

PREDATORS AS A POSSIBLE STRATEGY FOR CONTROLLING A *XYLELLA* EPIDEMIC?

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Abstract. In Southern Italy, since 2013, there has been an ongoing Olive Quick Decline Syndrome (OQDS) outbreak, due to the bacterium *Xylella fastidiosa*, which has caused a dramatic impact from both socio-economic and environmental points of view.

Current agronomic practices are mainly based on uprooting the sick olive trees and their surrounding ones, with later installment of olive cultivars more resistant to the bacterium infection. Unfortunately, both of these practices are having an undesirable impact on the environment and on the economy. Here, a spatially structured mathematical model has been proposed to include a predator *Zelus renardii* as a possible biocontrol agent of the *Xylella* epidemic. The fact that *Z. renardii* has been reported to be a generalist predator implies that its introduction is not an efficient control strategy to eradicate a *Xylella* epidemic. Instead, a specialist predator, whenever identified, would lead to the eventual eradication of a *Xylella* epidemic. In either cases it has been confirmed that a significant reduction of the weed biomass can lead to the eradication of the vector population, hence of a *Xylella* epidemic, independently of the presence of predators.

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1. INTRODUCTION

The etiological agent of the olive quick decline syndrome (OQDS), is the vector borne bacterial disease *Xylella fastidiosa*.

The main vector of *X. fastidiosa* in Southern Italy has been identified in the so called meadow spittlebug, i.e. *Philaenus spumarius* (Hemiptera, Aphrophoridae), a xylem sap-feeding specialist. Their juvenile forms (nymphs) develop on weeds or ornamental plants, confined in a foam shelter produced by themselves for protection from predators and temperature, while their adult forms move to the tree canopy [19], [1], [20], [12], [10].

Once a plant is infected, bacteria multiply within the xylem vessels and the flux of water results inhibited by a bacterial microfilm. Typical symptoms are leaf scorch, dieback of twigs, branches and even of the whole plant [9], [21]. Possible sanitation of infected olive trees has not been validated yet. Additional details have been reported in [3].

In a series of papers by the same research team ([7], [5], [3]), motivated by the outbreak of the Olive Quick Decline Syndrome (OQDS) occurring in Southern Italy, an epidemiological model has been proposed describing the evolution of the main three players, i.e. the insect vector, *P. spumarius*, the olive trees, and weeds. Since the feeding behaviour and metabolic processes are qualitatively similar for both nymphs and adults [13], we have considered only one stage of active vectors of the infection.

A detailed mathematical analysis and related numerical simulations, reported in the above mentioned papers, have shown the key role played by the removal of weeds (food resources for juvenile insects) from olive orchards and surrounding areas. In addition, as expected, the adoption of more resistant olive tree cultivars has been shown to be a good strategy, though less cost effective, in controlling the pathogen.

In recent papers ([14], [17]) *Zelus renardii* (Hemiptera, Reduviidae) has been identified as a predator of *P. spumarius* for a possible control of a *Xylella* epidemic. This fact has motivated an extension of the previous spatially structured mathematical model, presented in [3], to include the dynamical behavior of the predator. The fact that *Z. renardii* has been reported to be a generalist predator implies the choice of an Holling type III functional response of predation in the mathematical model. As a consequence, it has been shown that the introduction of *Z. renardii* as a predator of *P. spumarius* is not an efficient control strategy to eradicate a *Xylella* epidemic. Instead, a specialist predator or of a parasitoid, whenever identified, would lead to the eventual eradication of a *Xylella* epidemic; as a matter of fact, in this case the appropriate choice for the predation functional response would be an Holling type II.

In either cases it has been confirmed that a significant reduction of the weed biomass can lead to the eradication of the vector population, independently of the presence of predators. A relevant contribution of our approach consists of the restriction of measures of intervention (control) only to a subregion of the whole habitat of interest.

All of the above has been illustrated by a set of computational experiments, within a variety of different possible parameter scenarios.

A warning has emerged concerning weed cut in the relevant habitat: it may negatively affect the efficiency of a predator, due an expected modification of the ecosystem.

On the other hand the infusion of an alien predator may have a catastrophic impact on the relevant ecosystem (see e.g. <https://www.cno.it/zelus-renardii-il-suo-impiego-preoccupa-gli-scientiati>)

This work may suggest possible model-driven real experiments for providing accurate estimates of all parameters of the model. Once these would be made available, adequate computational experiments may provide predictions on the time behavior of the relevant agroecosystem, under various experimental scenarios.

Our final goal is to analyze optimal control problems which may possibly lead to the identification of an optimal choice of the control parameters and possibly an optimal subregion of intervention. This would require the definition of suitable cost functionals, including all active and passive costs of possible agronomic practices,

This is left to future investigations.

The paper is organized as follows. In Section 2 the mathematical model is presented. In Section 3 a possible bio-control strategy has been presented, based on the inflow of a predator in the relevant ecosystem. In Subsection 3.1 the impact of a specialist predator has been analyzed, while the impact of a generalist predator has been analyzed in Subsection 3.2. The mathematical difference between the two cases is represented by a different predation functional response: Holling type II for a specialist predator; Holling type III for a generalist predator.

In Section 4 numerical simulations are presented which confirm the analytical results. In the numerical simulations the relevant parameters have been taken from [7]. Section 5 contains some concluding remarks. A comparison result for parabolic equations is presented in the Appendix.

2. THE MATHEMATICAL MODEL

We will consider a spatially structured model including the population of vector insects (*Philaenus spumarius*), the population of predator insects (*Zelus renardii*) and the population of infected olive trees.

The relevant ecoagronomical habitat will be described by a spatial domain $\Omega \subset \mathbb{R}^2$.

- VECTOR - *Philaenus spumarius*.

As far as the vector insect population is concerned, we will denote by $s_1(x, t)$ the spatial density of susceptible individuals, and by $i_1(x, t)$ the spatial density of infected individuals. The spatial density of the total population of *P. spumarius* will then be

$$C_I(x, t) = s_1(x, t) + i_1(x, t).$$

In absence of predators, in [3] the evolution dynamics of $s_1(x, t)$ and $i_1(x, t)$ has been assumed to be described by the following equations, respectively, subject to suitable initial and boundary conditions (parameters are described in Table 1):

$$\frac{\partial s_1}{\partial t}(x, t) = d\Delta s_1(x, t) + rC_I(x, t)[M(x) - \chi s_1(x, t)] - ns_1(x, t) - \beta s_1(x, t) \int_{\Omega} k(x, x')i_2(x', t)dx' \quad (1)$$

$$\frac{\partial i_1}{\partial t}(x, t) = d\Delta i_1(x, t) - ni_1(x, t) - r\chi i_1(x, t)C_I(x, t) + \beta s_1(x, t) \int_{\Omega} k(x, x')i_2(x', t)dx', \quad (2)$$

for $x \in \Omega$ and $t > 0$.

As a consequence, the evolution equation for the total insect population $C_I(x, t)$, in absence of predators, is given by

$$\frac{\partial C_I}{\partial t}(x, t) = d\Delta C_I(x, t) + rC_I(x, t)[M(x) - \chi C_I(x, t)] - nC_I(x, t). \quad (3)$$

We have assumed that the carrying capacity of all insects depends upon the environmental parameter $M(x)$, $x \in \Omega$, that had been already introduced in [7], [3], [5] to describe the biomass of all other plants, such as bushes, ornamental plants, etc. that we will simply call *weeds*, either healthy or infected; they are required for the proper development of all insects in the relevant ecosystem.

For the explanation of all other terms in the above equations we refer to [3] (see also [8], [2], and references therein).

- PREDATORS

As far as the population of predators is concerned, it is denoted by $Z(x, t)$; its evolutionary dynamics, in absence of a population of *P. spumarius*, may be described as follows (parameters are described in Table 1):

$$\frac{\partial Z}{\partial t}(x, t) = d_2 \Delta Z(x, t) + r_2 Z(x, t) \left[1 - \frac{Z(x, t)}{K(x) + cM(x)} \right], \quad (4)$$

for $x \in \Omega, t > 0$, subject to suitable boundary and initial conditions.

