, \varpi_1)\ \tau_{ij}(\mathbf{x}', t) dV(\mathbf{x}'), \quad \tau^2_{ij} = l^2 \int_V \mathscr{G}_2(|\mathbf{x} - \mathbf{x}'|, \varpi_2)\ \nabla\tau_{ij}(\mathbf{x}', t) dV(\mathbf{x}'), \tag{2.3}$$

where $|\mathbf{x} - \mathbf{x}'|$ is the Euclidean norm of the vector $\mathbf{x}$, l is the material length scale parameter introduced to consider the effect of the strain gradient field, the functions $\mathscr{G}_1(|\mathbf{x} - \mathbf{x}'|, \varpi_1)$ and $\mathscr{G}_2(|\mathbf{x} - \mathbf{x}'|, \varpi_2)$ are two nonlocal attenuation kernels. The function $\mathscr{G}_1(|\mathbf{x} - \mathbf{x}'|, \varpi_1)$ presents the principal kernel function associated to the nonlocal impact, whereas the function $\mathscr{G}_2(|\mathbf{x} - \mathbf{x}'|, \varpi_2)$ is additional nonlocal function associated to the strain gradient effect. The nonlocal parameters $\varpi_i = e_i\varsigma_i$ $(i = 1,\ 2)$ introduced to consider the effect of nonlocal stress field will be detailed later. In elasticity, the nonlocal stress tensors τ^1_{ij} and τ^2_{ij} are given by

$$\tau^1_{ij} = A_{ijkl} \int_V \mathscr{G}_1(|\mathbf{x} - \mathbf{x}'|, \varpi_1) e_{kl}(\mathbf{x}', t) dV(\mathbf{x}'), \quad \tau^2_{ij} = l^2 A_{ijkl} \int_V \mathscr{G}_2(|\mathbf{x} - \mathbf{x}'|, \varpi_2)\ \nabla e_{kl}(\mathbf{x}', t) dV(\mathbf{x}').$$

According to Eringen [11, 12], it is possible to assume that each nonlocal kernel is the Green's function of a specific linear differential operator $\mathcal{R}_{\varpi_i}$ such that

$$\mathcal{R}_{\varpi_i}[\mathscr{G}_i(|\mathbf{x} - \mathbf{x}'|, \varpi_i)] = \delta(|\mathbf{x} - \mathbf{x}'|), \quad i = 1, 2. \tag{2.4}$$

One of the most common operators used in nonlocal theories is $R_{\varpi_i} = I - \varpi_i^2\nabla^2$, where I is the identity operator and ∇^2 being the Laplacian, for which the associated Green's functions $\mathscr{G}_i(|\mathbf{x} - \mathbf{x}'|, \varpi_i)$ can be found in the literature for different dimensional setting [11, 12]. For simplicity, assuming that $\varpi = \varpi_1 = \varpi_2$ and applying the operator $R_{\varpi} = I - \varpi^2\nabla^2$ to (2.2), we get the constitutive equation for the total stress in differential form

$$(I - \varpi^2\nabla^2)\sigma_{ij} = (I - l^2\nabla^2)\tau_{ij}. \tag{2.5}$$

In elasticity, (2.5) reads

$$(I - \varpi^2\nabla^2)\sigma_{ij} = (I - l^2\nabla^2)A_{ijkl}e_{kl}.$$

Then the equations of motion for the nonlocal strain gradient problem can be obtained by applying the operator $R_\varpi = I - \varpi^2\nabla^2$ to the governing (2.1) and by using (2.5)

$$(I - l^2\nabla^2)\tau_{ij,j} = \rho(I - \varpi^2\nabla^2)(\ddot{u}_i - G_i). \tag{2.6}$$

In the following subsection, we propose to apply this procedure to obtain the nonlocal strain gradient porous thermoelastic equations.

2.2 Nonlocal strain gradient porous thermoelastic equations.

From Ieşan [18], the local form of the balance equations for porous thermoelastic material are :

(i) Equations of motion:

$$\tau_{ij,j} = \rho(\ddot{u}_i - G_i), \tag{2.7}$$

(ii) Equations of equilibrated forces:

$$h_{i,i} + \mathrm{g} = \rho(J\ddot{\phi} - \chi), \tag{2.8}$$

(iii) Equation of the first law of thermodynamics:

$$\rho\dot{U}_0 = \tau_{ji}\dot{e}_{ij} - \mathrm{g}\dot{\phi} + h_i\dot{\phi}_{,i} + \rho s - \nabla\cdot\mathbf{q}, \tag{2.9}$$

where τ_{ij} is the classical local stress tensor, h_i are equilibrated stress components, g is the intrinsic equilibrated body force, J is the equilibrated inertia, χ is the external equilibrated body force, s is the external rate of supply of entropy per unit mass, $\mathbf{q} \equiv q_i$ is the heat flux vector across the surface of the body and U_0 is the specific (per unit mass) internal energy.

Introducing the Helmholtz free energy

$$\Psi = U_0 - \theta S \tag{2.10}$$

and eliminating U_0 between (2.9) and (2.10), one has the first law of thermodynamics denoted by

$$\rho\dot{\Psi} + \rho\dot{\theta}S + \rho\theta\dot{S} - \tau_{ji}\dot{e}_{ij} - h_k\dot{\phi}_{,k} + \mathrm{g}\dot{\phi} - \rho s + \nabla\cdot\mathbf{q} = 0, \tag{2.11}$$

where S is the entropy density, $\theta = T - T_0$ is the temperature change, T is the absolute temperature of the medium and T_0 is the reference uniform temperature of the body chosen such that $|\theta/T_0| \ll 1$. Let $\dot{S}_{int}$ represent the time rate of internal entropy production

$$\rho\dot{S}_{int} = \rho\dot{S} - \Big[\frac{\rho s}{\theta} - \nabla\cdot\mathbf{J}\Big], \tag{2.12}$$

where $\mathbf{J}$ is the entropy flux and in classical form it reads as $\mathbf{J} = \dfrac{\mathbf{q}}{T}$. The second law of thermodynamics states that the time rate of internal entropy production is nonnegative at all points in a material body V for all thermo-mechanical deformation processes, which may be expressed as

$$\rho\dot{S} - \frac{\rho s}{\theta} + \nabla\cdot\mathbf{J} \geq 0. \tag{2.13}$$

To consider the size effect of heat conduction, the entropy flux $\mathbf{J}$ is no longer given by the classical expression $\dfrac{\mathbf{q}}{T}$ but contains an extra contribution in $\mathbf{Q}$, where $\mathbf{Q}$ is the flux of heat flux. From [16, 23], when the trace of $\mathbf{Q}$ vanishes [36], i.e. $\mathrm{tr}(\mathbf{Q}) = 0$, the entropy flux may be generalized to be,

$$\mathbf{J} = \frac{\mathbf{q}}{T} + \mathbf{q} \cdot \mathbf{Q}. \tag{2.14}$$

Substituting (2.14) into (2.13), yields:

$$\rho\theta\dot{S} - \rho s + \nabla \cdot \mathbf{q} + \theta(\mathbf{q} \cdot \nabla\theta^{-1} + \nabla\mathbf{q} \cdot \mathbf{Q} + \mathbf{q} \cdot \nabla \cdot \mathbf{Q}) \geq 0 \tag{2.15}$$

which combines with (2.11) yields:

$$-\rho\dot{\Psi} - \rho\dot{\theta}S + \tau_{ji}\dot{e}_{ij} + h_k\dot{\phi}_{,k} - \mathrm{g}\dot{\phi} + \theta(\mathbf{q} \cdot \nabla\theta^{-1} + \nabla\mathbf{q} \cdot \mathbf{Q} + \mathbf{q} \cdot \nabla \cdot \mathbf{Q}) \geq 0. \tag{2.16}$$

In nonlocal theory, by assuming that the initial body is free from stresses and has zero intrinsic equilibrated force, we take that the response functions depend on the set of the independent variables $\mathbf{x}$ and $\mathbf{x}'$, respectively as,

$$Y = \{e_{ij}(\mathbf{x}), \phi(\mathbf{x}), \phi_{,i}(\mathbf{x}), \theta(\mathbf{x})\} \text{ and } Y' = \{e_{ij}(\mathbf{x}'), \phi(\mathbf{x}'), \phi_{,i}(\mathbf{x}'), \theta(\mathbf{x}')\} \tag{2.17}$$

where

$$e_{ij}(\mathbf{x}') = \frac{1}{2}\Big(\frac{\partial u_i(\mathbf{x}')}{\partial \mathbf{x}'_j} + \frac{\partial u_j(\mathbf{x}')}{\partial \mathbf{x}'_i}\Big)$$

and $u_i(\mathbf{x}')$ is the displacement vector at a reference point $\mathbf{x}'$ in the body.

The time rate of Ψ is,

$$\dot{\Psi} = \frac{\partial\widehat{\Psi}}{\partial e_{ij}(\mathbf{x})}\dot{e}_{ij}(\mathbf{x}) + \frac{\partial\widehat{\Psi}}{\partial\phi(\mathbf{x})}\dot{\phi}(\mathbf{x}) + \frac{\partial\widehat{\Psi}}{\partial\phi_{,k}(\mathbf{x})}\dot{\phi}_{,k}(\mathbf{x}) + \frac{\partial\widehat{\Psi}}{\partial\theta(\mathbf{x})}\dot{\theta}(\mathbf{x}) \tag{2.18}$$

that is $\Psi = \widehat{\Psi}(Y, Y')$. Substituting (2.18) into (2.16), we get

$$\begin{aligned}&(\tau_{ij} - \rho\frac{\partial\widehat{\Psi}}{\partial e_{ij}(\mathbf{x})})\dot{e}_{ij}(\mathbf{x}) + (h_k - \rho\frac{\partial\widehat{\Psi}}{\partial\phi_{,k}(\mathbf{x})})\dot{\phi}_{,k}(\mathbf{x}) - (\mathrm{g} + \rho\frac{\partial\widehat{\Psi}}{\partial\phi(\mathbf{x})})\dot{\phi}(\mathbf{x}) - \rho(S + \frac{\partial\widehat{\Psi}}{\partial\theta(\mathbf{x})})\dot{\theta}(\mathbf{x})\\ &+\theta\mathbf{q} \cdot (\nabla\theta^{-1} + \nabla \cdot \mathbf{Q}) + \theta\nabla\mathbf{q} \cdot \mathbf{Q} \geq 0.\end{aligned} \tag{2.19}$$

To satisfy (2.19), the following constitutive equations may be gotten,

$$\tau_{ij} = \rho\frac{\partial\widehat{\Psi}}{\partial e_{ij}(\mathbf{x})}, \quad h_k = \rho\frac{\partial\widehat{\Psi}}{\partial\phi_{,k}(\mathbf{x})}, \quad \mathrm{g} = -\rho\frac{\partial\widehat{\Psi}}{\partial\phi(\mathbf{x})}, \quad S = -\frac{\partial\widehat{\Psi}}{\partial\theta(\mathbf{x})} \tag{2.20}$$

and we are left with the following reduced dissipation inequality

$$\theta(\mathbf{q} \cdot \Omega\mathbf{q} + \nabla\mathbf{q} \cdot \mathbf{Q}) \geq 0, \tag{2.21}$$

where

$$\Omega\mathbf{q} = \nabla\theta^{-1} + \nabla \cdot \mathbf{Q}. \tag{2.22}$$

Following Guyer and Krumhansl [17], we choose $\Omega = 1/kT_0\theta^2$ and $\mathbf{Q} = \Omega\varpi^2\nabla\mathbf{q}$, where k is thermal conductivity and $\varpi = e_0\varsigma$ is a non-local parameter, ς being the internal characteristic length and e_0 is a material constant. The internal characteristic length ς is the interatomic distance, e.g., length of $C-C$ bond (0.142 nm in Carbon nanotube) [6, 7, 24]. By using $\nabla\theta^{-1}=-\dfrac{\nabla\theta}{\theta^2}$, we get

$$(I-\varpi^2\nabla^2)\mathbf{q}=-kT_0\nabla\theta. \tag{2.23}$$

This equation corresponds to the nonlocal heat conduction law proposed by Jun Yu *et al.* [21] (see equation (28)), by Yang and Chen [35] (see equation (5)) and by Tian and Xiong [36] (see Fig. 1).

Compared with the classical Fourier's law, an additional mixed-derivative term $-\varpi^2\nabla^2\mathbf{q}$ in (2.23) will contribute to the spatial nonlocality in heat conduction. It would be difficult to determine if such extension is better to represent the transient response of spatial nonlocality based on pure, theoretical discussions. Therefore, validation of the new model (2.23) will be performed based on experimental results. In fact this new model (2.23) is verified by the experimental results of thin nanowires as it captured the linear dependence of the effective thermal conductivity on their radiuses [34].

Combining (2.11) together with (2.19), we arrive at the linear entropy balance

$$\rho T_0\dot{S}=-\nabla\cdot\mathbf{q}+\rho s. \tag{2.24}$$

In nonlocal linear theory for centro-symmetric isotropic materials, Ψ can be given in the quadratic approximation as

$$\begin{aligned}\rho\widehat{\Psi} &= A_{ijkl}e_{ij}(\mathbf{x})e_{kl}(\mathbf{x}')+\xi\phi(\mathbf{x})\phi(\mathbf{x}')+A_{ij}\phi_{,i}(\mathbf{x})\phi_{,j}(\mathbf{x}')+B_{ij}\Big(e_{ij}(\mathbf{x})\phi(\mathbf{x}')+e_{ij}(\mathbf{x}')\phi(\mathbf{x})\Big)\\ &- \beta_{ij}\Big(e_{ij}(\mathbf{x})\theta(\mathbf{x}')+e_{ij}(\mathbf{x}')\theta(\mathbf{x})\Big)-c\theta(\mathbf{x})\theta(\mathbf{x}')-m\Big(\phi(\mathbf{x})\theta(\mathbf{x}')+\phi(\mathbf{x}')\theta(\mathbf{x})\Big),\end{aligned}$$

where A_{ijkl} is the tensor of elastic constants, A_{ij}, B_{ij} and ξ are functions which are typical in porous theories, c is the thermal coefficient, m is the is thermo-void coefficient and β_{ij} is the tensor of thermal expansion. All these constitutive coefficients are prescribed functions of $\mathbf{x}$ and $\mathbf{x}'$ and satisfy the following symmetries relations:

$$\begin{aligned}A_{ijkl}(\mathbf{x},\mathbf{x}') &= A_{klij}(\mathbf{x},\mathbf{x}')=A_{jikl}(\mathbf{x},\mathbf{x}'),\quad A_{ij}(\mathbf{x},\mathbf{x}')=A_{ji}(\mathbf{x},\mathbf{x}'),\\ B_{ij}(\mathbf{x},\mathbf{x}') &= B_{ji}(\mathbf{x},\mathbf{x}'),\quad \beta_{ij}(\mathbf{x},\mathbf{x}')=\beta_{ji}(\mathbf{x},\mathbf{x}').\end{aligned}$$

Following Eringen [11, 12], the constitutive relations are obtained from

$$\Gamma=\int_V\Big(\frac{\partial\Psi}{\partial Y}+(\frac{\partial\Psi}{\partial Y'})^s\Big)dV(\mathbf{x}') \tag{2.25}$$

where the superscript 's' represents the symmetry of that quantity with respect to interchange of $\mathbf{x}$ and $\mathbf{x}'$.

In the following, we replace the response function g by $F+\mathrm{g}$, where the function F is a dissipation function given by

$$F=-\int_V\zeta(\mathbf{x},\mathbf{x}')\dot{\phi}(\mathbf{x}')dV(\mathbf{x}'), \tag{2.26}$$

and ζ is a non-negative constitutive coefficient. The components of the set $\Upsilon=\{\tau_{ij},h_i,F+\mathrm{g},S\}$ are obtained from relations (2.17)–(2.26), for centro-symmetric materials, as

$$\tau_{ij} = \int_V\Big[A_{ijkl}(\mathbf{x},\mathbf{x}')e_{kl}(\mathbf{x}')+B_{ij}(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}')-\beta_{ij}(\mathbf{x},\mathbf{x}')\theta(\mathbf{x}')\Big]dV(\mathbf{x}'),$$

$$\begin{aligned} h_i &= \int_V \Big[A_{ij}(\mathbf{x},\mathbf{x}')\phi_{,j}(\mathbf{x}')\Big] dV(\mathbf{x}'), \\ \mathrm{g} &= -\int_V \Big[B_{ij}(\mathbf{x},\mathbf{x}')e_{ij}(\mathbf{x}') + \xi(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}') + \zeta(\mathbf{x},\mathbf{x}')\dot{\phi}(\mathbf{x}') - m(\mathbf{x},\mathbf{x}')\theta(\mathbf{x}')\Big] dV(\mathbf{x}'), \\ \rho S &= \int_V \Big[\beta_{ij}(\mathbf{x},\mathbf{x}')e_{ij}(\mathbf{x}') + m(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}') + c(\mathbf{x},\mathbf{x}')\theta(\mathbf{x}')\Big] dV(\mathbf{x}'), \end{aligned} \tag{2.27}$$

where g in $(2.27)_2$ is actually $F + \mathrm{g}$. For a centro-symmetric isotropic material, the constitutive coefficients reduce to

$$A_{ijkl} = \lambda(\mathbf{x},\mathbf{x}')\delta_{ij}\delta_{kl} + 2\mu(\mathbf{x},\mathbf{x}')\delta_{ik}\delta_{jl}, \quad A_{ij} = \gamma(\mathbf{x},\mathbf{x}')\delta_{ij}, \quad B_{ij} = b(\mathbf{x},\mathbf{x}')\delta_{ij}, \quad \beta_{ij} = \beta(\mathbf{x},\mathbf{x}')\delta_{ij},$$

where δ_{ik} is the Kronecker delta function, the coefficients $\lambda,\ \mu,\ \gamma,\ b,\ \beta,\ \xi,\ \zeta,\ c$ and m are functions of $|\mathbf{x}-\mathbf{x}'|$, that is, $\lambda = \lambda(|\mathbf{x}-\mathbf{x}'|),\ \mu = \mu(|\mathbf{x}-\mathbf{x}'|)$, etc. Thus, the constitutive relations (2.27) become

$$\begin{aligned} \tau_{ij} &= \int_V \Big[\lambda(|\mathbf{x}-\mathbf{x}'|)\delta_{ij}e_{kk}(\mathbf{x}') + 2\mu(|\mathbf{x}-\mathbf{x}'|)e_{ij}(\mathbf{x}') + b(|\mathbf{x}-\mathbf{x}'|)\delta_{ij}\phi(\mathbf{x}') - \beta(|\mathbf{x}-\mathbf{x}'|)\delta_{ij}\theta(\mathbf{x}')\Big] dV(\mathbf{x}'), \\ h_i &= \int_V \Big[\gamma(|\mathbf{x}-\mathbf{x}'|)\phi_{,i}(\mathbf{x}')\Big] dV(\mathbf{x}'), \\ \mathrm{g} &= -\int_V \Big[b(|\mathbf{x}-\mathbf{x}'|)e_{kk}(\mathbf{x}') + \xi(|\mathbf{x}-\mathbf{x}'|)\phi(\mathbf{x}') + \zeta(|\mathbf{x}-\mathbf{x}'|)\dot{\phi}(\mathbf{x}') - m(|\mathbf{x}-\mathbf{x}'|)\theta(\mathbf{x}')\Big] dV(\mathbf{x}'), \\ \rho S &= \int_V \Big[\beta(|\mathbf{x}-\mathbf{x}'|)e_{kk}(\mathbf{x}') + m(|\mathbf{x}-\mathbf{x}'|)\phi(\mathbf{x}') + c(|\mathbf{x}-\mathbf{x}'|)\theta(\mathbf{x}')\Big] dV(\mathbf{x}'). \end{aligned} \tag{2.28}$$

For most of the materials, the cohesive zone is very small, and within that zone the intermolecular forces decrease rapidly with distance from the reference point. Hence we consider that all constitutive coefficients attenuate with distance [12], e.g.

$$\lim_{(|\mathbf{x}-\mathbf{x}'|)\to 0} \lambda(|\mathbf{x}-\mathbf{x}'|) \to 0.$$

We also consider that all the constitutive coefficients attenuate the same degree and they attain their maxima at $\mathbf{x} = \mathbf{x}'$. Therefore, we can take the following relations between nonlocal and local coefficients

$$\begin{aligned} \frac{\lambda(|\mathbf{x}-\mathbf{x}'|)}{\lambda_0} &= \frac{\mu(|\mathbf{x}-\mathbf{x}'|)}{\mu_0} = \frac{\gamma(|\mathbf{x}-\mathbf{x}'|)}{\gamma_0} = \frac{b(|\mathbf{x}-\mathbf{x}'|)}{b_0} = \frac{\beta(|\mathbf{x}-\mathbf{x}'|)}{\beta_0} \\ &= \frac{\xi(|\mathbf{x}-\mathbf{x}'|)}{\xi_0} = \frac{\zeta(|\mathbf{x}-\mathbf{x}'|)}{\zeta_0} = \frac{m(|\mathbf{x}-\mathbf{x}'|)}{m_0} = \frac{c(|\mathbf{x}-\mathbf{x}'|)}{c_0} = \mathscr{G}(|\mathbf{x}-\mathbf{x}'|). \end{aligned} \tag{2.29}$$

Here, the quantities in the denominator are constant coefficients. The $\lambda_0,\ \mu_0$ are the well known Lame's constants, $\beta_0 = (3\lambda_0 + 2\mu_0)\alpha_t,\ \alpha_t$ is the coefficient of linear thermal expansion, $\gamma_0,\ b_0,\ \xi_0,\ \zeta_0$ and m_0 are constants corresponding to voids and c_0 is the thermal constant. Following Eringen [12], as in (2.4), we assume that the nonlocal kernel satisfies

$$(I - \varpi^2\nabla^2)\mathscr{G}(|\mathbf{x}-\mathbf{x}'|) = \delta(|\mathbf{x}-\mathbf{x}'|). \tag{2.30}$$

We will now use the same procedure used to get (2.5) and (2.6). Applying the operator $(I - \varpi^2\nabla^2)$ on the constitutive relations (2.28), owing to the relation (2.29) and the property (2.30), we obtain (after suppressing the subscript '0' from the constitutive coefficients)

$$(I - \varpi^2\nabla^2)\tau_{ij} = (I - l^2\nabla^2)\tau^l_{ij} = (I - l^2\nabla^2)\Big[\lambda\delta_{ij}e_{kk}(\mathbf{x}) + 2\mu e_{ij}(\mathbf{x}) + b\delta_{ij}\phi(\mathbf{x}) - \beta\delta_{ij}\theta(\mathbf{x})\Big],$$

$$
\begin{aligned}
(I-\varpi^2\nabla^2)\mathrm{g} &= (I-l^2\nabla^2)\mathrm{g}^l = (I-l^2\nabla^2)\Big[-be_{kk}(\mathbf{x})-\xi\phi(\mathbf{x})-\zeta\dot{\phi}(\mathbf{x})+m\theta(\mathbf{x})\Big],\\
(I-\varpi^2\nabla^2)h_i &= (I-l^2\nabla^2)h_i^l = (I-l^2\nabla^2)\gamma\phi_{,i}(\mathbf{x}),\\
(I-\varpi^2\nabla^2)\rho S &= (I-l^2\nabla^2)(\rho S)^l = (I-l^2\nabla^2)\Big[\beta e_{kk}(\mathbf{x})+m\phi(\mathbf{x})+c\theta(\mathbf{x})\Big], \qquad (2.31)
\end{aligned}
$$

where the quantities τ_{ij}^l, h_i^l, g^l and $(\rho S)^l$ correspond to the local porous thermoelastic materials.