We have assumed that the carrying capacity of the predators, in addition to a general parameter $K(x)$, depends upon the additional environmental parameter $M(x), x \in \Omega$, as mentioned above.

In order to describe mathematically the contribution of *P. spumarius* to the dynamics of the predator population, we have to take into account that according to what reported in [14], *Z. renardii* is a generalist predator, and its efficiency in the predation *P. spumarius* is rather low. In [22, p.83] (see also [4], [15, p.38]) it was evidenced that the functional response $g(C_I)$ of a generalist predator to a specific prey follows an S-shaped curve (Holling type III), according to which the predator dedicates its predation only to large quantities of any specific prey C_I . In our case this means that in Equation (4) as additional term describing predation it should be of the form

$$g_{III}(C_I(x, t))Z(x, t) = \frac{\gamma_1(C_I(x, t))^2}{1 + \gamma_2(C_I(x, t))^2} Z(x, t), \quad (5)$$

with $\gamma_1, \gamma_2 > 0$.

Remark 1. We may remark that in case the relevant predator acts as a specialist one for the *P. spumarius*, the functional response of predation may be taken of the Holling type II form (see [22]) as follows

$$g_{II}(C_I(x, t))Z(x, t) = \frac{\gamma_1 C_I(x, t)}{1 + \gamma_2 C_I(x, t)} Z(x, t), \quad (6)$$

with $\gamma_1, \gamma_2 > 0$.

Unfortunately, to our knowledge specialist predators for the *P. spumarius* have not been identified yet. Anyhow, for a matter of completeness, in this paper both cases have been analyzed.

As a consequence of the above, if we include the impact of predation in Equation (3) we have

$$\frac{\partial C_I}{\partial t}(x, t) = d \Delta C_I(x, t) + r C_I(x, t) [M(x) - \chi C_I(x, t)] - n C_I(x, t) - pg(C_I(x, t))Z(x, t),$$

where g is either g_{II} or g_{III} introduced in (5) and (6).

Then Equations (1), (2) can be rewritten as follows

$$\begin{aligned} \frac{\partial s_1}{\partial t}(x, t) = & d \Delta s_1(x, t) + r C_I(x, t) [M(x) - \chi s_1(x, t)] - n s_1(x, t) \\ & - pg(C_I(x, t))Z(x, t) \frac{s_1(x, t)}{C_I(x, t)} - \beta s_1(x, t) \int_{\Omega} k(x, x') i_2(x', t) dx' \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{\partial i_1}{\partial t}(x, t) = & d \Delta i_1(x, t) - n i_1(x, t) - pg(C_I(x, t))Z(x, t) \frac{i_1(x, t)}{C_I(x, t)} \\ & - r \chi i_1(x, t) C_I(x, t) + \beta s_1(x, t) \int_{\Omega} k(x, x') i_2(x', t) dx'. \end{aligned} \quad (8)$$

- OLIVE TREES

For the olive trees it is better to refer to their canopies, so that we may consider pruning and regrowth. Healthy trees (canopy) are produced at constant net regrowth rate q , get infected by contact with infected insects at rate ζ , or by human activities such as budding and grafting, at rate b . For trees, in view of their long survival, we can neglect natural mortality. Infected trees experience disease-related mortality μ and human-induced mortality ℓ due to pruning and logging.

We shall denote by $s_2(x, t)$ the spatial density of the biomass of healthy trees, and by $i_2(x, t)$ the spatial density of the biomass of infected trees. As a consequence if $C_T(x, t)$ denotes the spatial density of the total biomass of olive trees

TABLE 1. Model parameters.

Symbol	Description
r	<i>P. spumarius</i> birth rate
r_2	<i>Z. renardii</i> birth rate
χ	<i>P. spumarius</i> intraspecific competition rate
n	<i>P. spumarius</i> natural mortality rate
q	Healthy trees (canopy) regrowth rate
C	Trees carrying capacity parameter
$K(x)$	Baseline <i>Z. renardii</i> carrying capacity parameter
$M(x)$	Spatial distribution of the weed biomass
c	Scale parameter of the impact of the weed biomass on the carrying capacity of <i>Z. renardii</i>
ℓ	Elimination rate of trees by pruning or logging
b	Infection rate of trees by infected tools
μ	Infected trees mortality rate
α	Infected trees recovery rate
β	<i>P. spumarius</i> infection rate by infected trees
ζ	Trees infection rate by infected <i>P. spumarius</i>

$$C_T(x, t) = s_2(x, t) + i_2(x, t).$$

The spatial dynamics of trees is expressed by the following equations ([3]):

$$\frac{\partial s_2}{\partial t}(x, t) = (q - \ell)s_2(x, t) - s_2(x, t)\frac{C_T(x, t)}{C} - (\zeta i_1(x, t) + b l i_2(x, t))s_2(x, t) + \alpha i_2(x, t). \quad (9)$$

$$\begin{aligned} \frac{\partial i_2}{\partial t}(x, t) = & -\mu i_2(x, t) - l i_2(x, t) - i_2(x, t)\frac{C_T(x, t)}{C} \\ & + (\zeta i_1(x, t) + b l i_2(x, t))s_2(x, t) - \alpha i_2(x, t). \end{aligned} \quad (10)$$

Both equations (9) and (10) act in the spatial domain $\Omega \subset \mathbb{R}^2$, at times $t \in (0, +\infty)$, and are subject to suitable initial conditions.

Assumptions:

- $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary;
- $M \in L^\infty(\Omega)$;
- $K \in L^\infty(\Omega)$, $K(x) \geq k_0$ a.e. in Ω (here k_0 is a positive constant);
- $k \in L^\infty(\Omega \times \Omega)$, $k(x, x') \geq 0$ a.e. in $\Omega \times \Omega$; normalized to β ;
- $d, r, r_2, \chi, n, q, \ell, C, c, \mu, \zeta, b, \alpha, \beta$ are positive constants.

3. CONTROL, ERADICABILITY

Our aim is to analyze the behavior of the insect vector *P. spumarius* in presence of a predator, which may act as a possible control strategy for the eradication of a *Xylella* epidemic in an ecosystem. For the habitat of the relevant ecosystem, we take a bounded domain $\Omega \subset \mathbb{R}^2$, with a sufficiently smooth boundary $\partial\Omega$.

In the evolution equation for predators we introduce an additional term $u(x) \geq 0$, possibly acting only in a suitable subregion $\omega \subset \Omega$. This control term may be understood as a continuous infusion of predators at position $x \in \omega \subset \Omega$. When we do not specify otherwise we shall take $u \in L^\infty(\omega)$, $u(x) > 0$ a.e. $x \in \omega$.

Actually the control might also be time dependent, but for the time being we will take it constant in time. Hence the dynamics of the predator population would be described by

$$\frac{\partial Z}{\partial t}(x, t) = d_2 \Delta Z(x, t) + r_2 Z(x, t) \left[1 - \frac{Z(x, t)}{K(x) + cM(x)} \right] + g(C_I(x, t))Z(x, t) + \mathbb{I}_\omega(x)u(x). \quad (11)$$

Denote $Q = \Omega \times (0, +\infty)$ and $\Sigma = \partial\Omega \times (0, +\infty)$.

Let us consider only the populations of insects, C_I for the *P. spumarius*, and Z for its predator, for a general functional response of predation:

$$\begin{cases} \frac{\partial C_I}{\partial t} = d\Delta C_I + rC_I(M - \chi C_I) - nC_I - pg(C_I(x, t))Z(x, t), & (x, t) \in Q \\ \frac{\partial Z}{\partial t} = d_2 \Delta Z + r_2 Z \left(1 - \frac{Z}{K + cM} \right) + g(C_I(x, t))Z(x, t) + \mathbb{I}_\omega u, & (x, t) \in Q \\ \frac{\partial C_I}{\partial n}(x, t) = \frac{\partial Z}{\partial n}(x, t) = 0, & (x, t) \in \Sigma \\ C_I(x, 0) = C_{I0}(x), \quad Z(x, 0) = Z_0(x), & x \in \Omega. \end{cases} \quad (12)$$

Via Banach's fixed point theorem and taking into account the comparison principle for parabolic equations (see the Appendix), we may state the following theorem.