Applying the operator $R_\varpi = I-\varpi^2\nabla^2$ to the governing equations (2.7)–(2.8) and (2.24) and then plugging the constitutive relations (2.31) and (2.23) into the resulting equations and retaining only the terms of order $O(\nabla^2)$, we obtain

$$
\begin{aligned}
\rho(I-\varpi^2\nabla^2)\ddot{u}_i &= (I-l^2\nabla^2)\Big[\mu\nabla^2u_i+(\lambda+\mu)\nabla(\nabla\cdot u_i)+b\nabla\phi-\beta\nabla\theta\Big]+\rho(I-\varpi^2\nabla^2)G_i,\\
\rho J(I-\varpi^2\nabla^2)\ddot{\phi} &= (I-l^2\nabla^2)\Big[\gamma\nabla^2\phi-\xi\phi-b\nabla\cdot u_i-\zeta\dot{\phi}+m\theta\Big]+\rho(I-\varpi^2\nabla^2)\chi,\\
c(I-l^2\nabla^2)\dot{\theta} &= (I-l^2\nabla^2)\Big[-\beta\nabla\cdot\dot{u}_i-m\dot{\phi}\Big]+k\nabla^2\theta+\frac{\rho}{T_0}(I-\varpi^2\nabla^2)s. \qquad (2.32)
\end{aligned}
$$

Note that in the absence of nonlocality ($\varpi=0$) and the effect of the strain gradient field ($l=0$), these equations reduce to those of local isotropic thermoelasticity with voids [8, 13, 18]. When $\varpi\neq 0$ and $l=0$, these equations reduce to equations (22)-(24) given by Bachher and Sarkar [4] when the relaxation time τ_0 vanishes. The corresponding boundary and initial conditions come from the system of basic equations (2.32). Following Aouadi *et al.* [1, 2], Ieşan [19, 20] or Mindlin [28], one can derive the form of the corresponding boundary conditions. Here, this will be done in the one-dimensional framework.

Without loss of generality, we set $\rho=J=c=1$. In the one dimensional setting, the system (2.32) can be written in the domain $\Omega=(0,L)\times\mathbb{R}^+$:

$$
\begin{aligned}
(1-\varpi^2\partial_{xx})u_{tt}-(1-l^2\partial_{xx})\Big(au_{xx}+b\phi_x-\beta\theta_x\Big)+f_1(u,\phi,\theta)&=0,\\
(1-\varpi^2\partial_{xx})\phi_{tt}-(1-l^2\partial_{xx})\Big(\gamma\phi_{xx}-\xi\phi-bu_x-\zeta\phi_t+m\theta\Big)+f_2(u,\phi,\theta)&=0, \qquad (2.33)\\
(1-l^2\partial_{xx})\theta_t+(1-l^2\partial_{xx})\Big(\beta u_{xt}+m\phi_t\Big)-k\theta_{xx}+f_3(u,\phi,\theta)&=0,
\end{aligned}
$$

where $a=\lambda+2\mu>0$, and $f_1=-(1-\varpi^2\partial_{xx})G_1$, $f_2=-(1-\varpi^2\partial_{xx})\chi$, $f_3=-(1-\varpi^2\partial_{xx})\dfrac{s}{T_0}$ are sources forces. We assume the following boundary conditions

$$
\begin{aligned}
u(0,t) &= u(L,t)=u_{xx}(0,t)=u_{xx}(L,t)=0,\\
\phi(0,t) &= \phi(L,t)=\phi_x(0,t)=\phi_x(L,t)=0,\\
\theta_x(0,t) &= \theta_x(L,t)=0, \quad t>0, \qquad (2.34)
\end{aligned}
$$

and the initial conditions for $x\in(0,L)$

$$
u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \phi(x,0)=\phi_0(x),\ \phi_t(x,0)=\phi_1(x),\ \theta(x,0)=\theta_0(x). \qquad (2.35)
$$

Remark 2.1. *We assume that the internal mechanical energy density is a positive definite form. Thus, the constitutive coefficients (for the one-dimensional case) satisfy the conditions*

$$
a>0, \quad \gamma>0, \quad a\xi>b^2, \qquad (2.36)
$$

meanwhile if we assume that the dissipation is positive we need to impose that

$$k > 0 \quad and \quad \zeta > 0. \tag{2.37}$$

3 Well-posedness

3.1 Preliminaries

Throughout this paper, we use the standard Lebesgue space $L^2(0,L)$ and the Sobolev spaces $H^m(0,L) = W^{m,2}(0,L)$ $(1 \leq m \leq \infty)$ with their usual scalar products and norms. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the L^2- inner product and L^2- norm, respectively.

It is well known that the operator $A = -\partial_{xx}$ with Dirichlet boundary conditions is self-adjoint, positive operator on $L^2(0,L)$ with

$$\mathcal{D}(A) = H^2(0,L) \cap H_0^1(0,L), \qquad \mathcal{D}(A^{\frac{1}{2}}) = H_0^1(0,L). \tag{3.1}$$

Let us introduce the inertia operator $R_\epsilon = I + \epsilon^2 A$ with the domain $\mathcal{D}(R_\epsilon) = H^2(0,L) \cap H_0^1(0,L)$ when $\epsilon > 0$ and $L^2(\Omega)$ otherwise. Observe that, in our case $\epsilon = \varpi$ or $\epsilon = l$. In the following, we limit ourselves to the cases $\varpi \neq 0$ and $l \neq 0$. We could also consider the case $\varpi = l = 0$. As this case has already been studied (see e.g. [26] among others) and is not essential to the analysis of this article; thus it will not be considered here.

Then, one has [15]

$$\mathcal{D}(R_\epsilon^{1/2}) \equiv H_0^1(0,L) \tag{3.2}$$

with the inner products [3, 15]

$$\langle R_\epsilon u, v \rangle := \langle u, v \rangle_\epsilon = \langle u, v \rangle + \epsilon^2 \langle u_x, v_x \rangle, \quad \forall\ u,\ v \in H_0^1(0,L). \tag{3.3}$$

Observe that $(3.1)_2$ yields that [3, 15]

$$R_\epsilon \in \mathscr{L}(H_0^1(0,L), H^{-1}(0,L)) \text{ with } \langle R_\epsilon u, v \rangle_{H^{-1}(0,L) \times H_0^1(0,L)} = \langle u, v \rangle_{H_0^1(0,L)}, \tag{3.4}$$

where $H^{-1}(0,L)$ denotes the dual of $H_0^1(0,L)$.

Furthermore, the $H_0^1(0,L)-$ellipticity of R_ϵ and the Lax-Milgram theorem give us that R_ϵ is boundedly invertible, i.e.

$$R_\epsilon^{-1} \in \mathscr{L}(H^{-1}(0,L), H_0^1(0,L)). \tag{3.5}$$

Thus, for $\epsilon \neq 0$, we have

$$\|R_\epsilon^{-1} u\|_{H_0^1(0,L)} = \|u\|_{H^{-1}(0,L)}, \qquad \forall u \in H^{-1}(0,L). \tag{3.6}$$

Finally, R_ϵ, being positive definite, self-adjoint as an operator $R_\epsilon : L^2(0,L) \supset \mathcal{D}(R_\epsilon) \to L^2(0,L)$ (as A is), it then follows from (3.3) and $(3.4)_2$ that for $u,\ v \in H_0^1(0,L)$,

$$\|R_\epsilon^{1/2} u\|^2 := \|u\|_\epsilon^2 = \|u\|^2 + \epsilon^2 \|u_x\|^2 = \|u\|^2_{H_0^1(0,L)}. \tag{3.7}$$

Now we consider the following assumptions on the forcing terms f_j $(j = 1, 2, 3)$:

(i) There exists a nonnegative C^2 function $F: \mathbb{R}^3 \to \mathbb{R}$ satisfying

$$\nabla F = (f_1, f_2, f_3),$$

and there exists a positive constant β such that for any $u,\ \phi,\ \theta \in \mathbb{R}$

$$F(u, \phi, \theta) \geq -\beta(|u|^2 + |\phi|^2 + |\theta|^2). \tag{3.8}$$

(ii) There exist positive constants $p \geq 1$ and C_f such that for any $u,\ \phi,\ \theta \in \mathbb{R}$

$$|\nabla f_j(u, \phi, \theta)| \leq C_f(1 + |u|^{p-1} + |\phi|^{p-1} + |\theta|^{p-1}), \quad j = 1, 2, 3, \tag{3.9}$$

which gives us a positive constant C_F such that

$$F(u, \phi, \theta) \leq C_F(1 + |u|^{p+1} + |\phi|^{p+1} + |\theta|^{p+1}). \tag{3.10}$$

Remark 3.1. *a) There is a large class of forces satisfying* (3.8)–(3.10). *For instance,*

$$F(u, \phi, \theta) = a|u + \phi|^{p+1} + b|u + \phi|^p + c|\theta|^p, \quad a, b, c > 0.$$

b) Going forward, we will omit the symbol t in the coordinates of the solution $U(t)$, that is, we will just write $U = (u, \phi, u_t, \phi_t, \theta)$, it being understood that u, ϕ, v, ψ, θ depend on t.

c) In the following, we use the general notation $(n.m)_i$ where the 'i' refers to the ith equation of the system $(n.m)$.