Theorem 1. *For any $C_{I0}, Z_0 \in L^\infty(\Omega)$, $C_{I0}(x), Z_0(x) \geq 0$ a.e. $x \in \Omega$ and for any $u \in L^\infty(\omega)$, $u(x) \geq 0$ a.e. $x \in \omega$, we get that (12) has a unique nonnegative solution.*

Consider now the following eigenvalue problem.

$$\begin{cases} -d\Delta\varphi - rM\varphi + n\varphi = \lambda\varphi, & x \in \Omega \\ \frac{\partial\varphi}{\partial n}(x) = 0, & x \in \partial\Omega. \end{cases}$$

By Rayleigh's principle [6, p. 516] we have that

$$\tilde{\lambda}_1 = \inf_{\varphi \in H^1(\Omega), \|\varphi\|_{L^2(\Omega)} \neq 0} \frac{\int_\Omega (d|\nabla\varphi|^2 - rM|\varphi|^2 + n|\varphi|^2) dx}{\int_\Omega |\varphi|^2 dx} \quad (13)$$

is the principal eigenvalue for the above mentioned problem to which we may associate a positive eigenfunction $\tilde{\varphi}_1$. It is not difficult to show that, if $\tilde{\alpha}$ is a positive real number, the function

$$C_2(x, t) = \tilde{\alpha}\tilde{\varphi}_1(x) \exp(-\tilde{\lambda}_1 t)$$

is the unique solution to the following linear parabolic problem:

$$\begin{cases} \frac{\partial C}{\partial t} = d\Delta C + rMC - nC, & (x, t) \in Q \\ \frac{\partial C}{\partial n}(x, t) = 0, & (x, t) \in \Sigma \\ C(x, 0) = \tilde{\alpha}\tilde{\varphi}_1(x), & x \in \Omega \end{cases}$$

It is also clear that, if $\tilde{\lambda}_1 > 0$, then $C_2(\cdot, t) \rightarrow 0$ in $L^\infty(\Omega)$, as $t \rightarrow +\infty$.

Now, if we take $\tilde{\alpha}$ a sufficiently large constant, by comparison results for parabolic equations, we may state that

$$0 \leq C_I(x, t) \leq C_2(x, t) \quad \text{a.e. } x \in \Omega, \quad \forall t \geq 0.$$

As a consequence, we get that $C_I(\cdot, t) \rightarrow 0$ in $L^\infty(\Omega)$, as $t \rightarrow +\infty$.

On the other hand, if $\tilde{\lambda}_1 = 0$, then there exist two positive constants $\tilde{\alpha}, \bar{\alpha}$ such that

$$0 \leq C_I(x, t) \leq \frac{\tilde{\alpha}}{t + \bar{\alpha}} \tilde{\varphi}_1(x) \quad \text{a.e. } x \in \Omega, \quad \forall t \geq 0.$$

As a consequence, we get that $C_I(\cdot, t) \rightarrow 0$ in $L^\infty(\Omega)$, as $t \rightarrow +\infty$.

We may then state the following theorem.

Theorem 2. *If $\tilde{\lambda}_1 \geq 0$, then for any $u \in L^\infty(\omega)$, $u(x) \geq 0$ a.e. $x \in \omega$, we get that*

$$C_I(\cdot, t) \longrightarrow 0 \quad \text{in } L^\infty(\Omega), \text{ as } t \rightarrow +\infty.$$

This result expresses, in mathematical terms, the fact that for $\tilde{\lambda}_1 \geq 0$ the total population of vector insects goes extinct, independently of the presence of predators in the ecosystem.

It is clear that in additional presence of predators, $C_I(t)$ will decay to 0 even faster.

We may further notice that, according to (13) we may make $\tilde{\lambda}_1 \geq 0$ by reducing M , the weed biomass, which confirms the results of our previous papers ([7], [3], [5]).

Let now $\tilde{\lambda}_1 < 0$. In this case, in absence of predators we cannot claim that in general the vector population tends to zero, for large times. Additional investigations are required. In particular the qualitative behavior of System (12) is different in the two cases of specialist and generalist predators.

3.1. Specialist predators

For a specialist predator System (12) becomes

$$\begin{cases} \frac{\partial C_I}{\partial t} = d\Delta C_I + rC_I(M - \chi C_I) - nC_I - p\frac{\gamma_1 C_I}{1 + \gamma_2 C_I}Z, & (x, t) \in Q \\ \frac{\partial Z}{\partial t} = d_2\Delta Z + r_2Z\left(1 - \frac{Z}{K + cM}\right) + \frac{\gamma_1 C_I}{1 + \gamma_2 C_I}Z + \mathbb{I}_\omega u, & (x, t) \in Q \\ \frac{\partial C_I}{\partial n}(x, t) = \frac{\partial Z}{\partial n}(x, t) = 0, & (x, t) \in \Sigma \\ C_I(x, 0) = C_{I0}(x), \quad Z(x, 0) = Z_0(x), & x \in \Omega. \end{cases}$$

Consider the following nonlinear parabolic problem

$$\begin{cases} \frac{\partial Z_1}{\partial t} = d_2\Delta Z_1 + r_2Z_1\left(1 - \frac{Z_1}{K + cM}\right) + \mathbb{I}_\omega u, & (x, t) \in Q \\ \frac{\partial Z_1}{\partial n}(x, t) = 0, & (x, t) \in \Sigma \\ Z_1(x, 0) = 0, & x \in \Omega. \end{cases} \quad (14)$$

Since $u(x) > 0$ a.e. $x \in \omega$, we get that for any $t_0 > 0$, there exists $m_{t_0} > 0$ such that

$$Z_1(x, t_0) \geq m_{t_0}, \quad \text{a.e. } x \in \Omega.$$

Let us prove the following result

Theorem 3. *If $\tilde{\lambda}_1 < 0$, then the parabolic equation (14) admits a unique positive steady state Z^* , i.e. a unique positive solution of the following semilinear elliptic problem*

$$\begin{cases} d_2\Delta Z^* + r_2Z^*\left(1 - \frac{Z^*}{K + cM}\right) + \mathbb{I}_\omega u = 0, & x \in \Omega \\ \frac{\partial Z^*}{\partial n}(x) = 0, & x \in \partial\Omega. \end{cases} \quad (15)$$

Proof. Since $u(x) > 0$ a.e. $x \in \omega$, we get that for any $t_0 > 0$, there exists $m_{t_0} > 0$ such that

$$Z_1(x, t_0) \geq m_{t_0}, \quad \text{a.e. } x \in \Omega.$$

Consider a positive t_0 (fixed). Let $\tilde{Z}_{10}, \bar{Z}_{10} \in (0, +\infty)$, \tilde{Z}_{10} sufficiently small and \bar{Z}_{10} sufficiently large. The solutions \tilde{Z}_1 and \bar{Z}_1 to

$$\begin{cases} \frac{\partial Z}{\partial t} = d_2 \Delta Z + r_2 Z \left(1 - \frac{Z}{K + cM}\right) + \mathbb{I}_\omega u, & (x, t) \in \Omega \times (t_0, +\infty) \\ \frac{\partial Z}{\partial n}(x, t) = 0, & (x, t) \in \partial\Omega \times (t_0, +\infty) \\ Z(x, t_0) = \tilde{Z}_{10} / Z(x, t_0) = \bar{Z}_{10}, & x \in \Omega, \end{cases}$$

respectively, satisfy that

$$t \mapsto \tilde{Z}_1(\cdot, t) \text{ is increasing, } t \mapsto \bar{Z}_1(\cdot, t) \text{ is decreasing on } [t_0, +\infty),$$

$$0 < \tilde{Z}_1(x, t) \leq Z_1(x, t) \leq \bar{Z}_1(x, t) \quad \text{a.e. } x \in \Omega, \quad \forall t \geq t_0.$$

We may infer that

$$\tilde{Z}_1(\cdot, t) \longrightarrow Z_1^*(\cdot), \quad \bar{Z}_1(\cdot, t) \longrightarrow Z_2^*(\cdot) \quad \text{in } L^2(\Omega),$$

as $t \rightarrow +\infty$, and that $0 < Z_1^*(x) \leq Z_2^*(x)$ a.e. $x \in \Omega$. Moreover, we may infer that Z_i^* are positive solutions to (15). We immediately get that

$$r_2 \int_{\Omega} Z_1^* Z_2^* \frac{Z_2^* - Z_1^*}{K + cM} dx + \int_{\omega} u(Z_2^* - Z_1^*) dx = 0,$$

and so we conclude that $Z_1^*(x) = Z_2^*(x)$ a.e. $x \in \Omega$.

So, we may infer that there exists a unique positive solution to (15). A regularity result for the solutions to the elliptic equations implies that $Z^* \in H^2(\Omega) \subset C(\bar{\Omega})$. \square

It follows that the principal eigenvalue λ_1^* to

$$\begin{cases} -d_2 \Delta \varphi - r_2 \varphi \left(1 - \frac{Z^*}{K + cM}\right) = \lambda \varphi, & x \in \Omega \\ \frac{\partial \varphi}{\partial n}(x) = 0, & x \in \partial\Omega \end{cases}$$

is positive. On the other hand, by comparison we get that

$$0 \leq Z^*(x) - \tilde{Z}_1(x, t) \leq w(x, t) \quad \text{a.e. } x \in \Omega, \quad \forall t \geq t_0,$$

where w is the solution to

$$\begin{cases} \frac{\partial w}{\partial t} = d_2 \Delta w + r_2 w \left(1 - \frac{Z^*}{K + cM}\right), & (x, t) \in \Omega \times (t_0, +\infty) \\ \frac{\partial w}{\partial n}(x, t) = 0, & (x, t) \in \partial\Omega \times (t_0, +\infty) \\ w(x, t_0) = Z^*(x) - \tilde{Z}_{10} \geq 0, & x \in \Omega. \end{cases}$$

Since $\lambda_1^* > 0$ we get that

$$w(\cdot, t) \longrightarrow 0 \quad \text{in } L^\infty(\Omega), \quad \text{as } t \rightarrow +\infty$$

(at the rate of $\exp(-\lambda_1^* t)$), and so

$$\tilde{Z}_1(\cdot, t) \longrightarrow Z^*(\cdot) \quad \text{in } L^\infty(\Omega), \quad \text{as } t \rightarrow +\infty.$$

Analogously, we get that

$$\bar{Z}_1(\cdot, t) \longrightarrow Z^*(\cdot) \quad \text{in } L^\infty(\Omega), \quad \text{as } t \rightarrow +\infty.$$

We may then conclude that

$$Z_1(\cdot, t) \longrightarrow Z^*(\cdot) \quad \text{in } L^\infty(\Omega),$$

as $t \rightarrow +\infty$.

It follows that, for any $\varepsilon > 0$, there exists $t(\varepsilon) \geq 0$ such that

$$Z(x, t) \geq Z_1(x, t) > Z^*(x) - \varepsilon, \quad \text{a.e. } x \in \Omega, \quad \forall t \geq t(\varepsilon).$$

On the other hand it can be shown that $\forall \varepsilon > 0, \exists t_2(\varepsilon) \geq 0$, such that $C_I(x, t) \leq C^*(x) + \varepsilon$ a.e. $x \in \Omega$, $\forall t \geq t_2(\varepsilon)$, where C^* is the unique positive solution to

$$\begin{cases} d\Delta C + rC(M - \chi C) - nC = 0, & x \in \Omega \\ \frac{\partial C}{\partial n}(x) = 0, & x \in \partial\Omega. \end{cases}$$

Hence, for any $\varepsilon > 0$ sufficiently small we have that

$$p \frac{\gamma_1 C_I}{1 + \gamma_2 C_I} Z \geq p\gamma_1 C_I \frac{Z^* - \varepsilon}{1 + \gamma_2(C^* + \varepsilon)} \quad \text{a.e. } x \in \Omega,$$

$\forall t \geq \bar{t}(\varepsilon) = \max\{t(\varepsilon), t_2(\varepsilon)\}$.

If we consider C_ε the solution to

$$\begin{cases} \frac{\partial C}{\partial t} = d\Delta C + rC(M - \chi C) - nC_I - \frac{p\gamma_1(Z^* - \varepsilon)}{1 + \gamma_2(C^* + \varepsilon)}C, & (x, t) \in \Omega \times (\bar{t}(\varepsilon), +\infty) \\ \frac{\partial C}{\partial n}(x, t) = 0, & (x, t) \in \partial\Omega \times (\bar{t}(\varepsilon), +\infty) \\ C(x, \bar{t}(\varepsilon)) = C_I(x, \bar{t}(\varepsilon)), & x \in \Omega, \end{cases}$$

then the comparison principle for parabolic equations implies that

$$C_I(x, t) \leq C_\varepsilon(x, t) \quad \text{a.e. } x \in \Omega, \quad \forall t \geq \bar{t}(\varepsilon).$$

If the principal eigenvalue $\lambda_1(\varepsilon)$ to

$$\begin{cases} -d\Delta\varphi - rM\varphi + n\varphi + \frac{p\gamma_1(Z^* - \varepsilon)}{1 + \gamma_2(C^* + \varepsilon)}\varphi = \lambda\varphi, & x \in \Omega \\ \frac{\partial\varphi}{\partial n}(x) = 0, & x \in \partial\Omega, \end{cases}$$

$$\lambda_1(\varepsilon) = \inf_{\varphi \in H^1(\Omega), \|\varphi\|_{L^2(\Omega)} \neq 0} \frac{\int_{\Omega} \left(d|\nabla\varphi|^2 - rM|\varphi|^2 + n|\varphi|^2 + \frac{p\gamma_1(Z^* - \varepsilon)}{1 + \gamma_2(C^* + \varepsilon)}|\varphi|^2 \right) dx}{\int_{\Omega} |\varphi|^2 dx}, \quad (16)$$

is positive, then

$$C_\varepsilon(\cdot, t) \longrightarrow 0 \quad \text{in } L^\infty(\Omega)$$

as $t \rightarrow +\infty$, and so

$$C_I(\cdot, t) \longrightarrow 0 \quad \text{in } L^\infty(\Omega),$$

as $t \rightarrow +\infty$.

On the other hand, by Rayleigh's principle we get that

$$\lim_{\varepsilon \rightarrow 0^+} \lambda_1(\varepsilon) = \lambda_1(0).$$

To summarize, we get the following result

Theorem 4. *If $\tilde{\lambda}_1 < 0$ and if $\lambda_1(0) > 0$, then*

$$C_I(\cdot, t) \longrightarrow 0 \quad \text{in } L^\infty(\Omega), \text{ as } t \rightarrow +\infty.$$

Remark 2. According to Theorem 2, if $\tilde{\lambda}_1 \geq 0$ (which may occur for a small value of M , the weed biomass) the vector population vanishes for large times, independently of the presence of predators.

While, according to Theorem 4, for $\tilde{\lambda}_1 < 0$ eradication of the vector population is possible upon the additional information that $\lambda_1(0) > 0$.

The form of $\lambda_1(0)$ given by Rayleigh's formula (16) shows that there is a monotonicity with respect to Z^* ; on the other hand Z^* is monotonically increasing with respect to $\mathbb{I}_\omega u$ (and so in particular to ω , by inclusion).

Moreover, if $Ess \inf u \rightarrow +\infty$, we may conjecture that $Ess \inf Z^* \rightarrow +\infty$ and consequently $\lambda_1(0) \rightarrow +\infty$ (for $\omega = \Omega$ this can be proven to hold).

This means that, if the inflow of specialist predators in the relevant habitat is sufficiently large (or we extend the region of intervention ω), eradication of the total vector population is possible.

We may additionally state the following.