3.2 Well-posedness

In this subsection, we study the well-posedness of the problem (2.33)–(2.35). We set $v = u_t$ and $\psi = \phi_t$. First, let us consider the Hilbert space

$$\mathscr{H} := \big(H^2(0, L) \cap H_0^1(0, L)\big) \times H_0^2(0, L) \times (H_0^1(0, L))^2 \times H_*^1(0, L),$$

with the inner product

$$\begin{aligned} \langle U, U^* \rangle_{\mathscr{H}} &= \langle v, v^* \rangle_\varpi + \langle \psi, \psi^* \rangle_\varpi + \langle \theta, \theta^* \rangle_l + \gamma \langle \phi_x, \phi_x^* \rangle_l + a \langle u_x, u_x^* \rangle_l \\ &+ b \langle u_x, \phi^* \rangle_l + \xi \langle \phi, \phi^* \rangle_l + b \langle \phi, u_x^* \rangle_l, \end{aligned} \tag{3.11}$$

where $U = (u, \phi, v, \psi, \theta)$, $U^* = (u^*, \phi^*, v^*, \psi^*, \theta^*)$ and

$$L_*^2(0, L) = \Big\{ w \in L^2(0, L) \ : \ \int_0^L w(s) ds = 0 \Big\}, \quad H_*^1(0, L) = H^1(0, L) \cap L_*^2(0, L).$$

The corresponding norm in $\mathscr{H}$ is given by

$$\|U\|_{\mathscr{H}}^2 = \|v\|_\varpi^2 + \|\psi\|_\varpi^2 + \|\theta\|_l^2 + \gamma \|\phi_x\|_l^2 + a\|u_x\|_l^2 + \xi\|\phi\|_l^2 + 2b\Re e\langle u_x, \phi \rangle_l. \tag{3.12}$$

Note that $(2.36)_3$ leads to

$$a\|u_x\|_l^2 + \xi\|\phi\|_l^2 + 2b\Re e\langle u_x, \phi\rangle_l > 0. \tag{3.13}$$

Consequently $\|U\|_{\mathscr{H}}^2$ is nonnegative. Moreover, the induced norm $\|\cdot\|_{\mathscr{H}}$ is equivalent to the usual norm in the Hilbert space $\mathscr{H}$,

$$C_1\|U\|_{\mathscr{H}} \le \|u\|_{H^2} + \|\phi\|_{H^2} + \|\theta\|_{H^1} + \|v\|_{H^1} + \|\psi\|_{H^1} \le C_2\|U\|_{\mathscr{H}},$$

with positive constants C_1 and C_2. While the estimate below is obvious, the estimate above can be obtained by choosing $\varepsilon > 0$ sufficiently small such $(a-\varepsilon)(\xi-\varepsilon) > b^2$ and then

$$\begin{aligned} a\|u_x\|_l^2 + 2b\Re e\langle u_x, \phi\rangle_l + \xi\|\phi\|_l^2 &= (a-\varepsilon)\|u_x\|_l^2 + 2b\Re e\langle u_x, \phi\rangle_l + (\xi-\varepsilon)\|\phi\|_l^2 + \varepsilon(\|u_x\|_l^2 + \|\phi\|_l^2) \\ &\ge \varepsilon\|u_x\|_l^2 + \varepsilon\|\phi\|_l^2. \end{aligned} \tag{3.14}$$

In particular, there exists a positive constant ϱ such that

$$\|u\|^2 + \|\phi\|^2 + \|\theta\|^2 \le \varrho(a\|u_x\|_l^2 + \xi\|\phi\|_l^2 + \|\theta\|_l^2) \le \varrho\|U\|_{\mathscr{H}}^2, \tag{3.15}$$

where

$$\varrho\beta \le \frac{1}{4}. \tag{3.16}$$

Problem (2.33)–(2.35) can be written as a Cauchy problem

$$\begin{cases} \dfrac{dU(t)}{dt} + (\mathscr{A} + \mathscr{B})U(t) = \mathcal{F}(U(t)), \quad t > 0, \\ U(0) = U_0 = (u_0, \phi_0, u_1, \phi_1, \theta_0) \in \mathscr{H}, \end{cases} \tag{3.17}$$

where, $\mathscr{A} : \mathcal{D}(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$, $\mathscr{B} : \mathcal{D}(\mathscr{B}) \subset \mathscr{H} \to \mathscr{H}$ and $\mathcal{F} : \mathscr{H} \to \mathscr{H}$ are defined by

$$\mathscr{A}U = \begin{pmatrix} -v \\ -\psi \\ -R_\varpi^{-1} R_l\Big(au_{xx} + b\phi_x - \beta\theta_x\Big) \\ -R_\varpi^{-1} R_l\Big(\gamma\phi_{xx} - \xi\phi - bu_x + m\theta\Big) \\ \beta v_x + m\psi \end{pmatrix}, \quad \mathscr{B}U = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \zeta R_\varpi^{-1} R_l \psi \\ -kR_l^{-1}\theta_{xx} \end{pmatrix}, \tag{3.18}$$

$$\mathcal{F}(U(t)) = \begin{pmatrix} 0 \\ 0 \\ -R_\varpi^{-1} f_1(u,\phi,\theta) \\ -R_\varpi^{-1} f_2(u,\phi,\theta) \\ -R_l^{-1} f_3(u,\phi,\theta) \end{pmatrix}. \tag{3.19}$$

Recalling $A = -\partial_{xx}$ and $R_\varpi = A(\varpi^2 I + A^{-1}) = (\varpi^2 I + A^{-1})A$, then yields

$$R_\varpi^{-1} A = (\varpi^2 I + A^{-1})^{-1}. \tag{3.20}$$

Multiplying the expression,

$$I = \frac{1}{\varpi^2}(\varpi^2 I + A^{-1}) - \frac{1}{\varpi^2}A^{-1},$$

by $(\varpi^2 I + A^{-1})^{-1}$, then (3.20) becomes

$$R_\varpi^{-1}A = \frac{1}{\varpi^2}I - \frac{1}{\varpi^2}A^{-1}(\varpi^2 I + A^{-1})^{-1} = \frac{1}{\varpi^2}I - \frac{1}{\varpi^2}R_\varpi^{-1},$$

from which we derive that

$$R_\varpi^{-1}R_l = \frac{l^2}{\varpi^2}I + (1 - \frac{l^2}{\varpi^2})R_\varpi^{-1}.$$

From (3.5), we infer that

$$R_\varpi^{-1}R_l \in \mathscr{L}(H^{-1}(0,L), H_0^1(0,L)).$$

Based on the definition of $\mathscr{A}$ and $\mathscr{H}$, one can see that

$$\begin{aligned}
\mathcal{D}(\mathscr{A}) &= \Big\{U \in \mathscr{H} \mid u \in H^3(0,L) \cap H_0^1(0,L);\ v \in H^2(0,L) \cap H_0^1(0,L);\ \phi \in H^3(0,L) \cap H_0^2(0,L);\\
&\qquad \psi \in H_0^2(0,L);\ \theta \in H^2(0,L) \cap H_*^1(0,L);\ \gamma\phi_{xx} - bu_x + m\theta \in H_0^1(0,L)\ \ + \text{BC's}\Big\},\\
\mathcal{D}(\mathscr{B}) &= \Big\{u \in H^2(0,L) \cap H_0^1(0,L);\ \phi \in H_0^2(0,L);\ v \in H_0^1(0,L);\ \psi \in H^{-1}(0,L);\ \theta \in H^2(0,L);\\
&\qquad \theta_{xx} \in H^{-1}(0,L)\ \ + \text{BC's}\Big\},\\
\mathcal{D}(\mathscr{A} + \mathscr{B}) &= \mathcal{D}(\mathscr{A}) \cap \mathcal{D}(\mathscr{B}).
\end{aligned}$$

The phrase "BC's" means that u, ϕ and θ satisfy all the relevant boundary conditions (2.34). It is easy to see that $\mathcal{D}(\mathscr{A})$ is dense in $\mathscr{H}$.

In order to present the existence of local solutions, we start by recalling the definition of maximal monotone operators and the link with the generation of C_0-semigroup [5, 10]. An operator $\mathscr{A}$ is called monotone if

$$\langle \mathscr{A}U - \mathscr{A}\widetilde{U}, U - \widetilde{U}\rangle \geq 0, \qquad \forall\ U,\ \widetilde{U} \in \mathcal{D}(\mathscr{A}).$$

In addition, it is called maximal monotone if $(\mathscr{A} + I)\ :\ \mathscr{H} \to \mathscr{H}'$ is surjective, where $\mathscr{H}'$ denotes the dual of $\mathscr{H}$. An operator $\mathscr{B}\ :\ \mathscr{H} \to \mathscr{H}'$ is called hemicontinuous if

$$\lim_{\lambda \to 0}\langle \mathscr{B}(U + \lambda\widetilde{U}), U^*\rangle_{\mathscr{H}} = \langle \mathscr{B}(U), U^*\rangle_{\mathscr{H}}, \qquad \forall\ U,\ \widetilde{U},\ U^* \in \mathscr{H}.$$

So, if $\mathscr{B}$ is hemicontinuous, its maximal monotonicity follows.

Now, if $\mathscr{A}$ and $\mathscr{B}$ are both maximal monotone and $\operatorname{int}(\mathcal{D}(\mathscr{A})) \cap \mathcal{D}(\mathscr{B}) \neq \varnothing$, by Theorem 2.6 in [5], one can conclude that $\mathscr{A} + \mathscr{B}$ is maximal monotone.

Since the operator $\mathscr{A} + \mathscr{B}$ is maximal monotone in $\mathscr{H}$ and $\mathcal{D}(\mathscr{A})$ is densely defined in $\mathscr{H}$, the operator $\mathscr{A} + \mathscr{B}$ generates a C_0-semigroup of contractions on $\mathscr{H}$. The proof of this follows from the Lumer-Phillips corollary to the Hille-Yosida theorem [29].

We shall apply the theory of monotone operators and nonlinear semigroups [5, 10].

Lemma 3.1. *The operator $\mathscr{A} + \mathscr{B}$ is maximal monotone in $\mathscr{H}$.*

Proof. We divide the proof in two steps.