Corollary 1. *Under the assumptions of Theorem 4, for any $\varepsilon > 0$ sufficiently small we get that $\lambda_1(\varepsilon) > 0$ and so $C_I(\cdot, t) \rightarrow 0$ in $L^\infty(\Omega)$, as $t \rightarrow +\infty$ and the decay rate is that of $\exp(-\lambda_1(\varepsilon)t)$.*

3.2. Generalist predators

In this case System (12) has to be changed into the following one, in which an Holling type III response to predation occurs. In this case (C_I, Z) is the solution to

$$\begin{cases} \frac{\partial C_I}{\partial t} = d\Delta C_I + rC_I(M - \chi C_I) - nC_I - p \frac{\gamma_1 C_I^2}{1 + \gamma_2 C_I^2} Z, & (x, t) \in Q \\ \frac{\partial Z}{\partial t} = d_2 \Delta Z + r_2 Z \left(1 - \frac{Z}{K + cM}\right) + \frac{\gamma_1 C_I^2}{1 + \gamma_2 C_I^2} Z + \mathbb{I}_\omega u, & (x, t) \in Q \\ \frac{\partial C_I}{\partial n}(x, t) = \frac{\partial Z}{\partial n}(x, t) = 0, & (x, t) \in \Sigma \\ C_I(x, 0) = C_{I0}(x), \quad Z(x, 0) = Z_0(x), & x \in \Omega \end{cases}$$

We are considering the case in which $\tilde{\lambda}_1 < 0$, as in Theorem 4 concerning specialist predators. In that case we have been able to show that, under the additional assumption of a large inflow of predators in the relevant habitat, the vector population will eventually go extinct. For generalist predators the result is negative, as we will show in the following theorem.

Theorem 5. *If $\tilde{\lambda}_1 < 0$, then there is no $u \in L^\infty(\omega)$, $u(x) \geq 0$ a.e. $x \in \omega$ such that*

$$C_I(\cdot, t) \longrightarrow 0 \quad \text{in } L^\infty(\Omega), \text{ as } t \rightarrow +\infty.$$

Proof. We argue by contradiction. Suppose that there is a control u such that $C_I(\cdot, t) \rightarrow 0$ in $L^\infty(\Omega)$, as $t \rightarrow +\infty$, then for all $\varepsilon > 0$, there exists $t_3(\varepsilon) \geq 0$ such that

$$\varepsilon > C_I(x, t) \quad \text{a.e. } x \in \Omega, \text{ for all } t \geq t_3(\varepsilon).$$

Let us notice that the comparison principle for parabolic equations implies that

$$0 \leq Z(x, t) \leq \bar{Z}(x, t) \quad \text{a.e. } x \in \Omega, \forall t \geq 0,$$

where \bar{Z} is the unique solution to

$$\begin{cases} \frac{\partial Z}{\partial t} = d_2 \Delta Z + r_2 Z \left(1 - \frac{Z}{K + cM}\right) + \frac{\gamma_1}{\gamma_2} Z + \mathbb{I}_\omega u, & (x, t) \in Q \\ \frac{\partial Z}{\partial n}(x, t) = 0, & (x, t) \in \Sigma \\ Z(x, 0) = Z_0(x), & x \in \Omega. \end{cases}$$

We have that $\bar{Z} \in L^\infty(Q)$ and so there exists M^0 , a positive constant such that

$$\left| \frac{\gamma_1 C_I}{1 + \gamma_2 C_I^2} Z \right| \leq M^0 \quad \text{a.e. } (x, t) \in Q,$$

because the function $s \mapsto \frac{\gamma_1 s}{1 + \gamma_2 s^2}$ (defined on $[0, +\infty)$) is bounded. We may conclude (via the comparison principle) that

$$C_I(x, t) \geq C_{I1}(x, t) \quad \text{a.e. } x \in \Omega, \text{ for all } t \geq t_3(\varepsilon),$$

where C_{I1} is the unique solution to

$$\begin{cases} \frac{\partial C}{\partial t} = d \Delta C + rC(M - \chi C) - nC - p\varepsilon M^0 C, & x \in \Omega, t > t_3(\varepsilon) \\ \frac{\partial C}{\partial n}(x, t) = 0, & x \in \Omega, t > t_3(\varepsilon) \\ C(x, t_3(\varepsilon)) = C_I(x, t_3(\varepsilon)), & x \in \Omega. \end{cases}$$

Since $C_I(\cdot, t) \rightarrow 0$ in $L^\infty(\Omega)$, as $t \rightarrow 0$, we get that $\tilde{\lambda}_1 + p\varepsilon M^0$, the principal eigenvalue to

$$\begin{cases} -d \Delta \varphi - rM\varphi + n\varphi + p\varepsilon M^0 \varphi = \lambda \varphi, & x \in \Omega \\ \frac{\partial \varphi}{\partial n}(x) = 0, & x \in \partial \Omega \end{cases}$$

is positive. Since $\tilde{\lambda}_1 + p\varepsilon M^0 > 0$ for any $\varepsilon > 0$, we conclude that $\tilde{\lambda}_1 \geq 0$, which leads to a contradiction.

So, we may conclude that there is no control u such that $C_I(\cdot, t) \rightarrow 0$ in $L^\infty(\Omega)$, as $t \rightarrow +\infty$. \square

Let us evaluate, however, how much the population density C_I decays if a control u is implemented.

We have that

$$Z(x, t) \leq \bar{Z}(x, t) \quad \text{a.e. } x \in \Omega, \forall t \geq 0.$$

The comparison principle implies that

$$C_I(x, t) \geq C^I(x, t) \quad \text{a.e. } x \in \Omega, \forall t \geq 0,$$

where C^I is the unique solution to

$$\begin{cases} \frac{\partial C^I}{\partial t} = d \Delta C^I + rC^I(M - \chi C^I) - nC^I - p\gamma_1 \|\bar{Z}\|_{L^\infty(Q)} (C^I)^2, & (x, t) \in Q \\ \frac{\partial C^I}{\partial n}(x, t) = 0, & (x, t) \in \Sigma \\ C^I(x, 0) = C_{I0}(x), & x \in \Omega. \end{cases}$$

Hence, C_I cannot decay below C^I .

On the other hand, it can be shown that

$$C^I(\cdot, t) \longrightarrow \tilde{C}^*(\cdot) \quad \text{in } L^\infty(\Omega),$$

as $t \rightarrow +\infty$, where \tilde{C}^* is the positive solution to

$$\begin{cases} d\Delta C + rC \left(M - \left(\chi + \frac{p\gamma_1 \|\bar{Z}\|_{L^\infty(Q)}}{r} \right) C \right) - nC = 0, & x \in \Omega \\ \frac{\partial C}{\partial n}(x) = 0, & x \in \partial\Omega. \end{cases}$$

Hence, we may conclude that for any $\varepsilon > 0$, there exists a $T(\varepsilon) \geq 0$ such that

$$C_I(x, t) \geq \tilde{C}^*(x) - \varepsilon \quad \text{a.e. } x \in \Omega, \quad \forall t \geq T(\varepsilon).$$

The above results, in particular Theorem 2 has confirmed the peculiar role of the parameter M , representing the weed biomass existing in the relevant habitat: independently of the introduction of predators, cutting weeds has been confirmed as a cost effective practice for controlling a *Xylella* epidemic in olive orchards (see the outcomes of [3, 5, 7]).

It would be of great interest, from an agronomical point of view, to relate the mathematical threshold parameters, $\tilde{\lambda}_1$ and $\lambda_1(0)$, to some relevant parameters of intervention, such as $M(x)$ (the spatial distribution of the weed biomass) and $u(x)$ (the spatial distribution of the predators inflow). Unfortunately this is a nontrivial mathematical problem, from a purely analytical point of view. A practical response to this issue can be obtained via adequate computational experiments, as it has been reported in Section 4.

On the other hand, the values of $\tilde{\lambda}_1$ and $\lambda_1(0)$ depend, in addition to the possible parameters of intervention $M(x)$ and $u(x)$, upon intrinsic parameters of the relevant agro-ecosystem, which are not yet known numerically. As mentioned in the concluding remarks, it is hopeful that our results may stimulate additional field experiments aimed at the identification of real values of all relevant parameters.

4. NUMERICAL SIMULATIONS

The numerical strategy adopted to approximate the controlled system (Equations (7)-(11)) consists of the finite element method for space discretization and the finite difference method for time discretization. This procedure is the state-of-the-art for the solution of parabolic Partial Differential Equations (PDEs); see e.g. [18].

Space discretization. We first apply a standard Galerkin procedure to the weak formulations of the controlled system (Equations (7)- (11)). As computational domain Ω we have taken a two-dimensional slab of size $400 \times 80 \text{ Km}^2$, which mimics e.g. the whole region of Apulia in Southern Italy, from South (right hand side of the domain) to North (left hand side of the domain). The rectangular domain has been discretized by a uniform grid of 200×40 bi-linear finite elements ($Q1$), yielding a total amount of 8,241 discretization nodes. The stiffness matrix is computed exactly, whereas the mass matrix is obtained by applying the mass lumping technique.

Time discretization. After space discretization by finite elements, we obtain the semi-discrete problem that consists of a system of ordinary differential equations (ODEs). We solve these ODEs by employing a first order semi-implicit finite difference scheme, where the linear diffusion terms are approximated by Backward Euler, whereas the non-linear reaction terms are approximated by Forward Euler. As a result, at each time step it is required the solution of a linear system of algebraic equations of dimension $5 \times 8,241 = 41,205$ degrees of

freedom. The linear system is solved by Gaussian elimination with the built-in function of Matlab. The time step size is $\Delta t = 0.0002$ years.

For further details on the numerical discretization of parabolic problems we refer e.g. to [18].

4.1. Parameter calibration

In the computational experiments, we have considered the following five scenarios:

- (1) case A1: $M = 0.4$, $u = 2 \cdot 10^5$, Holling II;
- (2) case A2: $M = 0.4$, $u = 2 \cdot 10^5$, Holling III;
- (3) case B11: $M = 1$, $u = 2 \cdot 10^4$, Holling II;
- (4) case B12: $M = 1$, $u = 2 \cdot 10^5$, Holling II;
- (5) case B22: $M = 1$, $u = 2 \cdot 10^5$, Holling III.

The predation inflow u has been applied only in the region $\omega = \{x_1 > 250\}$.

In the Holling II response functional, we have set $\gamma_1 = 1500$, $\gamma_2 = 5$, whereas in the Holling III response functional we have set $\gamma_1 = 80$, $\gamma_2 = 0.3$.

The initial conditions are set as:

- $i_1(x, 0) = 20 \exp(-100(x_1 - 3.8)^2 - 100(x_2 - 0.4)^2)$;
- $s_1(x, 0) = 100 - i_1(x, 0)$;
- $i_2(x, 0) = 0$;
- $s_2(x, 0) = 50$;
- $Z(x, 0) = 0$.

Coherently with our mathematical analysis, in both cases A1 and A2, in which the level of weed biomass has been kept sufficiently small ($M=0.4$), the epidemic is stopped, independently of the specificity of the predators, either specialist or generalist, i.e. either Holling II or Holling III response functional, respectively. See Fig. 1 for the case A1, while the case A2 is not reported because it is analogous to case A1.

In cases B11 and B12, Holling II has been adopted as predation response functional. They differ only by the level of predation inflow u : smaller in case B11 with respect to case B12. In case B11 the epidemic propagates, see Fig. 2; in case B12 the epidemic is stopped (not shown, analogous to case A1, see Fig. 1).

This result highlights the fact that even in the case of predators with a narrow range of possible preys, or in the case of parasitoids, the amount of released control agents represents a crucial aspect that has to be taken into account when planning a control strategy based on their release in the environment. Here it is confirmed that a release of control agents below a certain threshold frustrates the possibility of success of the control strategy.

Given accurate experimental values of all parameters of the mathematical model of the relevant ecosystem, a suitable iteration of the computational experiments B11 and B12 may lead to the identification of the above mentioned threshold level for u ; it is clear that such threshold value will depend upon the selected region of intervention ω , too.

In last case B22, Holling III has been adopted as predation response functional: even for a “large” predator inflow u the epidemic persists, see Fig. 3.

This result represents clear evidence, based on the model presented here, that the use of a generalist predator is not effective in the control of a *Xylella* vector.

5. CONCLUDING REMARKS

Z. renardii has been recently suggested as a possible biocontrol agent of the *X. fastidiosa* vector, *P. spumarius*. Besides *Z. renardii* being a non-native European species, it was detected for the first time in Europe in 2011 [11], its use as a control agent could be questioned for different reasons, mainly: i. *Z. renardii* has been reported as generalist predator; ii. its impact on biocoenoses of the both native and invaded range is almost unknown; iii. to our knowledge, no mass-rearing facilities are currently present to support a massive rearing of this predator. Besides the previously reported critical aspects that must be considered for using *Z. renardii* as biocontrol

agent of *P. spumarius*, in the present work we have mathematically demonstrated that the use of this generalist predator for the control of *P. spumarius* cannot be considered as a viable solution to a *Xylella* epidemic since it will not be able to reach the expected results either in containing or in eradicating the epidemics. The control and the eventual eradication of the epidemics can be reached in the case that specialist predators, possessing a narrow range of hosts, or parasitoids will be released in the relevant habitat. Even in this study, as previously reported in [7], the reduction of the weed biomass of the relevant habitat is confirmed as the most effective agronomical practice, even in terms of costs, for controlling the *Xylella* epidemic in olive orchards.

Concerning the actual applicability of our results, we caution that they are far from being conclusive for *X. fastidiosa* subsp. *pauca* - *P. spumarius* - olive tree epidemics.

We may adopt here the following statement taken from [3], based on a recent paper ([16]):

“a model is only an approximate interpretation of reality and it is always wrong in some small or relevant elements. The destiny of the model presented here is to be rapidly improved thanks to novel knowledge coming from new observations and better assumptions. The Authors hope that many and more brilliant minds will read the present pages, will identify and highlight putative mistakes, will get inspiration for their research, and will produce better, more complete, and useful models..... If the speculations presented here on implications for surveillance, control, and therapy of [*Xylella*] will contribute, even only minimally, to save [olive trees]....., then the Authors have accomplished their small mission. ”

In particular, we hope that our results may drive the implementation of additional experiments aimed at the concrete evaluation of the model parameters.

It is desirable that, with additional features that make it more realistic, our model might provide the foundations for designing optimal control strategies for public decision makers.

Future investigations will be devoted to the solution of suitable optimal control problems, including the search of an optimal subregion of intervention ω .

6. APPENDIX

Here $\Omega \subset \mathbb{R}^2$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Denote $Q = \Omega \times (0, +\infty)$, $\Sigma = \partial\Omega \times (0, +\infty)$.

In this paper we deal with several reaction-diffusion systems of the following form

$$\begin{cases} \frac{\partial v_1}{\partial t}(x, t) = c_1 \Delta v_1(x, t) + f_1(x, v_1(x, t), v_2(x, t)) + \tilde{f}_1(x, t), & (x, t) \in Q \\ \frac{\partial v_2}{\partial t}(x, t) = c_2 \Delta v_2(x, t) + f_2(x, v_1(x, t), v_2(x, t)) + \tilde{f}_2(x, t), & (x, t) \in Q \\ \frac{\partial v_1}{\partial n}(x, t) = \frac{\partial v_2}{\partial n}(x, t) = 0, & (x, t) \in \Sigma \\ v_1(x, 0) = v_{10}(x), v_2(x, 0) = v_{20}(x), & x \in \Omega, \end{cases} \quad (17)$$

where c_1, c_2 are positive constants, $f_1, f_2 : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $\tilde{f}_1, \tilde{f}_2 : \Omega \times [0, +\infty) \rightarrow \mathbb{R}$.