Step 1: $\mathscr{A}$ is maximal monotone. Let us denote $U=(u,\phi,v,\psi,\theta)$, $\widetilde{U}=(\widetilde{u},\widetilde{\phi},\widetilde{v},\widetilde{\psi},\widetilde{\theta})\in\mathcal{D}(\mathscr{A})$. Using integration by parts and the boundary conditions, lead to

$$\begin{aligned}\langle\mathscr{A}U-\mathscr{A}\widetilde{U},U-\widetilde{U}\rangle_{\mathscr{H}} &= 2i\Im m\Big(a\langle u_x^1,v_x^1\rangle_l+b\langle\phi^1,v_x^1\rangle_l+b\langle u_x^1,\psi^1\rangle_l+\beta\langle v_x^1,\theta^1\rangle_l+\gamma\langle\phi_x^1,\psi_x^1\rangle_l\\ &+ \xi\langle\phi^1,\psi^1\rangle_l+m\langle\psi^1,\theta^1\rangle_l\Big),\end{aligned}$$

where $u^1=u-\widetilde{u},\ \phi^1=\phi-\widetilde{\phi},\ v^1=v-\widetilde{v},\ \psi^1=\psi-\widetilde{\psi},\ \theta^1=\theta-\widetilde{\theta}$. Then, we have

$$\Re e\langle\mathscr{A}U-\mathscr{A}\widetilde{U},U-\widetilde{U}\rangle_{\mathscr{H}}=0$$

and thereby $\mathscr{A}$ is monotone. In order to prove that $\mathscr{A}$ is maximal monotone, we need to prove that Range $(I+\mathscr{A})=\mathscr{H}$. We must prove that

$$U+\mathscr{A}U=U^*$$

has a solution $U=(u,\phi,v,\psi,\theta)\in\mathcal{D}(\mathscr{A})$ for any $U^*=(u^*,\phi^*,v^*,\psi^*,\theta^*)\in\mathscr{H}$. This equation leads to the system

$$\begin{cases}u-v=u^*\in H^2(0,L)\cap H_0^1(0,L),\\ \phi-\psi=\phi^*\in H_0^2(0,L),\\ v-R_\varpi^{-1}R_l\Big(au_{xx}+b\phi_x-\beta\theta_x\Big)=v^*\in H_0^1(0,L),\\ \psi-R_\varpi^{-1}R_l\Big(\gamma\phi_{xx}-\xi\phi-bu_x+m\theta\Big)=\psi^*\in H_0^1(0,L),\\ \theta+\beta v_x+m\psi=\theta^*\in H_*^1(0,L).\end{cases}\tag{3.21}$$

Adding $(3.21)_{1,2}$ to $(3.21)_{3,4}$ respectively, and then multiplying the results by R_ϖ and multiplying $(3.21)_5$ by R_l, we obtain the following system

$$\begin{cases}R_\varpi u-R_l(au_{xx}+b\phi_x-\beta\theta_x)=R_\varpi(u^*+v^*),\\ R_\varpi\phi-R_l(\gamma\phi_{xx}-\xi\phi-bu_x+m\theta)=R_\varpi(\phi^*+\psi^*),\\ R_l\theta+R_l(\beta u_x+m\phi)=R_l(\theta^*+m\phi^*+\beta u_x^*).\end{cases}\tag{3.22}$$

We denote the Hilbert space $\mathcal{H}=\big(H^2(0,L)\cap H_0^1(0,L)\big)\times H_0^2(0,L)\times H_*^1(0,L)$, then the problem (3.22) is equivalent to

$$B\Big((u,\phi,\theta),(w,\varphi,\vartheta)\Big)=\mathcal{L}(w,\varphi,\vartheta),\quad\forall\ (w,\varphi,\vartheta)\in\mathcal{H},\tag{3.23}$$

where the bilinear form $B\ :\ \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ and the linear form $\mathcal{L}\ :\ \mathcal{H}\to\mathbb{C}$ are defined by

$$\begin{aligned}B\Big((u,\phi,\theta),(w,\varphi,\vartheta)\Big) &= \langle u,w\rangle_\varpi+\langle\phi,\varphi\rangle_\varpi+\langle\theta,\vartheta\rangle_l+a\langle u_x,w_x\rangle_l+\gamma\langle\phi_x,\varphi_x\rangle_l+\xi\langle\phi,\varphi\rangle_l\\ &+ b\langle\phi,w_x\rangle_l+b\langle u_x,\varphi\rangle_l-\beta\langle\theta,w_x\rangle_l+\beta\langle u_x,\vartheta\rangle_l-m\langle\theta,\varphi\rangle_l+m\langle\phi,\vartheta\rangle_l,\\ \mathcal{L}(w,\varphi,\vartheta) &= \langle v^*+u^*,w\rangle_\varpi+\langle\psi^*+\phi^*,\varphi\rangle_\varpi+\langle\theta^*+m\phi^*+\beta u_x^*,\vartheta\rangle_l,\end{aligned}\tag{3.24}$$

for any $(w,\varphi,\vartheta)\in\mathcal{H}$. By assuming that the physical parameters are constant coefficients satisfying (2.36) and (2.37), one can show that the bilinear form $B(\cdot,\cdot)$ is bounded. From $((3.24))_1$, one can obtain easily that

$$\left|B\Big((u,\phi,\theta),(w,\varphi,\vartheta)\Big)\right|\leq C\|(u,\phi,\theta)\|_{\mathcal{H}}\|(w,\varphi,\vartheta)\|_{\mathcal{H}}$$

which implies that B is continuous. From (3.14), we obtain that

$$\begin{aligned}\Re e\ B\Big((u,\phi,\theta),(u,\phi,\theta)\Big) &= \|u\|_{\varpi}^2+\|\phi\|_{\varpi}^2+\|\theta\|_l^2+\gamma\|\phi_x\|_l^2+a\|u_x\|_l^2+\xi\|\phi\|_l^2+2b\Re e\langle u_x,\phi\rangle_l \\ &\geq \|u\|_{\varpi}^2+\|\phi\|_{\varpi}^2+\|\theta\|_l^2+\gamma\|\phi_x\|_l^2+\frac{1}{2}(a-\frac{b^2}{\xi})\|u_x\|_l^2+\frac{1}{2}(\xi-\frac{b^2}{a})\|\phi\|_l^2.\end{aligned}$$

By assuming $a\xi-b^2>0$, we get

$$a-\frac{b^2}{\xi}>0,\quad \xi-\frac{b^2}{a}>0,$$

then there exists a positive constant M_0 such that for any $(u,\phi,\theta)\in\mathcal{H}$, we have

$$\Re e\ B\Big((u,\phi,\theta),(u,\phi,\theta)\Big)\geq M_0\|(u,\phi,\theta)\|_{\Lambda}^2.$$

Thus, B is coercive. Since $\mathcal{L}$ is continuous, from the Lax-Milgram theorem, problem (3.23) admits a unique solution

$$(u,\phi,\theta)\in\mathcal{H}=\big(H^2(0,L)\cap H_0^1(0,L)\big)\times H_0^2(0,L)\times H_*^1(0,L). \tag{3.25}$$

Then, by (3.22) , we have

$$\begin{cases} -R_l\Big(au_{xx}+b\phi_x-\beta\theta_x\Big)=R_{\varpi}(u^*+v^*-u), \\ -R_l\Big(\gamma\phi_{xx}-\xi\phi-bu_x+m\theta\Big)=R_{\varpi}(\phi^*+\psi^*-\phi). \end{cases} \tag{3.26}$$

Since $l\neq 0$ and the operator R_l is an isomorphism of $H_0^1(0,L)$ onto $H^{-1}(0,L)$, then the application of regularity theory for the linear elliptic equations leads from $(3.26)_1$ that u_{xx}, ϕ_x and θ_x belong to $H_0^1(0,L)$ and from $(3.26)_2$ that $\gamma\phi_{xx}-bu_x+m\theta$ belongs to $H_0^1(0,L)$. Hence by (3.25) we infer that $u\in H^3(0,L)\cap H_0^1(0,L)$, $\phi\in H^3(0,L)\cap H_0^2(0,L)$ and $\theta\in H^2(0,L)\cap H_*^1(0,L)$.

Finally, from $(3.21)_{1-2}$, we get $(u_t,\phi_t)\in\big(H^2(0,L)\cap H_0^1(0,L)\big)\times H_0^2(0,L)$, then the vector $(u,\phi,u_t,\phi_t,\theta)\in\mathcal{D}(\mathscr{A})$. Therefore, Range $(\mathcal{I}+\mathscr{A})=\mathscr{H}$. This complete the proof of the maximal monotonicity of $\mathscr{A}$.

Step 2: $\mathscr{B}$ is maximal monotone. To prove that $\mathscr{B}$ is maximal monotone, by Theorem 2.4 in [5] we only need to prove that $\mathscr{B}$ is monotone and hemicontinuous. For any $U=(u,\phi,u_t,\phi_t,\theta)$, $\widetilde{U}=(\widetilde{u},\widetilde{\phi},\widetilde{u}_t,\widetilde{\phi}_t,\widetilde{\theta})\in\mathscr{H}$, by using (3.11) and $(3.18)_2$ we have

$$\langle\mathscr{B}U-\mathscr{B}\widetilde{U},U-\widetilde{U}\rangle_{\mathscr{H}}=k\|\theta_x-\widetilde{\theta}_x\|^2+\zeta\|\phi_t-\widetilde{\phi}_t\|_l^2\geqslant 0,$$

which implies that $\mathscr{B}$ is monotone. We observe that

$$\begin{aligned}\Big|\langle\mathscr{B}(U+\lambda\widetilde{U}),U^*\rangle_{\mathscr{H}}-\langle\mathscr{B}U,U^*\rangle_{\mathscr{H}}\Big| &\leq k|\lambda||\langle\widetilde{\theta}_{xx},\theta^*\rangle|+\zeta|\lambda||\langle\widetilde{\phi}_t,\phi_t^*\rangle_l| \\ &\leq |\lambda|\Big(k\|\widetilde{\theta}_x\|\|\theta_x^*\|+\zeta\|\widetilde{\phi}_t\|\|\phi_t^*\|+\zeta l^2\|\widetilde{\phi}_{xt}\|\|\phi_{xt}^*\|\Big)\overset{\lambda\to 0}{\longrightarrow}0.\end{aligned}$$

It follows that

$$\lim_{\lambda\to 0}\langle\mathscr{B}(U+\lambda\widetilde{U}),U^*\rangle_{\mathscr{H}}=\langle\mathscr{B}(U),U^*\rangle_{\mathscr{H}}.$$

Therefore, $\mathscr{B}$ is hemicontinuous. Since we have shown that $\mathscr{B}$ is monotone and hemicontinuous, we can apply the result in [5, Theorem 2.4] to conclude that $\mathscr{B}$ is maximal monotone.

Now, since $\mathscr{A}$ and $\mathscr{B}$ are both maximal monotone and $\mathrm{int}(\mathcal{D}(\mathscr{A}))\cap\mathcal{D}(\mathscr{B})\neq\varnothing$, by [5, Theorem 2.6] we conclude that $\mathscr{A}+\mathscr{B}$ is maximal monotone. $\square$

We are now in a position to give the definitions of mild solutions, strong solutions (according to Pazy [29], Chapter 6) and weak solutions to problem (2.33)–(2.35).

Definition 3.1. *(*i*) Let $T > 0$. A solution $U \in C([0,T];\mathscr{H})$ to the integral equation*

$$U(t) = e^{-(\mathscr{A}+\mathscr{B})t}U(0) + \int_0^t e^{-(\mathscr{A}+\mathscr{B})(t-s)}\mathcal{F}(U(s))ds \tag{3.27}$$

is called a mild solution to (3.17) on the interval $[0,T]$.