Definition 1. We say that (v_1, v_2) is a (weak) solution to (17) if $v_1, v_2 \in C([0, T]; L^2(\Omega)) \cap AC((0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap L^2_{loc}([0, T]; H^2(\Omega))$ for any $T \in (0, +\infty)$ and if

$$\begin{cases} \frac{\partial v_1}{\partial t}(x, t) = c_1 \Delta v_1(x, t) + f_1(x, v_1(x, t), v_2(x, t)) + \tilde{f}_1(x, t), & \text{a.e. } (x, t) \in Q \\ \frac{\partial v_2}{\partial t}(x, t) = c_2 \Delta v_2(x, t) + f_2(x, v_1(x, t), v_2(x, t)) + \tilde{f}_2(x, t), & \text{a.e. } (x, t) \in Q \\ \frac{\partial v_1}{\partial n}(x, t) = \frac{\partial v_2}{\partial n}(x, t) = 0, & \text{a.e. } (x, t) \in \Sigma \\ v_1(x, 0) = v_{10}(x), v_2(x, 0) = v_{20}(x), & \text{a.e. } x \in \Omega. \end{cases}$$

The existence and uniqueness of a solution with nonnegative components (v_1 and v_2) for any of the reaction-diffusion systems (including the semilinear parabolic equations) in the previous sections follow by using Banach's fixed point theorem and the next comparison result for the solutions of the linear parabolic equations.

Theorem 6. (*Comparison result for parabolic equation*). Let w_i ($i \in \{1, 2\}$) be nonnegative (weak) solutions to

$$\begin{cases} \frac{\partial w_i}{\partial t}(x, t) = c_0 \Delta w_i(x, t) + F(x, w_i(x, t)) + G_i(x, t), & (x, t) \in Q \\ \frac{\partial w_i}{\partial n}(x, t) = 0, & (x, t) \in \Sigma \\ w_i(x, 0) = w_{0i}(x), & x \in \Omega, \end{cases}$$

respectively, where $c_0 \in (0, +\infty)$ and $F \in L^2_{loc}(\bar{Q})$ satisfies that there exists $\theta \in \mathbb{R}$ such that for almost any $x \in \Omega$:

$$F(x, \cdot) \in C^1([0, +\infty)), \quad \frac{\partial F}{\partial w}(x, w) \leq \theta \quad \text{a.e. } (x, w) \in Q.$$

If $w_{01}, w_{02} \in L^2(\Omega)$,

$$w_{01}(x) \leq w_{02}(x) \quad \text{a.e. } x \in \Omega,$$

and $G_1, G_2 \in L^2_{loc}(\bar{Q})$,

$$G_1(x, t) \leq G_2(x, t) \quad \text{a.e. } (x, t) \in Q,$$

then

$$w_1(x, t) \leq w_2(x, t) \quad \text{a.e. } x \in \Omega, \quad \forall t \geq 0.$$

Proof. Function $w = w_2 - w_1$ is a solution to

$$\begin{cases} \frac{\partial w}{\partial t}(x, t) = c_0 \Delta w(x, t) + (F(x, w_2(x, t)) - F(x, w_1(x, t)) + G_2(x, t) - G_1(x, t)), & (x, t) \in Q \\ \frac{\partial w}{\partial n}(x, t) = 0, & (x, t) \in \Sigma \\ w(x, 0) = w_{02}(x) - w_{01}(x), & x \in \Omega. \end{cases} \quad (18)$$

If we multiply (18) by w^- and integrate over $(0, t) \times \Omega$ we get that

$$\begin{aligned} -\frac{1}{2} \int_{\Omega} |w^-(x, t)|^2 dx + \frac{1}{2} \int_{\Omega} |w^-(x, 0)|^2 dx &\geq c_0 \int_0^t \int_{\Omega} |\nabla w^-(x, s)|^2 dx ds \\ &+ \int_0^t \int_{\Omega} (F(x, w_2(x, t)) - F(x, w_1(x, t))) w^-(x, s) dx ds \\ &\geq \int_0^t \int_{\Omega} \frac{\partial F}{\partial w}(x, \xi(x, s)) w(x, s) w^-(x, s) dx ds \geq -\theta \int_0^t \int_{\Omega} |w^-(x, s)|^2 dx ds, \end{aligned}$$

for any $t \geq 0$. We have applied the mean value theorem. It follows that for any $t \geq 0$ we have that

$$\int_{\Omega} |w^-(x, t)|^2 dx \leq 2\theta \int_0^t \int_{\Omega} |w^-(x, s)|^2 dx ds,$$

and by Gronwall's inequality we get that

$$\int_{\Omega} |w^-(x, t)|^2 dx = 0.$$

We may infer that

$$w^-(x, t) = 0 \quad \text{a.e. } x \in \Omega, \quad \forall t \geq 0,$$

and get the conclusion. \square

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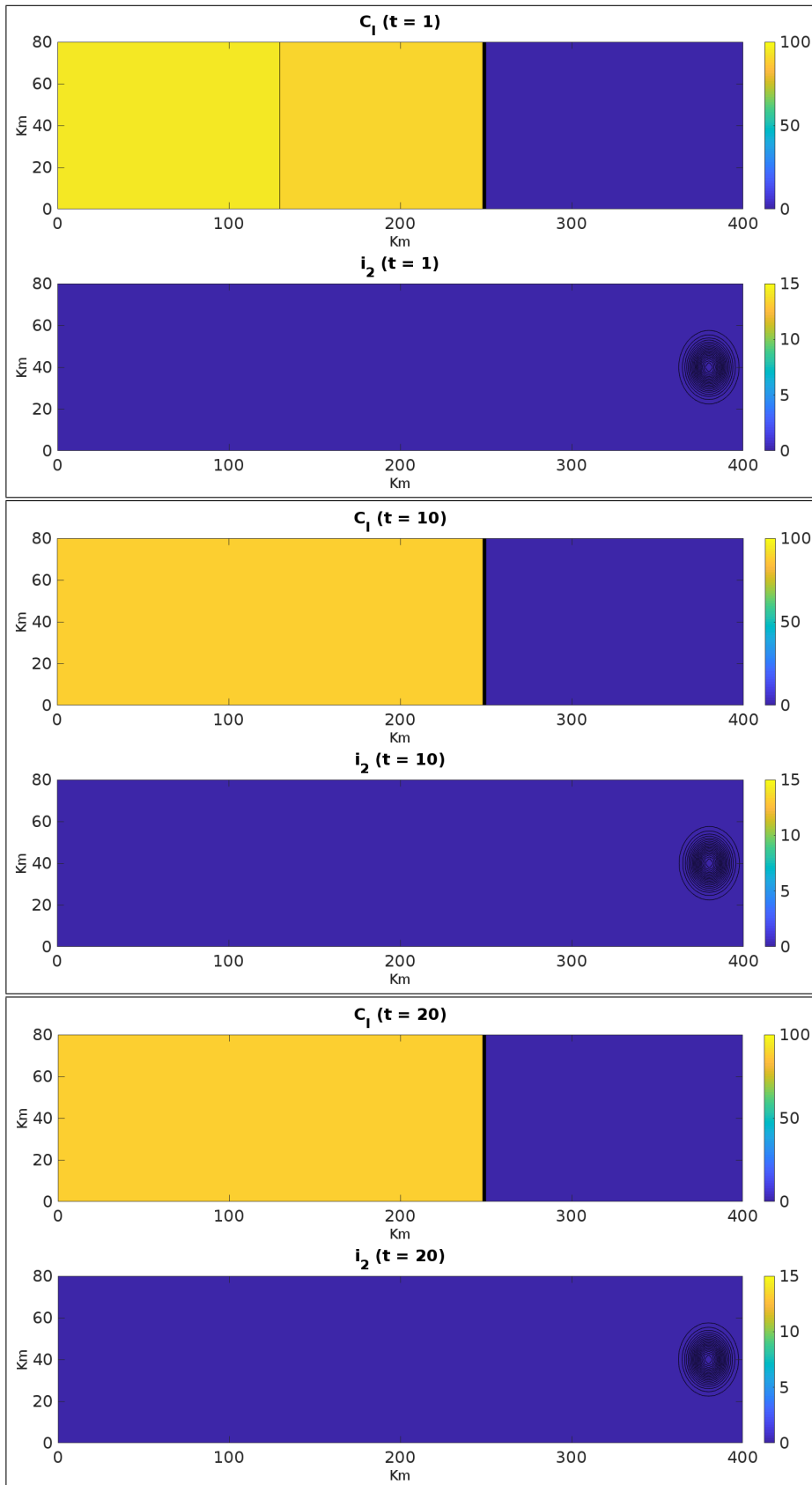


FIGURE 1. Case A1. Spatial distribution of the total insect population C_I , and of the infected trees i_2 , at three time frames. The epidemic is stopped.