*(*ii*) Let $T > 0$. A function $U : [0,T) \to \mathscr{H}$ is called strong solution to (3.17) on $[0,T)$, if U is continuous on $[0,T)$, continuously differentiable on $(0,T)$, with $U(t) \in \mathcal{D}(\mathscr{A}+\mathscr{B})$ for $t \in (0,T)$, and (3.17) is satisfied on $[0,T)$ almost everywhere.*

*(*iii*) Let $T > 0$ and initial data $(u_0, \phi_0, u_1, \phi_1, \theta_0) \in \mathscr{H}$. We say that a set of functions $(u, \phi, u_t, \phi_t, \theta)$ is a weak (or generalized) solution to* (2.33)–(2.35) *if $(u, \phi, u_t, \phi_t, \theta) \in C([0,T), \mathscr{H})$ satisfies (2.35) and the following identity in the sense of distributions*

$$\begin{aligned} &\langle u_{tt}, w\rangle_\varpi - \langle au_{xx} + b\phi_x - \beta\theta_x, w\rangle_l + \langle f_1(u,\phi,\theta), w\rangle + \langle \phi_{tt}, \varphi\rangle_\varpi - \langle \gamma\phi_{xx} - \xi\phi - bu_x - \zeta\phi_t + m\theta, \varphi\rangle_l \\ + \; &\langle f_2(u,\phi,\theta), \varphi\rangle + \langle \theta_t + \beta u_{xt} + m\phi_t, \vartheta\rangle_l + k\langle \theta_x, \vartheta_x\rangle + \langle f_3(u,\phi,\theta), \vartheta\rangle = 0, \end{aligned}$$

in $[0,L]\times[0,T]$ and for all $w \in H^2(0,L)\cap H_0^1(0,L)$, $\varphi \in H_0^2(0,L)$ and $\vartheta \in H_*^1(0,L)$.

We are now in a position to give the main result of this section.

Theorem 3.1. *We assume that assumptions* (3.8)–(3.10) *and* (3.16) *hold.*

*(*i*) Let $U_0 = (u_0, \phi_0, u_1, \phi_1, \theta_0) \in \mathscr{H}$, then problem* (2.33)–(2.35) *has a unique global mild solution satisfying $U \in C([0,\infty);\mathscr{H})$ given by*

$$U(t) = e^{-(\mathscr{A}+\mathscr{B})t}U(0) + \int_0^t e^{-(\mathscr{A}+\mathscr{B})(t-s)}\mathcal{F}(U(s))ds. \tag{3.28}$$

*(*ii*) The weak solutions depend continuously on the initial data in $\mathscr{H}$. More precisely, given any two weak solutions U^1 and U^2 are two mild solutions to problem* (2.33)–(2.35)*, then there exists a positive constant C depending on $U^1(0)$ and $U^2(0)$, such that*

$$\|U^1(t) - U^2(t)\|_{\mathscr{H}} \le e^{CT}\|U^1(0) - U^2(0)\|_{\mathscr{H}}, \qquad 0 \le t \le T. \tag{3.29}$$

*(*iii*) If $U_0 = (u_0, \phi_0, u_1, \phi_1, \theta_0) \in \mathcal{D}(\mathscr{A}+\mathscr{B})$, then the corresponding mild solution is strong, that is continuously differentiable, it takes values in $\mathcal{D}(\mathscr{A}+\mathscr{B})$ and it satisfies (3.17) in $\mathscr{H}$ for almost all $t \in [0,T]$.*

Proof. (i) We need to prove first that $\mathcal{F}(U) = (0, 0, -R_\varpi^{-1}f_1(u,\phi,\theta), -R_\varpi^{-1}f_2(u,\phi,\theta), -R_l^{-1}f_3(u,\phi,\theta))$ defined in (3.19) is locally Lipschitz in $\mathscr{H}$. Let us denote $U^1 = (u^1, \phi^1, u_t^1, \phi_t^1, \theta^1) \in \mathscr{H}$ and $U^2 = (u^2, \phi^2, u_t^2, \phi_t^2, \theta^2) \in \mathscr{H}$, such that $\|U^1\|_{\mathscr{H}}, \ \|U^2\|_{\mathscr{H}} \le R$, where $R > 0$ is a constant. Then from (3.6) we have

$$\begin{aligned} \|\mathcal{F}(U^1) - \mathcal{F}(U^2)\|_{\mathscr{H}} &\le \sum_{i=1}^{2} \left\| R_\varpi^{-1}\Big(f_i(u^1,\phi^1,\theta^1) - f_i(u^2,\phi^2,\theta^2)\Big)\right\|_{H_0^1(0,L)} \\ &+ \left\| R_l^{-1}\Big(f_3(u^1,\phi^1,\theta^1) - f_3(u^2,\phi^2,\theta^2)\Big)\right\|_{H_0^1(0,L)} \end{aligned}$$

$$\leq \quad C\sum_{i=1}^{3}\left\|f_i(u^1,\phi^1,\theta^1)-f_i(u^2,\phi^2,\theta^2)\right\|_{H^{-1}(0,L)}. \tag{3.30}$$

Using (3.9), we have for $i=1,2,3$,

$$\begin{aligned}\left|f_i(u^1,\phi^1,\theta^1)-f_i(u^2,\phi^2,\theta^2)\right|^2 &= \left|\nabla f_i(\varsigma(u^1,\phi^1,\theta^1)+(1-\varsigma)(u^2,\phi^2,\theta^2)\right|^2 |(u^1,\phi^1,\theta^1)-(u^2,\phi^2,\theta^2)|^2\\ &\leq C_f\left(|u^1|^{p-1}+|u^2|^{p-1}+|\phi^1|^{p-1}+|\phi^2|^{p-1}+|\theta^1|^{p-1}+|\theta^2|^{p-1}+1\right)^2\\ &.\left(|u^1-u^2|^2+|\phi^1-\phi^2|^2+|\theta^1-\theta^2|^2\right)\end{aligned} \tag{3.31}$$

where $0\leq\varsigma\leq 1$ and $C_f>0$ is a constant depending on the initial data. It follows from (3.31) and embedding $H^2(0,L)\cap H_0^1(0,L)\hookrightarrow H_0^1(0,L)\hookrightarrow L^\infty(0,L)$ and $H_0^2(0,L)\hookrightarrow H_0^1(0,L)\hookrightarrow L^\infty(0,L)$ that there exists a constant $C_R>0$ such that

$$\int_0^L\left|f_i(u^1,\phi^1,\theta^1)-f_i(u^2,\phi^2,\theta^2)\right|^2dx\leq C_R\left\|(u^1,\phi^1,\theta^1)-(u^2,\phi^2,\theta^2)\right\|^2\leq C_R\|U^1-U^2\|_{\mathscr{H}}^2$$

for $i=1,2,3$. Summing this estimation on i, we conclude that there exists $L_R>0$ such that

$$\|\mathcal{F}(U^1)-\mathcal{F}(U^2)\|^2\leq L_R\|U^1-U^2\|_{\mathscr{H}}^2.$$

Therefore, $\mathcal{F}$ satisfying the local Lipschitz condition.

Since $\mathscr{A}+\mathscr{B}$ is maximal monotone and, for each $t\in[0,\infty)$ fixed, $\mathcal{F}(\cdot):\mathscr{H}\to\mathscr{H}$ is locally Lipschitz. Then, by Theorem 7.2 in Chueshov *et al.* [10], for all $U_0\in\mathcal{D}(\mathscr{A}+\mathscr{B})$, there exists $t_{\max}\leq\infty$ and a unique strong solution U for (3.17) defined on the interval $[0,t_{\max})$. Moreover, if $U_0\in\mathscr{H}$, then (3.17) has a unique weak solution $U\in C\left([0,t_{\max}),\mathscr{H}\right)$, and such a solution satisfies $\lim\sup_{t\to t_{\max}}\|U(t)\|_{\mathscr{H}}=\infty$, provided $t_{\max}<\infty$.

Next we prove that the solution is global, that is, $t_{max}=\infty$. Taking the inner product of (3.17) and $U(t)\in\mathscr{H}$, we get that the total energy satisfies

$$\frac{d}{dt}\mathcal{E}(t)=-k\|\theta_x\|^2-\zeta\|\phi_t\|_l^2<0, \tag{3.32}$$

where

$$\mathcal{E}(t)=\frac{1}{2}\|U\|_{\mathscr{H}}^2+\int_0^L F(u,\phi,\theta)dx, \tag{3.33}$$

where $\|U\|_{\mathscr{H}}$ is defined by (3.12). From (3.32), we infer that

$$\mathcal{E}(t)\leq\mathcal{E}(0),\quad \forall t\geq 0. \tag{3.34}$$

By (3.8) and (3.15), we have

$$\int_0^L F(u,\phi,\theta)dx\geq-\beta\left(\|u\|^2+\|\phi\|^2+\|\theta\|^2\right)\geq-\beta\varrho\left(a\|u_x\|_l^2+\xi\|\phi\|_l^2+\|\theta\|^2\right)\geq-\beta\varrho\|U\|_{\mathscr{H}}^2. \tag{3.35}$$

Substituting (3.35) into (3.33), it follows that

$$\mathcal{E}(t)\leq(\frac{1}{2}-\beta\varrho)\|U\|_{\mathscr{H}}^2.$$

From (3.16), we deduce that $\frac{1}{4} \leq \beta_0 = \frac{1}{2} - \beta\varrho < \frac{1}{2}$ is positive. This implies that

$$\begin{aligned} \|U\|^2_{\mathscr{H}} &\leq \frac{1}{\beta_0}\mathcal{E}(t) \quad \text{(use (3.34))} \\ &\leq \frac{1}{\beta_0}\mathcal{E}(0). \end{aligned} \tag{3.36}$$

Hence, we conclude $\|U\|_{\mathscr{H}} < \infty$ for any $t \geq 0$, which implies that $t_{\max} = \infty$. This concludes the proof of item (i) of Theorem 3.1.

(ii) On the other hand let $T > 0$, for any $t \in (0, T)$, we consider two mild solutions $U^1(t)$ and $U^2(t)$ with initial data $U^1(0)$ and $U^2(0)$, respectively. Let us also assume that $\|U^i\|_{\mathscr{H}} \leq R$. Then, using (3.28),

$$\left\|U^1(t) - U^2(t)\right\|_{\mathscr{H}} \leq \left\|e^{-(\mathscr{A}+\mathscr{B})t}(U^1(0) - U^2(0))\right\|_{\mathscr{H}} + \int_0^t \left\|e^{-(\mathscr{A}+\mathscr{B})(t-s)}\Big(\mathcal{F}(U^1(s)) - \mathcal{F}(U^2(s))\Big)\right\|_{\mathscr{H}} ds.$$

Using the local Lipschitz property of $\mathcal{F}$ and (3.36), we obtain

$$\|U^1(t) - U^2(t)\|_{\mathscr{H}} \leq \|U^1(0) - U^2(0)\|_{\mathscr{H}} + C_{R,T}\int_0^t \|U^1(s) - U^2(s)\|_{\mathscr{H}} ds.$$

Then, the application of Gronwall's lemma gives (3.29) and the continuous dependence of the mild solution on the initial data follows immediately. This concludes the proof of item (ii) of Theorem 3.1.

(iii) Finally, as noticed above, from Theorem 1.5, p.192 in Pazy [29], any mild solution with initial data in $\mathcal{D}(\mathscr{A} + \mathscr{B})$ is strong. This proves the item (iii) of Theorem 3.1 and completes the proof. □

Remark 3.2. *Since $\mathscr{A} + \mathscr{B}$ is maximal monotone, then $-(\mathscr{A} + \mathscr{B})$ is maximal dissipative. Thus it generates a family $S(t) : \mathscr{H} \to \mathscr{H}$, $t \geq 0$, of nonlinear operators given by $S(t)U_0 = U(t)$ for any $U_0 \in \mathscr{H}$, where $U(t)$ is the unique weak solution of problem (2.33)–(2.35). It is not difficult to show that $S(t)$ satisfies the semigroup property. By using (3.29), one can establish that $S(t)$ is a continuous operator for all $t \geq 0$, and since $t \longmapsto S(t)U_0 = U(t)$ is continuous for all $U_0 \in \mathscr{H}$, it follows that $S(t)$ is a C_0−semigroup.*

4 Exponential stability and lack of analyticity

To show the exponential decay of solutions of (2.33)–(2.35), we use the following well-known result due to Gearhart-Herbst-Prüss-Huang for dissipative systems (see, e.g. [14, 30]).

Theorem 4.1. *Let $S(t) = e^{\mathcal{A}t}$ be a C_0–semigroup of contractions on Hilbert space $\mathcal{H}$. Then, $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} := \{i\lambda;\ \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A}), \qquad \limsup_{|\lambda|\to\infty} ||(i\lambda I - \mathcal{A})^{-1}||_{\mathscr{L}(\mathscr{H})} < \infty,\ \lambda \in \mathbb{R}, \tag{4.1}$$

where $\rho(\mathcal{A})$ is the resolvent set of the differential operator $\mathcal{A}$.

On the other hand, we use the following characterization of analytic semigroups (see, e.g. [14, 30]).

Theorem 4.2. *Let $S(t) = e^{\mathcal{A}t}$ be a C_0–semigroup of contractions on Hilbert space $\mathcal{H}$. Then, $S(t)$ is of analytic type if and only if*

$$i\mathbb{R} := \{i\lambda;\ \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A}), \qquad \limsup_{|\lambda|\to\infty} ||\lambda(i\lambda I - \mathcal{A})^{-1}||_{\mathscr{L}(\mathcal{H})} < \infty,\ \lambda \in \mathbb{R}. \tag{4.2}$$

4.1 Exponential stability

Our analysis is based on Theorem 4.1, which states that the semigroup is exponentially stable if and only if the imaginary axis belongs to the resolvent set (see assertion $(4.1)_1$) and the resolvent operator is uniformly bounded on the imaginary axis (see assertion $(4.1)_2$).

Lemma 4.1. *The condition* $(4.1)_1$ *holds, i.e.,* $i\mathbb{R} \subset \rho(\mathscr{A}+\mathscr{B})$.

Proof. According to the arguments given by Liu and Zheng ([25], page 25), the proof consists in three steps that can be found in many articles dealing with Theorem 4.1. We omit the details here for brevity because these steps are always the same (see e.g. [8, 26]). In the following we limit ourselves to the essential part which concerns our system. Therefore there exists a sequence of real numbers α_n, with $\alpha_n \to \chi$, $|\alpha_n| < |\chi|$, and a sequence of unit norm vectors in the domain of $\mathscr{A}+\mathscr{B}$, $U_n = (u_n, \phi_n, v_n, \psi_n, \theta_n)$, such that

$$\|(i\alpha_n I + \mathscr{A} + \mathscr{B})U_n\|_{\mathscr{H}} \to 0, \quad as\ n \to \infty. \tag{4.3}$$

Writing (4.3) term by term we get after some manipulations:

$$\begin{aligned} i\alpha_n u_n - v_n &\to 0 \text{ in } H^2(0,L)\cap H_0^1(0,L),\\ i\alpha_n \phi_n - \psi_n &\to 0 \text{ in } H_0^2(0,L),\\ i\alpha_n R_\varpi v_n - R_l\Big(a u_{nxx} + b\phi_{nx} - \beta\theta_{nx}\Big) &\to 0 \text{ in } H_0^1(0,L),\\ i\alpha_n R_\varpi \psi_n - R_l\Big(\gamma\phi_{nxx} - \xi\phi_n - b u_{nx} - \zeta\psi_n + m\theta_n\Big) &\to 0 \text{ in } H_0^1(0,L),\\ i\alpha_n R_l\theta_n + R_l\Big(\beta v_{nx} + m\psi_n\Big) - k\theta_{nxx} &\to 0 \text{ in } H_*^1(0,L). \end{aligned} \tag{4.4}$$

Since

$$\Re e < (i\alpha_n I + \mathscr{A} + \mathscr{B})U, U >_{\mathscr{H}} = -k\|\theta_{nx}\|^2 - \zeta\|\psi_n\|_l^2 \to 0, \tag{4.5}$$

we infer from (3.7) that $\theta_{nx},\ \psi_n,\ \psi_{nx} \to 0$ in $L^2(0,L)$. By $(4.4)_2$, we have $\phi_n,\ \phi_{nx} \to 0$ in $L^2(0,L)$ and from Poincaré's inequality, $\theta_n \to 0$ in $L^2(0,L)$.

Considering the inner product of $(4.4)_4$ times ϕ_n we obtain, after removing the terms which tend to zero, that

$$\gamma\|\phi_{nx}\|_l^2 + b\langle u_{nx}, \phi_n\rangle_l \to 0.$$

Hence, keeping in mind that $\phi_n,\ \phi_{nx} \to 0$ in $L^2(0,L)$ and the fact that $\|u_{nx}\|_l$ is bounded, we get

$$\|\phi_{nx}\|_l^2 \to 0.$$

Then $\phi_{nxx} \to 0$ in $L^2(0,L)$ and by $(4.4)_2$, $\psi_{nxx} \to 0$ in $L^2(0,L)$.

Next, we take the inner product of $(4.4)_4$ with u_{nx} we obtain, after removing the terms which tend to zero, that

$$-\gamma\langle\phi_{nxx}, u_{nx}\rangle_l + b\|u_{nx}\|_l^2 \to 0.$$

Next, we apply integration by parts for the first term of the aforementioned equation,

$$\gamma\langle\phi_{nx}, u_{nxx}\rangle_l - \gamma\Big[\phi_{nx}\overline{u}_{nx}\Big]_0^L - \gamma l^2\Big[\phi_{nxx}\overline{u}_{nxx}\Big]_0^L + b\|u_{nx}\|_l^2 \to 0.$$

As $\phi_{nx},\ \phi_{nxx} \to 0$ in $L^2(0,L)$ and u_{nxx} and u_{nxxx} are bounded (because of $(4.4)_{3,4}$), we have

$$-\gamma\Big[\phi_{nx}\overline{u}_{nx}\Big]_0^L - \gamma l^2\Big[\phi_{nxx}\overline{u}_{nxx}\Big]_0^L + b\|u_{nx}\|_l^2 \to 0.$$

The first and second boundary terms of the equation mentioned above vanish for the boundary conditions $(2.34)_{1,2}$. We then conclude that

$$\gamma\Big[\phi_{nx}\overline{u}_{nx}\Big]_0^L + \gamma l^2\Big[\phi_{nxx}\overline{u}_{nxx}\Big]_0^L \to 0,$$

and therefore

$$\|u_{nx}\|_l^2 \to 0. \tag{4.6}$$

Then, we take the inner product of $(4.4)_3$ with u_n,

$$\langle i\alpha_n v_n, u_n\rangle_{\varpi} + a\|u_{nx}\|_l^2 - b\langle\phi_{nx}, u_n\rangle_l - \beta\langle\theta_n, u_{nx}\rangle_l \to 0.$$

Using $(4.4)_1$, (4.6) and previous results, it is clear that

$$\|v_n\|_{\varpi}^2 \to 0.$$

Thus, we have a contradiction since U_n can not be of unit norm and, in consequence, the first condition of Theorem 4.1 holds. □

Lemma 4.2. *The condition* $(4.1)_2$ *holds, i.e.,* $\limsup_{|\alpha|\to\infty} ||(i\alpha I + \mathscr{A} + \mathscr{B})^{-1}||_{\mathscr{L}(\mathscr{H})} < \infty,\ \alpha \in \mathbb{R}.$

Proof. We prove this by a contradiction argument. Suppose that $(4.1)_2$ is not true. Then there exists a sequence α_n with $|\alpha_n| \to \infty$ and a sequence of complex functions $U_n = (u_n, \phi_n, v_n, \psi_n, \theta_n)$ in the domain of the operator $\mathscr{A}$ with unit norm such that (4.4) holds. Taking the real part of their inner product with U_n in the Hilbert space yields conditions (4.5). By (4.5) and the Poincaré inequality, we get

$$\theta_{nx},\ \theta_n,\ \psi_n,\ \psi_{nx}\ \to 0 \quad \text{in } L^2(0,L). \tag{4.7}$$

Taking the inner product of $(4.4)_5$ times $\alpha_n^{-1}u_{nx}$ yields

$$i\langle\theta_n, u_{nx}\rangle_l + i\beta\|u_{nx}\|_l^2 + m\langle\psi_n, \alpha_n^{-1}u_{nx}\rangle_l - k\langle\theta_{nxx}, \alpha_n^{-1}u_{nx}\rangle \to 0. \tag{4.8}$$

Since (4.7) holds, $\|u_{nx}\|_l$ and $\|\alpha_n^{-1}u_{nx}\|_l$ are bounded, we get after integrating by parts the last term,

$$i\beta\|u_{nx}\|_l^2 + k\langle\theta_{nx}, \alpha_n^{-1}u_{nxx}\rangle \to 0,$$

where the boundary condition $(2.34)_3$ is used. As $\theta_{nx} \to 0$ and since $\|\alpha_n^{-1}u_{nxx}\|$ is bounded (because $(4.4)_5$), we infer that

$$\|u_{nx}\|_l^2 \to 0, \tag{4.9}$$

and

$$\alpha_n^{-1}\|v_{nx}\|_l \to 0. \tag{4.10}$$

Taking the inner product of $(4.4)_3$ times v_n and dividing by α_n, one obtains that

$$i\|v_n\|_{\varpi}^2 + a\langle u_{nx}, \alpha_n^{-1}v_{nx}\rangle_l + b\langle \phi_n, \alpha_n^{-1}v_{nx}\rangle_l - \beta\langle \theta_n, \alpha_n^{-1}v_{nx}\rangle_l \to 0. \tag{4.11}$$

Since $\|u_{nx}\|_l$, $\|\phi_n\|_l$ and $\|\theta_n\|_l$ are bounded, we infer from (4.10) that

$$\|v_n\|_{\varpi} \to 0. \tag{4.12}$$

Taking the inner product of $(4.4)_4$ times ϕ_n yields

$$i\langle \psi_n, \alpha_n\phi_n\rangle_{\varpi} + \gamma\|\phi_{nx}\|_l^2 + \xi\|\phi_n\|_l^2 + b\langle u_{nx}, \phi_n\rangle_l + \zeta\langle \psi_n, \phi_n\rangle_l - m\langle \theta_n, \phi_n\rangle_l \to 0. \tag{4.13}$$

Since $\psi_n,\ \psi_{nx} \to 0$, we have from $(4.4)_1$ that $\alpha_n\phi_n,\ \alpha_n\phi_{nx} \to 0$. Moreover, as $\|\phi_n\|_l$ is bounded, in view of (4.7) and (4.9), we obtain that

$$\gamma\|\phi_{nx}\|_l^2 + \xi\|\phi_n\|_l^2 \to 0. \tag{4.14}$$

Since $\gamma,\ \xi > 0$ (see (2.36)), (4.14) implies that $\|\phi_{nx}\|_l$ and $\|\phi_n\|_l$ tend to zero in $L^2(0, L)$. But these behaviors contradict that the sequence U_n has norm unity. □