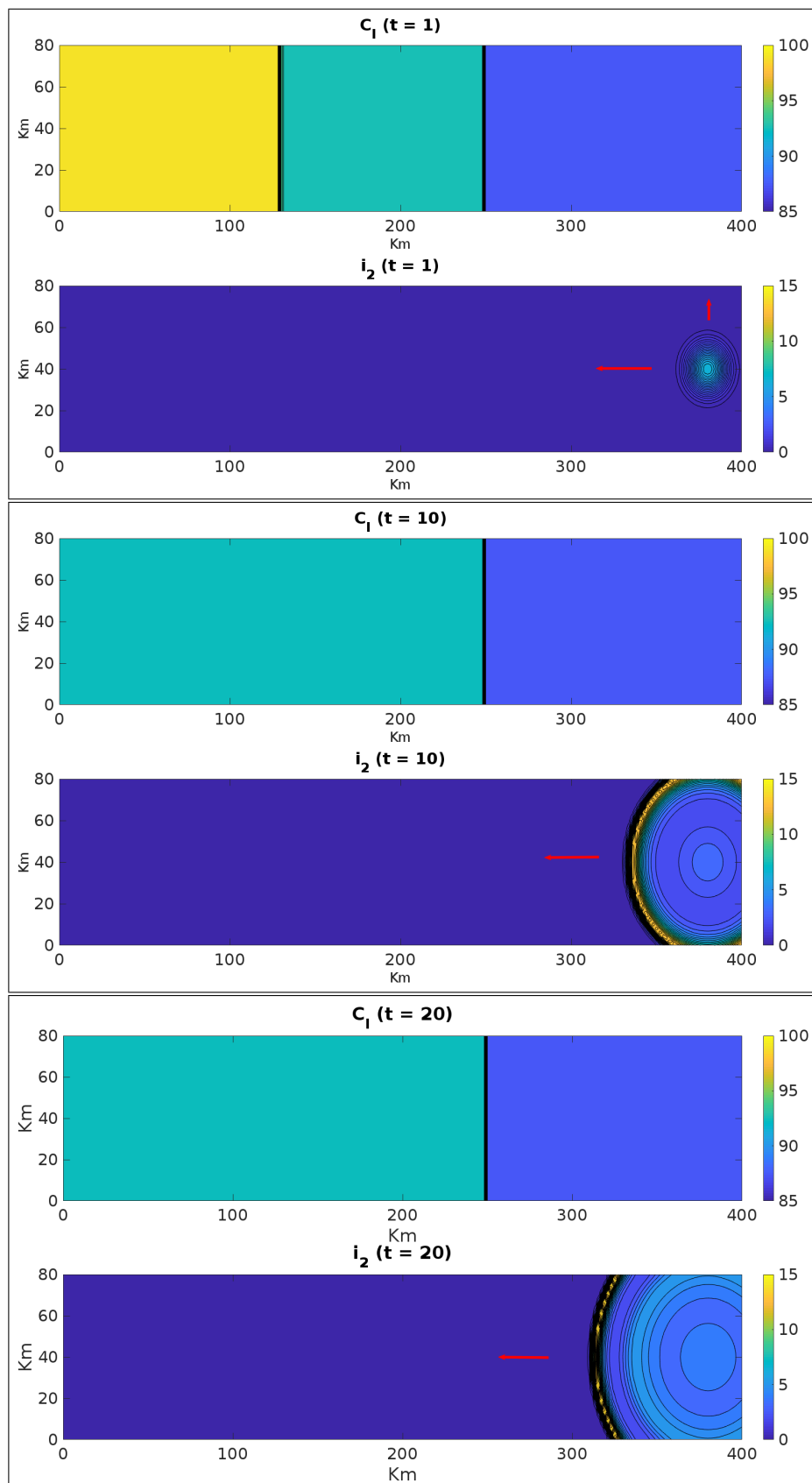


FIGURE 2. Case B11. Spatial distribution of the total insect population C_I , and of the infected trees i_2 , at three time frames. The epidemic propagates along the directions of the red arrows.

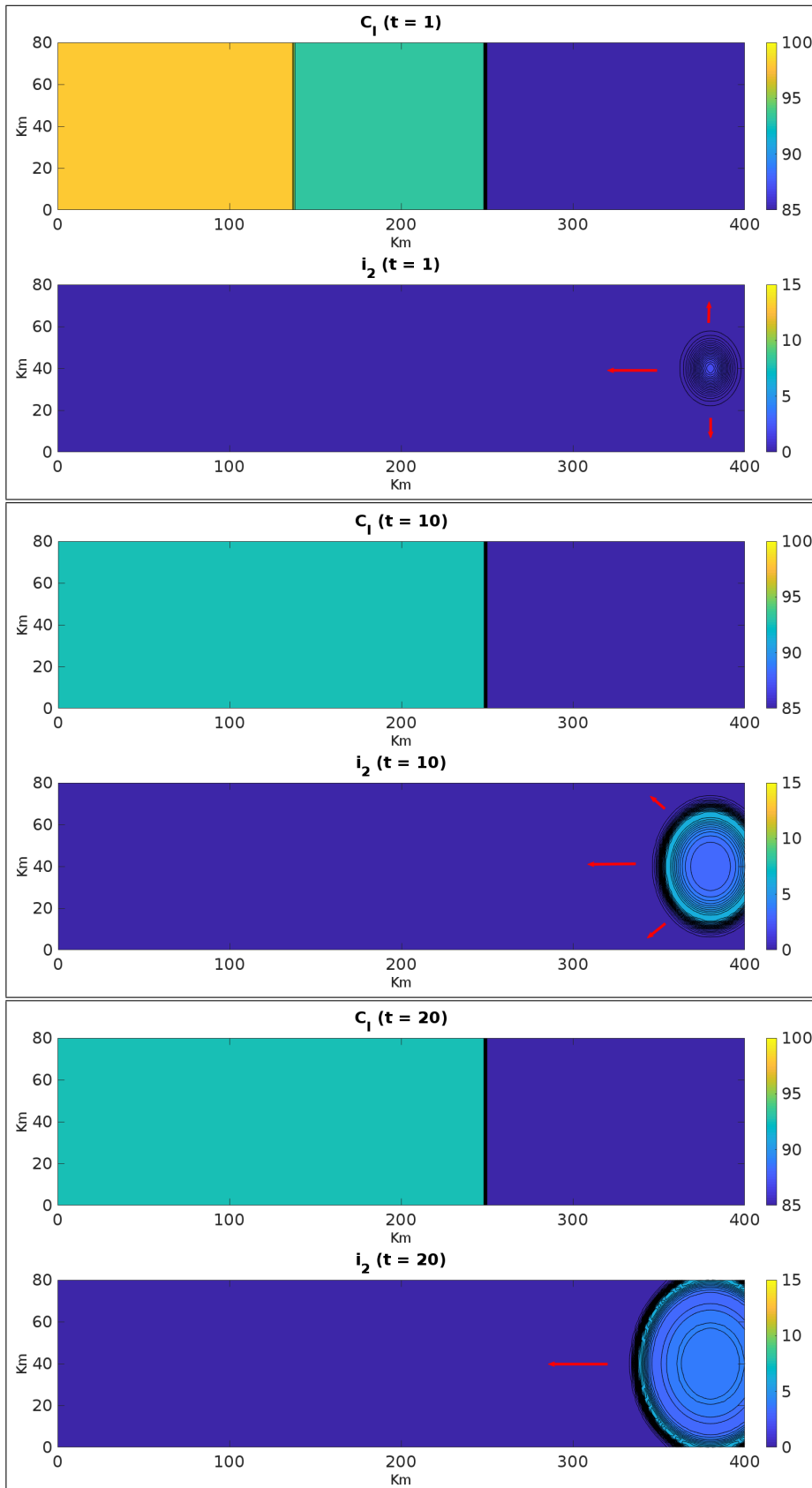


FIGURE 3. Case B22. Spatial distribution of the total insect population C_I , and of the infected trees i_2 , at three time frames. The epidemic propagates along the directions of the red arrows.