Now, we are in a position to prove the following

Lemma 4.3. *The operator $\mathscr{A} + \mathscr{B}$ generates a C_0-semigroup of contractions which is exponentially stable.*

Proof. From Lemmas 4.1 and 4.2, we can use the result due to Gearhart (see, e.g. [14, 30]) which states that a semigroup of contractions on a Hilbert space is exponentially stable if and only if conditions (4.1) hold. Thus the theorem is proved. □

4.2 Lack of analyticity

Often in one dimensional setting, the non-uniform boundedness of the resolvent operator is shown by giving an explicit sequence of exact solutions of the system. As the boundary conditions (2.34) are not adequate to propose exact solutions to (2.33), we will consider the particular case $f_i = 0,\ i = 1, 2, 3$ and $\gamma = 0$ to consider the following system in the domain $\Omega = (0, L) \times \mathbb{R}^+$:

$$\begin{aligned} (1 - \varpi^2\partial_{xx})u_{tt} - (1 - l^2\partial_{xx})\Big(au_{xx} + b\phi_x - \beta\theta_x\Big) &= 0, \\ (1 - \varpi^2\partial_{xx})\phi_{tt} - (1 - l^2\partial_{xx})\Big(-\xi\phi - bu_x - \zeta\phi_t + m\theta\Big) &= 0, \\ (1 - l^2\partial_{xx})\Big(\theta_t + \beta u_{xt} + m\phi_t\Big) - k\theta_{xx} &= 0, \end{aligned} \tag{4.15}$$

with the following boundary conditions

$$u(x, t) = u_{xx}(x, t) = \phi_x(x, t) = \theta_x(x, t) = 0, \quad \text{on} \ \ x = 0, L, \quad t > 0, \tag{4.16}$$

and the initial condition (2.35).

The problem (4.15), (4.16) and (2.35) is more suitable for the applied method and is well-posed over $\mathscr{K} = (H^2(0,L)\cap H_0^1(0,L))\times H_*^1(0,L)\times H_0^1(0,L)\times L_*^2(0,L)\times L_*^2(0,L)$. We will show the following

Theorem 4.3. *Let $S(t)$ be the C_0-semigroup of contractions on the Hilbert space $\mathscr{K}$ associated with the system* (4.15), (4.16) *and* (2.35). *Then $S(t)$ is not analytic.*

Proof. Taking into account Theorem 4.2 and Lemma 4.3, to prove our statement, we will argue by contradiction, that is, we will show that there exists a sequence of real number α_n with $\lim_{n\to\infty}|\alpha_n| = \infty$ and $U_n = (u_n, \phi_n, v_n, \psi_n, \theta_n) \in \mathcal{D}(\mathscr{A}_0+\mathscr{B})$ for $F_n = (f_{1n}, f_{2n}, f_{3n}, f_{4n}, f_{5n})^T \in \mathscr{K}$ with $\|F_n\|_{\mathscr{K}} < 1$ such that

$$i\alpha_n U_n + (\mathscr{A}_0+\mathscr{B})U_n = F_n,$$

where $\mathscr{A}_0$ is the linear operator associated to the system (4.15) given by (3.18) for $\gamma = 0$. The vector F_n is bounded in $\mathscr{K}$, but $\|U_n\|_{\mathscr{K}}$ tends to infinity. Rewriting the above spectral equation in term of its components, we have

$$\begin{cases} i\alpha_n u_n - v_n = f_{1n}, \\ i\alpha_n\phi_n - \psi_n = f_{2n}, \\ i\alpha_n R_\varpi v_n - R_l\Big(a u_{nxx} + b\phi_{nx} - \beta\theta_{nx}\Big) = R_\varpi f_{3n}, \\ i\alpha_n R_\varpi \psi_n - R_l\Big(-\xi\phi_n - b u_{nx} - \zeta\psi_n + m\theta_n\Big) = R_\varpi f_{4n}, \\ i\alpha_n R_l\theta_n + R_l\Big(\beta v_{nx} + m\psi_n\Big) - k\theta_{nxx} = R_l f_{5n}. \end{cases} \tag{4.17}$$

We choose for all $n\in\mathbb{N}$, $F_n = (0, 0, \sin(n\nu x), 0, 0)$. Then F_n is bounded in $\mathscr{K}$. On the basis of the boundary conditions (4.16) we can try solutions of the type

$$u_n = A_n\sin(n\nu x),\ \ \phi_n = B_n\cos(n\nu x),\ \ \theta_n = C_n\cos(n\nu x), \tag{4.18}$$

where $\nu = \dfrac{\pi}{L}$, A_n, B_n and C_n are unknown functions. Substituting (4.18) into (4.17), we find that A_n, B_n and C_n satisfy

$$\begin{cases} \Big(-\alpha_n^2\Xi_\varpi + a\Xi_l n^2\nu^2\Big)A_n + b\Xi_l n\nu B_n - \beta\Xi_l n\nu C_n = \Xi_\varpi, \\ b\Xi_l n\nu A_n - \Big(\alpha_n^2\Xi_\varpi - \Xi_l(\xi + i\alpha_n\zeta)\Big)B_n - m\Xi_l C_n = 0, \\ i\alpha_n\beta\Xi_l n\nu A_n + i\alpha_n m\Xi_l B_n + (i\alpha_n\Xi_l + kn^2\nu^2)C_n = 0, \end{cases} \tag{4.19}$$

where $\Xi_\epsilon = 1+\epsilon^2 n^2\nu^2$. We choose α_n such that

$$-\alpha_n^2\Xi_\varpi + a\Xi_l n^2\nu^2 = -\alpha_n^2(1+\varpi^2 n^2\nu^2) + a(1+l^2n^2\nu^2)n^2\nu^2 = 0,$$

which implies that

$$\alpha_n = \pm n\nu\sqrt{a}\sqrt{\frac{1+l^2n^2\nu^2}{1+\varpi^2n^2\nu^2}}.$$

Moreover, (4.19) leads to

$$A_n = \frac{N_n^1}{D_n},\quad B_n = \frac{N_n^2}{D_n},\quad C_n = \frac{N_n^3}{D_n},$$

where

$$\begin{aligned}
N_n^1 &= -\Xi_\varpi\Big[\Big(\alpha_n^2\Xi_\varpi - \Xi_l(\xi + i\alpha_n\zeta)\Big)\Big(i\alpha_n\Xi_l + kn^2\nu^2\Big) - i\alpha_n m^2\Xi_l^2\Big] \sim -i\alpha_n^3\Xi_l\Xi_\varpi^2 \text{ as } n \to \infty,\\
N_n^2 &= -\Xi_\varpi\Xi_l n\nu\Big(i\alpha_n b\Xi_l + kbn^2\nu^2 + i\alpha_n\beta m\Xi_l\Big) \sim -i\alpha_n(b+\beta m)\Xi_\varpi\Xi_l^2 n\nu \text{ as } n \to \infty,\\
N_n^3 &= i\alpha_n\Xi_\varpi\Xi_l n\nu\Big(mb\Xi_l + \beta\alpha_n^2\Xi_\varpi - \beta\Xi_l\xi - i\beta\Xi_l\alpha_n\zeta\Big) \sim i\alpha_n^3\Xi_\varpi^2\Xi_l n\nu\beta \text{ as } n \to \infty,\\
D_n &= -b(\Xi_l n\nu)^2\Big(i\alpha_n b\Xi_l + kbn^2\nu^2 + i\alpha_n\beta m\Xi_l\Big) - i\alpha_n\beta n^2\nu^2\Xi_l^2\Big(mb\Xi_l + \beta\alpha_n^2\Xi_\varpi - \beta\Xi_l\xi - i\beta\Xi_l\alpha_n\zeta\Big)\\
&\sim -i\alpha_n^3\beta^2 n^2\nu^2\Xi_l^2\Xi_\varpi \text{ as } n \to \infty.
\end{aligned}$$

The asymptotic behavior of the coefficients will give us

$$A_n \sim \Big(\frac{\varpi}{\beta l n\nu}\Big)^2, \qquad B_n \sim \frac{b\varpi^2 + \beta m\varpi^2}{an^3\nu^3 l^2\beta^2}, \quad C_n \sim \frac{\varpi^2}{\beta n\nu l^2}, \quad \text{as } n \to \infty.$$

This gives $u_n(x) \sim \Big(\frac{\varpi}{\beta l n\nu}\Big)^2 \sin(n\nu x)$ as $n \to \infty$. On the other hand, we have

$$\|U_n\|_{\mathscr{H}}^2 \geq al^2\|u_{nxx}\|^2 = al^2\Big(\frac{\varpi}{\beta l}\Big)^4\int_0^L |\sin(n\nu x)|^2 dx \sim \frac{aL\varpi^4}{2\beta^4 l^2}, \quad \text{as } n \to \infty. \tag{4.20}$$

Thus, we obtain $\lim_{\alpha_n\to\infty}\|\alpha_n U_n\|_{\mathscr{H}} = \infty$, which completes the proof of the theorem. □

Remark 4.1. *The case $\gamma = 0$ is considered only to use the boundary conditions* (4.16) *which is more adapted to the above method than (2.34). We note that, since the associated semigroup $S(t)$ is not analytic for $\gamma = 0$, then it is not analytic for $\gamma \neq 0$ either.*

5 Conclusion

We summarize the obtained results as follows:

(i) In this paper we have derived a nonlocal theory for thermoelastic materials with voids in the context nonlocal heat conduction combined with nonlocal Mindlin's strain gradient model. In particular, the nonlocal heat conduction (2.23) is derived to account for the heat conduction at nanoscale, which is verified by the experimental results of thin nanowires as it captured the linear dependence of the effective thermal conductivity on their radiuses [34]. By comparison with other nonlocal [4, 6, 7] or local strain gradient theories [1, 2, 19, 20], the model proposed in this paper is more reasonable in predicting the transient, porous thermoelastic response of nano-structural materials. This work, which has not been obtained in any reference yet, will be useful in predicting and better understanding the thermoelastic response of nano-structural materials which have attracted considerable attention due to their many important technological applications.

(ii) The well-posedness of the derived model in one-dimensional setting was proved. The exponential stability and analyticity issues were discussed as well. By an approach based on the Gearhart-Herbst-Prüss-Huang theorem, we prove that the associated semigroup is exponentially stable but not analytic. Hence the introduction of the nonlocal parameters ϖ and l together with porous and thermal dissipation make the solutions decay exponentially without the need to any damping dissipative mechanisms. This improve the applications of such a material to the real-world situations. Analyticity results are often of critical importance in application to control theory. To make the solutions analytic, we need to add to our system more dissipative mechanisms.

(iii) It should be pointed out that the extension of the exponential stability results from the one-dimensional case to higher dimensional case is nontrivial. This question is difficult from a mathematical point of view and remains open. The important question is whether or not this exponential stability will be preserved in multi-dimensional problem. Just in the one-dimensional case the thermal and porous dissipations become sufficiently strong to pull the mechanical energy rapidly. From Aouadi *et al.* [1, 2], one would expect that a frictional damping for the elastic component can lead to exponential stability in a multi-dimensional setting.

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